IRREGULAR AND MULTI–CHANNEL SAMPLING OF OPERATORS

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ABSTRACT. The classical sampling theorem for bandlimited functions has recently been generalized to apply to so-called bandlimited operators, that is, to operators with band-limited Kohn-Nirenberg symbols. Here, we discuss operator sampling versions of two of the most central extensions to the classical sampling theorem. In irregular operator sampling, the sampling set is not periodic with uniform distance. In multi-channel operator sampling, we obtain complete information on an operator by multiple operator sampling outputs.

1. INTRODUCTION

The so-called classical sampling theory addresses the problem of determining and reconstructing functions from countably many values that are attained on a discrete subset of the real line. The fundamental result in this theory is attributed to Whittaker, Kotel'nikov and Shannon. It asserts that a function bandlimited to an interval of length Ω can be recovered from the values of the function sampled regularly at Ω values per unit interval.

During the last few years the herein considered sampling theory for operators has been developed. It is motivated by the operator identification problem in communications engineering. There, the objective is to identify a channel operator from knowledge of the channel's action on a chosen input signal. A well known identification result states, for example, that any time–invariant channel operator is fully determined by its action on the Dirac impulse. Already in the 1960s, Kailath [12] and Bello [2] proclaimed that this simple identifiability result on time–invariant operators could be generalized to slowly time–varying operators, that is, to operators whose spreading functions are supported on sets of measure less than or equal to one. The spreading function is the symplectic Fourier transform of the operator's Kohn-Nirenberg symbol. The assertions of Kailath and Bello were confirmed in [16, 25].

The identifiability results in [16, 25] use weighted sums of regularly spaced delta impulses as identifiers for Hilbert–Schmidt operators with bandlimited Kohn–Nirenberg symbols. The discrete support of such a tempered distribution we shall refer to in the following as sampling set for operator sampling. Together with the fact that the classical sampling theorem can be seen as a special case of the identifiability results for bandlimited operators [16, 25, 27] — consider a bandlimited function as a multiplication operator which can be determined from its action on a regularly spaced sum of Dirac impulses — this has led to the development of the herein considered sampling theory for operators.

In this paper, we state and prove operator sampling versions of key generalizations of the classical sampling theorem for functions.

1

One central extension of the classical sampling theorem considers irregular or nonuniform sampling sets, see [20] and references therein. In practice, sampling with uniform distance is hardly realizable because of imperfections in data acquisition devices or perturbations during the collecting of data. As it is similarly challenging to generate regularly spaced sums of impulses for operator sampling, the consideration of irregularly placed impulses as identifiers for operators is natural. Here, we will give separation and Beurling density results for operator sampling which resemble corresponding results in the classical sampling theory for functions.

The feasibility of the classical sampling theory for very large bandwidth signals is limited by the sampling rates achievable in state of the art hardware. This problem is addressed through multi-channel sampling as pioneered by Papoulis [23]. Multi–channel sampling employs a number of samplers in parallel, thereby allowing the acquisition of samples to be carried out in each channel at a fraction of the sampling rate foreseen by the classical sampling theorem. Multi–channel sampling in the theory of operator sampling employs similarly the combination of multiple outputs from sampling procedures in order to reduce the rate at which impulses have to be sent. In addition, multi–channel sampling for operators, that is, multiple output sampling for operators, allows to identify operators whose Kohn–Nirenberg symbols are only bandlimited to an area of measure larger than one, as shown below. Note that in single output operator sampling, only operators with bandlimitations given by sets of measure less than one can be identified. Larger bandwidth of the Kohn–Nirenberg symbol cannot be compensated by an increase of the so-called sampling rate. [*]

We formulate and prove our results for Hilbert–Schmidt operators. Means for generalizing such results to non compact operators are outlined in [24, 27].

This paper is organized as follows. In Section 2, we give some background for operator sampling and introduce operator Paley-Wiener spaces. In Section 3 we state the uniform operator sampling result for operator Paley-Wiener spaces as given in [26]. We include a new proof of this result, a proof that allows for generalizations to the setting of irregular and multi-channel operator sampling. We give a generalization to operator classes which have not necessarily bandlimited Kohn–Nirenberg symbols. In Section 4, we develop irregular operator sampling for operator Paley-Wiener spaces. Also, we consider irregular sampling of operators whose Kohn-Nirenberg symbol is not bandlimited in view of Kramer's Lemma setting. Multi-channel operator sampling is discussed in Section 5.

2. Preliminaries

The Fourier transform on $L^2(\mathbb{R}^d)$ is densely defined by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(t)\, e^{-2\pi i t \cdot \xi} dt, \quad f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d).$$

Similarly, the symplectic Fourier transform $\mathcal{F}_s : L^2(\mathbb{R}^{2d}) \longrightarrow L^2(\mathbb{R}^{2d})$ is given by

$$\mathcal{F}_s f(t, \nu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x, \xi)\, e^{-2\pi i (\nu \cdot x - \xi \cdot t)} dx d\xi, \quad a.e.\ t, \nu \in \mathbb{R}.$$

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[*]Operator sampling is not simply an higher dimensional analogue of the 1-d Shannon sampling theorem. In the case of an operator acting on $L^2(\mathbb{R})$, the operator's 2–dimensional Kohn–Nirenberg symbol is to be determined from a signal defined on $\mathbb{R}$. No access to sample values of the Kohn–Nirenberg symbol is given, as is the case in 2-dimensional Shannon sampling theory.

For brevity of notation, we shall refrain from marking equalities and inequalities that hold in the L^2-sense with the customary *a.e.* whenever the context is unambiguous.

Let $S_\Omega = \prod_{k=1}^{d}[-\frac{\Omega_k}{2}, \frac{\Omega_k}{2}]$ with $\Omega = (\Omega_1, \Omega_2, \cdots, \Omega_d)$.

Definition 2.1. The Paley-Wiener space with bandwidth Ω is defined by

$$PW(S_\Omega) = \left\{ f \in L^2(\mathbb{R}^d) : \operatorname{supp} \hat{f} \subseteq S_\Omega \right\}.$$

It is known that if an expansion for $f \in PW(S_\Omega)$ converges in the norm of $PW(S_\Omega)$, then it converges pointwise and uniformly over $\mathbb{R}^d$. Note that $PW(S_\Omega)$ is isometrically isomorphic to $L^2(S_\Omega)$ due to Plancherel's theorem. For simplicity of notation, we shall denote by $L^2(S_\Omega)$ the subspace of $L^2(\mathbb{R}^d)$ which consists of $L^2(\mathbb{R}^d)$ functions supported on S_Ω.

The classical sampling theorem shown independently by Whittaker, Kotel'nikov and Shannon is generalized by the following *oversampling* theorem. It is based on possibly collecting samples more often than the sampling rate prescribes. The sampling rate, usually called Nyquist-Landau rate, for a function in $PW(S_\Omega)$ is defined to be the Lebesgue measure of the set S_Ω.

Here and in the following we use the notation

$$A(F) \asymp B(F), \quad F \in \mathcal{F},$$

if there exist positive constants c and C such that $cA(F) \leq B(F) \leq CA(F)$ for all objects F in the set $\mathcal{F}$.

Theorem 2.2. *For $\Omega, T > 0$ with $T_k\Omega_k < 1$ for all k and $\varphi \in PW(S_{\frac{2}{T}-\Omega})$ with $\hat{\varphi} = 1$ on S_Ω, we have a sampling expansion*

$$f(t) = T \sum_{n \in \mathbb{Z}^d} f(nT)\varphi(t - nT), \quad f \in PW(S_\Omega). \tag{1}$$

Moreover, (1) is stable in the sense that

$$\|f\|^2 \asymp \sum_{n \in \mathbb{Z}^d} |f(nT)|^2, \quad f \in PW(S_\Omega).$$

In practice, a stable sampling expansion guarantees that perturbations in the sampling output and in the reconstruction procedure are controlled by error bounds on the input function and vice versa.

The development of operator sampling necessitates the use of some rudimentary distribution theory. The space of distributions chosen here is the dual of the Feichtinger algebra $S_0(\mathbb{R}^d)$. The dual $S_0'(\mathbb{R}^d)$ is a Banach space with $\mathcal{S}(\mathbb{R}^d) \subsetneq S_0(\mathbb{R}^d) \subsetneq L^2(\mathbb{R}^d) \subsetneq S_0'(\mathbb{R}^d) \subsetneq \mathcal{S}'(\mathbb{R}^d)$, where $\mathcal{S}$ denotes the Schwartz class of rapidly decaying functions and $\mathcal{S}'$ its dual, the space of tempered distribution. There are several equivalent definitions of the Feichtinger algebra S_0 [6]. We choose the characterization of S_0 via the short time Fourier transform.

