Gabor frames in finite dimensions

Götz E. Pfander

Abstract Gabor frames have been extensively studied in time-frequency analysis over the last 30 years. They are commonly used in science and engineering to synthesize signals from, or to decompose signals into, building blocks which are localized in time and frequency. This chapter contains a basic and self-contained introduction to Gabor frames on finite-dimensional complex vector spaces. In this setting, we give elementary proofs of the central results on Gabor frames in the greatest possible generality; that is, we consider Gabor frames corresponding to lattices in arbitrary finite Abelian groups. In the second half of this chapter, we review recent results on the geometry of Gabor systems in finite dimensions: the linear independence of subsets of its members, their mutual coherence, and the restricted isometry property for such systems. We apply these results to the recovery of sparse signals, and discuss open questions on the geometry of finite-dimensional Gabor systems.

Key words: Gabor analysis on finite Abelian groups; linear independence, coherence, restricted isometry constants of Gabor frames; applications to compressed sensing, erasure channel error correction, channel identification.

1 Introduction

In his seminal 1946 paper "Theory of Communication", Dennis Gabor suggested the decomposition of the time-frequency *information area* of a communications channel into the smallest possible boxes that allow exactly one information-carrying coefficient to be transmitted per box [40]. He refers to Heisenberg's uncertainty principle to argue that the smallest time-frequency boxes are achieved using time-frequency shifted copies of *probability functions*, that is, of Gaussians. In summary, he pro-

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Götz E. Pfander
School of Engineering and Science, Jacobs University, 28759 Bremen, Germany, e-mail: g.pfander@jacobs-university.de.

poses transmitting the information-carrying complex-valued sequence $\{c_{nk}\}$ in the form of the signal

$$\psi(t) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{nk}\, e^{-\pi \frac{(t-n\Delta t)^2}{2(\Delta t)^2}}\, e^{2\pi i \frac{kt}{\Delta t}},$$

where the parameter $\Delta t > 0$ can be chosen depending on physical consideration and the application at hand. Denoting *modulation operators* by

$$M_\nu g(t) = e^{2\pi i \nu t} g(t), \quad \nu \in \mathbb{R},$$

and translation operators by

$$T_\tau g(t) = g(t-\tau), \quad \tau \in \mathbb{R},$$

Gabor proposed to transmit on the carriers $\{M_{k/\Delta t} T_{n\Delta t} g_0\}_{n,k\in\mathbb{Z}}$ where g_0 is the *Gaussian window function* $g_0(t) = e^{-\pi \frac{t^2}{2(\Delta t)^2}}$.

In the second half of the 20th century, the suggestion of Gabor, and in general the interplay of information density in time and in frequency, was studied extensively; see, for example, [23, 24, 32, 37, 59, 60, 61, 85]. This line of work focuses on functional analytic properties of function systems such as the ones suggested by Gabor. (Apart from these historical remarks, functional analysis will not play a role throughout this chapter.) Janssen, for instance, analyzed in detail in which sense $\{M_{k/\Delta t} T_{n\Delta t} g_0\}_{n,k\in\mathbb{Z}}$ can be used to represent functions and distributions.[1] He showed that while being complete in the Hilbert space of square integrable functions on the real line, the set suggested by Gabor is not a Riesz basis for this space [52]. Balian and Low then established independently from one another that any function φ which is *well concentrated* in time and in frequency does not give rise to a Riesz basis of the form $\{M_{k/\Delta t} T_{n\Delta t} \varphi\}_{n,k\in\mathbb{Z}}$ [5, 10, 11, 64]. This apparent failure of systems structured as suggested by Gabor was then rectified by resorting to the concept of frames that had been introduced by Duffin and Shaffer [29]. Indeed, $\{M_{k\Delta\nu} T_{n\Delta t} g_0\}_{n,k\in\mathbb{Z}}$ is a frame if $\Delta\nu < 1/\Delta t$ [65, 82, 83]. Since then the theory of Gabor systems has been intimately related to the theory of frames and many problems in frame theory find their origins in Gabor analysis. For example, the Feichtinger conjecture (see Section X.X and references therein), and so-called localized frames were first considered in the realm of Gabor frames [3, 4, 19, 47].

In engineering, Gabor's idea flourished over the last decade due to the increasing use of *orthogonal frequency division multiplexing* (OFDM) structured communication systems. Indeed, the carriers used in OFDM are $\{M_{k\Delta\nu} T_{n\Delta t} \varphi_0\}_{n\in\mathbb{Z},k\in K}$ where φ_0 is the characteristic function $\chi_{[0,1/\Delta\nu]}$ (or a mollified and/or cyclically extended copy thereof) and $K = \{-K_2, -K_2+1, \ldots, -K_1,\ K_1+1, \ldots K_2\}$ is introduced to respect transmission band limitations.

[1] Prior to the work of Gabor, von Neumann postulated that the function family which is now referred to as *Gaussian Gabor* system is complete [68] (see the respective discussions in [45, 48]).

While originally constructed on the real line, Gabor systems can be analogously defined on any locally compact Abelian group [20, 33, 36, 44]. Functions on finite Abelian groups form finite-dimensional vector spaces; hence, Gabor systems on finite groups have been studied first in the realm of numerical linear algebra. In particular, efficient matrix factorizations for the Gabor analysis, the Gabor synthesis, and the Gabor frame operator are discussed in the literature; see, for example, [6, 76, 77, 88].

Gabor systems on finite cyclic groups have also been studied numerically in order to better understand properties of Gabor systems on the real line. The relationship between Gabor systems on the real line, on the integers, and on cyclic groups are studied based on sampling and periodization arguments in [54, 55, 69, 86, 87].

Over the last two decades it became apparent that the structure of Gabor frames on finite Abelian groups allows for the construction of finite frames with remarkable geometric properties. Most noteworthy may be the fact that many equiangular frames have been constructed as Gabor frames (for references and details, see Section X.X.) Also, finite Gabor systems have been considered in the study of *constant amplitude zero autocorrelation* (CAZAC) sequences [8, 9, 42, 84] and to construct spreading sequences and error-correcting codes in radar and communications [50].

This chapter serves multiple purposes. In Sections 2 and 3 we give an elementary introduction to Gabor analysis on $\mathbb{C}^N$. Section 2 focuses on basic definitions and in Section 3 we describe the fundamental ideas that make Gabor frames useful to analyze or synthesize signals with varying frequency components.

In Section 4, we define and discuss Gabor frames on finite Abelian groups. The case of Gabor frames on general finite Abelian groups is only more technically involved than the setup chosen in Section 2. This is due to the fundamental theorem of finite Abelian groups: it states that every finite Abelian group is isomorphic to the product of cyclic groups.

We prove fundamental results for Gabor frames on finite Abelian groups in Section 5. The properties discussed are well-known, but the proofs contained in the literature involve non-trivial concepts from representation theory which we will replace with simple arguments from linear algebra.

Results in Section 5 are phrased for general finite Abelian groups, but we expect that some readers may want to skip Section 4 and simply assume in Sections 5–9 that the group G is cyclic as was done in Sections 2 and 3.

We discuss geometric properties of Gabor frames in Sections 6–9. In Section 6, we address the question of whether Gabor frames that are in general linear position, meaning any N vectors of a Gabor system are linearly independent in the underlying N-dimensional ambient space, can be constructed. As one of the byproducts of our discussion, we will establish the existence of a large class of unimodular tight Gabor frames which are maximally robust to erasures. In Section 7, we address the coherence of Gabor systems, and in Section 8 we state estimates for the probability that a randomly chosen Gabor window generates a Gabor frame which has useful *restricted isometry constants* (RIC). In Section 9, we state some results on Gabor frames in the framework of compressed sensing.

Throughout the chapter, we will not discuss multiwindow Gabor frames. For details on the structure of multiwindow Gabor frames, see [34, 63] and references therein.

2 Gabor frames for $\mathbb{C}^N$

For reasons that become apparent in Section 4, we index the components of a vector $x \in \mathbb{C}^N$ by $\{0,1,2,\ldots,N-2,N-1\}$, namely, by the N element *cyclic group* $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$. Moreover, to avoid algebraic operations on indices, we write $x(k)$ rather than x_k for the k-th component of the column vector x. That is, we write

$$x = \big(x_0,x_1,x_2,\ldots,x_{N-2},x_{N-1}\big)^T = \big(x(0),x(1),x(2),\ldots,x(N-2),x(N-1)\big)^T,$$

where x^T denotes the transpose of the vector x.

The *(discrete) Fourier transform* $\mathscr{F} : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ plays a fundamental role in Gabor analysis. It is given pointwise by

$$\mathscr{F}x(m) = \widehat{x}(m) = \sum_{n=0}^{N-1} x(n)\, e^{-2\pi i m n/N}, \quad m = 0,1,\ldots,N-1. \tag{1}$$

Throughout this chapter, operators are defined by their action on column vectors and we will not distinguish between an operator and its matrix representation with respect to the Euclidean basis $\{e_k\}_{k=0,1,\ldots,N-1}$ where $e_k(n) = \delta(k-n) = 1$ if $k=n$ and $e_k(n) = \delta(k-n) = 0$ else.

In matrix notation, the discrete Fourier transform (1) is represented by the *Fourier matrix* $W_N = (\omega^{-rs})_{r,s=0}^{N-1}$ with $\omega = e^{2\pi i/N}$. For example, we have

$$W_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix},\; W_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-2\pi i 1/6} & e^{-2\pi i 1/3} & e^{-2\pi i 1/2} & e^{-2\pi i 2/3} & e^{-2\pi i 5/6} \\ 1 & e^{-2\pi i 1/3} & e^{-2\pi i 2/3} & 1 & e^{-2\pi i 1/3} & e^{-2\pi i 2/3} \\ 1 & e^{-2\pi i 1/2} & 1 & e^{-2\pi i 3/6} & 1 & e^{-2\pi i 1/2} \\ 1 & e^{-2\pi i 2/3} & e^{-2\pi i 1/3} & 1 & e^{-2\pi i 2/3} & e^{-2\pi i 1/3} \\ 1 & e^{-2\pi i 5/6} & e^{-2\pi i 2/3} & e^{-2\pi i 1/2} & e^{-2\pi i 1/3} & e^{-2\pi i 1/6} \end{pmatrix}.$$

The *fast Fourier transform* (FFT) provides an efficient algorithm to compute matrix vector products of the form $W_N x$ [14, 22, 57, 78].

The most important properties of the Fourier transform are the *Fourier inversion formula* (2), the *Parseval–Plancherel formula* (3), and the *Poisson summation formula* (5).

Theorem 1. *The normalized harmonics* $\frac{1}{\sqrt{N}} e^{2\pi i m(\cdot)/N}$, $m = 0,1,\ldots,N-1$, *form an orthonormal basis of* $\mathbb{C}^N$ *and, hence, we have*

$$x = \tfrac{1}{N} \sum_{m=0}^{N-1} \widehat{x}(m)\, e^{2\pi i m(\cdot)/N}, \quad x \in \mathbb{C}^N, \tag{2}$$

and

$$\langle x,y\rangle = \frac{1}{N}\langle \widehat{x},\widehat{y}\rangle, \quad x,y\in\mathbb{C}^N. \tag{3}$$

Moreover, for natural numbers a and b with $ab=N$ we have

$$\sum_{n=0}^{b-1} e^{2\pi i amn/N} = \begin{cases} b, & \textit{if } m \textit{ is a multiple of } b,\\ 0, & \textit{otherwise,}\end{cases} \tag{4}$$

and

$$a\sum_{n=0}^{b-1} x(an) = \sum_{m=0}^{a-1}\widehat{x}(bm), \quad x\in\mathbb{C}^N. \tag{5}$$

Proof. We first prove (4). If m is a multiple of b, then $e^{2\pi i amn/N}=1$ for all $n=0,1,\ldots,b-1$, and (4) holds. Else, $z=e^{2\pi i am/N}\neq 1$. and using the geometric sum formula, we obtain

$$\sum_{n=0}^{b-1} e^{2\pi i amn/N} = \sum_{n=0}^{b-1} z^n = (1-z^b)/(1-z) = (1-1)/(1-z) = 0.$$

Setting $a=1$ in (4) implies the orthonormality of the normalized harmonics, in fact,

$$\left\langle \tfrac{1}{\sqrt{N}}e^{2\pi i m(\cdot)/N}, \tfrac{1}{\sqrt{N}}e^{2\pi i m'(\cdot)/N}\right\rangle = \tfrac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i(m-m')n/N} \overset{(4)}{=} \begin{cases} 1, & \text{if } m=m',\\ 0, & \text{otherwise,}\end{cases}$$

and the reconstruction formula (2) and Parseval–Plancherel (3) follow.

To obtain (5) and thereby complete the proof, we compute

$$\sum_{n=0}^{b-1} x(an) \overset{(2)}{=} \sum_{n=0}^{b-1}\tfrac{1}{N}\sum_{m=0}^{N-1}\widehat{x}(m)\,e^{2\pi i mn/N} = \tfrac{1}{N}\sum_{m=0}^{N-1}\widehat{x}(m)\sum_{n=0}^{b-1}e^{2\pi i mn/N} \overset{(4)}{=} \tfrac{b}{N}\sum_{m=0}^{a-1}\widehat{x}(mb)\,.$$

The Fourier inversion formula (2) shows that any x can be written as linear combination of harmonics. While $|x(n)|^2$ quantifies the energy of the signal x at time n, the so-called Fourier coefficient $\widehat{x}(m)$ indicates that the harmonic $e^{2\pi i m(\cdot)/N}$ is contained in x with energy $\frac{1}{N}|\widehat{x}(m)|^2$. Indeed, setting $x=y$ in (3) implies conservation of energy, namely

$$\sum_{n=0}^{N-1}|x(n)|^2 = \tfrac{1}{N}\sum_{m=0}^{N-1}|\widehat{x}(m)|^2, \quad x\in\mathbb{C}^N.$$

Mathematically speaking, Gabor analysis is centered on the interplay of the Fourier transform, translation operators, and modulation operators. The *cyclic shift operator* $T:\mathbb{C}^N\longrightarrow\mathbb{C}^N$ is given by

$$Tx = T\big(x(0), x(1), \ldots, x(N-1)\big)^T = \big(x(N-1), x(0), x(1), \ldots, x(N-2)\big)^T.$$

Translation T_k by $k \in \{0, 1, \ldots, N-1\}$ is given by

$$T_k x(n) = T^k x(n) = x(n-k), \quad n \in \{0, 1, \ldots, N-1\},$$

that is, T_k simply repositions the entries of x, for instance, $x(0)$ is the k-th entry of $T_k x$. Note that the difference $n-k$ is taken modulo N, this agrees with considering the indices of $\mathbb{C}^N$ as elements of the cyclic group $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$. In Section 4 we will consider Gabor frames for $\mathbb{C}^G$, that is, on the vector space where the components are indexed by a finite Abelian group G that is not necessarily cyclic.

Modulation operators $M_\ell : \mathbb{C}^N \longrightarrow \mathbb{C}^N$, $\ell = 0, 1, \ldots, N-1$, are given by

$$M_\ell x = \big(e^{2\pi i \ell 0/N} x(0), e^{2\pi i \ell 1/N} x(1), \ldots, e^{2\pi i \ell (N-1)/N} x(N-1)\big)^T, \quad x \in \mathbb{C}^N.$$

that is, the modulation operator M_ℓ simply performs a pointwise product of the input vector $x = x(\cdot)$ with the harmonic $e^{2\pi i \ell (\cdot)/N}$.

Translation operators are commonly referred to as *time-shift operators*. Moreover, modulation operators are *frequency shift operators*. Indeed, we have

$$\begin{aligned}\widehat{M_\ell x}(m) &= \mathscr{F} M_\ell x(m) = \sum_{n=0}^{N-1} \big(e^{2\pi i \ell n/N} x(n)\big)\, e^{-2\pi i m n/N} = \sum_{n=0}^{N-1} x(n)\, e^{-2\pi i (m-\ell) n/N} \\ &= \widehat{x}(m-\ell).\end{aligned}$$

Applying the Fourier inversion formula to both sides gives

$$M_\ell = \mathscr{F}^{-1} T_\ell \mathscr{F}.$$

A *time-frequency shift operator* $\pi(k,\ell)$ combines translation by k and modulation by ℓ, that is

$$\pi(k,\ell) : \mathbb{C}^N \longrightarrow \mathbb{C}^N, \quad x \mapsto \pi(k,\ell) x = M_\ell T_k x.$$

For example, for $G = \mathbb{Z}_4$ the operators T_1, M_2, and $\pi(1,3)$ are given by the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{2\pi i 3/4} & 0 & 0 \\ 0 & 0 & e^{2\pi i 2/4} & 0 \\ 0 & 0 & 0 & e^{2\pi i 1/4} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2\pi i 3/4} & 0 \\ 0 & 0 & 0 & e^{2\pi i 2/4} \\ e^{2\pi i 1/4} & 0 & 0 & 0 \end{pmatrix}.$$

The following observation greatly simplifies Gabor analysis on $\mathbb{C}^N$. Recall that the space of linear operators on $\mathbb{C}^N$ forms an N^2-dimensional Hilbert space with *Hilbert–Schmidt space* inner product given independently of the chosen orthonormal basis $\{e_n\}_{n=0,1,\ldots,N-1}$ by

$$\langle A, B\rangle_{HS} = \sum_{\widetilde{n}=0}^{N-1} \sum_{n=0}^{N-1} \langle A e_n, e_{\widetilde{n}}\rangle \overline{\langle B e_n, e_{\widetilde{n}}\rangle}.$$

Proposition 1. *The set of normalized time-frequency shift operators* $\{1/\sqrt{N}\,\pi(k,\ell)\}_{k,\ell=0,1,\ldots,N-1}$ *is an orthonormal basis for the Hilbert–Schmidt space of linear operators on* $\mathbb{C}^N$.

