Linear independence and coherence of Gabor systems in finite dimensional spaces

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Abstract:

This paper reviews recent results on the geometry of Gabor systems in finite dimensions. For example, we discuss the coherence of Gabor systems, the linear independence of subsets of Gabor systems, and the condition number of matrices formed by a small number of vectors from a Gabor system. We state a result on the recovery of signals that have a sparse representation in certain Gabor systems. The results listed here are obtained by the author in collaborations with Jim Lawrence, Felix Krahmer, Peter Rashkov, Jared Tanner, Holger Rauhut, and David Walnut linear independence

1. Introduction and Notation

The theory of Gabor systems in the Hilbert space of square integrable functions on the real line has received significant attention during the last ten to twenty years (see, for example, [4, 6, 8, 7] and references within). Much of the research concentrates on showing that certain Gabor systems are frames or Riesz bases for their closed linear span. The seemingly simpler concept of linear independence of vectors in a Gabor system was addressed in [10]. There, it was conjectured that any finite set of time–frequency shifted copies of a single square integrable function is linear independent. This conjecture still remains to be resolved.

In the last years, in part due to the emergence of the theory of compressed sensing and sparse signal recovery, the structure of Gabor systems in finite dimensional spaces has received increased attention. Such finite Gabor systems on finite Abelian groups are described below.

We let G denote a finite Abelian group. Its dual group $\widehat{G}$ consists of the group homomorphisms $\xi : G \mapsto S^1$. We have $\widehat{G} \subseteq \mathbb{C}^G = \{f : G \longrightarrow \mathbb{C}\}$, the latter being the space of complex valued functions on G. The support size of $f \in \mathbb{C}^G$ is $\|f\|_0 := |\{x : f(x) \neq 0\}|$. The Fourier transform of $f \in \mathbb{C}^G$ is normalized to be $\widehat{f}(\xi) = \sum_{x\in G} f(x)\,\overline{\xi(x)}$, $\xi \in \widehat{G}$.

Translation operators T_x, $x \in G$, and modulation operators M_ξ, $\xi \in \widehat{G}$, on $\mathbb{C}^G$ are unitary operators given by $(T_x f)(t) = f(t-x)$ and $(M_\xi f)(t) = f(t)\cdot\xi(t)$. Time-frequency shift operators $\pi(\lambda)$, $\lambda = (x,\xi) \in G\times\widehat{G}$, are the unitary operator on $\mathbb{C}^G$ represented by $\pi(\lambda)f = T_x \circ M_\xi f$, $\lambda = (x,\xi) \in G\times\widehat{G}$.

The system $\{\pi(\lambda)g : \lambda \in G\times\widehat{G}\} \subseteq \mathbb{C}^G$ is called (full) Gabor system with window $g \in \mathbb{C}^G$, it consists of $|G|^2$ vectors in a $|G|$ dimensional space.

The short-time Fourier transform with respect to g is given by

$$V_g f(\lambda) = \langle f, \pi(\lambda)g\rangle = \sum_{y\in G} f(y)\overline{g(y-x)\xi(y)}, \quad f \in \mathbb{C}^G, \lambda = (x,\xi) \in G\times\widehat{G}.$$

We shall not make a distinction between the linear mapping $V_g : \mathbb{C}^G \longrightarrow \mathbb{C}^{G\times\widehat{G}}$ and its matrix representation with respect to the Euclidean basis.

Full Gabor systems in finite dimensions share an important and very useful property: for any $g \neq 0$, the collection $\{\pi(\lambda)g\}_{\lambda\in G\times\widehat{G}}$ forms a uniform tight finite frame for $\mathbb{C}^G$ with frame bound $n^2\,\|g\|^2$, that is,

$$\sum_{\lambda\in G\times\widehat{G}} |\langle f, \pi(\lambda)g\rangle|^2 = n^2\,\|g\|^2\|f\|^2.$$

This is a simple consequence of the representation theory of the Weyl–Heisenberg group [9, 12].

In this paper we are concerned with properties of subsets of full Gabor systems. In Section 2, we consider the linear independence of subsets of $|G|$ elements of $\{\pi(\lambda)g\}_{\lambda\in G\times\widehat{G}}$. Recall that a finite set of vectors in $\mathbb{C}^G$ is in general linear position if any subset of at most $|G|$ of these vectors are linearly independent. While being a classical concept in mathematics, it is also relevant for communications, namely, for information transmission through a so-called erasure channel [2]. In fact, a frame $\mathcal{F} = \{x_k\}_{k=1}^m$ in $\mathbb{C}^n$ is called maximally robust to erasures if the removal of any $l \leq m-n$ vectors from $\mathcal{F}$ leaves a frame.