Definition 2.3. The Feichtinger algebra is defined by

$$S_0(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : V_g f(t, \nu) \in L^1(\mathbb{R}^{2d})\},$$

where $V_g f(t, \nu) = \langle f, M_\nu T_t g \rangle$ is the short-time Fourier transform of f with respect to the Gaussian $g(x) = e^{-\pi\|x\|^2}$. The norm on $S_0(\mathbb{R}^d)$ is given by $\|f\|_{S_0} = \|V_g f\|_{L^1}$.

In this paper we consider the sampling problem for Hilbert-Schmidt operators only.

Definition 2.4. The class of Hilbert-Schmidt operators $HS(L^2(\mathbb{R}^d))$ consists of bounded linear operators on $L^2(\mathbb{R}^d)$ which can be represented as integral operators of the form

$$Hf(x) = \int \kappa_H(x,t)f(t)dt, \quad f \in L^2(\mathbb{R}^d),$$

with kernel $\kappa_H \in L^2(\mathbb{R}^{2d})$.

The linear space of Hilbert-Schmidt operators $HS(L^2(\mathbb{R}^d))$ becomes a Hilbert space if it is endowed with the Hilbert space structure of $L^2(\mathbb{R}^d)$, that is, by

$$\langle H_1, H_2 \rangle_{HS} = \langle \kappa_{H_1}, \kappa_{H_2} \rangle_{L^2}.$$

In view of pseudodifferential operators, the Kohn-Nirenberg symbol σ_H [7, 14] of a Hilbert-Schmidt operator H is given by

$$\sigma_H(x,\xi) = \int \kappa_H(x, x-t)\, e^{-2\pi i t \cdot \xi} dt.$$

It leads to the operator representation

$$Hf(x) = \int \sigma_H(x,\xi)\hat{f}(\xi)\, e^{2\pi i x \cdot \xi} d\xi, \quad f \in L^2(\mathbb{R}^d).$$

In time-frequency analysis and communication engineering, the spreading function η_H of a Hilbert-Schmidt operator H is commonly considered. It is given by

$$\eta_H(t,\nu) = \int \kappa_H(x, x-t)\, e^{-2\pi i x \cdot \nu} dx,$$

and leads to

$$Hf(x) = \int\int \eta_H(t,\nu) M_\nu T_t f(x)\, dt d\nu, \quad f \in L^2(\mathbb{R}^d), \tag{2}$$

where the time-shift (translation) and frequency-shift (modulation) operators T_t and M_ν are defined by $T_t f(x) = f(x-t)$ and $\widehat{M_\nu f}(\gamma) = \hat{f}(\gamma-\nu)$, respectively. That is, a Hilbert-Schmidt operator H is a continuous superposition of translation and modulation operators with coefficient function η_H. The identity (2) is understood weakly, namely

$$\langle Hf, g \rangle = \int\int \eta_H(t,\nu)\langle M_\nu T_t f, g \rangle\, dt d\nu, \quad g \in L^2(\mathbb{R}^d).$$

As $\eta_H = \mathcal{F}_s \sigma_H$, operators with band-limited Kohn-Nirenberg symbols are operators whose spreading functions are compactly supported.

In communications, the time-varying operator H is also commonly represented by its time-varying impulse response $h_H(t,x)$ with

$$Hf(x) = \int h_H(t,x)f(x-t)dt,$$

where $h_H(t,x) = \kappa_H(x, x-t) = \int \eta_H(t,\nu)\, e^{2\pi i x \nu} d\nu$ a.e. Note that

$$\|H\|_{HS} = \|\kappa_H\|_{L^2} = \|h_H\|_{L^2} = \|\sigma_H\|_{L^2} = \|\eta_H\|_{L^2}.$$

Definition 2.5. The operator Paley-Wiener space of operators bandlimited to $S \subseteq \mathbb{R}^{2d}$ is

$$OPW(S) = \{H \in HS(L^2(\mathbb{R}^d)) : \operatorname{supp} \mathcal{F}_s \sigma_H \subseteq S\}.$$

In the literature, operators with $\operatorname{supp} \mathcal{F}_s \sigma_H \subseteq [a_1, b_1] \times \ldots \times [a_{2d}, b_{2d}]$ are commonly referred to as underspread or slowly time–varying operators if $\operatorname{volume}([a_1, b_1] \times \ldots \times [a_{2d}, b_{2d}] = (b_1 - a_1) \cdot \ldots \cdot (b_{2d} - a_{2d}) \leq 1$ and as overspread operators else (see [16] and references within).

We formulate the operator identification and sampling problems as follows. [†]

Definition 2.6. An operator class $\mathcal{H} \subseteq HS(L^2(\mathbb{R}^d))$ is identifiable if all $H \in \mathcal{H}$ extend to a domain containing a so–called identifier $f \in S_0'(\mathbb{R}^d)$ with

$$\|H\|_{HS} \asymp \|Hf\|_{L^2}, \quad H \in \mathcal{H}. \tag{3}$$

The operator class $\mathcal{H} \subseteq HS(L^2(\mathbb{R}^d))$ permits operator sampling if one can choose f in (3) with discrete support in $\mathbb{R}^d$ in the distributional sense. In that case, $\operatorname{supp} f$ is called sampling set for $\mathcal{H}$.

Note that $H \in OPW(S)$ with S compact can be extended to a bounded linear operator $H : S_0'(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ [5, 24]. This implies that proving identifiability of a Hilbert-Schmidt operator by an element in S_0' is equivalent to providing the lower bound A in (3), as an upper bound is given by $B = \|f\|_{S_0'}$ since

$$\|Hf\|_{L^2} \leq \|H\|_{\mathrm{op}} \|f\|_{S_0'} \leq \|H\|_{HS} \|f\|_{S_0'}, \quad H \in OPW(S).$$

Given a separable Hilbert space X, a sequence of elements $\{f_k\}_{k \in \mathbb{Z}}$ in X is called a frame for X if

$$\sum_{k \in \mathbb{Z}} |\langle f, f_k \rangle|^2 \asymp \|f\|_X^2, \quad f \in X.$$

To each frame $\{f_k\}_{k \in \mathbb{Z}}$ for X exists a so-called dual frame $\{\widetilde{f}_k\}_{k \in \mathbb{Z}}$ of $\{f_k\}_{k \in \mathbb{Z}}$ for X with

$$f = \sum_{k \in \mathbb{Z}} \langle f, f_k \rangle \widetilde{f}_k = \sum_{k \in \mathbb{Z}} \langle f, \widetilde{f}_k \rangle f_k, \quad f \in X.$$

Moreover, a frame which does not form a frame if we remove any element from it is called a Riesz basis, or, also, exact frame. A sequence $\{f_k\}_{k \in \mathbb{Z}}$ is called a Riesz sequence if it is a Riesz basis for $\overline{\operatorname{span}}\{f_k\}_{k \in \mathbb{Z}}$ [4, 8, 15].

3. Uniform sampling of Hilbert-Schmidt operators

Theorem 2.2 states that a bandlimited, square integrable function can be reconstructed by its values sampled at a sufficiently dense sampling grid. In this section, we first consider the sampling problem for operators whose Kohn-Nirenberg symbols are bandlimited in the time-frequency plane. We shall state and prove all theorems for $d = 1$ for convenience.

Operators with rectangular bandlimitation on their Kohn-Nirenberg symbols are the starting point of operator sampling [26, 27],

Theorem 3.1. *For $\Omega, T, T' > 0$ and $0 < T'\Omega \leq T\Omega \leq 1$, choose $\varphi \in PW([-(\frac{1}{T} - \frac{\Omega}{2}), \frac{1}{T} - \frac{\Omega}{2}])$ with $\hat{\varphi} = 1$ on $[-\frac{\Omega}{2}, \frac{\Omega}{2}]$ and $r \in L^\infty(\mathbb{R})$ with $\operatorname{supp} r \subset [-T + T', T]$ and $r = 1$ on $[0, T']$. Then $OPW([0, T'] \times [-\frac{\Omega}{2}, \frac{\Omega}{2}])$ permits operator sampling as*

$$\|H\|_{HS} = \sqrt{T}\, \|H \sum_{n \in \mathbb{Z}} \delta_{nT}\|_{L^2}, \quad H \in OPW([0, T'] \times [-\frac{\Omega}{2}, \frac{\Omega}{2}]),$$

[†]See [16] for a more general concept of operator identification.

and operator reconstruction is possible by means of

$$h_H(t, x) = r(t)T \sum_{n \in \mathbb{Z}} (H \sum_{k \in \mathbb{Z}} \delta_{kT})(t + nT)\varphi(x - t - nT).$$

In [16], the proof of the identifiability of $OPW([0, T'] \times [-\frac{\Omega}{2}, \frac{\Omega}{2}])$ is based on the unitarity of the Zak transform. For clarity and to indicate directions for generalizations of this theorem, we prove Theorem 3.1 through elementary orthonormal basis expansions based on Fourier series.