Proof. Consider $A=(a_{\widetilde{n}n})$ and $B=(b_{\widetilde{n}n})$ as matrices with respect to the Euclidean basis. We have

$$\langle (a_{\widetilde{n}n}),(b_{\widetilde{n}n})\rangle_{HS}=\sum_{\widetilde{n}=0}^{N-1}\sum_{n=0}^{N-1} a_{\widetilde{n}n}\overline{b_{\widetilde{n}n}}\,.$$

Clearly, $\langle \pi(k,\ell),\pi(\widetilde{k},\widetilde{\ell})\rangle_{HS}=0$ if $k\neq\widetilde{k}$ as the matrices $\pi(k,\ell)$ and $\pi(\widetilde{k},\widetilde{\ell})$ have then disjoint support. Moreover, Theorem 1 implies

$$\langle 1/\sqrt{N}\,\pi(k,\ell),1/\sqrt{N}\,\pi(k,\widetilde{\ell})\rangle_{HS}=\langle 1/\sqrt{N}\,e^{2\pi i\ell(\cdot)/N},\ 1/\sqrt{N}\,e^{2\pi i\widetilde{\ell}(\cdot)/N}\rangle=\delta(\ell-\widetilde{\ell}).$$

We now define Gabor systems on $\mathbb{C}^N$. For $\varphi\in\mathbb{C}^N\setminus\{0\}$ and $\Lambda\subseteq\{0,1,\ldots,N-1\}\times\{0,1,\ldots,N-1\}$ we call

$$(\varphi,\Lambda)=\{\pi(k,\ell)\varphi\}_{(k,\ell)\in\Lambda}$$

the *Gabor system* generated by the *window function* φ and Λ. A Gabor system which spans $\mathbb{C}^N$ is a frame and is referred to as *Gabor frame*.

For instance, the Gabor system $(\,(1,2,3,4)^T,\{0,1,2,3\}\times\{0,1,2,3\}\,)$ in $\mathbb{C}^4$, consists of the columns in the matrix

$$\left(\begin{array}{rrrr|rrrr|rrrr|rrrr}
1 & 1 & 1 & 1 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2\\
2 & 2i & -2 & -2i & 1 & i & -1 & -i & 4 & 4i & -4 & -4i & 3 & 3i & -3 & -3i\\
3 & -3 & 3 & -3 & 2 & -2 & 2 & -2 & 1 & -1 & 1 & -1 & 4 & -4 & 4 & -4\\
4 & -4i & -4 & 4i & 3 & -3i & -3 & 3i & 2 & -2i & -2 & 2i & 1 & -i & -1 & i
\end{array}\right),$$

while the elements of $(\,(1,2,3,4,5,6)^T,\{0,2,4\}\times\{0,3\}\,)$ are listed in

$$\left(\begin{array}{rr|rr|rr}
1 & 1 & 5 & 5 & 3 & 3\\
2 & 2i & 6 & 6i & 4 & 4i\\
3 & 3 & 1 & 1 & 5 & 5\\
4 & 4i & 2 & 2i & 6 & 6i\\
5 & 5 & 3 & 3 & 1 & 1\\
6 & 6i & 4 & 4i & 2 & 2i
\end{array}\right).$$

The *short-time Fourier transform* $V_\varphi:\mathbb{C}^N\longrightarrow\mathbb{C}^{N\times N}$ with respect to the window $\varphi\in\mathbb{C}^N\setminus\{0\}$ is given by

$$V_\varphi x(k,\ell)=\langle x,\pi(k,\ell)\varphi\rangle=\mathscr{F}(x\,T_k\varphi)(\ell)=\sum_{n=0}^{N-1}x(n)\overline{\varphi(n-k)}e^{-2\pi i\ell n/N},\ \ x\in\mathbb{C}^N,$$

[33, 34, 45, 46]. Observe that $V_\varphi x(k,\ell)=\mathscr{F}(xT_k\varphi)(\ell)$ indicates that the short-time Fourier transform on $\mathbb{C}^N$ can be efficiently computed using a fast Fourier transform (FFT). This representation also indicates why short-time Fourier transforms are commonly referred to as *windowed Fourier transforms*: a window function φ centered at 0 is translated by k, the pointwise product with x selects a portion of x centered at k, and this portion is analyzed using a (fast) Fourier transform.

The short-time Fourier transform treats time and frequency almost symmetrically. In fact, using Parseval–Plancherel we obtain

$$\begin{aligned} V_\varphi x(k,\ell) &= \langle x, \pi(k,\ell)\varphi\rangle = \langle \widehat{x}, \widehat{M_\ell T_k \varphi}\rangle = \langle \widehat{x}, T_\ell M_{-k}\widehat{\varphi}\rangle \\ &= e^{-2\pi i k\ell/N}\langle \widehat{x}, M_{-k}T_\ell\widehat{\varphi}\rangle = e^{-2\pi i k\ell/N} V_{\widehat{\varphi}}\widehat{x}(\ell,-k), \quad x\in\mathbb{C}^N. \end{aligned} \tag{6}$$

While the short-time Fourier transform plays a distinct role in Gabor analysis on the real line — it is defined on $\mathbb{R}\times\widehat{\mathbb{R}}$ while Gabor frames are indexed by discrete subgroups of $\mathbb{R}\times\widehat{\mathbb{R}}$ — in the finite-dimensional setting, the short-time Fourier transform reduces to the analysis map with respect to the *full Gabor system* $(\varphi,\{0,1,\ldots,N-1\}\times\{0,1,\ldots,N-1\})$, that is, a Gabor system with $\Lambda=\{0,1,\ldots,N-1\}\times\{0,1,\ldots,N-1\}$. Hence, the inversion formula for the short-time Fourier transform

$$\begin{aligned} x(n) &= \frac{1}{N\,\|\varphi\|_2^2}\sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} V_\varphi x(k,\ell)\,\varphi(n-k)\,e^{-2\pi i\ell n/N} \\ &= \frac{1}{N\,\|\varphi\|_2^2}\sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} \langle x,\pi(k,\ell)\varphi\rangle\;\pi(k,\ell)\varphi(n), \qquad x\in\mathbb{C}^N, \end{aligned} \tag{7}$$

simply states that for all $\varphi\neq 0$, the system $(\varphi,\{0,1,\ldots,N-1\}\times\{0,1,\ldots,N-1\})$ is an $N\|\varphi\|^2$-tight Gabor frame. Equation (7) is a trivial consequence of Corollary 2 below. It characterizes tight Gabor frames (φ,Λ) for the case that summation over $\{0,1,\ldots,N-1\}\times\{0,1,\ldots,N-1\}$ in (7) is replaced by summation over a subgroup Λ of $\mathbb{Z}_N\times\mathbb{Z}_N=\{0,1,\ldots,N-1\}\times\{0,1,\ldots,N-1\}$.

Not all Gabor frames are tight, meaning the dual frame of a frame (φ,Λ) is not necessarily (φ,Λ). The following outstanding property of Gabor frames assures that the canonical dual frame of a Gabor frame is again a Gabor frame. A similar property does not hold for other similarly structured frames, for example, canonical dual frames of wavelet frames are in general not wavelet frames.

Proposition 2. *The canonical dual frame of a Gabor frame (φ,Λ) with frame operator S is the Gabor frame $(S^{-1}\varphi,\Lambda)$.*

Proof. We will show that $\pi(k,\ell)\circ S = S\circ\pi(k,\ell)$ for all $(k,\ell)\in\Lambda$. Then, $S^{-1}\circ\pi(k,\ell)=\pi(k,\ell)\circ S^{-1}$ and the members of the dual frame of (φ,Λ) are of the form $\pi(k,\ell)\big(S^{-1}\varphi\big)$, $(k,\ell)\in\Lambda$. Hence, the following elementary computation completes the proof:

$$
\begin{aligned}
S \circ \pi(k,\ell)x(n) &= \sum_{\widetilde{k}=0}^{N/a-1}\sum_{\widetilde{\ell}=0}^{N/b-1} \langle \pi(k,\ell)x,\, \pi(\widetilde{k},\widetilde{\ell})\varphi\rangle\, \pi(\widetilde{k},\widetilde{\ell})\varphi \\
&= \sum_{\widetilde{k}=0}^{N/a-1}\sum_{\widetilde{\ell}=0}^{N/b-1}\sum_{\widetilde{n}=0}^{N-1} e^{2\pi i\ell b\widetilde{n}/N} x(\widetilde{n}-ka)\, e^{-2\pi i\widetilde{\ell}b\widetilde{n}/N}\overline{\varphi(\widetilde{n}-\widetilde{k}a)}\, e^{-2\pi i\widetilde{\ell}bn/N}\varphi(n-\widetilde{k}a) \\
&= \sum_{\widetilde{k}=0}^{N/a-1}\sum_{\widetilde{\ell}=0}^{N/b-1}\sum_{\widetilde{n}=0}^{N-1} x(\widetilde{n})\, e^{-2\pi i(\widetilde{\ell}-\ell)b(\widetilde{n}+ka)/N}\overline{\varphi(\widetilde{n}-(\widetilde{k}-k)a)}\, e^{-2\pi i\widetilde{\ell}bn/N}\varphi(n-\widetilde{k}a) \\
&= \sum_{\widetilde{k}=0}^{N/a-1}\sum_{\widetilde{\ell}=0}^{N/b-1}\sum_{\widetilde{n}=0}^{N-1} x(\widetilde{n})\, e^{-2\pi i\widetilde{\ell}b\widetilde{n}/N}\overline{\varphi(\widetilde{n}-\widetilde{k}a)}\; e^{-2\pi i(\widetilde{\ell}+\ell)bn/N}\varphi(n-(\widetilde{k}+k)a)\, e^{2\pi i\ell bka/N} \\
&= \sum_{\widetilde{k}=0}^{N/a-1}\sum_{\widetilde{\ell}=0}^{N/b-1} \langle x,\, \pi(\widetilde{k},\widetilde{\ell})\varphi\rangle\, \pi(k,\ell)\pi(\widetilde{k},\widetilde{\ell})\varphi \\
&= \pi(k,\ell)\circ Sx(n)\,.
\end{aligned}
$$

3 Gabor frames as time-frequency analysis tool

As discussed in Section 1, Gabor systems were introduced to efficiently utilize communication channels. In this section, we will focus on a second fundamental application of Gabor systems; it concerns the time-frequency analysis of signals that are dominated by few components that are concentrated in time and/or frequency.

The Fourier transform's ability to separate a signal into its frequency components provides a powerful tool in science and mathematics. Many signals, however — for example, speech and music — have frequency contributions which appear only during short time intervals. The Fourier transform of a piano sonata may provide information on which notes dominate the score, but it falls short of enabling us to write down the score of the sonata that is needed to reproduce it on a piano. Gabor analysis addresses this shortcoming by providing information on which frequencies appear in a signal at which times.

Recall that $(\varphi, \{0,1,\ldots,N-1\}\times\{0,1,\ldots,N-1\})$ is an $N\|\varphi\|^2$-tight Gabor frame. Assuming $\|\varphi\|^2 = 1/N$, we obtain

$$
\sum_{n=0}^{N-1}|x(n)|^2 = \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1}|V_\varphi x(k,\ell)|^2 = \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1}|\mathscr{F}(xT_k\varphi)(\ell)|^2, \quad x\in\mathbb{C}^N,
$$

that is, the short-time Fourier transform distributes the energy of x on the time-frequency grid $\{0,1,\ldots,N-1\}\times\{0,1,\ldots,N-1\}$. Equation (6) implies that

$$
|V_\varphi x(k,\ell)| = |\langle x, M_\ell T_k\varphi\rangle| = |\langle \widehat{x}, M_{-k}T_\ell\widehat{\varphi}\rangle| \le \min\big\{\langle |x|, T_k|\varphi|\rangle,\ \langle |\widehat{x}|, T_\ell|\widehat{\varphi}|\rangle\big\}.
$$

Hence, any φ with φ and $\widehat{\varphi}$ being well localized at 0, meaning $|\varphi(n)|$, $|\widehat{\varphi}(m)|$ are small for n,m and $N-n, N-m$ large, implies that the energy captured in the *spec-*

trogram value $SPEC_\varphi(k,\ell) = |V_\varphi x(k,\ell)|^2$ is only large if frequencies close to ℓ have a large presence in x around time k. Unfortunately, *Heisenberg's uncertainty principle* implies that φ and $\widehat{\varphi}$ cannot be simultaneously arbitrarily well localized at 0. The simplest realization of this principle is the following result attributed to Donoho and Stark [28, 67]. In the following, we set $\|x\|_0 = |\{n:\ x(n) \neq 0\}|$.

Proposition 3. *Let* $x \in \mathbb{C}^N \setminus \{0\}$*, then* $\|x\|_0 \cdot \|\widehat{x}\|_0 \geq N$.

Proof. For $x \in \mathbb{C}^N$, $x \neq 0$, and $A = \max\{|\widehat{x}(m)|,\ m = 0,1,\ldots N-1\} \neq 0$, we compute

$$NA^2 \leq N\left(\sum_{n=0}^{N-1} |x(n)|\right)^2 \leq N\|x\|_0 \sum_{n=0}^{N-1} |x(n)|^2 = \|x\|_0 \sum_{m=0}^{N-1} |\widehat{x}(m)|^2 \leq \|x\|_0 \|\widehat{x}\|_0 A^2.$$

Theorem 12 below strengthens Proposition 3 in the case that N is prime.

To illustrate the use of Gabor frames in time-frequency analysis, we will use various Gabor windows to analyze the multicomponent signal $x \in \mathbb{C}^{200}$ given by

$$\begin{aligned} x(n) = {} & \chi_{\{0,\ldots,49\}}(n)\,\sin(2\pi 20n/200) + \chi_{\{150,\ldots,199\}}(n)\sin(2\pi 50(n-150)/200) \\ & + \chi_{\{50,\ldots,149\}}(n)\,\sin\big(2\pi\,(30(n-50)^2/200^2 + 20(n-50)/200)\big) \\ & + 1.2\,\chi_{\{80,\ldots,99\}}(n)\,\big(1+\cos(2\pi\,(10n/200-1/2)\big)\,\cos(2\pi 60n/200) \\ & + 1.2\,\chi_{\{60,\ldots,79\}}(n)\,\big(1+\cos(2\pi\,(10n/200-1/2)\big)\,\cos(2\pi 50n/200) \\ & + .5\,\chi_{\{100,\ldots,199\}}(n)\,\big(1+\cos(2\pi\,(2n/200-1/2)\big)\,\cos(2\pi 20n/200) \\ & + \chi_{\{20,\ldots,31\}}(n)\,\big(1+\cos(2\pi\,(12n/200-1/2)\big)\,\cos(2\pi 20n/200) \\ & + 1.1\,\chi_{\{100,\ldots,109\}}(n)\,\big(1+\cos(2\pi\,(20n/200-1/2)\big),\ n = 0,1,\ldots,199, \quad (8) \end{aligned}$$

where $\chi_A(n) = 1$ if $n \in A$ and 0 else. The signal and its Fourier transform are displayed in Figure 1. Note that x is real-valued, so its Fourier transform has even symmetry. As we will also use real-valued window functions below, we obtain short-time Fourier transforms which are symmetric in frequency and it suffices to display $SPEC_\varphi$ in Figures 2–9 only for frequencies 0 to 100.[2]

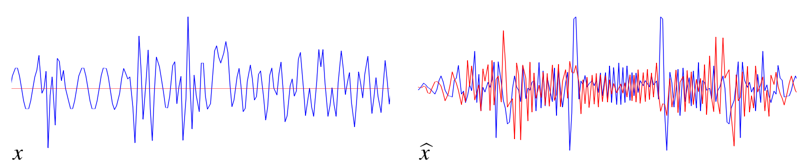

Fig. 1 The test signal x given in (8) and used in Figures 2–6 and Figure 9 as well as its Fourier transform. Here and in the following, the real part of a signal is given in blue and its imaginary part is given in red.

[2] Our treatment is unit-free. The reader may assume that n counts seconds, then m counts hertz, or n represents milliseconds, then m represents megahertz.

In Figures 2 and 3, we use orthogonal Gabor systems generated by characteristic functions. In Figure 2 we choose as Gabor window the normalized characteristic function given by $\varphi(n) = 1/\sqrt{20}$ for $n = 191, 192, \ldots, 199, 0, 1, \ldots 10$ and $\varphi(n) = 0$ for $n = 11, 12, \ldots, 190$. The spectrogram $SPEC_\varphi x = |V_\varphi x|^2$ in Figures 2 shows that the signal has as dominating frequency 20 in the beginning and frequency 50 towards the end , with a linear transition in between. In addition, the five additional frequency clusters of x appear at 5 different time instances.

The picture shows some vertical ringing artifacts. These are due to the sidelobes of the Fourier transform $\widehat{\varphi}$ of φ. They imply that components well localized in frequency have an effect on $|V_\varphi x(k,\ell)|^2$ for a large range of ℓ.