Moreover, we consider the coherence of Gabor systems in Section 3. We state probabilistic estimates of the coherence of a full Gabor system with respect to a randomly generated window. In Section 4, we consider the condition number of matrices formed by a small subset of a Gabor system.

The results presented below were obtained over the last few years in collaboration with Jim Lawrence and David Walnut [12], Felix Krahmer and Peter Rashkov [11], and Holger Rauhut and Jared Tanner [14, 13].

2. Gabor systems in general linear position

The following simple observations illustrate the usefulness of Gabor systems which are in general linear position.

Proposition 1 *[11, 12] For $g \in \mathbb{C}^G \setminus \{0\}$, the following are equivalent:*

1. *$\{\pi(\lambda)g\}_{\lambda \in G\times\widehat{G}}$ are in general linear position.*
2. *For all $f \in \mathbb{C}^G \setminus \{0\}$ we have $\|V_g f\| \geq |G|^2 - |G| + 1$.*
3. *For all $f \in \mathbb{C}^G$, $V_g f$ is completely determined by its values on any set Λ with $|\Lambda| = n$.*
4. *$\{\pi(\lambda)g\}_{\lambda \in G\times\widehat{G}}$ is maximally robust to erasures.*
5. *The $|G| \times |G|^2$ matrix V_g has the property that every minor of order n is nonzero.*

Corollary 2 *[12] If $\{\pi(\lambda)g\}_{\lambda \in G\times\widehat{G}}$ are in general linear position, then $\|g\|_0 = |G|$ and $\|\widehat{g}\|_0 = |G|$.*

Unfortunately, not each finite Abelian groups G permits the existence of a vector $g \in \mathbb{C}^G$ satisfying one and therefore all conditions listed in Proposition 1. For example, for the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, no such g exists [11]. The situation is different for $G = \mathbb{Z}_p$. Recall that E is of full measure if the Lebesgue measure of $\mathbb{C}^G \setminus E$ is 0.

Theorem 3 *[12] If $|G|$ is prime, that is, $G = \mathbb{Z}_p$, p prime, then there is a dense open set E of full measure in $\mathbb{C}^G$ such that for every $g \in E$, the elements of the full Gabor system $\{\pi(\lambda)g\}_{\lambda \in G\times\widehat{G}}$ are in general linear position. That is, for almost all g we have $\|V_g f\| \geq |G|^2 - |G| + 1$ for all $f \neq 0$.*

Rudimentary numerical experiments encourage us to ask the following question.

Question 4 *[12] For G cyclic, that is, $G = \mathbb{Z}_n$, $n \in \mathbb{N}$, exists $g \in \mathbb{C}^G$ so that the conclusions of Proposition 1, and, therefore, $\|V_g f\| \geq |G|^2 - |G| + 1$, $f \in \mathbb{C}^G$, hold*

In fact, for $|G|$ prime, Theorem 3 can be strengthened.

Theorem 5 *[11] Let $G = \mathbb{Z}_p$, p prime. For almost every $g \in \mathbb{C}^G$, we have*

$$\|V_g f\|_0 \geq |G|^2 - \|f\|_0 + 1 \tag{1}$$

for all $f \in \mathbb{C}^G \setminus \{0\}$. Moreover, for $1 \leq k \leq |G|$ and $1 \leq l \leq |G|^2$ with $k + l \geq |G|^2 + 1$ there exists f with $\|f\|_0 = k$ and $\|V_g f\|_0 = l$.

Proposition 6 *[11] If $|G|$ is not prime, then V_g has zero minors for all $g \in \mathbb{C}^G$. Hence, there is no $g \in \mathbb{C}^G$ such that (1) holds for all $f \in \mathbb{C}^G$.*

Numerical experiments for Abelian groups of order less than or equal to 8, as well as our result for all cyclic groups of prime order, indicate that the following question might have an affirmative answer.