Proof. For almost every $t \in \mathbb{R}$, we have $\eta_H(t, \cdot) \in L^2[-\frac{\Omega}{2}, \frac{\Omega}{2}] \subseteq L^2[-\frac{1}{2T}, \frac{1}{2T}]$ and, by expanding $\eta_H(t, \nu)$ with respect to the orthonormal basis $\{\sqrt{T} e^{-2\pi i(t+nT)\nu}\}_{n \in \mathbb{Z}}$ of $L^2[-\frac{1}{2T}, \frac{1}{2T}]$, we obtain

$$
\begin{aligned}
\eta_H(t, \nu) &= \sum_{n \in \mathbb{Z}} \langle \eta_H(t, \nu), \sqrt{T} e^{-2\pi i(t+nT)\nu} \rangle \sqrt{T} e^{-2\pi i(t+nT)\nu} \\
&= T \sum_{n \in \mathbb{Z}} h_H(t, t + nT) e^{-2\pi i(t+nT)\nu}, \quad \nu \in [-\tfrac{\Omega}{2}, \tfrac{\Omega}{2}], \quad (4)
\end{aligned}
$$

so

$$\eta_H(t, \nu) = T\hat{\varphi}(\nu) \sum_{n \in \mathbb{Z}} h_H(t, t + nT) e^{-2\pi i(t+nT)\nu}, \quad \nu \in \mathbb{R},$$

with φ chosen to satisfy $\varphi \in PW([-(\frac{1}{T} - \frac{\Omega}{2}), \frac{1}{T} - \frac{\Omega}{2}])$ and $\hat{\varphi} = 1$ on $[-\frac{\Omega}{2}, \frac{\Omega}{2}]$. Then, for almost every $t \in \mathbb{R}$

$$
\begin{aligned}
h_H(t, x) &= T \sum_{n \in \mathbb{Z}} h_H(t, t + nT) \int \hat{\varphi}(\nu) e^{-2\pi i(t+nT)\nu} e^{2\pi i\nu x} d\nu \\
&= T \sum_{n \in \mathbb{Z}} h_H(t, t + nT) \varphi(x - t - nT), \quad x \in \mathbb{R}. \quad (5)
\end{aligned}
$$

On the other hand, we have $(H \sum_k \delta_{kT})(x) = \sum_k h_H(x - kT, x) \in L^2(\mathbb{R})$ so that $(H \sum_k \delta_{kT})(t + nT) = \sum_k h_H(t + nT - kT, t + nT)$. Now, $r(t)(H \sum_{k \in \mathbb{Z}} \delta_{kT})(t + nT) = h_H(t, t + nT)$ for $t \in [0, T']$ since supp $h_H(\cdot, x) \subseteq [0, T'] \subseteq [0, T]$. With (5), this gives

$$h_H(t, x) = r(t)T \sum_{n \in \mathbb{Z}} (H \sum_{k \in \mathbb{Z}} \delta_{kT})(t + nT)\varphi(x - t - nT), \quad x \in \mathbb{R}, \text{ a.e. } t \in \mathbb{R}, \quad (6)$$

a formula which contains the identifier $\sum_{k \in \mathbb{Z}} \delta_{kT}$ for $OPW([0, T'] \times [-\frac{\Omega}{2}, \frac{\Omega}{2}])$. Note that the series in (6) converges pointwise and uniformly over $\mathbb{R}$ in x.

Moreover, since (4) was an orthonormal basis expansion, Parseval's identity gives

$$\|\eta_H(t, \cdot)\|_{L^2}^2 = T \sum_{n \in \mathbb{Z}} |h_H(t, t + nT)|^2 = T \sum_{n \in \mathbb{Z}} |r(t)(H \sum_{k \in \mathbb{Z}} \delta_{kT})(t + nT)|^2, \text{ a.e. } t \in \mathbb{R}$$

so that

$$\|h_H(t, \cdot)\|_{L^2}^2 = T \sum_{n \in \mathbb{Z}} |r(t)(H \sum_{k \in \mathbb{Z}} \delta_{kT})(t + nT)|^2, \text{ a.e. } t \in \mathbb{R},$$

and

$$\begin{aligned}
\|h_H\|_{L^2}^2 &= \int_0^{T'} \|h_H(t,\cdot)\|_{L^2}^2 dt = T \int_0^{T'} \sum_{n\in\mathbb{Z}} |h_H(t,t+nT)|^2 dt \\
&= T \int_0^T \sum_{n\in\mathbb{Z}} |h_H(t,t+nT)|^2 dt = T \int_0^T \sum_{n\in\mathbb{Z}} |(H\sum_{k\in\mathbb{Z}}\delta_{kT})(t+nT)|^2 dt \\
&= T \int_{\mathbb{R}} |(H\sum_{k\in\mathbb{Z}}\delta_{kT})(t)|^2 dt = T\|H\sum_{k\in\mathbb{Z}}\delta_{kT}\|_{L^2}^2
\end{aligned}$$

as $h_H(t,t+nT)$ and $(H\sum_k \delta_{kT})(t+nT)$ vanish on the interval $[T',T]$. The operator class $OPW([0,T']\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ is identifiable by $\sum_{k\in\mathbb{Z}}\delta_{kT}$ as we showed $\|H\|_{HS} = \|h_H\|_{L^2} = \sqrt{T}\|H\sum_{k\in\mathbb{Z}}\delta_{kT}\|_{L^2}$. $\qquad\square$

One crucial ingredient in the proof above is the fact that for each t, the set $\mathcal{E}_t = \{\sqrt{T} e^{-2\pi i(t+nT)\nu}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\frac{1}{2T},\frac{1}{2T}]$. Note that the functionals corresponding to $\mathcal{E}_t$ depend on t which is necessary to associate $h_H(t,t+nT)$ with $Hg(t+nT)$ for some identifier g. Another important ingredient is the support condition on $h_H(\cdot,x)$. It guarantees that no aliasing in the infinite summation $H(\sum_{k\in\mathbb{Z}}\delta_{kT})(t+nT)$ takes place as for each t the sum is reduced to a single non zero summand $h_H(t,t+nT)$.

We assume in Theorem 3.1 that the area of the rectangle $[0,T']\times[-\frac{\Omega}{2},\frac{\Omega}{2}]$ is less than or equal to 1. This assumption coincides with the one given in Kailath's conjecture for identifiability of such operator classes [12]. For $1 < T'\Omega \le T\Omega$, perfect reconstruction of $h_H(t,x)$ from its samples is not possible since the sampling rate $\frac{1}{T}$ is strictly less than the Nyquist-Landau rate Ω for $h_H(t,\cdot)$. In this case, not only operator sampling, but also operator identification by any tempered distribution as single input signal is not possible as shown in [28].

Now we extend Theorem 3.1 to the case where $h_H(t,\cdot)$ lies in a shift-invariant space other than the Paley-Wiener space. Given a Riesz sequence $\{\varphi(\cdot-nT)\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$, let

$$V_T(\varphi) = \overline{\text{span}}\{\varphi(\cdot-nT)\} = \Big\{\sum_{n\in\mathbb{Z}} c_n\varphi(\cdot-nT) \,:\, \{c_n\}_{n\in\mathbb{Z}}\in l^2\Big\}.$$

Let $\mathcal{H}_{T,\varphi} \subseteq HS(L^2(\mathbb{R}))$ consist of integral operators H with $h_H \in L^2[0,T]\otimes V_T(\varphi)$. Note that different from the operator Paley-Wiener setup, not each such operator maps boundedly S_0' to L^2.

We require the shift-invariant space $V_T(\varphi)$ to be a reproducing kernel Hilbert space [9].

Definition 3.2. A Hilbert space X of complex-valued functions on a given domain $D \ne \emptyset$ is a reproducing kernel Hilbert space if there exists a kernel $k(s,t)$ defined on $D\times D$ satisfying $k(\cdot,t)\in X$ for all $t\in D$ and $f(t) = \langle f(\cdot),k(\cdot,t)\rangle_X$ for all $f\in X$ and $t\in D$. Such a function $k(s,t)$ is called a reproducing kernel.