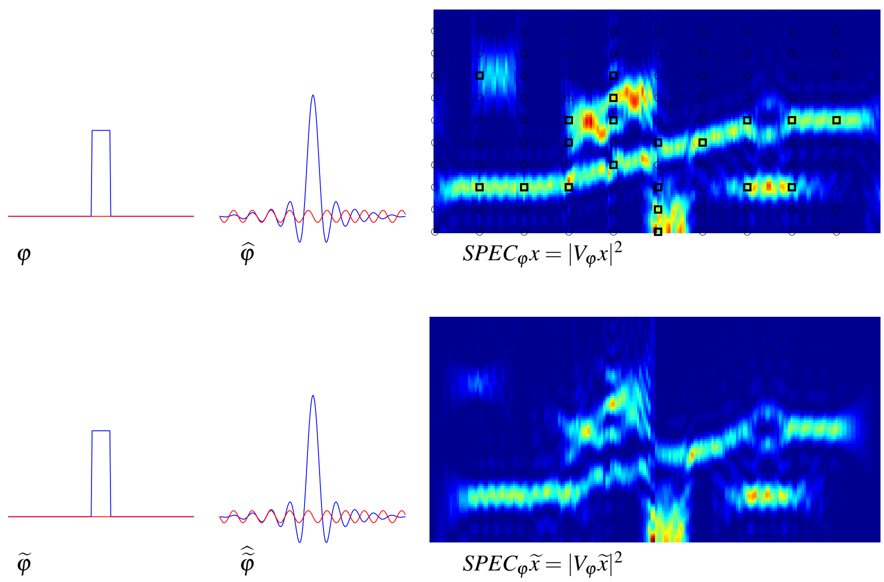

Fig. 2 Gabor frame analysis of the multicomponent signal displayed in Figure 1. We use the Gabor system (φ, Λ) with $\varphi(n) = 1/\sqrt{20}$ for $n = 191, 192, \ldots, 199, 0, 1, \ldots 10$ and $\varphi(n) = 0$ for $n = 11, 12, \ldots, 190$. The Gabor system forms an orthonormal basis of $\mathbb{C}^{200}$ and is therefore self-dual, that is $\varphi = \widetilde{\varphi}$. We display φ, $\widehat{\varphi}$, $\widetilde{\varphi}$, $\widehat{\widetilde{\varphi}}$ as well as the spectrogram of x and of its approximation $\widetilde{x}$. The circles on $SPEC_\varphi x$ depict Λ; they mark frame coefficients of the frame (φ, Λ). The squares denote the 20 biggest frame coefficients which are then used to construct the approximation $\widetilde{x}$ to x.

The values of the short-time Fourier transform $V_\varphi x$ allow us to reconstruct x using (7). Doing so requires the use of N^2 coefficients to reconstruct a signal in $\mathbb{C}^N$. Clearly, it is more efficient to use only the values of $V_\varphi x$ on a lattice Λ that allows for (φ, Λ) being a frame of cardinality not exceeding the dimension of the ambient space N.

In Figure 2, we circle the values of $|V_\varphi x(k,\ell)|^2$ with $(k,\ell) \in \Lambda = \{0, 20, \ldots, 180\} \times \{0, 10, \ldots, 190\}$. It is easy to see that (φ, Λ) is an orthonormal basis; hence, we can reconstruct the signal x using only values of the short-time Fourier transform that correspond to the circled values. Note that, in general, whenever (φ,Λ) is a frame with dual frame $(\widetilde{\varphi},\Lambda)$, we can reconstruct x by means of

$$x = \sum_{(k,\ell)\in\Lambda} \langle x, \pi(k,\ell)\varphi\rangle\, \pi(k,\ell)\widetilde{\varphi}.$$

In many applications, though, one would like to reduce the amount of information that is first stored and then used to reproduce the signal to below the dimension N of the ambient space. Rather than reproducing x perfectly, we are satisfied to obtain an approximation

$$\widetilde{x} = \sum_{(k,\ell)\in\Lambda} R(\langle x, \pi(k,\ell)\varphi\rangle)\, \pi(k,\ell)\widetilde{\varphi},$$

which captures the key features of x.

Here, we illustrate the effect of a rather simplistic compression algorithm. Namely, we use only the 40 largest coefficients (20 in the depicted half of the spectrogram) to produce an approximation $\widetilde{x}$ to x. That is, $R(\langle x,\pi(k,\ell)\varphi\rangle) = \langle x,\pi(k,\ell)\varphi\rangle$ for the 40 largest coefficients and $R(\langle x,\pi(k,\ell)\varphi\rangle) = 0$ else. The locations in time and frequency of the chosen coefficients are marked by squares.

Graphic comparisons of $\widetilde{x}$ with x and of $\widehat{\widetilde{x}}$ with $\widehat{x}$ are not very useful. Instead, we compare the spectrogram of $\widetilde{x}$ withe the spectrogram of the original signal x. This demonstrates well the effect of our compression procedure; most of the features of x are in fact preserved.

The setup chosen to generate Figure 3 differs from the one used to obtain Figure 2 only in the choice of window function φ. Here, we choose a wider window function, which leads to a better localized $\widehat{\varphi}$. In detail, we choose $\varphi(n) = 1/\sqrt{40}$ for $n = 181, 192, \ldots, 199, 0, 1, \ldots 20$ and $\varphi(n) = 0$ for $n = 21, 22, \ldots, 180$. As lattice we choose $\Lambda = \{0, 40, 80, \ldots, 160\} \times \{0, 5, 10, \ldots, 195\}$ and observe that (φ,Λ) is again an orthonormal basis.

Comparing the spectrogram of x in Figure 3 with the one of x in Figure 2, we observe a reduced ringing effect and slightly better localization in frequency at the price of losing localization in time. Unfortunately, a comparison of $SPEC_\varphi x$ with $SPEC_\varphi \widetilde{x}$ shows that the canonical choice of lattice seems not to work well in conjunction with our compression algorithm. The large gaps between lattice notes in time causes part of the frequency transition not to be preserved by our simplistic compression algorithm.

In Figures 4–6 we choose as window functions Gaussians. In Figure 4 we choose

$$\varphi(n) = c\, e^{-(n/6)^2}$$

where c normalizes φ and as lattice $\Lambda = \{0, 8, 16, \ldots, 192\} \times \{0, 20, 40, \ldots, 180\}$. For Figure 5 we select

$$\varphi(n) = c\, e^{-(n/14)^2}$$

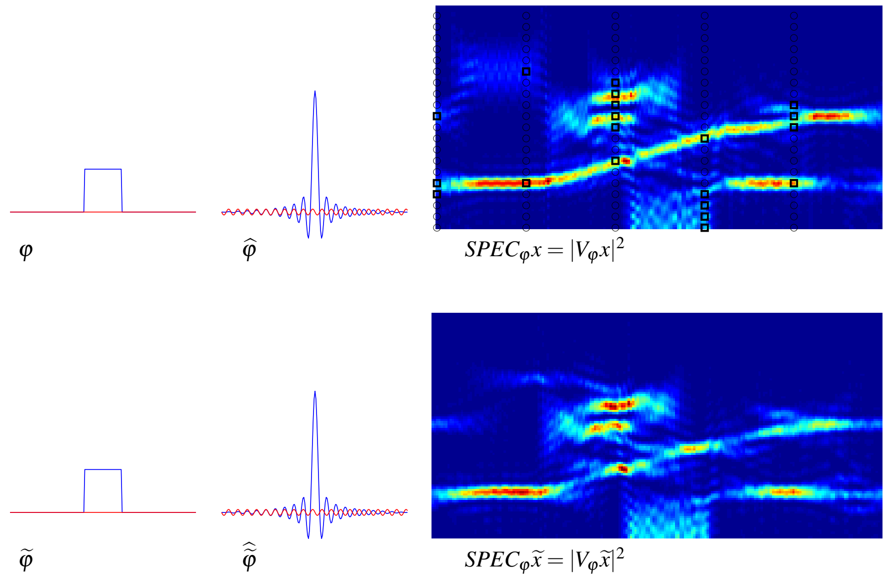

Fig. 3 Gabor frame analysis of the multicomponent signal displayed in Figure 1. We use the orthonormal Gabor system (φ,Λ) with $\varphi(n) = 1/\sqrt{40}$ for $n = 181, 192, \ldots, 199, 0, 1, \ldots 20$ and $\varphi(n) = 0$ for $n = 21, 12, \ldots, 180$. We display φ, $\widehat{\varphi}$, $\widetilde{\varphi}$, $\widehat{\widetilde{\varphi}}$, $SPEC_{\varphi}x$ and $SPEC_{\varphi}\widetilde{x}$. The circles on $SPEC_{\varphi}x$ mark frame coefficients of the frame (φ,Λ), the squares denote the 20 coefficients used to construct $\widetilde{x}$.

where c again normalizes φ. We let $\Lambda = \{0, 20, 40, \ldots, 180\} \times \{0, 8, 16, \ldots, 192\}$. We perform the same naive compression procedure used above to obtain Figures 2 and 3. Note that the lattices in Figures 5 and 6 contain 250 elements, and in fact, the Gabor frame (φ,Λ) is overcomplete.

Choosing a Gaussian window function has the benefit of removing the sidelobes and of providing an easily readable spectrogram. But our compression procedure is harmed by two facts. First of all, we are now picking 40 out of 250 coefficients; these are clustered in the dominating area, so secondary time-frequency components of x are also overlooked. Clearly, our algorithm does not benefit from the redundancy of the Gabor frame in use. Second, the good localization in frequency of φ implies that some of the components fall between lattice values. Therefore, they are overlooked.

A comparison of Figures 4 and 5 shows again the tradeoff between good time and good frequency resolution.

In Figure 6 we choose the same Gaussian window as in Figure 4, but a lattice which is not the product of two lattices in $\{0, 1, \ldots, 199\}$. In fact, we have

$$\Lambda = \{0, 40, \ldots, 160\} \times \{0, 8, \ldots, 192\} \cup \{20, 60, 100, 140, 180\} \times \{4, 12, 20, \ldots, 196\}.$$

But deviating from rectangular lattices offers little help. Moreover, even though we are choosing a lattice of the same redundancy, namely, we choose a frame with

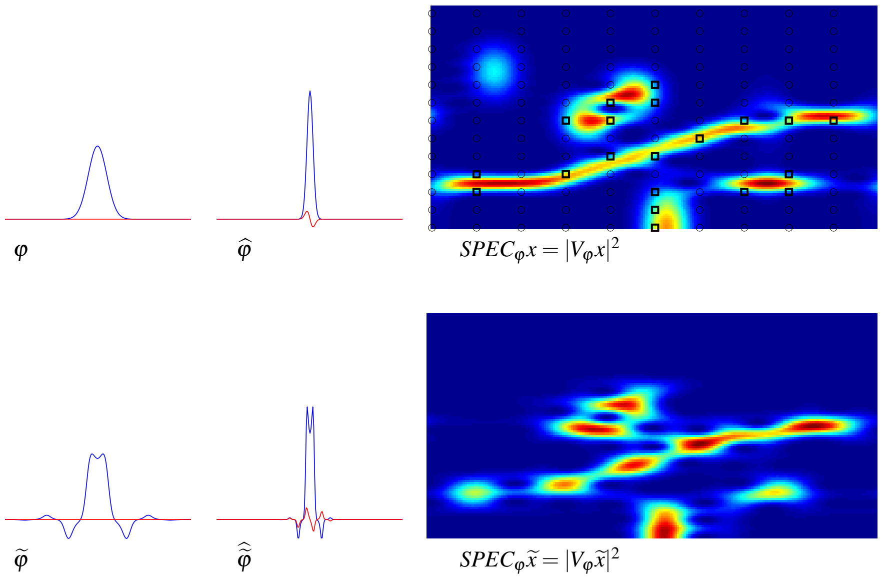

Fig. 4 Gabor frame analysis of the signal in Figure 1. As Gabor window we choose a normalized version of the Gaussian $\varphi(n) = c\,e^{-(n/6)^2}$, $n = 0, 1, \ldots, 199$. We display again φ, $\widehat{\varphi}$, $\widetilde{\varphi}$, $\widehat{\widetilde{\varphi}}$, $SPEC_{\varphi}x$ and $SPEC_{\varphi}\widetilde{x}$, where Λ is marked on $SPEC_{\varphi}x$ by circles. As before, the squares denote the 20 largest coefficients. Unmarked frame coefficients are not used to construct $\widetilde{x}$.

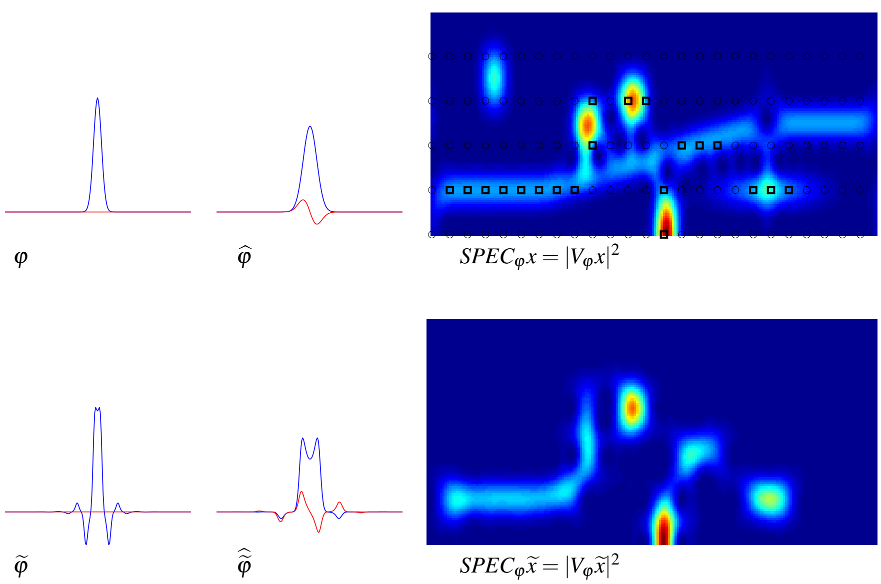

Fig. 5 Here, we use as Gabor window a normalized version of $\varphi(n) = c\,e^{-(n/14)^2}$, $n = 0, 1, \ldots, 199$. As before, φ, $\widehat{\varphi}$, $\widetilde{\varphi}$, $\widehat{\widetilde{\varphi}}$, $SPEC_{\varphi}x$ and $SPEC_{\varphi}\widetilde{x}$ are shown, Λ as well as the 20 largest coefficients used to construct $\widetilde{x}$ are marked on $SPEC_{\varphi}x$.

250 elements in a 200-dimensional space, the dual window has poor frequency localization. This significantly reduces the quality of reconstruction when using the compressed version $\widetilde{x}$ of the signal x, as the dual window used for synthesis smears out the frequency signature of the signal.

Similar discussions on the use of Gabor frames to analyze discrete one-dimensional signals and discrete images can be found in [21, 51, 66, 70, 86, 87].

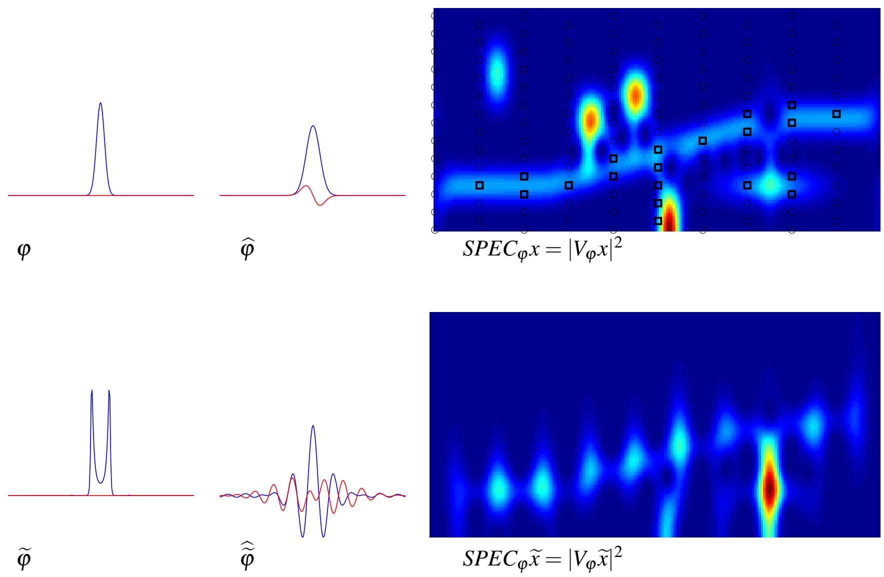

Fig. 6 We use the same window function as in Figure 4, but a different lattice. This changes the displayed dual window $\widetilde{\varphi}$ and its Fourier transform $\widehat{\widetilde{\varphi}}$. $SPEC_\varphi x$ and $SPEC_\varphi \widetilde{x}$ vary greatly. The lattice Λ and its 20 largest coefficients are marked as in Figures 2–5 above.

4 Gabor analysis on finite Abelian groups

In Section 2 we defined Gabor systems in $\mathbb{C}^N$. Implicitly we considered vectors in $\mathbb{C}^N$ as vectors defined on the *cyclic group* $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$. For example, the translation operator T_k was defined by $T_k x(n) = x(n-k)$ where $n-k$ was taken modulus N, that is, n and k were considered to be elements in the cyclic group $\mathbb{Z}_N$.

In this section, we will develop Gabor systems with an arbitrary finite Abelian group G in place of $\mathbb{Z}_N$. We thereby obtain results on Gabor systems on the finite-dimensional vector space

$$\mathbb{C}^G = \{x : G \longrightarrow \mathbb{C}\},$$

that is, $\mathbb{C}^G$ is a $|G|$-dimensional vector space with vector entries indexed by elements in the group G. We will continue to write $\mathbb{C}^N$ rather than $\mathbb{C}^{\mathbb{Z}_N}$ if $G=\mathbb{Z}_N$.

The group structure of the index set G allows us to define unitary *translation operators* $T_k : \mathbb{C}^G \longrightarrow \mathbb{C}^G$, $k \in G$, by

$$T_k x(n) = x(n-k), \quad n \in G.$$

Modulation operators on $\mathbb{C}^G$ are pointwise products with characters on the finite Abelian group G. A *character* $\xi \in \mathbb{C}^G$ is a group homomorphism mapping G into the multiplicative group $S^1 = \{z \in \mathbb{C} : \ |z| = 1\}$ [7, 56, 81, 91]. The set of characters on G forms a group under pointwise multiplication. This group is called the *dual group* of G and is denoted by $\widehat{G}$.

