

Remarks on multivariate Gaussian Gabor frames

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Abstract

We report on initial findings on Gabor systems with multivariate Gaussian window. Unlike the existing characterisation for dimension one in terms of lattice density, our results indicate that the behavior of Gaussians in higher-dimensional Gabor systems is intricate and further exploration is a valuable and challenging task.

Keywords. Gabor frames and Riesz bases, sampling in Bargmann-Fock spaces, Beurling density

1 Introduction

Gabor's seminal paper [Gab46] claimed that every function in $L^2(\mathbb{R})$ can be well represented as a series of translated and modulated copies of the Gaussian $\mathbf{g}_1(x) = e^{-\pi|x|^2}$. In detail, he suggested that for every $f \in L^2(\mathbb{R})$ there exists a sequence of scalars $c_{kl}(f)$ such that

$$f(t) = \sum_{k,l \in \mathbb{Z}} c_{kl}(f) e^{2\pi i l t} \mathbf{g}_1(t - k). \quad (1)$$

But while its central role in analysis and its wide spectrum of nice analytic properties, for example, optimal time–frequency concentration, make the Gaussian a natural candidate to be a window function for so-called Gabor systems, it is now well established that any series representation of the form (1) only converges to f in a distributional sense, and not in the L^2 -norm [Jan81]. Today, the spanning properties of the Gabor system $(\mathbf{g}_1, a\mathbb{Z} \times b\mathbb{Z}) = \{e^{2\pi i b l t} \mathbf{g}_1(t - a k)\}$, $a, b > 0$, are fully understood, for example, the system suggested by Gabor turns out to be overcomplete ([Lyu92, SW92] and Theorem 6 below).

Multivariate Gaussian and general Gabor systems though are far from being understood. While, for example, it is known that for any g , the Gabor system $(g, \Lambda) = \{e^{2\pi i \omega t} g(t - x) : (x, \omega) \in \Lambda\}$ is not a frame for $L^2(\mathbb{R}^d)$ if the set Λ has density less than 1, nontrivial positive results for given window functions such as the Gaussian are scarce in the literature [Hei07]. Simultaneously to our work,

Gröchenig has started to study of multivariate Gaussian Gabor systems. His focus though is on so-called complex lattices [Grö10]. Here, we provide some results that illustrate the intricate structure of Gaussian Gabor frames in higher dimensions for real lattices.

2 Gabor frames

We denote by \mathfrak{g}_d the d -dimensional normalized Gaussian $2^{\frac{d}{4}}e^{-\pi\|x\|^2}$. It is clear that $\mathfrak{g}_d = \otimes_{d \text{ times}} \mathfrak{g}_1$. A *translation* or *time shift* is the operator $(T_x f)(t) = f(t - x)$, $x \in \mathbb{R}^d$, and a *modulation* or *frequency shift* is the operator $(M_\omega f)(t) = e^{2\pi i \langle \omega, t \rangle} f(t)$, $\omega \in \widehat{\mathbb{R}}^d$. A *time-frequency shift* is then

$$(\pi(\lambda)f)(t) = (M_\omega T_x f)(t) = e^{2\pi i \langle \omega, t \rangle} f(t - x), \quad \lambda = (x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Definition 1 Let $\Lambda \subset \mathbb{R}^{2d}$ be a discrete set. A *Gabor system* (g, Λ) for $L^2(\mathbb{R}^d)$ is the set of all time-frequency shifts of the window function g by $\lambda = (x, \omega) \in \Lambda$, that is, $(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$.

Note that it is an easy consequence from Fourier analysis that $(\chi_{[0,1)}, \mathbb{Z} \times \mathbb{Z})$ is an orthonormal basis for $L^2(\mathbb{R})$. For $(\chi_{[0,1)}, \Lambda)$, with $\Lambda \neq \mathbb{Z} \times \mathbb{Z}$, the situation is already quite delicate as shown in [Jan03].

While Gabor orthonormal bases are useful, the so-called Balian-Low Theorem implies that they have a crucial shortcoming. Namely, if (g, Λ) is an orthonormal basis for $L^2(\mathbb{R})$, then g can not be well-localized in both time and frequency, in fact, we then have

$$\int |xg(x)|^2 dx \int |\xi \widehat{g}(\xi)|^2 d\xi = \infty.$$

Consequently, in Gabor analysis we resort to consider frames and Riesz bases.