For example, if φ is a complex-valued integrable function well-defined everywhere in $\mathbb{R}$ and satisfies

$$\sum_{n\in\mathbb{Z}} |\varphi(t+n)|^2 < \infty, \quad t\in[0,1],$$

then $V_1(\varphi)$ is a reproducing kernel Hilbert space [13]. Alternatively, if φ is continuous and belongs to the Wiener amalgam space $W(L^\infty,l^1)$, that is, to the subspace

of $L^2(\mathbb{R})$ defined by the norm

$$\|\varphi\|_{W(L^\infty, l^1)} = \sum_{n\in\mathbb{Z}} \sup_{t\in[0,1]} |\varphi(t+n)| < \infty,$$

then $V_1(\varphi)$ is a reproducing kernel Hilbert space as well [1].

Theorem 3.3. *Assume that $V_T(\varphi)$ is a reproducing kernel Hilbert space and its reproducing kernel $k(s,t)$ satisfies the condition that $\{k(\cdot, t+nT)\}_{n\in\mathbb{Z}}$ is a frame for $V_T(\varphi)$ for each $t \in [0,T]$. Then $\sum_{k\in\mathbb{Z}} \delta_{kT}$ identifies $\mathcal{H}_{T,\varphi}$, that is,*

$$\|H\|_{HS} \asymp \|H\sum_{k\in\mathbb{Z}}\delta_{kT}\|_{L^2}, \quad H \in \mathcal{H}_{T,\varphi}.$$

The reconstruction of operators is possible by

$$h_H(t,x) = \chi_{[0,T]}(t) \sum_{n\in\mathbb{Z}} (H\sum_{k\in\mathbb{Z}}\delta_{kT})(t+nT)k_n^*(x,t),$$

where $\{k_n^*(\cdot, t)\}_{n\in\mathbb{Z}}$ is a dual frame of $\{k(\cdot, t+nT)\}_{n\in\mathbb{Z}}$ for a.e. $t \in [0,T]$.

Proof. Since $h_H(t,\cdot) \in V_T(\varphi)$ and $\{k(\cdot, t+nT)\}_{n\in\mathbb{Z}}$ is a frame for $V_T(\varphi)$ for each $t \in [0,T]$,

$$h_H(t,x) = \sum_{n\in\mathbb{Z}} \langle h_H(t,\cdot), k(\cdot, t+nT)\rangle k_n^*(x,t)$$

where $\{k_n^*(\cdot, t)\}_{n\in\mathbb{Z}}$ is a dual frame of $\{k(\cdot, t+nT)\}_{n\in\mathbb{Z}}$. Since $k(s,t)$ is a reproducing kernel of $V_T(\varphi)$,

$$h_H(t,x) = \sum_{n\in\mathbb{Z}} h_H(t, t+nT)k_n^*(x,t)$$

and

$$\|h_H(t,\cdot)\|_{L^2}^2 \asymp \sum_{n\in\mathbb{Z}} |h_H(t, t+nT)|^2. \tag{7}$$

Formally, we have $(H\sum_{k\in\mathbb{Z}}\delta_{kT})(t+nT) = \sum_{k\in\mathbb{Z}} h_H(t+nT-kT, t+nT)$ so that

$$r(t)(H\sum_{k\in\mathbb{Z}}\delta_{kT})(t+nT) = h_H(t, t+nT)$$

where $r(t) = \chi_{[0,T]}(t)$. Together with (7), we note $H\sum_{k\in\mathbb{Z}}\delta_{kT} \in L^2(\mathbb{R})$ and we conclude

$$\|h_H\|_{L^2}^2 = \|H\|_{HS}^2 \;\asymp\; \int_0^T \sum_{n\in\mathbb{Z}} |(H\sum_{k\in\mathbb{Z}}\delta_{kT})(t+nT)|^2 dt = \|H\sum_{k\in\mathbb{Z}}\delta_{kT}\|_{L^2}^2.$$

$\square$

Example 3.4. Let $T = 1$ and take $\varphi(t) = \chi_{[0,1)}(t)$. Then $V_1(\varphi)$ is a reproducing kernel Hilbert space and its reproducing kernel $k(s,t)$ allows $\{k(\cdot, t+n)\}_{n\in\mathbb{Z}}$ to be a frame, in fact, an orthonormal basis for $V_1(\varphi)$ since $k(s,t) = \sum_{n\in\mathbb{Z}} \chi_{[n,n+1)}(s)\chi_{[n,n+1)}(t)$. Hence, for $t \in [0,1]$,

$$k(s,t+n) = \sum_{m\in\mathbb{Z}} \chi_{[m,m+1)}(s)\chi_{[m,m+1)}(t+n) = \sum_{m\in\mathbb{Z}} \chi_{[m,m+1)}(s)\delta_{n,m} = \chi_{[n,n+1)}(s).$$

We conclude that $\sum_k \delta_k$ identifies $\mathcal{H} = \{H \in HS(L^2(\mathbb{R})) : h_H \in L^2[0,1] \otimes V_1(\varphi)\}$. For $H \in \mathcal{H}$, the kernel $\kappa_H(x,t)$ is a step function along diagonals $x = t + c$, $c \in \mathbb{R}$.

4. Irregular sampling of Hilbert-Schmidt operators

First, we provide the background on irregular sampling of functions that is needed to develop operator sampling results based on irregular sampling sets.

4.1. Irregular Sampling in Paley-Wiener spaces.

Definition 4.1. Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{R}$, with $\lambda_k < \lambda_{k+1}$, $k \in \mathbb{Z}$.

(1) Λ is a set of sampling, also referred to as stable sampling set, for $PW([-\frac{\Omega}{2}, \frac{\Omega}{2}])$ if

$$\|f\|_{L^2}^2 \asymp \sum_{k\in\mathbb{Z}} |f(\lambda_k)|^2, \quad f \in PW([-\tfrac{\Omega}{2}, \tfrac{\Omega}{2}]).$$

(2) Λ is a set of interpolation for $PW([-\frac{\Omega}{2}, \frac{\Omega}{2}])$ if the interpolation or moment problem

$$f(\lambda_k) = c_k, \quad k \in \mathbb{Z},$$

has a solution in $PW([-\frac{\Omega}{2}, \frac{\Omega}{2}])$ for every $\{c_k\} \in l^2(\mathbb{Z})$.

(3) Λ is uniformly discrete if,

$$(\lambda_{k+1} - \lambda_k) \geq \delta > 0, \quad k \in \mathbb{Z}.$$

In this case δ is called a separation constant.

(4) Λ is relatively uniformly discrete if Λ is a finite union of uniformly discrete sets.

(5) The upper and lower Beurling densities are defined, respectively, by

$$D^+(\Lambda) = \limsup_{h\to\infty} \frac{n^+(h)}{h} \quad \text{and} \quad D^-(\Lambda) = \liminf_{h\to\infty} \frac{n^-(h)}{h},$$

where for $h > 0$, $n^+(h)$ and $n^-(h)$ are the largest number and smallest number of points from Λ in $[x - \frac{h}{2}, x + \frac{h}{2})$, $x \in \mathbb{R}$, respectively. If $D^+(\Lambda) = D^-(\Lambda)$, then we say that Λ has uniform Beurling density $D(\Lambda) = D^+(\Lambda) = D^-(\Lambda)$.

We recall necessary and sufficient conditions on the Beurling density of a set $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ for its nonharmonic sequence to be a frame or a Riesz sequence for $L^2[-\frac{\Omega}{2}, \frac{\Omega}{2}]$ [29].

Theorem 4.2.

(1) Λ is a set of sampling for $PW([-\frac{\Omega}{2}, \frac{\Omega}{2}])$ if and only if $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is a frame for $L^2[-\frac{\Omega}{2}, \frac{\Omega}{2}]$. Moreover, for Λ being relatively uniformly discrete, a necessary condition for $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ to be a frame for $L^2[-\frac{\Omega}{2}, \frac{\Omega}{2}]$ is $D^-(\Lambda) \geq \Omega$, and a sufficient condition is $D^-(\Lambda) > \Omega$.

(2) Λ is a set of interpolation for $PW([-\frac{\Omega}{2}, \frac{\Omega}{2}])$ if and only if $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is a Riesz sequence in $L^2[-\frac{\Omega}{2}, \frac{\Omega}{2}]$. Moreover, for Λ being uniformly discrete, a necessary condition for $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ to be a Riesz sequence in $L^2[-\frac{\Omega}{2}, \frac{\Omega}{2}]$ is $D^+(\Lambda) \leq \Omega$, and a sufficient condition is $D^+(\Lambda) < \Omega$.

In general, it is highly non-trivial to determine whether a set $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\frac{\Omega}{2}, \frac{\Omega}{2}]$. A famous affirmative result was given by Kadec [11].

Theorem 4.3. *(Kadec's 1/4-theorem)* Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$. If there is $L \geq 0$ such that

$$|\lambda_k - \tfrac{k}{\Omega}| \leq L < \tfrac{1}{4\Omega}, \quad k \in \mathbb{Z}, \tag{8}$$

then $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\frac{\Omega}{2},\frac{\Omega}{2}]$, and $\frac{1}{4\Omega}$ is the best possible constant for (8) to hold.