In summary, for $\xi \in \widehat{G}$, the *modulation operator* on $M_\xi : \mathbb{C}^G \longrightarrow \mathbb{C}^G$ is given by

$$M_\xi x(n) = \xi(n) x(n), \quad n \in G.$$

For $\lambda = (k,\xi) \in G \times \widehat{G}$, we define the *time-frequency shift operator* $\pi(\lambda)$ by

$$\pi(\lambda) : \mathbb{C}^G \longrightarrow \mathbb{C}^G, \quad x \mapsto \pi(\lambda)x = \pi(k,\xi)x = M_\xi T_k x = \xi(\cdot)\, x(\cdot - k)\,.$$

We are now in position to define Gabor systems on $\mathbb{C}^G$ where G is a finite Abelian group with dual group $\widehat{G}$. Let Λ be a subset of the product group $G \times \widehat{G}$ and let $\varphi \in \mathbb{C}^G \setminus \{0\}$. The respective *Gabor system* is then given by

$$(\varphi, \Lambda) = \{\pi(\lambda)\varphi\}_{\lambda \in \Lambda}\,.$$

A Gabor system which spans $\mathbb{C}^G$ is a frame and is called a *Gabor frame*. In many cases, we will consider Gabor systems with Λ being a subgroup of $G \times \widehat{G}$.

The *short-time Fourier transform* $V_\varphi : \mathbb{C}^G \longrightarrow \mathbb{C}^{G\times\widehat{G}}$ with respect to the window $\varphi \in \mathbb{C}^G$ is given by

$$V_\varphi x(k,\xi) = \langle x, \pi(k,\xi)\varphi\rangle = \mathscr{F}(x T_k \varphi)(\xi) = \sum_{n\in G} x(n)\overline{\varphi(n-k)\langle \xi, x\rangle}, \quad x \in \mathbb{C}^G,$$

[33, 34, 45, 46]. The inversion formula for the short-time Fourier transform

$$x(n) = \frac{1}{|G|\,\|\varphi\|_2^2} \sum_{(k,\xi)\in G\times\widehat{G}} V_\varphi x(k,\xi)\, \varphi(n-k)\langle \xi, k\rangle\,, \qquad x \in \mathbb{C}^G,$$

holds for all $\varphi \neq 0$, as we will see in Corollary 2 below. As in the case $G = \mathbb{Z}_N$, we conclude that the system $(\varphi, G\times\widehat{G})$ is a $|G|\cdot\|\varphi\|^2$-tight Gabor frame (see Section XX).

Before continuing our discussion of Gabor systems on finite Abelian groups in Section 4.2, we will prove the harmonic analysis results that lie at the basis of Gabor analysis on finite Abelian groups.

4.1 Harmonic analysis on finite Abelian groups

As mentioned above, a character on a finite Abelian group is a group homomorphism mapping G into the multiplicative circle group $S^1 = \{z \in \mathbb{C},\ |z| = 1\}$. The set of characters is denoted by $\widehat{G}$, which is a finite Abelian group under pointwise multiplication, meaning with composition $(\xi_1 + \xi_2)(n) = \xi_1(n)\xi_2(n)$.

In order to describe explicitly characters on finite Abelian groups, we will combine simple results on characters on cyclic groups with the *fundamental theorem of finite Abelian groups*. It states that every finite Abelian group is isomorphic to the product of cyclic groups.

Theorem 2. *For every finite Abelian group G exist $N_1, N_2, \dots, N_d$ with*

$$G \cong \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \ldots \times \mathbb{Z}_{N_d}. \tag{9}$$

The factorization and the number of factors in (9) *is not unique, but there exists a unique set of primes $\{p_1, \dots, p_d\}$ and a unique set of natural numbers $\{r_1, \dots, r_d\}$ so that* (9) *holds with $N_1 = p_1^{r_1}$, $N_2 = p_2^{r_2}, \dots, N_d = p_d^{r_d}$.*

Proof. For our purpose it is only relevant that a factorization as given in (9) exists. We will outline an inductive proof of this fact.

Recall that $|G|$ is called the order of the group G, $\langle n \rangle$ denotes the group generated by $n \in G$, and the order of $n \in G$ is $|\langle n \rangle|$.

If $|G| = 1$ then $G = \{0\}$ and the claim holds trivially. Suppose that all groups of order $|G| < N$ satisfy (9). Let now G be given with $|G| = N$. We need to distinguish two cases.

If $N = p^s$ with p prime, choose $n \in G$ with maximal order. If its order is $|G|$, then $G = \langle n \rangle$ and $G \cong \mathbb{Z}_N$. If its order is less than $|G|$, then a short sequence of algebraic arguments shows that there exists H with $G \cong \langle n \rangle \times H$. We obtain (9) for G by applying the induction hypothesis to H.

If $N = rp^s$ with p prime, $r \geq 2$ relatively prime with p, and $s \geq 1$. Then

$$G \cong \{n : \text{ the order of } n \text{ is a power of } p\} \times \{n : \text{ the order of } n \text{ is not divisible by } p\}$$

can be shown to be a factorization of G into two subgroups of smaller order and we can again apply the induction hypothesis.

As mentioned above, representations of finite groups as products of cyclic groups are not unique; for example, we have $\mathbb{Z}_{KL}$ isomorphic to $\mathbb{Z}_K \times \mathbb{Z}_L$ if (and only if) K and L are relatively prime.

Any group isomorphism induces a group isomorphism between the respective dual groups. Theorem 2 therefore implies that for our study of characters on general finite Abelian groups it suffices to study characters on products of cyclic groups. Hence, we may assume

$$G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \ldots \times \mathbb{Z}_{N_d}$$

in the following.

Observe that for the cyclic group $G = \mathbb{Z}_N = \{0,1,\ldots,N-1\}$, a character ξ is fully determined by $\xi(1)$. Since

$$1 = \xi(0) = \xi(N) = \xi(1+\ldots+1) = \xi(1)^N,$$

we have $\xi(1) \in \{e^{2\pi i m/N},\, m = 0,1,\ldots,N-1\}$. We conclude that $\widehat{\mathbb{Z}_N}$ contains exactly N characters; they are

$$\xi_m = \big(e^{2\pi i m(\cdot)0/N}, e^{2\pi i m(\cdot)1/N}, e^{2\pi i m(\cdot)2/N}, \ldots, e^{2\pi i m(\cdot)(N-1)/N}\big)^T, \quad m = 0,1,\ldots,N-1\,.$$

The modulation operators for cyclic groups that are defined abstractly here therefore coincide with the definition of modulation operators on $\mathbb{C}^N$ given in Section 2.

Observe that under pointwise multiplication, the group of characters $\widehat{\mathbb{Z}_N}$ is cyclic and has N elements, that is, $\widehat{\mathbb{Z}_N} \cong \mathbb{Z}_N$, a fact that we will use below.

For $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \ldots \times \mathbb{Z}_{N_d}$, observe that any character ξ on G induces a character on the component groups $\mathbb{Z}_{N_1}$, $\mathbb{Z}_{N_2}$, $\ldots$, $\mathbb{Z}_{N_d}$. Hence, we can associate to any character ξ on G an $m = (m_1, m_2, \ldots, m_d)$ with

$$\xi(e_r) = \xi\big((0,\ldots,0,1,0,\ldots 0)\big) = e^{2\pi i m_r/N_1}, \quad r = 1,\ldots,d.$$

Clearly, as ξ is a group homomorphism, it is fully described by m and we have

$$\begin{aligned} \xi(n_1,n_2,\ldots,n_d) &= \xi_{m_1}(n_2)\ldots\xi_{m_1}(n_d) \\ &= e^{2\pi i m_1 n_1/N_1}\, e^{2\pi i m_2 n_2/N_2} \cdot \ldots \cdot e^{2\pi i m_d n_d/N_d} \\ &= e^{2\pi i \left(m_1 n_1/N_1 + m_2 n_2/N_2 + \ldots m_d n_d/N_d\right)}\,. \end{aligned} \tag{10}$$

For notational simplicity, we will identify ξ with the derived m and write

$$\langle m,n\rangle = \xi(n) = e^{2\pi i \left(m_1 n_1/N_1 + m_2 n_2/N_2 + \ldots m_d n_d/N_d\right)}\,. \tag{11}$$

We observe that

$$\widehat{G} = \big(\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \ldots \times \mathbb{Z}_{N_d}\big)^{\widehat{}} \cong \widehat{\mathbb{Z}_{N_1}} \times \widehat{\mathbb{Z}_{N_2}} \times \ldots \times \widehat{\mathbb{Z}_{N_d}}.$$

Clearly, then $\widehat{\widehat{G}} \cong \widehat{G} \cong G$; in addition, G can be canonically identified with $\widehat{\widehat{G}}$ by means of the group homomorphism $n : m \mapsto \langle m,n\rangle$, thereby justifying the duality notation used in (11).

In the finite Abelian group setting, the *Fourier transform* $\mathscr{F} : \mathbb{C}^G \longrightarrow \mathbb{C}^{\widehat{G}}$ is given by

$$\begin{aligned}\mathscr{F}x(m) = \widehat{x}(m) &= \sum_{n\in G} x(n)\,\langle m,n\rangle \\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\ldots\sum_{n_d=0}^{N_d-1} x(n_1,n_2,\ldots,n_d)\,e^{-2\pi i\left(m_1n_1/N_1+m_2n_2/N_2+\ldots m_dn_d/N_d\right)}, \\ &\qquad\qquad m=(m_1,m_2,\ldots m_d)\in\widehat{G}.\end{aligned}$$

Theorem 1 above implies that the normalized characters on $\mathbb{Z}_N$ form an orthonormal basis of $\mathbb{C}^N$. Combining this with (10) shows that the normalized characters on any finite Abelian group G form an orthonormal system of cardinality $|G| = N_1\cdot\ldots\cdot N_d = \dim\mathbb{C}^G$. We conclude that the normalized characters form an orthonormal basis of $\mathbb{C}^G$. This simple observation generalizes (2) and (3) to the general finite Abelian group setting. For example, the *Fourier inversion formula* (2) becomes

$$\begin{aligned}x(n) &= \frac{1}{|G|}\sum_{m\in\widehat{G}}\widehat{x}(m)\,\overline{\langle m,n\rangle} \\ &= \frac{1}{|G|}\sum_{m_1=0}^{N_1-1}\sum_{m_2=0}^{N_2-1}\ldots\sum_{m_d=0}^{N_d-1}\widehat{x}(m_1,m_2,\ldots,m_d)\,e^{2\pi i\left(m_1n_1/N_1+m_2n_2/N_2+\ldots m_dn_d/N_d\right)}, \\ &\qquad\qquad n=(n_1,n_2,\ldots n_d)\in G.\end{aligned}$$

To state and prove the *Poisson summation formula* (13) for the Fourier transform on $\mathbb{C}^G$, we define for any subgroup H of G the *annihilator subgroup*

$$H^\perp = \{m\in\widehat{G}:\ \langle m,n\rangle = 1 \text{ for all } n\in H\}.$$

Clearly, $H^\perp$ is a subgroup of $\widehat{G}$. In Gabor and harmonic analysis, discrete subgroups of G are commonly referred to as *lattices* and their annihilators as their *dual lattices*.

Theorem 3. *Let H be a subgroup (lattice) of G and let $H^\perp$ be its annihilator subgroup (dual lattice). Then*

$$\sum_{n\in H}\langle m,n\rangle = \begin{cases}|H|, & \text{if } m\in H^\perp\\ 0, & \text{otherwise}\end{cases}, \qquad \sum_{m\in H^\perp}\langle m,n\rangle = \begin{cases}|H^\perp|, & \text{if } n\in H\\ 0, & \text{otherwise}\end{cases}, \tag{12}$$

and

$$|H^\perp|\sum_{n\in H}x(n) = \sum_{m\in H^\perp}\widehat{x}(m), \quad x\in\mathbb{C}^G. \tag{13}$$

Proof. Let $m\in\widehat{G}$. Then $n\mapsto\langle m,n\rangle$ for $n\in H$ defines a character on H. This character is identical or orthogonal to the trivial character on H, namely, $0: n\mapsto 1$ for $n\in H$, hence

$$\sum_{n\in H}\langle m,n\rangle = \sum_{n\in H}\langle m,n\rangle\overline{\langle 0,n\rangle} = \begin{cases}|H|, & \text{if } m=0 \text{ on } H\\ 0, & \text{otherwise.}\end{cases} = \begin{cases}|H|, & \text{if } m\in H^\perp\\ 0, & \text{otherwise.}\end{cases}$$

The second equality in (12) follows from the first equality in (12) by observing that $H^\perp$ is a subgroup of $\widehat{G}$ and that $(H^\perp)^\perp \subseteq \widehat{\widehat{G}}$ can be canonically identified with $H \subseteq G$.

The interchange of summation argument used to obtain (5) in Theorem 1 can be used again to prove (13).

The fact that $G \cong \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \ldots \times \mathbb{Z}_{N_d}$ for any finite Abelian group G implies that the discrete Fourier matrix W_G can be expressed as the Kronecker product of the Fourier matrices for the cyclic groups $\mathbb{Z}_{N_1}, \mathbb{Z}_{N_2}, \ldots, \mathbb{Z}_{N_d}$, that is, $W_G = W_{N_1} \otimes W_{N_2} \otimes \ldots \otimes W_{N_d}$. For example, we have

$$W_{\mathbb{Z}_2\times\mathbb{Z}_2} = W_{\mathbb{Z}_2} \otimes W_{\mathbb{Z}_2} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

4.2 Examples of and further remarks on Gabor systems on finite Abelian groups

In Section 4.1 it was shown that the study of finite Abelian groups coincides with the study of finite products of cyclic groups. Moreover, we described in detail characters on products of cyclic groups and thereby modulation operators on such groups.

For example, for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, the operators $T_{(1,0)}$, and $M_{(1,1)}$ are in matrix form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\pi((1,0),(1,1))$ is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proposition 1 generalizes to the following result.

Proposition 4. *The normalized time-frequency shift operators* $\{1/\sqrt{|G|}\ \pi(\lambda)\}_{\lambda\in G\times\widehat{G}}$ *form an orthonormal basis for the space of linear operators on* $\mathbb{C}^G$ *equipped with the Hilbert–Schmidt inner product.*

Proof. This follows from direct computation or by simply using the fact that the tensors of orthonormal bases form an orthonormal basis of the tensor space.

Consider again $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then

$$G \times \widehat{G} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \widehat{\mathbb{Z}_2 \times \mathbb{Z}_2} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \widehat{\mathbb{Z}_2} \times \widehat{\mathbb{Z}_2} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

and the Gabor system $((1,2,3,4)^T, \mathbb{Z}_2\times\mathbb{Z}_2 \times \mathbb{Z}_2\times\mathbb{Z}_2)$ consists of the columns of

$$\left(\begin{array}{rrrr|rrrr|rrrr|rrrr} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 2 & -2 & 2 & -2 & 1 & -1 & 1 & -1 & 4 & -4 & 4 & -4 & 3 & -3 & 3 & -3 \\ 3 & 3 & -3 & -3 & 4 & 4 & -4 & -4 & 1 & 1 & -1 & -1 & 2 & 2 & -2 & -2 \\ 4 & -4 & -4 & 4 & 3 & -3 & -3 & 3 & 2 & -2 & -2 & 2 & 1 & -1 & -1 & 1 \end{array}\right).$$

Note that the Gabor system above is not the tensor product of two Gabor systems on the finite Abelian group $\mathbb{Z}_2$. This is because $(1,2,3,4)^T$ is not a simple tensor, that is, does not have the form $v\otimes w$ for $v,w\in\mathbb{C}^{\mathbb{Z}_2}$. Certainly, Gabor systems on product groups can be generated by tensoring Gabor systems on the component groups; that is, for finite Abelian groups G_1 and G_2 with subsets $\Lambda_1\subseteq G_1$ and $\Lambda_2\subseteq G_2$, and $\varphi_1\in\mathbb{C}^{G_1}$ and $\varphi_2\in\mathbb{C}^{G_2}$, we obtain the $\mathbb{C}^{G_1\times G_2}$ Gabor system

$$(\varphi_1,\Lambda_1)\otimes(\varphi_2,\Lambda_2)=(\varphi_1\otimes\varphi_2,\Lambda_1\times\Lambda_2);$$

see, for example, [21, 31].

Every Gabor system (φ,Λ), $\varphi\neq 0$, with $\Lambda=G\times\widehat{G}$ is a tight frame for $\mathbb{C}^G$, but certainly other algebraic and geometric properties of (φ,Λ) depend on the group G and the window function φ, as we will discuss below.

5 Elementary properties of Gabor frames and of the Gabor frame operator

In this section we derive the central properties of Gabor frames for $\mathbb{C}^G$. Throughout this chapter, the reader may choose to assume $\mathbb{C}^G=\mathbb{C}^N=\mathbb{C}^{\{0,1,\dots,N-1\}}$ as considered in Section 2. Indeed, Section 2 reflects the special case $G=\widehat{G}=\mathbb{Z}_N=\{0,1,\dots,N-1\}$.

Gabor frames are derived from group frames as described in Definition X.X in Section X, a fact responsible for the Gabor system $(\varphi,G\times\widehat{G})$ being a tight frame for all $\varphi\in\mathbb{C}^G\setminus\{0\}$ (see Section X.X and [33, 34, 44, 45]). Gabor frames (φ,Λ) with Λ being a subgroup of $G\times\widehat{G}$ share a number of remarkable properties that are rooted in the fact that $\pi: G\times\widehat{G}\longrightarrow\mathscr{L}(\mathbb{C}^G,\mathbb{C}^G)$, $\lambda\mapsto\pi(\lambda)$, is a so-called projective representation [34]. (It is, in fact, up to isomorphisms, the only irreducible faithful projective representation of $G\times\widehat{G}$ on $\mathbb{C}^G$ [34].)

The results proven below have been derived in the setting of general finite Abelian groups in [33] and [34]. There, the authors use nontrivial facts from representation theory. Our aim remains to give a self-contained treatment of Gabor frames in finite dimensions, so we present elementary linear algebra proofs instead.

The following simple observation forms the foundation for most fundamental results in Gabor analysis. In abstract terms, (14) and (15) represent the previously mentioned fact that π is a projective representation.