Definition 2 A family of functions $\{\phi_k\}_{k \in \mathbb{Z}} \subset L^2(\mathbb{R}^d)$ with

$$A\|f\|_2^2 \leq \sum_k |\langle f, \phi_k \rangle|^2 \leq B\|f\|_2^2, \quad f \in L^2(\mathbb{R}^d), \quad (2)$$

for positive A and B is called a *frame* for $L^2(\mathbb{R}^d)$. The constants A and B are called, respectively, a *lower frame bound* and an *upper frame bound* of the frame $\{\phi_k\}$.

Definition 3 A family of functions $\{e_k\}_{k \in \mathbb{Z}} \subset L^2(\mathbb{R}^d)$ with

$$A\|c\|_2^2 \leq \left\| \sum_k c_k e_k \right\|_2^2 \leq B\|c\|_2^2, \quad c \in \ell_0^2(\mathbb{R}^d),$$

for positive A and B is called a *Riesz basis*.

For a detailed description of frame and Riesz basis theory we refer to [Grö01, Chr03].

To consider lattices, or more general countable sets in \mathbb{R}^{2d} , we state the definition of density and summarize its role in Gabor analysis. Let $\mathfrak{B}_d(R)$ denote the l^2 -ball in \mathbb{R}^d centered at 0 and with radius R .

Definition 4 *The lower and upper Beurling densities of $\mathcal{M} \subset \mathbb{R}^d$ are given by, respectively,*

$$D^-(\mathcal{M}) = \liminf_{R \rightarrow \infty} \inf_{z \in \mathbb{R}^d} \frac{|\mathcal{M} \cap \{\mathfrak{B}_d(R) + z\}|}{\pi R^d},$$

$$D^+(\mathcal{M}) = \limsup_{R \rightarrow \infty} \sup_{z \in \mathbb{R}^d} \frac{|\mathcal{M} \cap \{\mathfrak{B}_d(R) + z\}|}{\pi R^d}.$$

Whenever $D^-(\mathcal{M}) = D^+(\mathcal{M})$, then \mathcal{M} is said to have a uniform Beurling density, denoted by $D(\mathcal{M}) = D^-(\mathcal{M}) = D^+(\mathcal{M})$. \mathcal{M} is uniformly separated if $\inf\{|\lambda - \mu| : \lambda \neq \mu \in \mathcal{M}\} > 0$. If \mathcal{M} is a full-rank lattice, that is, $\mathcal{M} = AZ^d$, $\det A \neq 0$, then \mathcal{M} is uniformly separated and $D(\mathcal{M}) = \frac{1}{\det A}$.

Theorem 5 (Density theorem) *Let $g \in L^2(\mathbb{R}^d)$ and let Λ be a full-rank lattice.*

1. *If $D(\Lambda) < 1$, then (g, Λ) is incomplete in $L^2(\mathbb{R}^d)$.*
2. *If (g, Λ) is a frame for $L^2(\mathbb{R}^d)$, then $D(\Lambda) \geq 1$.*
3. *If (g, Λ) is a Riesz basis for its closed linear span, then $D(\Lambda) \leq 1$.*

Thus, if (g, Λ) is an orthonormal basis, then $D(\Lambda) = 1$.

The results listed in Theorem 5 have roots in various papers; they are nicely summarized in [Hei07]. In the one-dimensional case, Gaussian Gabor frames and Riesz bases are well characterized [Lyu92, SW92].

Theorem 6 *Let $\Lambda \subset \mathbb{R}^2$. The Gabor system (\mathbf{g}_1, Λ) is a frame if and only if there exists a uniformly separated $\Lambda' \subset \Lambda$ such that $1 < D^-(\Lambda') \leq D^+(\Lambda) < \infty$. If Λ is uniformly separated and $D^+(\Lambda) < 1$, then (\mathbf{g}_1, Λ) is a Riesz basis of a proper subspace of $L^2(\mathbb{R})$.*

In their proofs, Lyubarski and Seip-Wallsten used methods from complex analysis; the connection between Gaussian Gabor frames and complex analysis is described below.