Question 7 *[11] For every cyclic group G and almost every $g \in \mathbb{C}^G$, does*

$$\begin{aligned}&\big\{(\|f\|_0, \|V_g f\|_0),\ f \in \mathbb{C}^G \setminus \{0\}\big\}\\ &= \big\{(\,\|f\|_0\,,\, \|\widehat{f}\|_0 + |G|^2 - |G|\,),\ f \in \mathbb{C}^G \setminus \{0\}\big\}\end{aligned}$$

hold?

The following result improves on Theorem 5. It allows for the construction of Gabor based equal norm tight frames of p^2 elements in $\mathbb{C}^n$, $n \leq p$. To our knowledge, the only previously known equal norm tight frames that are maximally robust to erasures are so-called harmonic frames (see Conclusions in [2]).

Proposition 8 *[11] There exists a unimodular $g \in \mathbb{C}^{\mathbb{Z}_p}$, p prime, that is, a g with $|g(x)| = 1$ for all $x \in G$ satisfying the conclusions of Theorem 5.*

To construct an equal norm tight frame, we choose a $g \in (S^1)^p$ satisfying the conclusions of Proposition 8. We remove $p - n$ components of the equal norm tight frame $\{\pi(\lambda)g\}_{\lambda \in G\times\widehat{G}}$. The resulting frame remains an equal norm tight frame which is maximally robust to erasure. Note that this frame is not a Gabor frame proper. Reducing the number of vectors in the frame to $m \leq p^2$ vectors leaves an equal norm frame which is maximally robust to erasure but which might not be tight. With the restriction to frames with p^2 elements, p prime, we have shown the existence of Gabor frames which share the usefulness of harmonic frames when it comes to transmission of information through erasure channels.

Background and more details on frames and erasures can be found in [2, 15] and the references cited therein.

Note that Theorem 5 has as direct consequence

Theorem 9 *[11] Let $g \in \mathbb{C}^{\mathbb{Z}_p}$, p prime, satisfy the conclusion of Theorem 5. Then any $f \in \mathbb{C}^{\mathbb{Z}_p}$ with $\|f\|_0 \leq \frac{1}{2}|\Lambda|$, $\Lambda \subset \mathbb{Z}_p \times \widehat{\mathbb{Z}_p}$, is uniquely determined by Λ and $r_\Lambda V_g f$.*

Here, only the support size of f is known. No additional information on the support of f is required to determine f.

In terms of sparse representations, we consider the question whether any vector $f = \sum\limits_{\lambda \in \Lambda} c_\lambda \pi(\lambda) g$ can be determined by a few entries of f in case that $|\Lambda|$ is small.

Theorem 10 *[11] Let $g \in \mathbb{C}^{\mathbb{Z}_p}$, p prime, satisfy the conclusion of Theorem 5. Then any $f \in \mathbb{C}^{\mathbb{Z}_p}$ with $f = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda) g$, $\Lambda \subset \mathbb{Z}_p \times \widehat{\mathbb{Z}_p}$ is uniquely determined by B and $r_B f$ whenever $|B| \geq 2|\Lambda|$.*

Note that similar to before, the efficient recovery of f from $2|\Lambda|$ samples of f in Theorem 10 does not require knowledge of Λ.

The question asking how to recover f from a small number of entries of f efficiently will be briefly addressed with Theorem 14

3. Coherence of Gabor systems

In the following we restrict our attention to cyclic groups $G = \mathbb{Z}_n, n \in \mathbb{N}$. We consider the so-called Alltop window h^A [15] with entries

$$h^A(x) \;=\; \frac{1}{\sqrt{n}} e^{2\pi i x^3/n}, \quad x = 0, \ldots, n-1, \qquad (2)$$

and the randomly generated window h^R with entries

$$h^R(x) \;=\; \frac{1}{\sqrt{n}} \epsilon_x, \quad x = 0, \ldots, n-1, \qquad (3)$$

where the ϵ_x are independent and uniformly distributed on the torus $\{z \in \mathbb{C}, |z| = 1\}$.
For $\|h\|_2 = 1$, the coherence of a full Gabor systems is

$$\mu \;=\; \max_{(\ell,p)\neq(\ell',p')} |\langle M_\ell T_p h, M_{\ell'} T_{p'} h\rangle|. \qquad (4)$$

In [16] it is shown that the coherence of $\{\pi(\lambda)h^A : \lambda \in \mathbb{Z}_n \times \widehat{\mathbb{Z}}_n\} \subseteq \mathbb{C}^n$ given in (2) satisfies

$$\mu \;=\; \frac{1}{\sqrt{n}} \qquad (5)$$

for n prime. This is close to optimal since as the lower bound for the coherence of frames with n^2 elements in $\mathbb{C}^n$ is $\mu \geq \frac{1}{\sqrt{n+1}}$ [16].
Unfortunately, the coherence (4) of h^A applies only for n prime. For arbitrary n we now consider the random window h^R.