4.2. **Irregular sampling in Operator Paley-Wiener spaces.**

Definition 4.4. A sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ in $\mathbb{R}$ is a set of sampling for an operator class $\mathcal{H}$, if for some $\{c_k\}_{k\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, we have $\sum_{k\in\mathbb{Z}} c_k\delta_{\lambda_k} \in S_0'(\mathbb{R})$ and $\sum_{k\in\mathbb{Z}} c_k\delta_{\lambda_k}$ identifies $\mathcal{H}$.

Operator sampling is operator identification with discretely supported identifiers. Consequently, irregular operator sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ is a-priori possible only if $T\Omega \le 1$ [16, 25].

Theorem 4.5. $T\Omega \le 1$ is a necessary condition for the existence of a sampling set for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$.

Analogous to Theorem 4.2, we have the following result.

Theorem 4.6. If $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$, $\lambda_{k+1} > \lambda_k$, is uniformly discrete, then a necessary condition for Λ being a set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ is $D^-(\Lambda) \ge \Omega$ and a sufficient condition is $D^-(\Lambda) > \Omega$ and $\lambda_{k+1} - \lambda_k \ge T$.

Proof. Assume that Λ is a set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ with $D^-(\Lambda) < \Omega$. Then there exist C and $\tilde{C} > 0$ such that

$$\|H\|_{HS}^2 \le C\|\sum_{k\in\mathbb{Z}}\kappa_H(\cdot,\lambda_k)\|^2 \le \tilde{C}\sum_{k\in\mathbb{Z}}\|\kappa_H(\cdot,\lambda_k)\|^2, \quad H \in OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}]),$$

since for each x there exist at most $\lfloor\frac{T}{\delta}\rfloor + 1$ nonzero summands above where $\delta = \inf_k(\lambda_{k+1} - \lambda_k)$. But as $D^-(\Lambda) < \Omega$, there exists $f \in PW([-\frac{\Omega}{2},\frac{\Omega}{2}])$ with $\|f\| = 1$ and $\sum_k |f(\lambda_k)|^2 \le 1/(2\tilde{C})$. Defining H by $\kappa_H(x,y) = f(y)$ for $0 \le x - y \le T$, $\kappa_H = 0$ else, we have $\kappa_H(x,\lambda_k) = f(\lambda_k)$ for $\lambda_k \le x \le \lambda_k + T$ and 0 else.

We have

$$T = \|H\|_{HS}^2 \le \tilde{C}\sum_{k\in\mathbb{Z}}\|\kappa_H(x,\lambda_k)\|^2 = \tilde{C}\sum_{k\in\mathbb{Z}}T|f(\lambda_k)|^2 \le \tilde{C}T/2\tilde{C} = T/2,$$

a contradiction.

Now we shall establish the sufficient condition for Λ to be a set of sampling. As Λ is uniformly discrete and $D^-(\Lambda) > \Omega$, $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is a frame for $L^2[-\frac{\Omega}{2},\frac{\Omega}{2}]$. Theorem 4.2 (1) implies that Λ is a set of sampling for $PW([-\frac{\Omega}{2},\frac{\Omega}{2}])$ and for $t \in \mathbb{R}$,

$$\|h_H(t,\cdot)\|_{L^2}^2 \asymp \sum_{n\in\mathbb{Z}}|h_H(t,t+\lambda_n)|^2, \quad H \in OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}]).$$

As $\lambda_{k+1} - \lambda_k \ge T$, $k \in \mathbb{Z}$, we conclude that Λ is a set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ since for $H \in OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$,

$$\|H\|_{HS}^2 = \|h_H\|_{L^2}^2 \;=\; \int_0^T \|h_H(t,\cdot)\|^2 dt \;\asymp\; \int_0^T \sum_{n\in\mathbb{Z}}|h_H(t,t+\lambda_n)|^2 dt$$

$$= \int_0^T \sum_{n\in\mathbb{Z}}|(H\sum_{k\in\mathbb{Z}}\delta_{\lambda_k})(t+\lambda_n)|^2 dt \;\le\; \|H\sum_{k\in\mathbb{Z}}\delta_{\lambda_k}\|_{L^2}^2.$$

$\square$

In the remainder of this section, we shall discuss the separation condition $(\lambda_{k+1} - \lambda_k) \ge T$, $k \in \mathbb{Z}$, on the sampling sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ for

$OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$. This condition rules out aliasing in the sense that for each x, the sum $\sum_{n\in\mathbb{Z}} h_H(x-\lambda_n, x) = \sum_{n\in\mathbb{Z}} \kappa_H(x, \lambda_n)$ has only one nonzero summand. Consequently, the sufficient condition on $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ for being a set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ given in Theorem 4.6 implies

$$\Omega \leq D^-(\Lambda) \leq D^+(\Lambda) \leq \frac{1}{T}$$

so that $T\Omega \leq 1$. Theorem 4.5 shows that this is not an additional restriction on operator Paley-Wiener spaces to allow for irregular operator sampling.

Corollary 4.7. *If $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}, \lambda_{k+1} > \lambda_k$, satisfies $|\lambda_k - kT| \leq L < \frac{T}{4}$ for some $L \geq 0$ and $\lambda_{k+1}-\lambda_k \geq T$, $k \in \mathbb{Z}$, then Λ is a set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$, $T\Omega \leq 1$.*

Proof. Since $|\lambda_k - kT| \leq L < \frac{T}{4}$, $\{e^{-2\pi i\lambda_k\xi}\}$ is a Riesz basis for $L^2[-\frac{1}{2T},\frac{1}{2T}]$ by Theorem 4.3. If $T\Omega < 1$, the result follows directly from Theorem 4.6. The case $T\Omega = 1$ follows analogously. $\square$

Note that the hypothesis on Λ in Corollary 4.7 is satisfied if and only if $\lambda_k = kT + \epsilon_k$ with

$$-\frac{1}{4} < -L \leq \ldots \leq \epsilon_{-2} \leq \epsilon_{-1} \leq \epsilon_0 \leq \epsilon_1 \leq \epsilon_2 \leq \ldots \leq L < \frac{1}{4}.$$

Theorem 4.8. *Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$, $\lambda_{k+1} > \lambda_k$, be a set of sampling for $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ and let $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ be a Riesz basis for $L^2[-\frac{\Omega}{2},\frac{\Omega}{2}]$, then $(\lambda_{k+1} - \lambda_k) \geq T$, $k \in \mathbb{Z}$.*

Proof. It is easy to see that if $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\frac{\Omega}{2},\frac{\Omega}{2}]$ and $\sum_{k\in\mathbb{Z}} c_k\delta_{\lambda_k}$ is an identifier of $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$, then $c_k \neq 0$ for all $k \in \mathbb{Z}$.

Assume $\lambda_{l+1} - \lambda_l < T$ for some l. For such l, set

$$\kappa_H(x, \lambda_k) \quad = 0 \quad \text{if } k \neq l, l+1$$
$$\text{and} \quad \kappa_H(x, \lambda_l) \quad = \begin{cases} c_{l+1} & \text{if } \lambda_{l+1} \leq x \leq \lambda_l + T \\ 0 & \text{otherwise} \end{cases}$$
$$\text{and} \quad \kappa_H(x, \lambda_{l+1}) \quad = \begin{cases} -c_l & \text{if } \lambda_{l+1} \leq x \leq \lambda_l + T \\ 0 & \text{otherwise.} \end{cases}$$

The freedom of choice of values for $\kappa_H(x, \lambda_k)$ is justified by Theorem 4.2 (2). Then $H \sum_{k\in\mathbb{Z}} c_k\delta_{\lambda_k}(x) = \sum_{k\in\mathbb{Z}} c_k\kappa_H(x, \lambda_k) = 0$ for all $x \in \mathbb{R}$, but as $c_l, c_{l+1} \neq 0$, we have $\kappa_H \neq 0$ and therefore $H \neq 0$. $\square$

Example 4.9. Theorem 4.8 implies that $\{\lambda_n\}_{n\in\mathbb{Z}} = \{2n\}_{n\in\mathbb{Z}} \cup \{2n + \alpha\}_{n\in\mathbb{Z}}$ with $0 < \alpha < 1$ is not a set of sampling for $OPW([0,1]\times[-\frac{1}{2},\frac{1}{2}])$.

The condition $(\lambda_{k+1} - \lambda_k) \geq T$, $k \in \mathbb{Z}$ is not necessary for operator sampling of $OPW([0,T]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ if $\{e^{-2\pi i\lambda_k\xi}\}_{k\in\mathbb{Z}}$ is not a Riesz basis but a frame for $L^2[-\frac{\Omega}{2},\frac{\Omega}{2}]$.