The *short-time Fourier transform*, also called *continuous Gabor transform* or *windowed Fourier transform* is defined formally by

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i \omega t} \overline{g(t-x)} dt = \langle f, M_\omega T_x g \rangle = \langle \widehat{f}, T_\omega M_{-x} \widehat{g} \rangle$$

If $\|g\|_2 = 1$, for example, if $g = \mathbf{g}_d$, then the short-time Fourier transform is a unitary operator, so $\|V_g f\|_{L^2(\mathbb{R}^{2d})}^2 = \|f\|^2$. In the Gabor case, the frame property (2) becomes a sampling set condition: there exist constants $A, B > 0$ such that

$$A\|V_g f\|_{L^2(\mathbb{R}^{2d})}^2 \leq \sum_{\lambda \in \Lambda} |V_g f(\lambda)|^2 \leq B\|V_g f\|_{L^2(\mathbb{R}^{2d})}^2, \quad f \in L^2(\mathbb{R}^d).$$

The Bargmann-Fock space is defined by

$$\mathcal{F}(\mathbb{C}^d) = \left\{ F - \text{entire with } \|F\|_{\mathcal{F}} = \left(\int_{\mathbb{C}^d} |F(z)|^2 e^{-\pi|z|^2} dz \right)^{\frac{1}{2}} < \infty \right\}.$$

The Bargmann transform B maps $L^2(\mathbb{R}^d)$ unitarily onto $\mathcal{F}(\mathbb{C}^d)$ by

$$B : f \mapsto Bf : z \mapsto 2^{\frac{1}{4}} \int f(t) e^{2\pi i t z - \pi t^2 - \pi \frac{z^2}{2}} dt, \quad z \in \mathbb{C}^d.$$

With this notation it is easy to see that

$$V_{\mathbf{g}_d} f(x, -\xi) = e^{2\pi i x \xi} Bf(x + i\xi) e^{-\frac{\pi}{2}|x + i\xi|^2},$$

[Lyu92, SW92]. This demonstrates that (\mathbf{g}_d, Λ) is a frame if and only if Λ is a sampling set for \mathcal{F} . In $d = 1$ this was used to prove Theorem 6 [SW92, Lyu92]. But in higher dimension it appears as if as little is known about sampling in Bargmann-Fock spaces as is known about multivariate Gaussian Gabor frames (see [Grö10] for a more detailed discussion of this).

3 Gaussian Gabor frames for $L^2(\mathbb{R}^d)$

The easiest way to create frames for $L^2(\mathbb{R}^d)$, $d \geq 2$, is to take tensor products of lower-dimensional frame systems. For n lattices $\Lambda_1, \dots, \Lambda_n$ of the same dimension, we set $\odot_{i=1}^n \Lambda_i = \{(x_1, \dots, x_n) \times (\omega_1, \dots, \omega_n) : (x_i, \omega_i) \in \Lambda_i\}$.

Lemma 7 *Let (g, Λ_1) and (h, Λ_2) be frames for $L^2(\mathbb{R}^d)$. Then $(g \otimes h, \Lambda_1 \odot \Lambda_2)$ is a frame for $L^2(\mathbb{R}^{2d})$.*

For a simple proof we refer to [PR10].

Proposition 8 *Let $\Lambda = \mathbb{Z}^2 \times \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mathbb{Z}^2$. If $a < 1$ and $b < 1$, then (\mathbf{g}_2, Λ) is a frame for $L^2(\mathbb{R}^2)$. If $a = b = 1$, (\mathbf{g}_2, Λ) is complete in $L^2(\mathbb{R}^2)$, but not a frame. If $a > 1$ or $b > 1$, then (\mathbf{g}_2, Λ) is incomplete.*

Note that here $D(\Lambda) = \frac{1}{ab}$. Below we shall use the Zak transform which is defined via the series

$$Zf(x, \omega) = \sum_{k \in \mathbb{Z}^d} f(x - k) e^{2\pi i k \omega}, \quad (x, \omega) \in \mathbb{R}^{2d}.$$

For a detailed presentation of the properties of the Zak transform we refer to [Grö01, Chr03].

Proof of Proposition 8. If $a < 1, b < 1$, then Theorem 6 implies $(\mathfrak{g}_1, \mathbb{Z} \times a\mathbb{Z})$ and $(\mathfrak{g}_1, \mathbb{Z} \times b\mathbb{Z})$ are frames for $L^2(\mathbb{R})$. Lemma 7 implies that $(\mathfrak{g}_2, \Lambda)$ is a frame for $L^2(\mathbb{R}^2)$.