Proposition 5. *For $\lambda,\mu\in G\times\widehat{G}$ exists $c_{\lambda,\mu},c_{\mu,\lambda}$ in $\mathbb{C}$, $|c_{\lambda,\mu}|=|c_{\mu,\lambda}|=1$, with*

$$\pi(\lambda)\pi(\mu)=c_{\lambda,\mu}\pi(\lambda+\mu)=c_{\lambda,\mu}\overline{c_{\mu,\lambda}}\pi(\mu)\pi(\lambda) \tag{14}$$

and

$$\pi(\lambda)^{-1} = \pi(\lambda)^* = c_{\lambda,\lambda}\pi(-\lambda). \tag{15}$$

If Λ is a subgroup of $G\times\widehat{G}$, then the time-frequency shifts $\pi(\mu)$, $\mu\in\Lambda$, commute with the (φ,Λ) Gabor frame operator

$$S:\mathbb{C}^G\longrightarrow\mathbb{C}^G,\quad x\mapsto\sum_{\lambda\in\Lambda}\langle x,\pi(\lambda)\varphi\rangle\,\pi(\lambda)\varphi$$

for every $\varphi\in\mathbb{C}^G$.

Proof. For $G=\mathbb{Z}_N$, a direct computation shows that $c_{(k,\ell)(\widetilde{k},\widetilde{\ell})}=e^{-2\pi ik\widetilde{\ell}/N}$. This implies (14) and (15) in the case of cyclic groups. The general case follows from the fact that any finite Abelian group is the product of cyclic groups, and the fact that time–frequency shift operators on $\mathbb{C}^G$ are tensor products of time-frequency shift operators on $\mathbb{C}^{\mathbb{Z}_N}$.

To show that $S\pi(\mu)=\pi(\mu)S$ for $\mu\in\Lambda$, we compute

$$\begin{aligned}\pi(\mu)^*S\pi(\mu)x &= \sum_{\lambda\in\Lambda}\langle\pi(\mu)f,\pi(\lambda)\varphi\rangle\pi(\mu)^*\pi(\lambda)\varphi\\ &= \sum_{\lambda\in\Lambda}\langle x,c_{\mu,\mu}\pi(-\mu)\pi(\lambda)\varphi\rangle c_{\mu,\mu}\pi(-\mu)\pi(\lambda)\varphi\\ &= |c_{\mu,\mu}|^2\sum_{\lambda\in\Lambda}\langle x,c_{\mu(-\lambda)}\pi(\lambda-\mu)\varphi\rangle c_{\mu(-\lambda)}\pi(\lambda-\mu)\varphi\\ &= \sum_{\lambda\in\Lambda}\langle x,\pi(\lambda-\mu)\varphi\rangle|c_{\mu(-\lambda)}|^2\pi(\lambda-\mu)\varphi\\ &= \sum_{\lambda\in\Lambda}\langle x,\pi(\lambda)\varphi\rangle\pi(\lambda)\varphi\;=\;Sx.\end{aligned}$$

The substitution in the last step utilizes the fact that $\mu\in\Lambda$ and Λ is a group.

As first consequence of Proposition 5, we derive *Janssen's representation* (17) of the Gabor frame operator [53].

To this end, define the *adjoint subgroup* of the subgroup $\Lambda\subseteq G\times\widehat{G}$ to be

$$\Lambda^\circ=\{\mu\in G\times\widehat{G}:\ \pi(\lambda)\pi(\mu)=\pi(\mu)\pi(\lambda)\ \text{ for all }\lambda\in\Lambda\}.$$

Similarly to $\left(\Lambda^\perp\right)^\perp=\Lambda$, we have $\left(\Lambda^\circ\right)^\circ=\Lambda$. For illustrative purposes, we depict some lattices, their duals, and their adjoints in Figure 7.

Theorem 4. *Let Λ be a subgroup of $G\times\widehat{G}$ and let $\varphi,\widetilde{\varphi}\in\mathbb{C}^G$. Then*

$$\sum_{\lambda\in\Lambda}\langle x,\pi(\lambda)\varphi\rangle\pi(\lambda)\widetilde{\varphi}=|\Lambda|/|G|\sum_{\mu\in\Lambda^\circ}\langle\widetilde{\varphi},\pi(\mu)\varphi\rangle\pi(\mu)x,\quad x\in\mathbb{C}^G. \tag{16}$$

In particular, the (φ,Λ) Gabor frame operator S has the form

$$S=|\Lambda|/|G|\sum_{\mu\in\Lambda^\circ}\langle\varphi,\pi(\mu)\varphi\rangle\,\pi(\mu). \tag{17}$$

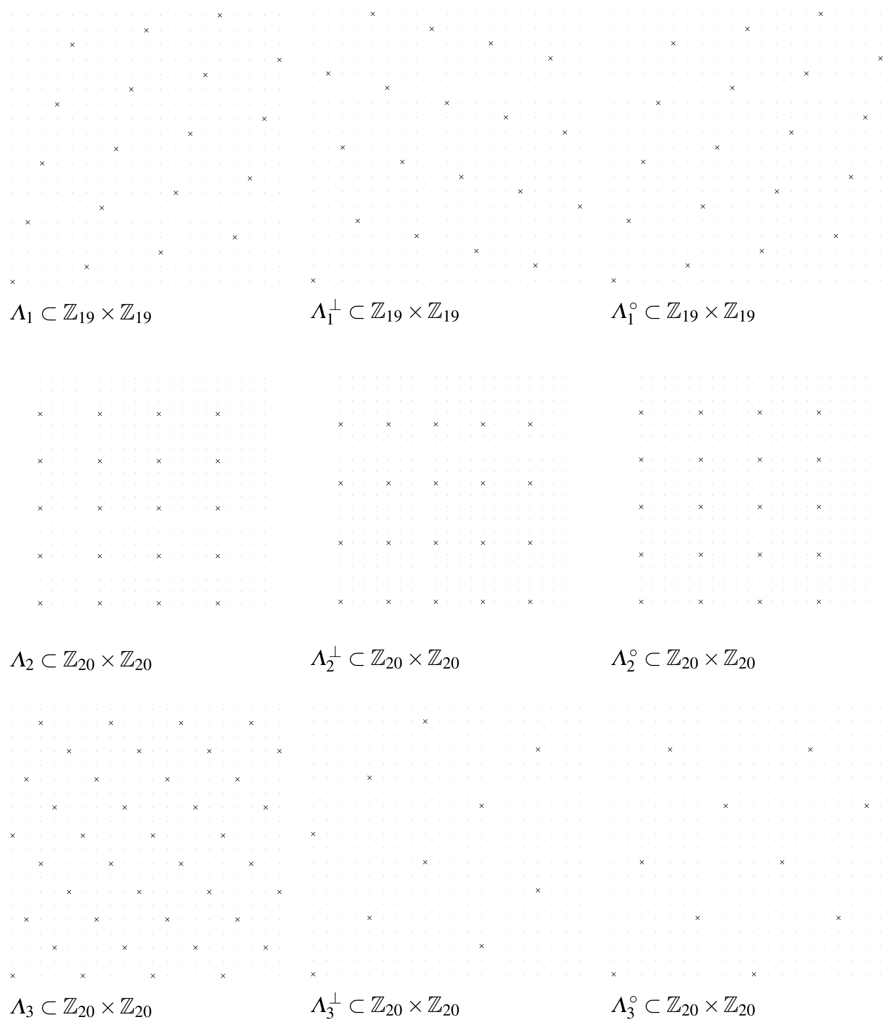

Fig. 7 Examples of lattices, their dual lattices, and their adjoint lattices. The lattice $\Lambda_1 \subset \mathbb{Z}_{19} \times \mathbb{Z}_{19}$ is the smallest subgroup of $\mathbb{Z}_{19} \times \mathbb{Z}_{19}$ containing $(1,4)$, $\Lambda_2 \subset \mathbb{Z}_{20} \times \mathbb{Z}_{20}$ is generated by $(1,2)$, and $\Lambda_3 \subset \mathbb{Z}_{20} \times \mathbb{Z}_{20}$ is the subgroup generated by the set $\{(1,4),(0,10)\}$.

Setting $K = \{k : \ (k,\ell) \in \Lambda \text{ for some } \ell \in \widehat{G}\}$, we note that the matrix representing the frame operator with respect to the Euclidean orthonormal basis has support in the union of $|K|$ (off) diagonals. Walnut's representation (21) below will give additional insight on the canonical matrix representation of Gabor frame operators.

Proof. Recall Proposition 4, namely the fact that $\{1/\sqrt{|G|}\ \pi(\lambda)\}_{\lambda \in G \times \widehat{G}}$ forms an orthonormal basis for the space of linear operators on $\mathbb{C}^G$ which is equipped with the Hilbert–Schmidt inner product. Hence, for $\varphi, \widetilde{\varphi} \in \mathbb{C}^G$, the operator

$$S : x \mapsto \sum_{\lambda \in \Lambda} \langle x, \pi(\lambda)\varphi \rangle \pi(\lambda)\widetilde{\varphi}$$

has a unique representation

$$S = \sum_{\mu \in G \times \widehat{G}} \eta_\mu \pi(\mu).$$

Applying Proposition 5 gives for any $\lambda \in \Lambda$

$$\sum_{\mu \in G \times \widehat{G}} \eta_\mu \pi(\mu) = S = \pi(\lambda)^* S \pi(\lambda) = \sum_{\mu \in G \times \widehat{G}} \eta_\mu \pi(\lambda)^* \pi(\mu) \pi(\lambda).$$

Equations (14) and (15) in Proposition 5 imply that $\pi(\lambda)^*\pi(\mu)\pi(\lambda)$ is a scalar multiple of $\pi(\mu)$. As the coefficients η_μ, $\mu \in G \times \widehat{G}$, are unique, we have for each $\mu \in G \times \widehat{G}$ either $\eta_\mu = 0$ or $\pi(\lambda)^*\pi(\mu)\pi(\lambda) = \pi(\mu)$ for all $\lambda \in \Lambda$, that is, $\mu \in \Lambda^\circ$. We conclude that $\eta_\mu = 0$ if $\mu \notin \Lambda^\circ$.

It remains to show that for $\mu \in \Lambda^\circ$, we have $\eta_\mu = |\Lambda|/|G|\ \langle \widetilde{\varphi}, \pi(\mu)\varphi \rangle$. To this end, note that the rank one operator $x \mapsto \langle x, \varphi \rangle \widetilde{\varphi}$ is represented by the matrix $\widetilde{\varphi}\,\overline{\varphi}^T$. Its Hilbert-Schmidt inner product with a matrix M satisfies $\langle \widetilde{\varphi}\,\overline{\varphi}^T, M \rangle_{HS} = \langle \widetilde{\varphi}, M\varphi \rangle$. Consequently, for $\mu \in \Lambda^\circ$, we have

$$\begin{aligned}
\eta_\mu &= 1/|G|\ \langle S, \pi(\mu) \rangle_{HS} = 1/|G| \sum_{\lambda \in \Lambda} \langle \pi(\lambda)\widetilde{\varphi}\,\overline{\pi(\lambda)\varphi}^T,\ \pi(\mu) \rangle_{HS} \\
&= 1/|G| \sum_{\lambda \in \Lambda} \langle \pi(\lambda)\widetilde{\varphi},\ \pi(\mu)\pi(\lambda)\varphi \rangle = 1/|G| \sum_{\lambda \in \Lambda} \langle \pi(\lambda)\widetilde{\varphi},\ \pi(\lambda)\pi(\mu)\varphi \rangle \\
&= 1/|G| \sum_{\lambda \in \Lambda} \langle \widetilde{\varphi},\ \pi(\mu)\varphi \rangle = |\Lambda|/|G|\ \langle \widetilde{\varphi},\ \pi(\mu)\varphi \rangle .
\end{aligned}$$

Taking inner products of the left hand and the right hand side of (16) with $\widetilde{x} \in \mathbb{C}^G$ shows that Janssen's representation implies the so-called *fundamental identity in Gabor analysis* (FIGA) (18) below, see also [35, 45].

Corollary 1. *Let Λ be a subgroup of $G \times \widehat{G}$. Then*

$$\sum_{\lambda \in \Lambda} V_\varphi x(\lambda)\ \overline{V_{\widetilde{\varphi}} \widetilde{x}(\lambda)} = |\Lambda|/|G| \sum_{\lambda \in \Lambda^\circ} V_\varphi \widetilde{\varphi}(\lambda)\ \overline{V_x \widetilde{x}(\lambda)}, \quad x, \widetilde{x}, \varphi, \widetilde{\varphi} \in \mathbb{C}^G. \tag{18}$$

An additional important consequence of Proposition 5 is the fact that the canonical duals of Gabor frames are again Gabor frames, that is, the canonical dual frame of a Gabor frame inherits the time-frequency structure of the original frame.

Theorem 5. *Let Λ be a subgroup of $G \times \widehat{G}$, and let the Gabor system (φ, Λ) span $\mathbb{C}^G$. The canonical dual frame of (φ, Λ) has the form $(\widetilde{\varphi}, \Lambda)$, that is, for appropriate $\widetilde{\varphi} \in \mathbb{C}^G$ we have*

$$x = \sum_{\lambda \in \Lambda} \langle x, \pi(\lambda)\widetilde{\varphi} \rangle \pi(\lambda)\varphi = \sum_{\lambda \in \Lambda} \langle x, \pi(\lambda)\varphi \rangle \pi(\lambda)\widetilde{\varphi}, \quad x \in \mathbb{C}^G.$$

Proof. Proposition 5 states that the (φ,Λ) frame operator

$$S:\mathbb{C}^G \longrightarrow \mathbb{C}^G, \quad x \mapsto \sum_{\lambda\in\Lambda} \langle x, \pi(\lambda)\varphi\rangle \pi(\lambda)\varphi,$$

and, consequently, its inverse S^{-1}, commutes with $\pi(\mu)$ for $\mu \in \Lambda$. Hence, the elements of the canonical dual frame of (φ,Λ) are of the form

$$\gamma_\lambda = S^{-1}\pi(\lambda)\varphi = \pi(\lambda)S^{-1}\varphi = \pi(\lambda)\widetilde{\varphi}, \quad \lambda \in \Lambda.$$

For overcomplete Gabor frames, that is, Gabor frames which span $\mathbb{C}^G$ and that have cardinality larger than $N = |G|$, the dual window is not unique. In fact, choosing dual frames different of the canonical dual frame may allow to reduce the computational complexity needed to compute the coefficients of a Gabor expansion [88].

Gabor frames $(\widetilde{\varphi},\Lambda)$ that are dual to (φ,Λ) are characterized by the following *Wexler–Raz criterion* (see [34, 95] and references therein). It is a direct consequence of Theorem 4.

Theorem 6. *Let Λ be a subgroup of $G\times\widehat{G}$. For the Gabor systems (φ,Λ) and $(\widetilde{\varphi},\Lambda)$, we have*

$$x = \sum_{\lambda\in\Lambda} \langle x, \pi(\lambda)\widetilde{\varphi}\rangle\, \pi(\lambda)\varphi, \quad x \in \mathbb{C}^G, \tag{19}$$

if and only if

$$\langle \varphi, \pi(\mu)\widetilde{\varphi}\rangle = |G|/|\Lambda|\ \delta_{\mu,0}, \quad \mu \in \Lambda^\circ. \tag{20}$$

Proof. Equation (19) implies that the operator $S: x \mapsto \sum_{\lambda\in\Lambda} \langle x, \pi(\lambda)\varphi\rangle \pi(\lambda)\varphi$ is the identity, that is, by Theorem 4 we have

$$\pi(0) = Id = S = |\Lambda|/|G| \sum_{\mu\in\Lambda^\circ} \langle \varphi, \pi(\mu)\widetilde{\varphi}\rangle \pi(\mu).$$

As the operators $\{\pi(\mu)\}$ are linearly independent by Proposition 4, we conclude that $|\Lambda|/|G|\ \langle \varphi, \pi(\mu)\widetilde{\varphi}\rangle = \delta_{\mu,0}$ which is (20).

The reverse implication follows trivially from Janssen's representation.

Corollary 2. *If Λ is a subgroup of $G\times\widehat{G}$, then (φ,Λ) is a tight frame for $\mathbb{C}^G$ if and only if (φ,Λ°) is an orthogonal set.*

Proof. The result follows from choosing $\widetilde{\varphi} = \varphi$ in (19) and (20).

Moreover, the Wexler–Raz criterion Theorem 6 implies the following *Ron–Shen duality* result [34, 80].

Theorem 7. *Let Λ be a subgroup of $G\times\widehat{G}$. The system (φ,Λ) is a frame for $\mathbb{C}^G$ if and only if (φ,Λ°) is a linear independent set.*

Proof. If (φ,Λ) is a frame, then Theorem 6 implies the existence of a dual window $\widetilde{\varphi}$ with $\langle\pi(\lambda)\varphi,\pi(\mu)\widetilde{\varphi}\rangle=\delta_{\lambda,\mu}$ for $\lambda,\mu\in\Lambda^\circ$. But then $0=\sum_{\lambda\in\Lambda^\circ}c_\lambda\pi(\lambda)$ implies for $\mu\in\Lambda^\circ$ that

$$0=\langle\sum_{\lambda\in\Lambda^\circ}c_\lambda\pi(\lambda)\varphi,\pi(\mu)\widetilde{\varphi}\rangle=\sum_{\lambda\in\Lambda^\circ}c_\lambda\langle\pi(\lambda)\varphi,\pi(\mu)\widetilde{\varphi}\rangle=c_\mu\langle\pi(\mu)\varphi,\pi(\mu)\widetilde{\varphi}\rangle,$$

and we conclude $c_\mu=0$ for all $\mu\in\Lambda^\circ$. Hence, (φ,Λ°) is linearly independent.

On the other hand, if (φ,Λ°) is a linear independent set, then exists a unique vector $\widetilde{\varphi}$ in span$\{\pi(\mu)\varphi\}_{\mu\in\Lambda^\circ}$ which is orthogonal to span$\{\pi(\mu)\varphi\}_{\mu\in\Lambda^\circ\setminus\{0\}}$ and $\langle\varphi,\pi(\mu)\widetilde{\varphi}\rangle=\delta_{\mu,0}$ for all $\mu\in\Lambda^\circ$. Theorem 6 implies that (φ,Λ) is a frame.