Example 4.10. The tempered distribution $\sum_{k\in\mathbb{Z}}(-1)^k\delta_{\frac{k}{2}}$ identifies $OPW([0,1] \times [-\frac{1}{2},\frac{1}{2}])$ [25].

To illustrate the rigidity of operator sampling in comparison to function sampling, we add the following simple example.

Example 4.11. Let $\Lambda_r = \{\lambda_k\}_{k \in \mathbb{Z}}$ be given by $\lambda_k = k$ for $k \neq 0$ and $\lambda_0 = r \in \mathbb{R}$. The set Λ_r is a set of sampling for $PW([-\frac{1}{2}, \frac{1}{2}])$ if and only if $r \notin \mathbb{Z} \setminus \{0\}$. To see this, note that as $\{e^{2\pi i k \xi}\}_{k \in \mathbb{Z}}$ is a Riesz basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$, so is $\{e^{2\pi i k \xi}\}_{k \neq 0} \cup \{e^{2\pi i r \xi}\}$ if $r \notin \mathbb{Z} \setminus \{0\}$. By Theorem 4.8, $\Lambda_r = \{\lambda_k\}_{k \in \mathbb{Z}}$ is a set of sampling for $OPW([0,1] \times [-\frac{1}{2}, \frac{1}{2}])$ if and only if $r = 0$.

4.3. **An operator version of Kramer's Lemma.** Kramer's lemma plays a cru-cial role in the proofs of a number of important sampling theorems. For example, it allows for sampling series expansions for functions which are integral transforms of type other than Fourier one. For example, Bessel-Hankel, Legendre, Jacobi, Laguerre, Gegenbauer, Chebyschev, prolate spheroidal, and Hermite transforms can be considered, where each transform is defined as an integral transform whose kernel is its special function [9, 21, 31]. In particular, if a function on $\mathbb{R}^2$ has a circular symmetry, then a multi-dimensional Fourier transform can be reduced to a one-dimensional Bessel-Hankel transform [22].

Theorem 4.12. *(Kramer's Lemma) Let $I \subseteq \mathbb{R}$ be a bounded interval and $k(\cdot, t) \in L^2(I)$ for each fixed t in $D \subseteq \mathbb{R}$. If there is a sampling sequence $\{t_n\}_{n \in \mathbb{Z}}$ in D such that $\{k(\xi, t_n)\}_{n \in \mathbb{Z}}$ forms a frame for $L^2(I)$, then for any $f(t) = \langle F, \overline{k(\cdot, t)} \rangle_{L^2(I)}$, $F \in L^2(I)$, we have*

$$f(t) = \sum_{n \in \mathbb{Z}} f(t_n) S_n(t)$$

where the reconstruction functions S_n are given by

$$S_n(t) = \int_I \tilde{k}_n(\xi) k(\xi, t) d\xi,$$

with $\{\tilde{k}_n(\xi)\}_{n \in \mathbb{Z}}$ a dual frame of $\{\overline{k(\xi, t_n)}\}_{n \in \mathbb{Z}}$.

The original Kramer's lemma assumed that $\{k(\xi, t_n)\}_{n \in \mathbb{Z}}$ is an orthonormal ba-sis for $L^2(I)$ [9, 17]. However, we can easily see that the result extends to the case where $\{k(\xi, t_n)\}_{n \in \mathbb{Z}}$ forms a frame. We remark that the classical sampling theorem addressing f in $PW([-\frac{1}{2}, \frac{1}{2}])$ is given by Kramer's lemma if we take $k(\xi, t) = e^{2\pi i \xi t}, F = \hat{f}$ and use the fact that $\{e^{2\pi i n \xi}\}_{n \in \mathbb{Z}}$ is an orthonormal ba-sis for $L^2[-\frac{1}{2}, \frac{1}{2}]$.

Let $I \subseteq \mathbb{R}$ be bounded. For

$$\mathcal{K} : L^2(I) \longrightarrow L^2(\mathbb{R}), \quad F \longmapsto \langle F(\cdot), k(x, \cdot) \rangle_{L^2(I)}, \tag{9}$$

bounded, set

$$\begin{aligned} \mathcal{H}^k(S) \quad = \quad & \big\{ H \in HS(L^2(\mathbb{R})) \\ & : h_H(t, x) = \langle \zeta_H(t, \cdot), k(x, \cdot) \rangle_{L^2}, \ \zeta_H \in L^2(\mathbb{R} \times I), \ \operatorname{supp} \zeta_H \subseteq S \big\}. \end{aligned}$$

Clearly, for $k(x, \nu) = e^{-2\pi i \nu x}$ we have $\mathcal{H}^k(S) = OPW(S)$.

Theorem 4.13. *Let $H \in \mathcal{H}^k([0, d] \times I)$, $d > 0$, I in $\mathbb{R}$ bounded. If there is a set $\{y_n\}_{n \in \mathbb{Z}}$ in $\mathbb{R}$, such that $\{k(t + y_n, \nu)\}_{n \in \mathbb{Z}}$ forms a frame for $L^2(I)$ for every $t \in [0, d]$, and $d' = \inf(y_{n+1} - y_n) \geq d$, then exists $c > 0$ with*

$$\|H(\sum_{k \in \mathbb{Z}} \delta_{y_k})\|_{L^2} \geq c \|H\|_{HS}, \quad H \in \mathcal{H}^k([0, d] \times I). \tag{10}$$

If the map $\mathcal{K}$ in (9) is bounded below, then

$$\|H(\sum_{k\in\mathbb{Z}}\delta_{y_k})\|_{L^2} \asymp \|H\|_{HS}, \quad H \in \mathcal{H}^k([0,d]\times I), \tag{11}$$

and operator reconstruction is possible as

$$h_H(t,x) = r(t)\sum_{n\in\mathbb{Z}}(H\sum_{k\in\mathbb{Z}}\delta_{y_k})(t+y_n)\phi_n(t,x),$$

where $\phi_n(t,x)$ is given by

$$\phi_n(t,x) = \int_I k_n^*(t,\nu)\overline{k(x,\nu)}d\nu$$

with $\{k_n^*(t,\nu)\}_{n\in\mathbb{Z}}$ being a dual frame of $\{k(t+y_n,\nu)\}_{n\in\mathbb{Z}}$ for each t and $r \equiv 1$ on $[0,d]$ with $\operatorname{supp} r \subseteq [d-d',d']$.

Proof. As $\inf_k(y_{k+1}-y_k) = d' \geq d$ we have

$$r(t)(H\sum_{k\in\mathbb{Z}}\delta_{y_k})(t+y_n) = r(t)\sum_{k\in\mathbb{Z}}h_H(t+y_n-y_k,t+y_n) = h_H(t,t+y_n), \quad t\in\mathbb{R},$$

where $r \equiv 1$ on $[0,d]$ and $\operatorname{supp} r \subseteq [d-d',d']$. Consequently,

$$
\begin{aligned}
\|\zeta_H\|_{L^2}^2 &= \int \|\zeta_H(t,\cdot)\|_{L^2}^2\,dt \asymp \int \|\{\langle\zeta_H(t,\cdot),k(t+y_n,\cdot)\rangle_{L^2}\}\|_{l^2}^2\,dt\\
&= \int \|\{h_H(t,t+y_n)\}\|_{l^2}^2\,dt = \int\sum_{n\in\mathbb{Z}}|h_H(t,t+y_n)|^2 dt\\
&= \int\sum_{n\in\mathbb{Z}}|r(t-y_n)(H\sum_{k\in\mathbb{Z}}\delta_{y_k})(t)|^2 dt = \sum_{n\in\mathbb{Z}}\int_{y_n}^{y_{n+1}}|H\sum_{k\in\mathbb{Z}}\delta_{y_k}(t)|^2 dt\\
&= \int |H\sum_{k\in\mathbb{Z}}\delta_{y_k}(t)|^2 dt = \|H\sum_{k\in\mathbb{Z}}\delta_{y_k}\|^2, \quad H\in\mathcal{H}^k([0,d]\times I).
\end{aligned}
$$

We used the fact that $r(t-y_n)(H\sum_{k\in\mathbb{Z}}\delta_{y_k})(t) = (H\sum_{k\in\mathbb{Z}}\delta_{y_k})(t)$ for $t\in[y_n,y_n+d]$ and $r(t-y_n)(H\sum_{k\in\mathbb{Z}}\delta_{y_k})(t) = 0$ for $t\in[y_n+d,y_{n+1})$, $n\in\mathbb{Z}$.

Equations (10) and (11) follow from the fact that $\mathcal{K}$ is bounded and the hypothesis that $\mathcal{K}$ is bounded below, respectively.