To show completeness of the Gabor system for $a = b = 1$, we observe that for $(x, \omega) = (x_1, x_2, \omega_1, \omega_2)$

$$Z_{\mathfrak{g}_2}(x, \omega) = Z_{\mathfrak{g}_1}(x_1, \omega_1) \cdot Z_{\mathfrak{g}_1}(x_2, \omega_2)$$

Because $(\mathfrak{g}_1, \mathbb{Z} \times \mathbb{Z})$ is complete in $L^2(\mathbb{R})$, but not a frame, according to Proposition 9.4.3 in [Chr03], $Z_{\mathfrak{g}_1}$ vanishes on a set of measure zero in $[0, 1]^2$. Hence, the Zak transform $Z_{\mathfrak{g}_2}$ vanishes only on a set of zero measure in $[0, 1]^4$. According to Proposition 9.4.3 in [Chr03], $(\mathfrak{g}_2, \mathbb{Z}^2 \times \mathbb{Z}^2)$ is complete. Furthermore, since $\mathfrak{g}_2 \in \mathcal{S}(\mathbb{R}^2)$, its Zak transform is continuous. Hence, it is not bounded away from 0 almost everywhere. Proposition 8.3.2 in [Grö01] implies that the Gabor system $(\mathfrak{g}_2, \mathbb{Z}^2 \times \mathbb{Z}^2)$ is not a frame for $L^2(\mathbb{R}^2)$.

If $a > 1$ or $b > 1$, say $b > 1$, then $(\mathfrak{g}_1, \mathbb{Z} \times b\mathbb{Z})$ is incomplete in $L^2(\mathbb{R})$. Hence we can choose $f_1 \in L^2(\mathbb{R}^d), f_1 \neq 0$ such that $V_{\mathfrak{g}_1} f_1(m_1, bm_1) = 0$ for all $(m_1, n_1) \in \mathbb{Z}^2$. Then for any $f_2 \in L^2(\mathbb{R}), f_2 \neq 0$, the STFT

$$V_{\mathfrak{g}_2}(f_1 \otimes f_2)(m_1, m_2, an_1, bn_2) = V_{\mathfrak{g}_1}(m_1, an_1)V_{\mathfrak{g}_1}(m_2, bn_2) = 0. \quad (3)$$

But $f_1 \otimes f_2 \neq 0$, so $(\mathfrak{g}_2, \mathbb{Z} \times \mathbb{Z} \times a\mathbb{Z} \times b\mathbb{Z})$ is incomplete. \square

Proposition 9 *Let $\Lambda = \mathbb{Z}^2 \times \begin{pmatrix} a & a \\ -b & b \end{pmatrix} \mathbb{Z}^2$. Then the Gabor system $(\mathfrak{g}_2, \Lambda)$ is a frame for $L^2(\mathbb{R}^2)$ if $a, b < \frac{1}{2}$. If $a = b = \frac{1}{2}$, then $(\mathfrak{g}_2, \Lambda)$ is complete but not a frame for $L^2(\mathbb{R}^2)$. If $a, b > \frac{1}{2}$, then $(\mathfrak{g}_2, \Lambda)$ is incomplete.*

Note that here $D(\Lambda) = \frac{1}{2ab}$, so $(\mathfrak{g}_2, \Lambda)$ is *a-priori* not a frame if $2ab > 1$.

Proof. For $\Lambda = \mathbb{Z}^2 \times \begin{pmatrix} a & a \\ -b & b \end{pmatrix} \mathbb{Z}^2, F = f_1 \otimes f_2 \in L^2(\mathbb{R}^2)$, we have

$$V_{\mathfrak{g}_2} F(m_1, m_2, a(n_1 + n_2), b(n_2 - n_1)) = V_{\mathfrak{g}_1} f_1(m_1, a(n_1 + n_2)) \cdot V_{\mathfrak{g}_1} f_2(m_2, b(n_2 - n_1))$$

If n_1, n_2 are of the same parity, then $n_1 \pm n_2$ is always even, otherwise, $n_1 \pm n_2$ is odd. Hence, if $a, b > \frac{1}{2}$, then we can choose a nonzero $f_1 \in L^2(\mathbb{R})$ such that $V_{\mathfrak{g}_1} f_1(m_1, a(n_1 + n_2)) = 0$, for all m_1 and all (n_1, n_2) with $n_1 - n_2$ even, and a nonzero $f_2 \in L^2(\mathbb{R})$ such that $V_{\mathfrak{g}_1} f_2(m_2, b(n_2 - n_1)) = 0$, for all m_2 , and all (n_1, n_2) with $n_1 - n_2$ odd, since the densities of the respective lattices in \mathbb{R}^2 are greater than 1. Then $F = f_1 \otimes f_2 \neq 0$ but

$$V_{\mathfrak{g}_2} F(m_1, m_2, a(n_1 + n_2), b(n_2 - n_1)) = 0, \quad \forall m_1, m_2, n_1, n_2,$$

implying incompleteness of $(\mathfrak{g}_2, \Lambda)$ for all $a, b > \frac{1}{2}$.