Theorem 11 *[14] Let $n \in \mathbb{N}$ and choose a random window h^R with entries*

$$h^R(x) \;=\; \frac{1}{\sqrt{n}} \epsilon_x, \quad x = 0, \ldots, n-1,$$

where the ϵ_x are independent and uniformly distributed on the torus $\{z \in \mathbb{C}, |z| = 1\}$. Let μ be the coherence of the associated Gabor dictionary (4), then for $\alpha > 0$ and n even,

$$\mathbb{P}\big(\mu \geq \frac{\alpha}{\sqrt{n}}\big) \leq 4n(n-1)e^{-\alpha^2/4},$$

while for n odd,

$$\mathbb{P}\big(\mu \geq \frac{\alpha}{\sqrt{n}}\big) \leq 2n(n-1)\left(e^{-\frac{n-1}{n}\alpha^2/4} + e^{-\frac{n+1}{n}\alpha^2/4}\right). \qquad (6)$$

Up to the constant factor α, the coherence in Theorem 11 comes close to the lower bound $\mu \geq \frac{1}{\sqrt{n+1}}$ with high probability. (The probability depends on α).

4. Conditioning of submatrices of V_g

For applications such as sparse signal recovery, not only linear independence of subsets of Gabor systems is required. It is rather needed, that small subsets of Gabor systems form well-conditioned matrices.
Throughout this section, we let $\Psi = V_g \in \mathbb{C}^{n\times n^2}$ with $g = h^R$ being the randomly generated unimodular window described in (3). For $\Lambda \subseteq G\times\widehat{G}$ we denote by Ψ_Λ the matrix consisting only of those columns indexed by $\lambda \in \Lambda$.

Theorem 12 *[13] Let $\varepsilon, \delta \in (0,1)$ and $|\Lambda| = S$. Suppose that*

$$S \leq \frac{\delta^2 n}{4e(\log(S/\varepsilon) + c)} \qquad (7)$$

with $c = \log(e^2/(4(e-1))) \approx 0.0724$. Then $\|I_\Lambda - \Psi_\Lambda^* \Psi_\Lambda\| \leq \delta$ with probability at least $1-\varepsilon$; in other words the minimal and maximal eigenvalues of $\Psi_\Lambda^*\Psi_\Lambda$ satisfy $1-\delta \leq \lambda_{\min} \leq \lambda_{\max} \leq 1+\delta$ with probability at least $1-\varepsilon$.

Remark 13 *[13] Assuming equality in condition (7) and solving for ε we deduce*

$$\begin{aligned}\mathbb{P}\big(\|I_\Lambda - \Psi_\Lambda^*\Psi_\Lambda\| > \delta\big) \;&\leq\; \frac{e^2}{4(e-1)} S \exp\left(-\frac{\delta^2 n}{4eS}\right)\\ &=\; CS\exp\left(-\frac{\delta^2 n}{4eS}\right)\end{aligned}$$

with $C \approx 1.075$.

Theorem 12 allows us to guarantee the successful use of efficient algorithms to determine $f = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g$ from a few entries of f in case that $|\Lambda|$ is small. Here, we will concentrate on algorithms based on Basis Pursuit. Basis Pursuit seeks the solution of the convex problem

$$\min_x \|x\|_1 \quad \text{subject to } \Psi_g x = y, \qquad (8)$$

where $\|x\|_1 = \sum_{\lambda\in\mathbb{Z}_n^2} |x_\lambda|$ is the ℓ_1-norm of x. Efficient convex optimization techniques for Basis Pursuit can be found in [1, 3, 5].

Theorem 14 *[13] Assume x is an arbitrary S-sparse coefficient vector. Choose the random unimodular Gabor window $g = h^R$ defined in (3), that is, with random entries independently and uniformly distributed on the torus $\{z \in \mathbb{C}, |z| = 1\}$. Assume that*

$$S \leq C\frac{n}{\log(n/\varepsilon)} \qquad (9)$$

for some constant C. Then with probability at least $1-\varepsilon$ Basis Pursuit (8) recovers x from $y = \Psi x = \Psi_g x$.

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