Moreover, we have, for $\nu\in I$,

$$\zeta_H(t,\nu) = \sum_{n\in\mathbb{Z}}\langle\zeta_H(t,\nu),k(t+y_n,\nu)\rangle_{L^2}k_n^*(t,\nu) = \sum_{n\in\mathbb{Z}}h_H(t,t+y_n)k_n^*(t,\nu),$$

where $\{k_n^*(t,\nu)\}_{n\in\mathbb{Z}}$ is a dual frame of $\{k(t+y_n,\nu)\}_{n\in\mathbb{Z}}$ for each t. Multiplying $\overline{k(x,\nu)}$ and integrating with respect to ν on both sides, we have for fixed $t\in[0,d]$

$$h_H(t,x) = \sum_{n\in\mathbb{Z}}h_H(t,t+y_n)\phi_n(t,x), \quad x\in\mathbb{R},$$

where $\phi_n(t,x) = \int_I k_n^*(t,\nu)\overline{k(x,\nu)}d\nu$. $\qquad\square$

Note that a set of sampling $\{y_n\}_{n\in\mathbb{Z}}$ in Theorem 4.13 is uniformly discrete and a separation constant d' is greater than or equal to d. On the other hand, the condition that $\{k(t+y_n,\nu)\}_{n\in\mathbb{Z}}$ forms a frame for $L^2(I)$ indicates that the sampling points should not be too sparse.

Remark 4.14. Unlike Kramer's Lemma for functions, we do not have any explicit example for operator sampling other than the $OPW(S)$ case. Sampling theorems based on various kinds of orthogonal polynomials generally do not satisfy all hypotheses in Theorem 4.13. For instance, for the Bessel-Hankel transform, we have $k(x, \nu) = \sqrt{x\nu} J_n(x\nu)$ where J_n is the Bessel function of the first kind of order n. Taking λ_k as the k-th positive root of $J_n(x)$, one can, in fact, obtain a sampling expansion induced from Bessel-Hankel transform [9]. However, the kernel $k(x, \nu) = \sqrt{x\nu} J_n(x\nu)$ does not allow for $\{k(t + y_n, \nu)\}_{n \in \mathbb{Z}}$ being a frame for each t.

In Theorem 3.1 we used the fact that $\{m(\nu)e^{-2\pi i n \nu}\}_{n \in \mathbb{Z}}$ forms an orthonormal basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$ for any function m satisfying $|m(\nu)| = 1$. In general, we have the following.

Proposition 4.15. *For $D, I \subseteq \mathbb{R}$ and $k(x, \nu)$ defined on $\mathbb{R} \times I$, let $k(x, \nu)$ satisfy $k(t + x, \nu) = m(t, \nu)k(x, \nu)$ for some m such that $0 < \|m(t, \cdot)\|_0 \leq \|m(t, \cdot)\|_\infty < \infty$ for all $t \in D$. If $\{k(y_n, \nu)\}_{n \in \mathbb{Z}}$ is a frame for $L^2(I)$, then $\{k(t + y_n, \nu)\}_{n \in \mathbb{Z}}$ is also a frame for $L^2(I)$ for all $t \in D$.*

For example, the Fourier kernel $k(x, \nu) = e^{-2\pi i x \nu}$ and the Hilbert transform kernel $k(x, \nu) = -i \operatorname{sgn}(\nu) e^{-2\pi i x \nu}$ satisfy the hypotheses of Proposition 4.15.

5. Multi-channel sampling for Hilbert-Schmidt operators

In classical multi-channel sampling, a signal is reconstructed using discrete values from the outputs of N different time–invariant operators applied to a single input signal. Generally, each of the N outputs is sampled at $\frac{1}{N}$-th of the Nyquist-Landau rate of the input signal. For example, when the signal's bandwidth is Ω Hz, then we should collect at least Ω samples per second. But if we design N channel filters appropriately, then it suffices to obtain Ω/N samples per second from each channel and combine the samples in order to reconstruct the signal. This allows us to reduce sampling rate requirements on sampling hardware at the cost of employing multiple samplers.

A number of important theorems, for example, on periodic nonuniform sampling, on derivative sampling and on samples of Hilbert transforms can be explained in the framework of multi-channel sampling [3, 9]. In [10], multi-channel sampling has been developed for abstract Hilbert space, allowing each channel output to be sampled at different, irregular points.

In Sections 3 and 4, it has been shown that a slowly time-varying/underspread operator is identifiable by a single channel output while in [16, 24] it is shown that an overspread operator is not identifiable in this sense. However, in this section we shall show that overspread operators may be recoverable from multiple channel outputs. In addition, we seek to reduce the rate at which delta impulses are produced for channel identification.

Throughout this section, we shall consider Hilbert-Schmidt operators whose Kohn-Nirenberg symbols are bandlimited to rectangular domains.

Theorem 5.1. *For $M, N \in \mathbb{N}$, $OPW([0, N] \times [-\frac{M}{2}, \frac{M}{2}])$ permits multi-channel operator sampling as*

$$\|H\|_{HS}^2 = \frac{1}{M^2 N} \sum_{j=0}^{MN-1} \|H(\sum_{n \in \mathbb{Z}} e^{2\pi i j n / MN} \delta_{\frac{n}{M}})\|^2, \quad H \in OPW([0, N] \times [-\frac{M}{2}, \frac{M}{2}]).$$

Proof. Suppose that $\eta_H \in L^2([0,N] \times [-\frac{M}{2}, \frac{M}{2}])$ for some $M, N \in \mathbb{N}$. Consider an orthonormal basis $\{\frac{1}{\sqrt{M}} e^{-2\pi i (t + \frac{n}{M}) \nu}\}_{n \in \mathbb{Z}}$ for $L^2[-\frac{M}{2}, \frac{M}{2}]$, $t \in [0, N]$. Then

$$
\begin{aligned}
\eta_H(t, \nu) &= \sum_{n \in \mathbb{Z}} \langle \eta_H(t, \nu), \tfrac{1}{\sqrt{M}} e^{-2\pi i (t + \frac{n}{M}) \nu} \rangle \tfrac{1}{\sqrt{M}} e^{-2\pi i (t + \frac{n}{M}) \nu} \\
&= \sum_{n \in \mathbb{Z}} h_H(t, t + \tfrac{n}{M}) \tfrac{1}{M} e^{-2\pi i (t + \frac{n}{M}) \nu}, \quad \nu \in [-\tfrac{M}{2}, \tfrac{M}{2}],
\end{aligned}
$$

and

$$
h_H(t, x) = \sum_{n \in \mathbb{Z}} h_H(t, t + \tfrac{n}{M}) \operatorname{sinc} M(x - t - \tfrac{n}{M}), \ x \in \mathbb{R}, \ a.e. \ t \in [0, N].
$$

Since $\{\sqrt{M} \operatorname{sinc} M(\cdot - t - \frac{n}{M})\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $PW([-\frac{M}{2}, \frac{M}{2}])$, Parseval's identity gives

$$
\|h_H(t, \cdot)\|_2^2 = \tfrac{1}{M} \sum_{n \in \mathbb{Z}} |h_H(t, t + \tfrac{n}{M})|^2, \quad t \in [0, N].
$$

Hence, we have

$$
\begin{aligned}
\|h_H\|_{L^2(\mathbb{R}^2)}^2 &= \int_0^N \|h_H(t, \cdot)\|_{L^2(\mathbb{R})}^2 dt = \tfrac{1}{M} \int_0^N \sum_{n \in \mathbb{Z}} |h_H(t, t + \tfrac{n}{M})|^2 dt \\
&= \tfrac{1}{M} \sum_{r=0}^{MN-1} \int_{\frac{r}{M}}^{\frac{r+1}{M}} \sum_{n \in \mathbb{Z}} |h_H(t, t + \tfrac{n}{M})|^2 dt \\
&= \tfrac{1}{M} \int_0^{\frac{1}{M}} \sum_{r=0}^{MN-1} \sum_{n \in \mathbb{Z}} |h_H(t + \tfrac{r}{M}, t + \tfrac{r}{M} + \tfrac{n}{M})|^2 dt \\
&= \tfrac{1}{M} \int_0^{\frac{1}{M}} \sum_{r=0}^{MN-1} \sum_{n \in \mathbb{Z}} |h_H(t + \tfrac{r}{M}, t + \tfrac{n}{M})|^2 dt.
\end{aligned}
$$