We note further that

$$\begin{aligned} \Lambda &= \{(m_1, m_2, 2ak_1, 2bk_2)^T : m_1, m_2, k_1, k_2 \in \mathbb{Z}\} \\ &\cup \{(m_1, m_2, 2ak_1 + a, 2bk_2 + b)^T : m_1, m_2, k_1, k_2 \in \mathbb{Z}\}. \end{aligned}$$

If $a, b = \frac{1}{2}$, the system $(\mathfrak{g}_2, \Lambda)$ is complete, because it is the union of two complete systems. However, it is not a frame for $L^2(\mathbb{R}^2)$: we can choose $\epsilon > 0$ and $f_1, f_2 \in L^2(\mathbb{R})$ with unit norm such that

$$\sum_{k, l \in \mathbb{Z}} |V_{\mathfrak{g}_1} f_1(k, l)|^2 < \epsilon, \quad \sum_{k, l \in \mathbb{Z}} |V_{\mathfrak{g}_1} f_2(k, l + \frac{1}{2})|^2 < \epsilon.$$

Then letting $F = f_1 \otimes f_2$, it is not difficult to see that

$$\begin{aligned} & \sum_{m_1, m_2, n_1, n_2} |V_{\mathfrak{g}_2} F(m_1, m_2, \frac{1}{2}(n_1 + n_2), \frac{1}{2}(n_2 - n_1))|^2 \\ &= \sum_{\substack{m_1, m_2, n_1, n_2 \\ 2|n_1 - n_2}} |V_{\mathfrak{g}_2} F(m_1, m_2, \frac{1}{2}(n_1 + n_2), \frac{1}{2}(n_2 - n_1))|^2 \\ &\quad + \sum_{\substack{m_1, m_2, n_1, n_2 \\ 2 \nmid n_1 - n_2}} |V_{\mathfrak{g}_2} F(m_1, m_2, \frac{1}{2}(n_1 + n_2), \frac{1}{2}(n_2 - n_1))|^2 \\ &\leq \sum_{k, l \in \mathbb{Z}} |V_{\mathfrak{g}_1} f_1(k, l)|^2 \cdot \sum_{k, l \in \mathbb{Z}} |V_{\mathfrak{g}_1} f_2(k, l)|^2 \\ &\quad + \sum_{k, l \in \mathbb{Z}} |V_{\mathfrak{g}_1} f_1(k, l + \frac{1}{2})|^2 \cdot \sum_{k, l \in \mathbb{Z}} |V_{\mathfrak{g}_1} f_2(k, l + \frac{1}{2})|^2 \leq 2C\epsilon, \end{aligned}$$

where C is the norm of the $\ell^2(\mathbb{Z}^2)$ -valued bounded analysis operator $D : f \mapsto \{V_{\mathfrak{g}_1} f(\lambda) : \lambda \in \mathbb{Z}^2\}$, see [Grö01], Proposition 12.2.5. This implies that no lower frame bound exists for $(\mathfrak{g}_2, \Lambda)$.

If $a, b < \frac{1}{2}$, then $(\mathfrak{g}_2, \Lambda)$ is a frame for $L^2(\mathbb{R}^2)$, because it is the union of two frames for $L^2(\mathbb{R}^2)$. \square

Remark: Unfortunately, the cases $a > \frac{1}{2}, b < \frac{1}{2}$ or $a < \frac{1}{2}, b > \frac{1}{2}$ are not answered by Proposition 9.