We close this section with a general version of *Walnut's representation* of the Gabor frame operator in the finite-dimensional setting.

Theorem 8. *For a subgroup Λ of $G\times\widehat{G}$, set $H_0=\{\ell:\ (0,\ell)\in\Lambda\}$ and $K=\{k:(k,\ell)\in\Lambda$ for some $\ell\}$. For each $k\in K$ choose an ℓ_k with $(k,\ell_k)\in\Lambda$. The (φ,Λ) Gabor frame operator matrix $(S_{\widetilde{n},n})$ satisfies*

$$S_{\widetilde{n},n}=|H_0|\ \chi_{H_0^\perp}(\widetilde{n}-n)\sum_{k\in K}\varphi(\widetilde{n}-k)\,\overline{\varphi(n-k)}\ \langle\ell_k,\widetilde{n}-n\rangle \tag{21}$$

where $H_0^\perp=\{\ell\in G:\ \langle\ell,k\rangle=1$ for all $k\in H_0\}$ denotes the annihilator subgroup of H_0. If $\Lambda=\Lambda_1\times\Lambda_2$, then (21) *reduces to*

$$S_{\widetilde{n}n}=|\Lambda_1|\ \chi_{\Lambda_2^\perp}(\widetilde{n}-n)\sum_{k\in\Lambda_1}\varphi(\widetilde{n}-k)\,\overline{\varphi(n-k)}\,. \tag{22}$$

Proof. For $k\in K$, let H_k denote the k-section of Λ, that is, $H_k=\{\ell:\ (k,\ell)\in\Lambda$ for some $\ell\in\widehat{G}\}$. Clearly, $\ell,\widetilde{\ell}\in H_k$ if and only if $\widetilde{\ell}-\ell\in H_0$. Hence, $H_k=H_0+\ell_k$ for any $\ell_k\in H_k\subseteq\widehat{G}$.

We compute

$$\begin{aligned}
S_{\widetilde{n},n}&=\sum_{\lambda\in\Lambda}\pi(\lambda)\varphi(\widetilde{n})\big(\pi(\lambda)\varphi(n)\big)^*\\
&=\sum_{k\in K}\sum_{\ell\in H_k}\varphi(\widetilde{n}-k)\,\langle\ell,\widetilde{n}\rangle\ \overline{\varphi(n-k)\,\langle\ell,n\rangle}\\
&=\sum_{k\in K}\varphi(\widetilde{n}-k)\,\overline{\varphi(n-k)}\ \sum_{\ell\in H_0}\langle\ell+\ell_k,\widetilde{n}-n\rangle\\
&=\sum_{k\in K}\varphi(\widetilde{n}-k)\,\overline{\varphi(n-k)}\ \langle\ell_k,\widetilde{n}-n\rangle\ \sum_{\ell\in H_0}\langle\ell,\widetilde{n}-n\rangle\\
&\overset{(12)}{=}\sum_{k\in K}\varphi(\widetilde{n}-k)\,\overline{\varphi(n-k)}\ \langle\ell_k,\widetilde{n}-n\rangle\ |H_0|\ \chi_{H_0^\perp}(\widetilde{n}-n).
\end{aligned}$$

Equation (22) follows directly from (21) by observing that $K=\Lambda_1$, $H_0=H_k=\Lambda_2$, and $\ell_k=0$ for $k\in\Lambda_1$.

Equation (22) implies that for real-valued φ the frame operator S for $(\varphi, \Lambda_1 \times \Lambda_2)$ restricts to $\mathbb{R}^G$ and, in particular, the dual frame generating window $\gamma = S^{-1}\varphi$ is then real-valued as well. The band structure of Gabor frame operators that is displayed in (21) and (22) is also observed in Janssen's representation (17). It shows that at most $|H_0^\perp|\,|G| = |G|/|H_0|$ entries of S are nonzero. This observation is in particular valuable if H_0, respectively Λ_2, is a large subgroup of $\widehat{G}$.

6 Linear independence

A traditional and frequent task in Gabor analysis on the real line is to show that a given Gabor system is a Riesz basis in, or a frame for, the Hilbert space of square integrable functions $L^2(\mathbb{R})$. Simple linear independence of Gabor systems in $L^2(\mathbb{R})$ was first considered by Heil, Ramanathan and Topiwala [49]. Their conjecture that the members of every Gabor system are linearly independent in $L^2(\mathbb{R})$ remains open to this date. In fact, it is unknown whether for all window functions φ in $L^2(\mathbb{R})$, the four functions in

$$\{\varphi(t),\ \varphi(t-1),\ e^{2\pi i t}\varphi(t),\ e^{2\pi i\sqrt{2}t}\varphi(t-\sqrt{2})\}$$

are linearly independent [25, 49].

In finite dimensions, a family of vectors is a Riesz basis for its span if and only if the vectors are linearly independent. Similarly, a family of vectors is a frame if and only if they span the finite-dimensional ambient space. Clearly, the dimension of the ambient space limits the number of linearly independent vectors, and in this section, we address the question of whether the vectors of a Gabor system in $\mathbb{C}^G$ are in *general linear position*. That is, we ask which Gabor frames (φ, Λ) have the property that every selection of less than or equal to $|G| = \dim \mathbb{C}^G$ vectors from (φ, Λ) are linearly independent.

As before, for a vector x in a finite-dimensional space let

$$\|x\|_0 = |\operatorname{supp} x|$$

count the nonzero entries of x. Also, recall that the *spark* of a matrix M is given by $\min\big\{\|c\|_0,\ c \neq 0,\ Mc = 0\big\}$. Rephrasing the above, we ask the question: for which φ and Λ is the spark of the (φ, Λ) synthesis operator equal to $|G|+1$? Note that in complementary work, upper bounds on the spark of certain Gabor synthesis operators were obtained [96].

Before stating the main results from [58, 62], we will motivate the here presented line of work by describing its relevance to information transmission in erasure channels and in operator identification [58]. As a byproduct of our analysis, we obtain a large family of unimodular tight frames that are maximally robust to erasures [18].

In generic communication systems, information in the form of a vector $x \in \mathbb{C}^G$ is not transmitted directly. First, it is coded in a way that allows for the recovery of

x at the receiver, regardless of errors that may be introduced by the communications channel. To achieve some robustness against errors, we can choose a frame $\{\varphi_k\}_{k\in K}$ for $\mathbb{C}^G$ and transmit x in the form of coefficients $\{\langle x,\varphi_k\rangle\}_{k\in K}$. At the receiver, a dual frame $\{\widetilde{\varphi}_k\}$ of $\{\varphi_k\}$ can be used to recover x via the frame reconstruction formula $x=\sum_k\langle x,\varphi_k\rangle\widetilde{\varphi}_k$.

In the case of an *erasure channel*, some of the transmitted coefficients may be lost. If only the coefficients $\{\langle x,\varphi_k\rangle\}_{k\in K'}$, $K'\subseteq K$, are received, then the original vector x can still be recovered[3] if and only if the subset $\{\varphi_k\}_{k\in K'}$ remains a frame for $\mathbb{C}^G$. Of course, this requires $|K'|\geq|G|=\dim\mathbb{C}^G$.

Definition 1. A frame $\Phi=\{\varphi_k\}_{k\in K}$ in $\mathbb{C}^G$ is maximally robust to erasures if the removal of any $L\leq|K|-|G|$ vectors from $\mathscr{F}$ leaves a frame.

By definition, a frame is maximally robust to erasures if and only if the frame vectors are in general linear position.

Another important application is the problem of identifying linear time-varying operators.

Definition 2. A linear space of operators $\mathscr{H}\subseteq\{H:\mathbb{C}^G\to\mathbb{C}^G,\ H\text{ linear}\}$ is identifiable with identifier φ if the linear map $E_\varphi:\mathscr{H}\to\mathbb{C}^G, H\mapsto H\varphi$, is injective.

A time-varying communication channel is frequently modeled as a linear combination of time-frequency shift operators. The idea behind this model is that the transmitted signal reaches the receiver through a small number of paths, each path causing a path-specific delay k, a path-specific frequency shift ℓ (due to Doppler effects), and a path-specific gain factor $c_{k,\ell}$. If we have *a priori* knowledge of the time-frequency shifts Λ caused by the paths the signals travel, then we aim to obtain knowledge of the gain factors, that is, we aim to identify operators from the class

$$\mathscr{H}_\Lambda=\Big\{\sum_{\lambda\in\Lambda}c_\lambda\pi(\lambda),\ c_\lambda\in\mathbb{C}\Big\},\quad \Lambda\subseteq G\times\hat{G}.$$

Clearly, knowing the channel is a crucial prerequisite for a successful transmission of information.

Often, the time delays and the modulation parameters are not known, but we may have an upper bound on the number of paths the signal may travel to the receiver. Then, we aim to identify the class of operators

$$\mathscr{H}_s=\{\sum_{\lambda\in\Lambda}c_\lambda\pi(\lambda),c_\lambda\in\mathbb{C},\ \Lambda\in G\times\hat{G}\ \text{ with }\ |\Lambda|\leq s\}\,. \tag{23}$$

The following result relates the concepts discussed above.

Theorem 9. *The following are equivalent for* $\varphi\in\mathbb{C}^G\backslash\{0\}$*:*

1. The Gabor system $(\varphi,G\times\widehat{G})$ *is in general linear position.*

[3] Here we assume that the receiver knows which coefficients have been erased and which coefficients have been received.

2. *The Gabor system $(\varphi, G\times\widehat{G})$ forms an equal norm tight frame which is maximally robust to erasures.*
3. *For all $x\in\mathbb{C}^G\setminus\{0\}$, $\|V_\varphi x\|_0 \geq |G|^2-|G|+1$.*
4. *For all $x\in\mathbb{C}^G$, $V_\varphi x$ and, therefore, x is completely determined by its values on any set Λ with $|\Lambda|=|G|$.*
5. *$\mathscr{H}_\Lambda$ is identifiable by φ if and only if $|\Lambda|\leq|G|$.*

If $|G|$ is even, then Statements 1–5 are equivalent to Statement 6 below, for $|G|$ odd, Statements 1–5 imply Statement 6:

6. *$\mathscr{H}_s$ is identifiable by φ if and only if $s\leq|G|/2$.*

Proof. The equivalence of Statements 1–5 follow from standard linear algebra arguments [58, 62]. Note in addition that to deduce Statement 2 from any of the other statements, we can use that *a priori* $(\varphi, G\times\widehat{G})$ is an equal norm tight frame as long as $\varphi\neq 0$.

For illustrative purposes, we give below a proof of Statement 1 implies Statement 6. Assume that the vectors in $(\varphi, G\times\widehat{G})$ are in general position and $s\leq|G|/2$. Then $H\varphi=\widetilde{H}\varphi$ for $H,\widetilde{H}$ implies

$$0=\sum_{\lambda\in\Lambda}c_\lambda\pi(\lambda)\varphi-\sum_{\widetilde{\lambda}\in\widetilde{\Lambda}}\widetilde{c}_{\widetilde{\lambda}}\pi(\widetilde{\lambda})\varphi.$$

Note that the right hand side is a linear combination of elements from $(\varphi,\Lambda\cup\widetilde{\Lambda})\subseteq(\varphi,G\times\widehat{G})$ with $|(\varphi,\Lambda\cup\widetilde{\Lambda})|=|\Lambda\cup\widetilde{\Lambda}|\leq 2|G|/2=|G|$. Statement 1 implies linear independence of $(\varphi,\Lambda\cup\widetilde{\Lambda})$, hence, all coefficients are 0 or cancel out. We conclude that $H=\widetilde{H}$.

A similar argument shows that, in general, $\mathscr{H}_s$ is not identifiable if $s>|G|/2$.

Theorem 9 leads to the question of whether a φ satisfying Statements 1–6 in Theorem 9 exists. For the special case $|G|$ prime, the answer is affirmative [58, 62].

Theorem 10. *If $G=\mathbb{Z}_p$, p prime, then exists φ in $\mathbb{C}^G$ such that Statements 1–6 in Theorem 9 are satisfied. Moreover, we can choose the vector φ to be unimodular.*

Proof. A complete proof is given in [62]. It is non-trivial and we will only recall some central ideas that are used in it.

Consider the Gabor window consisting of p complex variables $z_0,z_1,\dots,z_{p-1}$. Take $\Lambda\subseteq G\times\widehat{G}$ with $|\Lambda|=p$ and form a matrix from the p vectors in the Gabor system (z,Λ). The determinant of the matrix is a homogeneous polynomial P_Λ in $z_0,z_1,\dots,z_{p-1}$ of degree p. We have to show that $P_\Lambda\neq 0$. This is achieved by observing that at least one monomial appears in the polynomial P_Λ with a coefficient which is not 0. Indeed, it can be shown that there exists at least one monomial whose coefficient is the product of minors of the Fourier matrix W_p. We can apply Chebotarev's theorem on roots of unity (see Theorem 12). It states that every minor of the Fourier matrix W_p, p prime, is nonzero [30, 39, 90], a property that does not hold for groups with $|G|$ composite. Hence, $P_\Lambda\neq 0$.

We conclude that for each $\Lambda \subseteq G \times \widehat{G}$ with $|\Lambda| = p$, the determinant P_Λ vanishes only on the non-trivial algebraic variety $E_\Lambda = \{z = (z_0, z_1, \ldots z_{p-1}) : \; P_\Lambda(z) = 0\}$. E_Λ has Lebesgue measure 0; hence, any generic φ, that is,

$$\varphi \in \mathbb{C}^G \setminus \big(\bigcup_{\Lambda \subseteq G \times \widehat{G},\ |\Lambda| = p} E_\Lambda \big)$$

generates a Gabor system $(\varphi, G \times \widehat{G})$ in general linear position.

To show that we can choose an unimodular φ, it suffices to demonstrate that the set of unimodular vectors is not contained in $\bigcup_{\Lambda \subseteq G \times \widehat{G},\ |\Lambda| = p} E_\Lambda$ [58].

Theorem 10 is complemented by the following simple observation.

Theorem 11. *If $G = \mathbb{Z}^2 \times \mathbb{Z}^2$, then exists no φ in $\mathbb{C}^G$ such that the vectors in $(\varphi, G \times \widehat{G})$ are in general linear position.*

Proof. For a generic $\varphi = (c_0, c_1, c_2, c_3)^T$, we compute the determinant of the matrix with columns φ, $\pi((0,0),(1,0))\,\varphi$, $\pi((1,1),(0,0))\,\varphi$, and $\pi((1,1),(0,1))\varphi$; that is

$$\begin{aligned}
\det \begin{pmatrix} c_0 & c_0 & c_3 & c_3 \\ c_1 & c_1 & c_2 & -c_2 \\ c_2 & -c_2 & c_1 & c_1 \\ c_3 & -c_3 & c_0 & -c_0 \end{pmatrix} &= \det \begin{pmatrix} 0 & 2c_0 & 0 & 2c_3 \\ 0 & 2c_1 & 2c_2 & 0 \\ 2c_2 & 0 & 0 & 2c_1 \\ 2c_3 & 0 & 2c_0 & 0 \end{pmatrix} \\
&= -16\, c_0 \det \begin{pmatrix} 0 & c_2 & 0 \\ c_2 & 0 & c_1 \\ c_3 & c_0 & 0 \end{pmatrix} - 16\, c_3 \det \begin{pmatrix} 0 & c_1 & c_2 \\ c_2 & 0 & 0 \\ c_3 & 0 & c_0 \end{pmatrix} \\
&= -c_0 c_1 c_2 c_3 + c_0 c_1 c_2 c_3 = 0.
\end{aligned}$$

We conclude that for all φ, the four vectors φ, $\pi((0,0),(1,0))\,\varphi$, $\pi((1,1),(0,0))\,\varphi$, and $\pi((1,1),(0,1))\varphi$ are linearly dependent.

In [58], numerical results show that a vector which satisfies Statement 2, and therefore all statements in Theorem 9 for $G = \mathbb{Z}_4, \mathbb{Z}_6$, exists (see Figure 8). This observation leads to the following open question [58].

Question 1. For $G = \mathbb{Z}_N$, $N \in \mathbb{N}$, does there exist a window φ in $\mathbb{C}^G$ with $(\varphi, G \times \widehat{G})$ in general linear position?

The numerical procedure applied to resolve the cases $G = \mathbb{Z}_4$ and $\mathbb{Z}_6$ is unfortunately not applicable to larger groups of composite order. In fact, to answer Question 1 for the group $G = \mathbb{Z}_8$ numerically would require the computation of 64 choose 8, which is 4,426,165,368 determinants of 8 by 8 matrices. (Using symmetries, the amount of computation can be reduced, but not enough to allow for a numerical solution of the problem at hand.)

The proof of Theorem 10 outlined above is not constructive. In fact, with the exception of small primes 2,3,5,7, we cannot test numerically whether a given vector φ satisfies the statements in Theorem 9. Again, a naive direct approach to check whether the system $(\varphi, \mathbb{Z}_{11} \times \widehat{\mathbb{Z}_{11}})$ is in general linear position requires the computation of 121 choose 11, that is 1,276,749,965,026,536 determinants of 11 by 11 matrices.

Question 2. For $G=\mathbb{Z}_p$, p prime, does there exist an explicit construction of φ in $\mathbb{C}^G$ such that the vectors in $(\varphi, G\times\widehat{G})$ are in general linear position?

The truth of the matter is that for $G=\mathbb{Z}_p$, p prime, it is known that almost every vector φ generates a system $(\varphi, G\times\widehat{G})$ in general linear position, but aside from groups of order less than or equal to 7, not a single vector φ with $(\varphi, G\times\widehat{G})$ in general linear position is known.