Generalizing the ideas underlying Proposition 9 leads to the following result for lattices Λ with a particular subgroup structure:

Theorem 10 *Let $\odot_{i=1}^d A_i \mathbb{Z}^2$ be a subgroup of $\Lambda \subset \mathbb{R}^{2d}$ of index n . If there exist natural numbers $l_i, 1 \leq i \leq d$, such that $\sum_{i=1}^d l_i = n$ and $l_i < \det A_i$, then the system $(\mathfrak{g}_d, \Lambda)$ is incomplete in $L^2(\mathbb{R}^d)$.*

Proof. We split the n cosets of $\odot_{i=1}^d A_i \mathbb{Z}^2$ into d groupings $\Delta_1, \dots, \Delta_d$ such that $|\Delta_i| = l_i$. Δ_i contains coset representatives denoted by $[\tau]$. We have

$$\Lambda = \bigcup_{i=1}^d \bigcup_{[\tau] \in \Delta_i} \{A_1 \mathbb{Z}^2 \times \dots \times A_d \mathbb{Z}^2\} + [\tau],$$

The short-time Fourier transform of the tensor product $\otimes_{i=1}^d f_i$ factorizes, namely

$$V_{\mathfrak{g}_d}(\otimes_{i=1}^d f_i)(x, \omega) = \prod_{i=1}^d V_{\mathfrak{g}_1} f_i(x_i, \omega_i),$$

where $(x_i, \omega_i) \in A_i\mathbb{Z}^2 + [\tau_i]$, $[\tau_i]$ being the coset representative of $A_i\mathbb{Z}^2$ in the restriction of $\odot_{i=1}^d A_i\mathbb{Z}^2$ to $A_i\mathbb{Z}^2$. As the density of the set

$$U_i = \bigcup_{[\tau] \in \Delta_i} A_i\mathbb{Z}^d + [\tau], \quad 1 \leq i \leq d$$

is $l_i D(A_i) < 1$, Theorem 6 applies and non-zero functions $f_i \in L^2(\mathbb{R})$ can be chosen so that $V_{\mathfrak{g}_1} f_i(x_i, \omega_i) = 0$, for all $(x_i, \omega_i) \in U_i$. Then as in Proposition 9 we conclude that $V_{\mathfrak{g}_d}(\otimes_{i=1}^d f_i)$ vanishes on all of Λ , but $\otimes_{i=1}^d f_i \neq 0$. Hence, this Gabor system is incomplete in $L^2(\mathbb{R}^d)$. \square

Remark: If Λ satisfies the hypothesis of Theorem 10, then the density theorem implies incompleteness if $D(\Lambda) = n \prod_{i=1}^d \frac{1}{\det A_i} < 1$, that is, if $\prod_{i=1}^d \det A_i > n$. Hence, for Theorem 10 to be relevant, we need to combine the condition $\prod_{i=1}^d \det A_i \leq n$ with the condition $\det A_i > l_i$ and $\sum_{i=1}^d l_i = n$. This leads to

$$\prod_{i=1}^d l_i < \sum_{i=1}^d l_i. \quad (4)$$

Assuming without loss of generality the order $l_1 \geq l_2 \geq \dots \geq l_d > 0$, we divide (4) by l_1 and observe that then $\prod_{i=2}^d l_i < d$. As all l_i are positive integers, we conclude that $l_2 = l_3 = l_4 = \dots = l_d = 1$ and $l_1 = n - d + 1$.

Note that Theorem 10 implies the incompleteness asserted in Proposition 9 for $a, b > \frac{1}{2}$ because $(\mathbb{Z} \times 2a\mathbb{Z}) \odot (\mathbb{Z} \times 2b\mathbb{Z})$ is a subgroup of Λ of index 2 and $l_1 = l_2 = 1 < 2a, 2b$. Similarly, we can deduce the following result.

Corollary 11 *Let $\Lambda = \begin{pmatrix} ak & a \\ 0 & b \end{pmatrix} \mathbb{Z}^2 \times \mathbb{Z}^2$, $k \in \mathbb{N}$. Then the Gabor system $(\mathfrak{g}_2, \Lambda)$ is incomplete if there exists $l \in \mathbb{N}$ such that $a > \frac{l}{k}$, $b > \frac{k-l}{k}$.*

Proof. The subgroup $(ak\mathbb{Z} \times \mathbb{Z}) \odot (bk\mathbb{Z} \times \mathbb{Z})$ has index k in Λ . If there exist l_1, l_2 such that $l_1 < ak, l_2 < bk$ and $l_1 + l_2 = k$, the result follows by Theorem 10. \square

Remark: The range of parameters k, l , where the condition from Corollary 11 is stronger than the density condition ($abk > 1$) is quite small if $k \geq 5$: the only values of l for which

$$\frac{1}{k} > ab > \frac{l}{k} \cdot \frac{k-l}{k}$$

are $l = 1, k - 1$ because always $2(k - 2) > k$.

We can combine the results above to obtain further examples.

Proposition 12 *Let $\Lambda = \mathbb{Z}^3 \times \begin{pmatrix} a & a & 0 \\ -b & b & 0 \\ 0 & 0 & c \end{pmatrix} \mathbb{Z}^3$. Then the Gabor system $(\mathfrak{g}_3, \Lambda)$ is a frame for $L^2(\mathbb{R}^3)$ if $a, b < \frac{1}{2}, c < 1$. If $a = b \leq \frac{1}{2}, c = 1$, $(\mathfrak{g}_3, \Lambda)$ is complete, but not a frame for $L^2(\mathbb{R}^3)$. If $a, b > \frac{1}{2}$ or $c > 1$, $(\mathfrak{g}_3, \Lambda)$ is incomplete.*

Proof. We choose $F = f_1 \otimes f_2 \otimes f_3 \in L^2(\mathbb{R}^3)$ in order to apply a tensor argument as (3). When $a, b > \frac{1}{2}$, the claim follows immediately from Proposition 9.

When $c > 1$, it suffices to choose f_3 which is in the orthogonal complement of $\{T_m M_{cn} \mathfrak{g}_1 : m, n \in \mathbb{Z}\}$ and repeat the same line of reasoning.

Whenever $a, b < \frac{1}{2}, c < 1$, then $(\mathfrak{g}_3, \Lambda)$ is a frame, because it is the product of two frames (see Lemma 7 and Proposition 9). \square

Remark: The herein presented results extend to some (but not all) lattices which are symplectically identical to those listed. For a symplectic transformation M with associated metaplectic operator $\mu(M)$, the spanning properties of $(\mu(M)g, M\Lambda)$ are equivalent to those of (g, Λ) [Fol89, Grö01]. Furthermore, the metaplectic operators $\mu(M)$ associated to symplectic matrices M of the form $\begin{pmatrix} B & 0 \\ 0 & (B^*)^{-1} \end{pmatrix}$, B unitary, respectively $\begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$, are dilation by the unitary matrix B , respectively the Fourier transform [Fol89, Grö01]. Both leave the Gaussian invariant, hence, for such M , $(\mathfrak{g}_d, M\Lambda)$ is a frame if and only if $(\mathfrak{g}_d, \Lambda)$ is.

The results presented only scratch the surface of the theory of multivariate Gaussian Gabor frames. But we hope that together with [Grö10], they will motivate further study on the subject.

References

- [Chr03] O. Christensen. *An Introduction to Frames and Riesz bases*. Birkhäuser, Boston, 2003.
- [Fol89] G.B. Folland. *Harmonic Analysis in Phase Space*. Annals of Math. Studies. Princeton Univ. Press, Princeton, NJ, 1989.
- [Gab46] D. Gabor. Theory of communication. *J.IEE (London)*, 93(3):429–457, November 1946.
- [Grö01] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Birkhäuser, Boston, 2001.
- [Grö10] K. Gröcheing. Multivariate Gabor frames and sampling of entire functions in several variables. *Preprint*, 2010.
- [Hei07] C. Heil. History and evolution of the density theorem for Gabor frames. *Journal of Fourier Analysis and Applications*, 13(2):113–166, April 2007.
- [Jan81] A.J.E.M. Janssen. Gabor representation of generalized functions. *J.Math.Anal.Appl.*, 83(2):377–394, 1981.
- [Jan03] A.J.E.M. Janssen. Zak transforms with few zeros and the tie. In H.G. Feichtinger and T. Strohmer, editors, *Advances in Gabor analysis*, pages 31–70. Birkhäuser, 2003.
- [Lyu92] Y. Lyubarskii. Frames in the Bargmann space of entire functions. *Adv. Soviet Math.*, 1992.

- [PR10] G.E. Pfander and P. Rashkov. A geometric construction of smooth windows for Gabor frames. Technical report, Jacobs University, 2010.
- [SW92] K. Seip and R. Wallsten. Sampling and interpolation in the Bargmann-Fock space II. *J. Reine Angew. Math*, 1992.