As illustrated by Theorem 9, a positive answer to Questions 1 and 2 would have far-reaching applications. For example, to our knowledge, the only previously known *equal norm tight frames that are maximally robust to erasures* are so-called harmonic frames, that is, frames consisting of columns of Fourier matrices where some rows have been removed. (See, for example, the conclusions section in [18]). Similarly, Theorem 10 together with Theorem 9 provides us with equal norm tight frames with p^2 elements in $\mathbb{C}^N$ for $N\le p$: we can choose a unimodular φ satisfying the conclusions of Theorem 10 and remove uniformly $p-N$ components of the equal norm tight frame $(\varphi, G\times\widehat{G})$ in order to obtain an equal norm tight frame for $\mathbb{C}^N$ which is maximally robust to erasure. Obviously, the removal of components does not leave a Gabor frame proper. Alternatively, eliminating some vectors from a Gabor frame satisfying the conclusions of Theorem 10 leaves an equal norm Gabor frame which is maximally robust to erasure but which might not be tight.

We want to point out that a positive answer to Question 1 would imply the generalization of sampling of operator results that hold on the space of square integrable functions on the real line to operators defined on square integrable functions on Euclidean spaces of higher dimensions [75].

In the remainder of this section, we describe an observation that might be helpful to establish a positive answer to Question 1. *Chebotarev's theorem* can be phrased in the form of an uncertainty principle, that is, as a manifestation of the principle that x and $\widehat{x}$ cannot both be well localized at the same time [90]. Recall that $\|x\|_0=|\operatorname{supp} x|$.

Theorem 12. *For $G=\mathbb{Z}_p$, p prime, we have*

$$\|x\|_0+\|\widehat{x}\|_0\ge |G|+1=p+1,\quad x\in\mathbb{C}^G\backslash\{0\}.$$

The corresponding *time-frequency uncertainty* result for the short-time Fourier transform is the following [58, 62].

Theorem 13. *Let $G=\mathbb{Z}_p$, p prime. For appropriately chosen $\varphi\in\mathbb{C}^G$,*

$$\|x\|_0+\|V_\varphi x\|_0\ge |G\times\widehat{G}|+1=p^2+1,\quad x\in\mathbb{C}^G\setminus\{0\}\,.$$

Theorems 12 and 13 are sharp in the sense that all pairs (u,v) satisfying the respective bound will correspond to the support size pair of a vector and its Fourier transform, respectively its short-time Fourier transform. In particular, for almost every φ, we have that for all $1\le u\le |G|$, $1\le v\le |G|^2$ with $u+v\ge |G|^2+1$ there exists x with $\|x\|_0=u$ and $\|V_\varphi x\|_0=v$. Comparing Theorems 12 and 13, we observe that for $a,b\in\mathbb{Z}_p$, the pair of numbers (a,p^2-b) can be realized as $(\|x\|_0,\|V_\varphi x\|_0)$

if and only if $(a, p-b)$ can be realized as $(\|x\|_0, \|\widehat{x}\|_0)$. This observation leads to the following question [58].

Question 3. [62] For G cyclic, that is, $G=\mathbb{Z}_N$, $N\in\mathbb{N}$, exists φ in $\mathbb{C}^G$ such that

$$\big\{(\|x\|_0, \|V_\varphi x\|_0),\ x\in\mathbb{C}^G\big\} = \big\{(\|x\|_0, \|G\|^2-|G|+\|\widehat{x}\|_0),\ x\in\mathbb{C}^G\big\}\,?$$

Figure 8 compares the achievable support size pairs $(\|x\|_0, \|V_\varphi x\|_0)$, φ chosen appropriately, and $(\|x\|_0, \|\widehat{x}\|_0)$ for the groups $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$.

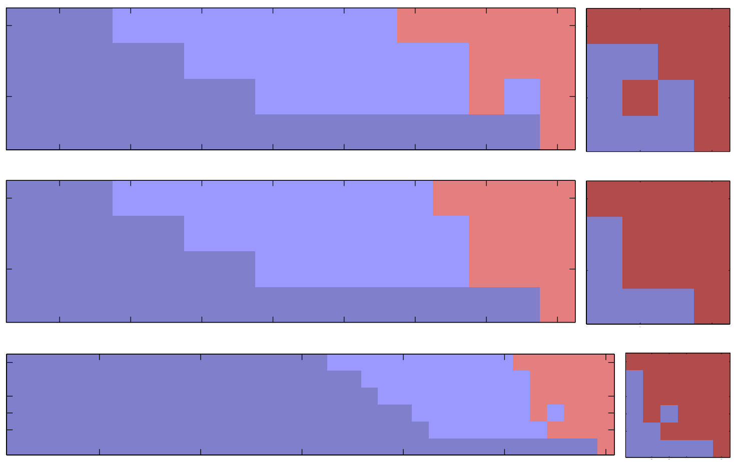

Fig. 8 The set $\big\{(\|x\|_0, \|V_\varphi x\|_0),\ x\in\mathbb{C}^G\backslash\{0\}\big\}$ for appropriately chosen $\varphi\in\mathbb{C}^G\backslash\{0\}$ for $G=\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_6$. For comparison, the right column shows the set $\big\{(\|x\|_0, \|\widehat{x}\|_0),\ x\in\mathbb{C}^G\backslash\{0\}\big\}$. Dark red/blue implies that it is proven analytically in [58] that the respective pair (u,v) is achieved / is not achieved, where φ is a generic window. Light red/blue implies that it was shown numerically that the respective pair (u,v) is achieved / is not achieved.

Note that any vector φ satisfying Statements 1–6 in Theorem 9 has the property that $\|\varphi\|_0=\|\widehat{\varphi}\|_0=|G|$ [62]. For arbitrary $\varphi\neq 0$, it is easily observed that

$$\|V_\varphi x\|_0 \geq |G|, \quad x\in\mathbb{C}^G, \tag{24}$$

and stronger qualitative statements on $\|V_\varphi x\|_0$ depending on $\|\varphi\|_0, \|\widehat{\varphi}\|_0, \|x\|_0, \|\widehat{x}\|_0$ are provided in [58].

Ghobber and Jaming obtained quantitative versions of (24) and Theorem 13. For example, the result below estimates the energy of x that can be captured by a small number of components of $V_\varphi x$ [41].

Theorem 14. *Let $G=\mathbb{Z}_N$, $N\in\mathbb{N}$. For φ with $\|\varphi\|=1$ and $\Lambda\subseteq G\times\widehat{G}$ with $|\Lambda|<|G|=N$, we have*

$$\sum_{\lambda\in\Lambda}|V_\varphi x(\lambda)|^2 \leq \big(1-(1-|\Lambda|/|G|)^2/8\big)\,\|x\|^2, \quad x\in\mathbb{C}^G.$$

7 Coherence

The analysis of the coherence of Gabor systems has a two-fold motivation. First of all, many equiangular frames have been constructed as Gabor frames and, second, a number of algorithms aimed at solving underdetermined system $Ax=b$ for a sparse vector x succeed if the coherence of columns in A is sufficiently small; see Section 8 and [27, 43, 92, 93, 94].

The *coherence* of a unit norm frame $\Phi=\{\varphi_k\}$ is given by

$$\mu(\Phi)\,=\,\max_{k\neq\ell}|\langle\varphi_k,\phi_\ell\rangle|.$$

That is, the coherence of a unit norm frame $\Phi=\{\varphi_k\}$ is the cosine of the smallest angle between elements from the frame. A unit norm frame $\Phi=\{\varphi_k\}$ with $|\langle\varphi_k,\varphi_{\widetilde{k}}\rangle|=\text{constant}$ for $k\neq\widetilde{k}$ is called an *equiangular frame*. It is easily seen that, among all unit norm frames with K elements in $\mathbb{C}^N$, the equiangular frames are those with minimal coherence.

If $\|\varphi\|=1$, then the Gabor system (φ,Λ) is unit norm and, if Λ is a subgroup of $G\times\widehat{G}$, then Proposition 5 implies that the coherence of (φ,Λ) is

$$\mu(\varphi,\Lambda)\,=\,\max_{\lambda\in\Lambda\setminus\{0\}}|\langle\varphi,\pi(\lambda)\varphi\rangle|=\max_{\lambda\in\Lambda\setminus\{0\}}|V_\varphi\varphi(\lambda)|.$$

In frame theory, it is a well-known fact that for any unit norm frame Φ of K vectors in $\mathbb{C}^N$, we have

$$\mu(\Phi)\,\geq\,\sqrt{\frac{K-N}{N(K-1)}}; \tag{25}$$

see, for example, [89] and references therein. For tight frames, (25) follows from a simple estimate of the magnitude of the off-diagonal entries of the Gram matrix $(\langle\varphi_k,\varphi_{\widetilde{k}}\rangle)$:

$$\begin{aligned}(K-1)K\,\mu(\Phi)^2 &\geq \sum_{k\neq\widetilde{k}}|\langle\varphi_k,\,\varphi_{\widetilde{k}}\rangle|^2=\sum_{k=1}^{K}\Big(-|\langle\varphi_k,\,\varphi_k\rangle|^2+\sum_{\widetilde{k}=1}^{K}|\langle\varphi_k,\,\varphi_{\widetilde{k}}\rangle|^2\Big)\\ &=\sum_{k=1}^{K}\big(-1+\tfrac{K}{N}\|\varphi_k\|^2\big)=\tfrac{K^2}{N}-K\,.\end{aligned} \tag{26}$$

This computation also shows that any tight frame with equality in (25) is equiangular. Note that equiangularity necessitates $K\leq N^2$, a result which holds for all unit norm frames [89].

The Gabor frame $(\varphi,G\times\widehat{G})$ has $|G|^2$ elements, and, hence, (25) simplifies to

$$\mu(\varphi,G\times\widehat{G})\,\geq\,\sqrt{\frac{|G|^2-|G|}{|G|(|G|^2-1)}}=\sqrt{\frac{|G|-1}{|G|^2-1}}=1/\sqrt{|G|+1}.$$

Alltop considered the window $\varphi_A \in \mathbb{C}^p$, $p \geq 5$ prime, with entries

$$\varphi_A(k) = p^{-1/2}\, e^{2\pi i k^3/p}, \quad k = 0,1,\ldots,p-1. \tag{27}$$

For the *Alltop window* function, we have [1, 89]

$$\mu(\varphi_A, \mathbb{Z}_p \times \widehat{\mathbb{Z}_p}) = 1/\sqrt{p}$$

which is close to the optimal lower bound $1/\sqrt{p+1}$. In fact, φ_A being unimodular implies that $(\varphi_A, G\times\widehat{G})$ is the union of $|G|$ orthonormal bases. A minor adjustment to the argument in (26) shows that whenever Φ is the union of N orthonormal bases for $\mathbb{C}^N$, we have necessarily $\mu(\Phi) \geq 1/\sqrt{N}$.

The Alltop window for $G = \mathbb{Z}_N$, N not prime, does not guarantee good coherence. For illustrative purposes, we display $|V_{\varphi_A}\varphi_A(\lambda)| = |\langle \varphi_A, \pi(\lambda)\varphi_A\rangle|$, $\lambda \in \mathbb{Z}_N \times \widehat{\mathbb{Z}_N}$, for $N = 6,7,8$,

$$\begin{pmatrix} 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&0&0&0\\ u&u&u&u&u&u&u\\ u&u&u&u&u&u&u\\ u&u&u&u&u&u&u\\ u&u&u&u&u&u&u\\ u&u&u&u&u&u&u\\ u&u&u&u&u&u&u \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&.5&0&.5&0&.5&0&.5\\ 1/\sqrt{2}&0&0&0&1/\sqrt{2}&0&0&0\\ 0&.5&0&.5&0&.5&0&.5\\ 0&0&0&0&1&0&0&0\\ 0&.5&0&.5&0&.5&0&.5\\ 1/\sqrt{2}&0&0&0&1/\sqrt{2}&0&0&0\\ 0&.5&0&.5&0&.5&0&.5 \end{pmatrix}, \tag{28}$$

where $u = 1/\sqrt{7} \approx 0.3880$.

Gabor systems $(\varphi_A, G\times\widehat{G})$ employing the Alltop window for $G = \mathbb{Z}_N$, $N \in \mathbb{N}$, were also analyzed numerically in [2] in terms of chirp sensing codes. In fact, the *frames of chirps* considered there are of the form

$$\Phi_{\text{chirps}} = \big\{\phi_\lambda(x) = \phi_{(k,\ell)}(x) = e^{2\pi i k x^2/N}\, e^{2\pi i \ell x/N}, \quad \lambda = (k,\ell) \in G\times\widehat{G}\big\}.$$

We have

$$\begin{aligned} \pi(k,\ell)\varphi_A(x) &= e^{2\pi i \ell x/N}\, e^{2\pi i (x-k)^3/N} = e^{2\pi i \ell x/N}\, e^{2\pi i (x^3 - 3x^2k + 3xk^2 - k^3)/N} \\ &= e^{-2\pi i k^3/N}\, e^{2\pi i x^3/N}\, e^{2\pi i (\ell - k^2)x/N}\, e^{-2\pi i 3kx^2/N} \\ &= \overline{e^{2\pi i k^3/N}\varphi_A(x)}\, \phi_{(3k,\ell-k^2)}(x), \end{aligned}$$

and if N is not divisible by 3, then Φ_{chirps} is, aside from renumbering, the unitary image of a Gabor frame with Alltop window. Hence, for N not divisible by 3, coherence results on $(\varphi_A, G\times\widehat{G})$ are identical to coherence results on Φ_{chirp}. Also, the restricted isometry constants (see Section 8) for $(\varphi_A, G\times\widehat{G})$ and Φ_{chirp} are identical for the same reason.

As an alternative to the Alltop sequence, J.J. Benedetto, R.L. Benedetto, and Woodworth used results from number theory such as Andre Weil's exponential sum bounds to estimate the coherence of Gabor frames based on Björk sequences as Gabor window functions [8, 12, 13]. Note that any Björk sequence φ_B is a *constant amplitude zero autocorrelation* (CAZAC) sequence, and, therefore, we have

$$\langle T_k\varphi_B, \varphi_B\rangle = 0 = \langle M_\ell \varphi_B, \varphi_B\rangle, \quad (k,\ell) \in G \times \widehat{G}.$$

Accounting again for the zero entries in the CAZAC Gabor frame Gram matrices, we observe that the smallest achievable coherence is $1/\sqrt{|G|-1}$.

For $p \geq 5$ prime with $p = 1 \bmod 4$, the *Björk sequence* $\varphi_B \in \mathbb{C}^{\mathbb{Z}_p}$ is given by

$$\varphi_B(x) = \frac{1}{\sqrt{p}} \begin{cases} 1, & \text{for } x = 0, \\ e^{i \arccos\left(1/(1+\sqrt{p})\right)}, & x = m^2 \bmod p \text{ for some } m = 1, \dots, p-1, \\ e^{-i \arccos\left(1/(1+\sqrt{p})\right)}, & \text{otherwise,} \end{cases}$$

and for $p \geq 3$ prime with $p = 3 \bmod 4$, we set

$$\varphi_B(x) = \frac{1}{\sqrt{p}} \begin{cases} e^{i \arccos\left((1-p)/(1+p)\right)/p}, & x \neq m^2 \bmod p \text{ for all } m = 0, 1, \dots, p-1, \\ 1, & \text{otherwise.} \end{cases}$$

Then [8]

$$\mu(\varphi_B, \mathbb{Z}_p \times \widehat{\mathbb{Z}_p}) < \frac{2}{\sqrt{p}} + \begin{cases} \frac{4}{p}, & p = 1 \bmod 4; \\ \frac{4}{p^{3/2}}, & p = 3 \bmod 4. \end{cases}$$

In comparison to (28) , the rounded values of $|V_{\varphi_B}\varphi_B(\lambda)| = |\langle \varphi_B, \pi(\lambda)\varphi_B\rangle|$, $\lambda \in \mathbb{Z}_N \times \widetilde{\mathbb{Z}_N}$ for $N = 7$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2955 & 0.3685 & 0.5991 & 0.1640 & 0.4489 & 0.4354 \\ 0 & 0.3685 & 0.1640 & 0.4354 & 0.2955 & 0.5991 & 0.4489 \\ 0 & 0.5991 & 0.4354 & 0.3685 & 0.4489 & 0.2955 & 0.1640 \\ 0 & 0.1640 & 0.2955 & 0.4489 & 0.3685 & 0.4354 & 0.5991 \\ 0 & 0.4489 & 0.5991 & 0.2955 & 0.4354 & 0.1640 & 0.3685 \\ 0 & 0.4354 & 0.4489 & 0.1640 & 0.5991 & 0.3685 & 0.2955 \end{pmatrix}.$$

To study the generic behavior of the coherence of Gabor systems $\mu(\varphi, \mathbb{Z}_N \times \widehat{\mathbb{Z}_N})$ for $N \in \mathbb{N}$, we turn to random windows. To this end, we let ε denote a random variable uniformly distributed on the torus $\{z \in \mathbb{C}, |z| = 1\}$. For $N \in \mathbb{N}$, we let φ_R be the random window function with entries

$$\varphi_R(x) = \frac{1}{\sqrt{N}} \varepsilon_x, \quad x = 0, \dots, N-1, \tag{29}$$

where the ε_x are independent copies of ε. In short, φ_R is a *normalized random Steinhaus sequence*.

For $N = 8$, the rounded values of $|V_{\varphi_R}\varphi_R(\lambda)|$, $\lambda \in \mathbb{Z}_N \times \widetilde{\mathbb{Z}_N}$, for a sample φ_R, are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1915 & 0.5266 & 0.3831 & 0.1418 & 0.1269 & 0.4575 & 0.5410 & 0.0341 \\ 0.0520 & 0.2736 & 0.2872 & 0.7912 & 0.2384 & 0.1880 & 0.0741 & 0.3411 \\ 0.3712 & 0.5519 & 0.2569 & 0.2757 & 0.5049 & 0.3123 & 0.2200 & 0.1215 \\ 0.0968 & 0.2423 & 0.6019 & 0.2632 & 0.1005 & 0.2632 & 0.6019 & 0.2423 \\ 0.3712 & 0.1215 & 0.2200 & 0.3123 & 0.5049 & 0.2757 & 0.2569 & 0.5519 \\ 0.0520 & 0.3411 & 0.0741 & 0.1880 & 0.2384 & 0.7912 & 0.2872 & 0.2736 \\ 0.1915 & 0.0341 & 0.5410 & 0.4575 & 0.1269 & 0.1418 & 0.3831 & 0.5266 \end{pmatrix}.$$

Here and in the following, $\mathbb{E}$ denotes expectation and $\mathbb{P}$ the probability of an event. In this context, a slight adjustment of the proof of Proposition 4.6 in [58] implies that for p prime

$$\mathbb{P}\big((\varphi_R, \mathbb{Z}_p \times \widehat{\mathbb{Z}_p}) \text{ is a unimodular tight frame maximally robust to erasures}\big) = 1.$$

The following result on the expected coherence of Gabor systems is given in [73]. Aside from the factor α, the coherence in Theorem 15 resembles with high probability the coherence $1/\sqrt{N}$ of the Alltop window and in this sense is close to the lower coherence bound $1/\sqrt{N+1}$.

Theorem 15. *Let $N \in \mathbb{N}$ and let φ_R be the random vector with entries*

$$\varphi_R(x) = \frac{1}{\sqrt{N}}\varepsilon_x, \quad x = 0, \ldots, N-1, \tag{30}$$

where the ε_x are independent and uniformly distributed on the torus $\{z \in \mathbb{C}, |z| = 1\}$. Then for $\alpha > 0$ and N even,

$$\mathbb{P}\Big(\mu(\varphi_R, \mathbb{Z}_N \times \widehat{\mathbb{Z}_N}) \ \geq \frac{\alpha}{\sqrt{N}}\Big) \leq 4N(N-1)e^{-\alpha^2/4},$$

while for N odd,

$$\mathbb{P}\Big(\mu(\varphi_R, \mathbb{Z}_N \times \widehat{\mathbb{Z}_N}) \ \geq \frac{\alpha}{\sqrt{N}}\Big) \leq 2N(N-1)\left(e^{-\frac{N-1}{N}\alpha^2/4} + e^{-\frac{N+1}{N}\alpha^2/4}\right).$$

For example, a window $\varphi \in \mathbb{C}^{10,000}$ chosen according to (30) generates a Gabor frame with coherence less than $8.6/\sqrt{10,000} = 0.086$ with probability exceeding $10,000 \cdot 9,999 \cdot e^{-8.6^2/4} \approx 0.0671$. Note that our result does not guarantee the existence of a Gabor frame for $\mathbb{C}^{10,000}$ with coherence 0.085. The Alltop window, though, provides us with a Gabor frame for $\mathbb{C}^{9,973}$ with coherence ≈ 0.0100.

Proof. The result is proven in full in [73]; here, we will simply give an outline of the proof in the case that N is even.

To estimate $\langle \varphi_R, \pi(\lambda)\varphi_R \rangle = \langle \varphi_R, M_\ell T_k \varphi_R \rangle$ for $\lambda = (k, \ell) \in G \times \widehat{G} \setminus \{0\}$, note first that if $k = 0$, then $\langle \varphi_R, M_\ell \varphi_R \rangle = \langle |\varphi_R|^2, M_\ell 1 \rangle = 0$ for $\ell \neq 0$.

For the case $k \neq 0$, choose first $\omega_q \in [0,1)$ in $\varepsilon_q = e^{2\pi i \omega_q}$ and observe that

$$\overline{\langle \varphi_R, \pi(\lambda)\varphi_R \rangle} = \langle \pi(\lambda)\varphi_R, \varphi_R \rangle = \frac{1}{N} \sum_{q \in G} e^{2\pi i \frac{q\ell}{N}} \varepsilon_{q-p} \overline{\varepsilon_q} = \frac{1}{N} \sum_{q \in G} e^{2\pi i \left(\omega_{q-p} - \omega_q + \frac{q\ell}{N} \right)}.$$

The random variables

$$\delta_q^\lambda = e^{2\pi i \left(k_{q-p} - \omega_q + \frac{q\ell}{n} \right)},$$

are uniformly distributed on the torus $\mathbb{T}$, but they are not jointly independent. As demonstrated in [73], these random variables can be split into two subsets of jointly independent random variables $\Lambda_1, \Lambda_2 \subseteq G$ with $|\Lambda_1| = |\Lambda_2| = N/2$.

The complex Bernstein inequality [94, Proposition 15], [71], implies that for an independent sequence $\varepsilon_q, q = 0, \dots, N-1$, of random variables that are uniformly distributed on the torus, we have

$$\mathbb{P}\left(\left| \sum_{q=0}^{N-1} \varepsilon_q \right| \geq Nu \right) \leq 2e^{-Nu^2/2}. \tag{31}$$

Using the pigeonhole principle and the inequality (31) leads to

$$\begin{aligned} \mathbb{P}\left(|\langle \pi(\lambda)\varphi_R, \varphi_R \rangle| \geq t\right) &\leq \mathbb{P}\big(\big| \sum_{q \in \Lambda^1} \delta_q^{(p,\ell)} \big| \geq Nt/2\big) + \mathbb{P}\big(\big| \sum_{q \in \Lambda^2} \delta_q^{(p,\ell)} \big| \geq Nt/2\big) \\ &\leq 4\exp(-Nt^2/4). \end{aligned}$$

Applying the union bound over all possible $\lambda \in G \times \widehat{G} \setminus \{(0,0)\}$ and choosing $t = \alpha/\sqrt{N}$ concludes the proof.

Remark 1. A Gabor system (φ, Λ) which is in general linear position, which has small coherence, or which satisfies the restricted isometry property, is generally not useful for time-frequency analysis as described in Section 3. Recall that in order to obtain meaningful spectrograms of time-frequency localized signals, we chose windows which were well localized in time and in frequency, that is, we chose windows so that $V_\varphi \varphi(k,\ell) = \langle \varphi, \pi(k,\ell)\varphi \rangle$ was small for k, ℓ far from 0 (in the cyclic group $\mathbb{Z}_N$). To achieve a good coherence, though, we attempt to seek out φ such that $V_\varphi \varphi(k,\ell)$ is close to being a constant function on all of the time-frequency plane.

To illustrate how inappropriate it is to use windows as discussed in Section 6–Section 9, we perform in Figure 9 the analysis carried out in Figures 2–6 with a window chosen according to (30).

8 Restricted isometry constants

The coherence of a unit norm frame measures the smallest angle between two distinct elements of the frame. In the theory of compressed sensing, it is crucial to understand the geometry of subsets of frames that contain a small number of elements, but more than just two elements. The coherence of unit norm frames can be

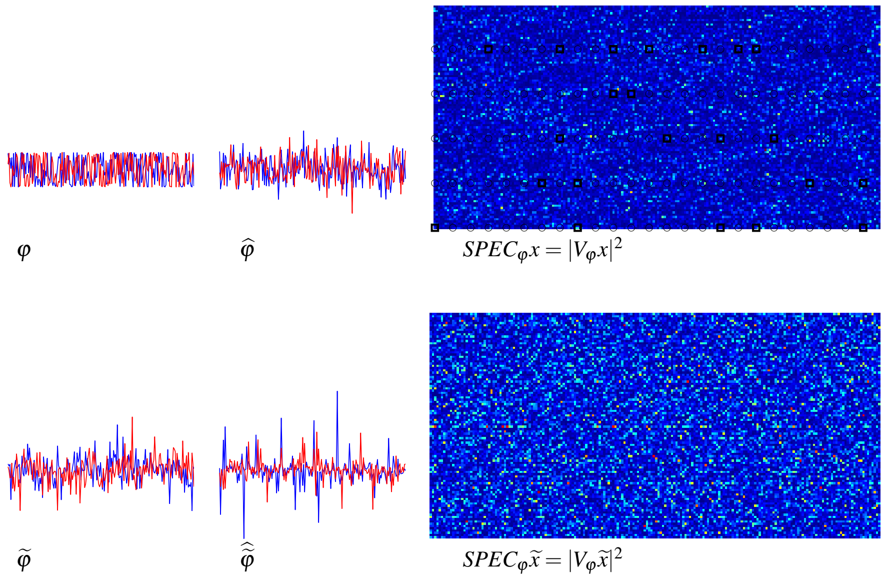

Fig. 9 We carry out the same analysis of the signal in Figure 1 as in Figures 2– 6. The Gabor system uses as window $\varphi = \varphi_R$ given in (30). The functions φ, $\widetilde{\varphi}$, are both not localized to an area in time or in frequency; in fact, this serves as an advantage in compressed sensing. We display again only the lower half of the spectrogram of x and of its approximation $\widetilde{x}$. Both are of little use. The lattice used is given by $\Lambda = \{0, 8, 16, \dots, 192\} \times \{0, 20, 40, \dots, 180\}$ and is marked by circles. Those of the 40 biggest frame coefficients in the part of the spectrogram shown are marked by squares.

used to control the behavior of small subsets, but the compressed sensing results achieved in this manner are rather weak. To capture the geometry of small families of vectors, the concept of *restricted isometry constants* (RICs) has been developed to obtain useful results in the area of compressed sensing [15, 16, 38, 79].

The *restricted isometry constant* $\delta_s(\Phi) = \delta_s$, $2 \le s \le N$, of a frame Φ of M elements in $\mathbb{C}^N$, is the smallest $0 < \delta_s < 1$ that satisfies

$$(1-\delta_s)\sum_{i=1}^{M}|c_i|^2 \le \Big\|\sum_{i=1}^{M}c_i\varphi_i\Big\|_2^2 \le (1+\delta_s)\sum_{i=1}^{M}|c_k|^2 \quad \text{for all } c \text{ with } \|c\|_0 \le s. \qquad (32)$$

A simple computation shows that the coherence of a unit norm frame Φ satisfies $\mu(\Phi) = \delta_2(\Phi)$.

Statement (32) implies that every subfamily of s vectors forms a Riesz system with Riesz bounds $(1-\delta_s)$, $(1+\delta_s)$. In particular, the existence of a restricted isometry constant implies that any s vectors in Φ are linearly independent.

Frames with small restricted isometry constants for s sufficiently large are difficult to construct. A trick to bypass the problem of having to do an intricate study of all possible selections of s vectors from a frame Φ with M elements, $M >> s$, is to introduce randomness in the definition of the frame. For example, if each component of each vector in a frame is generated independently by a fixed random process, then every family of s vectors is structured identically and the probability that a target δ_s fails can be estimated using a union bound argument.

To obtain results on restricted isometry constants of generic Gabor systems, we will choose as window function again φ_R, namely, the normalized random Steinhaus sequence defined in (29). The following is the main result in [74].

Theorem 16. *Let $G=\mathbb{Z}_N$ and let φ_R be a normalized Steinhaus sequence.*

1. *The expectation of the restricted isometry constant δ_s of $(\varphi_R, G\times\widehat{G})$, $s\leq N$, satisfies*

$$\mathbb{E}\,\delta_s \leq \max\left\{C_1\sqrt{\frac{s^{3/2}}{N}}\log s\sqrt{\log N},\ C_2\frac{s^{3/2}\log^{3/2}N}{N}\right\},$$

where $C_1, C_2>0$ are universal constants.

2. *For $0\leq\lambda\leq 1$, we have*

$$\mathbb{P}(\delta_s \geq \mathbb{E}[\delta_s]+\lambda) \leq e^{-\lambda^2/\sigma^2}, \quad \textit{where } \sigma^2 = \frac{C_3 s^{\frac{3}{2}}\log N\log^2 s}{N}$$

with $C_3>0$ being a universal constant.

The result remains true when generating the entries of φ by any Gaussian or subgaussian random variable. In particular, the result holds true if the entries of φ are generated with a Bernoulli process; in this case, the Shannon entropy of the generated $N\times N^2$ matrix is remarkably small, namely, N bits [74].

9 Gabor synthesis matrices for compressed sensing

The problem of determining a signal in a high-dimensional space by combining *a priori* nonlinear information on a vector or on its Fourier transform with a small number of linear measurements appears frequently in the natural sciences and engineering. Here, we will address the problem of determining a vector $F\in\mathbb{C}^M$ by N linear measurements under the assumption that

$$\|F\|_0 = |\{n:\ F(n)\neq 0\}| \leq s, \quad s\ll N\ll M.$$

This topic is treated in general terms in Section X.X; we will focus entirely on the case where the linear measurements are achieved through the application of a Gabor frame synthesis matrix.

In detail, with T^*_φ denoting the $(\varphi, G\times\widehat{G})$ synthesis operator and

$$\Sigma_s = \{F\in\mathbb{C}^{G\times\widehat{G}}:\ \|F\|_0\leq s\},$$

we ask the question for which s, every vector $F \in \Sigma_s \subseteq \mathbb{C}^{G\times\widehat{G}}$ can be recovered efficiently from
$$T_\varphi^* F = \sum_{\lambda\in G\times\widehat{G}} F_\lambda \pi(\lambda)\varphi \in \mathbb{C}^G.$$

The problem of finding the sparse vector $F \in \Sigma_s$ from $T_\varphi^* F$ is identical to the problem of identifying $\mathscr{H}_s$ as defined in (23) from the observation of $H\varphi = \sum_{\lambda\in G\times\widehat{G}} \eta_\lambda \pi(\lambda)\varphi$. This holds since $\{\pi(\lambda)\}_{\lambda\in G\times\widehat{G}}$ is a linear independent set in the space of linear operators on $\mathbb{C}^G$, and, hence, the coefficient vector η is in one-to-one correspondence with the respective channel operator [73].

In addition, the problem at hand can be rephrased as follows. Suppose we know that a vector $x \in \mathbb{C}^G$ has the form $x = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)\varphi$ with $|\Lambda| \le s$, that is, x is the linear combination of at most s frame elements from (φ,Λ). Can we compute the coefficients c_λ? Obviously, x can be expanded in $(\varphi, G\times\widehat{G})$ in many ways, for example, using

$$x = \sum_{\lambda\in\Lambda} \langle x, \pi(\lambda)\widetilde{\varphi}\rangle \pi(\lambda)\varphi \tag{33}$$

where $(\widetilde{\varphi},\Lambda)$ is a dual frame of (φ,Λ). The coefficients in (33) are optimal in the sense that they have the lowest possible ℓ^2-norm. In this section, though, the goal is to find the expansion involving the fewest non-zero coefficients.

Theorem 10 implies that for $G = \mathbb{Z}_p$, p prime, there exists φ with the elements of $(\varphi, G\times\widehat{G})$ being in general linear position. Consequently, if $s \le p/2$, then T_φ^* is injective on Σ_s and recovering F from $T_\varphi^* F$ is always possible, but this may not be computationally feasible, as every one of the $|G\times\widehat{G}|$ choose s possible subsets of $G\times\widehat{G}$ sets of F would have to be considered as support sets of F.

To obtain a numerically feasible problem, we have to reduce s, and indeed, for small s, the literature contains a number of criteria on the so-called measurement matrix M to allow the computation of F from MF by algorithms such as *basis pursuit* (BP) (see (X.X) in Section X.X) and *orthogonal matching pursuit* (OMP) (see (X.X) in Section X.X).

It is known that the success of basis pursuit and orthogonal matching pursuit for small s can be guaranteed if the coherence of the columns of a measurement matrix is small, in our setting, if $\mu(\varphi, G\times\widehat{G}) < 1/(2s-1)$ [26, 92]. In fact, combining this result with our coherence results in Section 7 — in particular, the coherence of the Alltop frame $(\varphi_A, G\times\widehat{G})$ for $G = \mathbb{Z}_p$, p prime — leads to the following results [73].

Theorem 17. *Let $G = \mathbb{Z}_p$, p prime, and let φ_A be the Alltop window given in* (27). *If $s < \frac{\sqrt{p}+1}{2}$ then Basis Pursuit recovers F from $T_{\varphi_A}^* F$ for every $F \in \Sigma_s \subseteq G\times\widehat{G}$.*

In the case of Steinhaus sequences, Theorem 15 implies the following theorem [73].

Theorem 18. *Let $G = \mathbb{Z}_N$, N even. Let φ_R be the random unimodular window in* (29). *Let $t > 0$ and*

$$s \leq \frac{1}{4}\sqrt{\frac{N}{2\log N + \log 4 + t}} + \frac{1}{2}\,.$$

Then with probability $1-e^{-t}$*, Basis Pursuit recovers* F *from* $T^*_{\varphi_R}F$ *for every* $F \in \Sigma_s$

Note that in Theorems 17 and 18, the number of measurements N required to guarantee the recovery of every s-sparse vector scales as s^2 in. This can be improved upon if we are satisfied to recover an s-sparse vector with high probability [72].

Theorem 19. *Let* $G=\mathbb{Z}_N$*,* $N\in\mathbb{N}$*. There exists* $C>0$ *so that whenever* $s\leq CN/\log(N/\varepsilon)$*, the following holds: for* $F\in\Sigma_s$ *choose* φ_R *according to* (30)*, then with probability at least* $1-\varepsilon$ *Basis Pursuit recovers* F *from* $T^*_{\varphi_R}F$*.*

Clearly, in Theorem 19 s scales as $N/\log(N)$, but we recover the vector F only with high probability.

The estimates on the restricted isometry constants in Theorem 16 imply that with high probability the Gabor synthesis matrix $T^*_{\varphi_R}$ guarantees that Basis Pursuit recovers $F\in\Sigma_s$ from $T^*_{\varphi_R}F$ if s is of the order $N^{2/3}/\log^2 N$ [74]. This follows from the fact that Basis Pursuit recovers $F\in\Sigma_s$ if $\delta_{2s}(\varphi_R, G\times\widehat{G}) \leq 3/(4+\sqrt{6})$ [15, 17].

Numerical simulations show that the recoverability guarantees given above are rather pessimistic. In fact, the performance of Gabor synthesis matrices with Alltop window φ_A and with random window φ_R as measurement matrices seem to perform similarly well as, for example, random Gaussian matrices [73].

For related Gabor frame results aimed at recovering signals that are only well approximated by s-sparse vectors, see [72, 73, 74].

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