Since $(H \sum_n \frac{1}{\sqrt{MN}} \delta_{\frac{n}{M}})(t + \frac{k}{M}) = \sum_n \frac{1}{\sqrt{MN}} h_H(t + \frac{k}{M} - \frac{n}{M}, t + \frac{k}{M})$, for fixed $t \in [0, \frac{1}{M}]$, we have

$$
r(t) \sum_{n \in \mathbb{Z}} \tfrac{1}{\sqrt{MN}} h_H(t + \tfrac{k}{M} - \tfrac{n}{M}, t + \tfrac{k}{M}) = \tfrac{1}{\sqrt{MN}} \sum_{r=0}^{MN-1} h_H(t + \tfrac{r}{M}, t + \tfrac{k}{M}),
$$

where $r(t) = \chi_{[0, \frac{1}{M}]}(t)$. Similarly, consider $(H \sum_n \frac{1}{\sqrt{MN}} \omega_j^n \delta_{\frac{n}{M}})(x)$ where $\omega_j = e^{2\pi i j / MN}$, $j = 0, 1, \cdots, MN - 1$. Then

$$
(H \sum_n \tfrac{1}{\sqrt{MN}} \omega_j^n \delta_{\frac{n}{M}})(t + \tfrac{k}{M}) = \sum_n \tfrac{\omega_j^n}{\sqrt{MN}} h_H(t + \tfrac{k}{M} - \tfrac{n}{M}, t + \tfrac{k}{M}).
$$

For fixed $t \in [0, \frac{1}{M}]$,

$$
r(t) \sum_{n \in \mathbb{Z}} \tfrac{\omega_j^n}{\sqrt{MN}} h_H(t + \tfrac{k}{M} - \tfrac{n}{M}, t + \tfrac{k}{M}) = \sum_{r=0}^{MN-1} \tfrac{\omega_j^{k-r}}{\sqrt{MN}} h_H(t + \tfrac{r}{M}, t + \tfrac{k}{M}).
$$

Consider the system of linear equations

$$r(t) \begin{bmatrix} (H \sum_n \frac{1}{\sqrt{MN}} \delta_{\frac{n}{M}})(t+\frac{k}{M}) \\ (H \sum_n \frac{\omega_1^n}{\sqrt{MN}} \delta_{\frac{n}{M}})(t+\frac{k}{M}) \\ \vdots \\ (H \sum_n \frac{\omega_{MN-1}^n}{\sqrt{MN}} \delta_{\frac{n}{M}})(t+\frac{k}{M}) \end{bmatrix} = A_k \begin{bmatrix} h_H(t,t+\frac{k}{M}) \\ h_H(t+\frac{1}{M},t+\frac{k}{M}) \\ \vdots \\ h_H(t+\frac{MN-1}{M},t+\frac{k}{M}) \end{bmatrix}$$

where all matrices A_k are unitary $MN \times MN$ DFT matrices with entries given by $(A_k)_{j,l} = \frac{1}{\sqrt{MN}} e^{2\pi i (j-1)(k-(l-1))/MN}$. Since all A_k's are unitary, we have

$$\left\| \begin{bmatrix} \|\{(H \sum_n \frac{1}{\sqrt{MN}} \delta_{\frac{n}{M}})(t+\frac{k}{M})\}_k\|_{l^2} \\ \|\{(H \sum_n \frac{\omega_1^n}{\sqrt{MN}} \delta_{\frac{n}{M}})(t+\frac{k}{M})\}_k\|_{l^2} \\ \vdots \\ \|\{(H \sum_n \frac{\omega_{MN-1}^n}{\sqrt{MN}} \delta_{\frac{n}{M}})(t+\frac{k}{M})\}_k\|_{l^2} \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} \|\{h_H(t,t+\frac{k}{M})\}_k\|_{l^2} \\ \|\{h_H(t+\frac{1}{M},t+\frac{k}{M})\}_k\|_{l^2} \\ \vdots \\ \|\{h_H(t+\frac{MN-1}{M},t+\frac{k}{M})\}_k\|_{l^2} \end{bmatrix} \right\|^2,$$

where $\|\cdot\|$ denotes the $L^2[0,\frac{1}{M}]^{MN}$ norm. Therefore

$$\begin{aligned} \|h_H\|^2 &= \frac{1}{M} \int_0^{\frac{1}{M}} \sum_{k\in\mathbb{Z}} \sum_{j=0}^{MN-1} |(H \sum_{n\in\mathbb{Z}} \frac{\omega_j^n}{\sqrt{MN}} \delta_{\frac{n}{M}})(t+\frac{k}{M})|^2 dt \\ &= \frac{1}{M} \int_{\mathbb{R}} \sum_{j=0}^{MN-1} |(H \sum_{n\in\mathbb{Z}} \frac{\omega_j^n}{\sqrt{MN}} \delta_{\frac{n}{M}})(t)|^2 dt \\ &= \frac{1}{M^2 N} \sum_{j=0}^{MN-1} \|H \sum_{n\in\mathbb{Z}} \omega_j^n \delta_{\frac{n}{M}}\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

$\square$

Clearly, the matrices A_k can be replaced by appropriate sequences of matrices whose norms are bounded above and away from zero.

Theorem 5.2. *Let $M, N \in \mathbb{N}$ and $\{c_{j,n}\}_{j=1,n\in\mathbb{Z}}^{MN}$ bounded with A_k, $k \in \mathbb{Z}$, invertible with $\|A_k^{-1}\| \leq C < \infty$ where $(A_k)_{j,l=1,\dots,MN} = c_{j,k-l+1}$. For $f_j = \sum_{n\in\mathbb{Z}} c_{j,n} \delta_{\frac{n}{M}}$, $1 \leq j \leq MN$, we have*

$$\|H\|_{HS}^2 \asymp \sum_{j=1}^{MN} \|Hf_j\|_{L^2}^2, \quad H \in OPW([0,N]\times[-\frac{M}{2},\frac{M}{2}]).$$

The assumption in Theorem 5.2 is satisfied if for all j, $\{c_{j,n}\}_{n\in\mathbb{Z}}$ is $MN-$periodic and $A_{j,0}$ is invertible with $\|A_{j,0}^{-1}\|$ also uniformly bounded.

Now we consider periodic nonuniform sampling as first proposed by Yen [30]. We first recall periodic nonuniform sampling theorem for functions in $PW([-\frac{\Omega}{2}, \frac{\Omega}{2}])$.

Theorem 5.3. *There exists a Riesz basis $\{S_j(t-\frac{nN}{\Omega})\}_{j=1,n\in\mathbb{Z}}^N$ for $PW([-\frac{\Omega}{2}, \frac{\Omega}{2}])$ such that*

$$f(t) = \sum_{j=1}^{N} \sum_{n\in\mathbb{Z}} f\left(\frac{nN}{\Omega} + \alpha_j\right) S_j\left(t - \frac{nN}{\Omega}\right), \quad f \in PW([-\frac{\Omega}{2}, \frac{\Omega}{2}]),$$

where $0 \leq \alpha_j < \frac{N}{\Omega}$, $1 \leq j \leq N$, and $\alpha_i \neq \alpha_j$ for $i \neq j$.

We show that $OPW([0,N]\times[-\frac{M}{2},\frac{M}{2}])$ is identifiable by MN identifiers which are given by delta-trains whose supports are periodically nonuniformly distributed.

Theorem 5.4. *For $M, N \in \mathbb{N}$, and $0 \le \alpha_1 < \alpha_2 < \ldots < \alpha_{MN} < N$, we have*

$$\|H\|_{HS}^2 \asymp \sum_{j=1}^{MN} \|H(\sum_{n\in\mathbb{Z}} \delta_{nN+\alpha_j})\|^2, \quad H \in OPW([0,N]\times[-\tfrac{M}{2},\tfrac{M}{2}]).$$

Proof. If we apply Theorem 5.3 to $h_H(t,\cdot) \in PW([-\frac{M}{2},\frac{M}{2}])$ with MN channels, then we obtain

$$h_H(t,x) = \sum_{j=1}^{MN} \sum_{n\in\mathbb{Z}} h_H(t, t+nN+\alpha_j)\, \varphi_j(x-t-nN),\ x\in\mathbb{R},\ a.e.\ t\in[0,N],$$

and

$$\|h_H(t,\cdot)\|^2 \asymp \sum_{j=1}^{MN} \sum_{n\in\mathbb{Z}} |h_H(t, t+nN+\alpha_j)|^2,\ a.e.\ t\in[0,N],$$

where $\{\varphi_j(x-t-nN)\}_{j=1,n\in\mathbb{Z}}^{MN}$ is a Riesz basis for $PW([-\frac{M}{2},\frac{M}{2}])$ for each $t\in[0,N]$. Therefore, we have

$$\begin{aligned}
\|H\|_{HS}^2 = \|h_H\|_{L^2}^2 \quad &\asymp\quad \int_0^N \sum_{j=1}^{MN} \sum_{n\in\mathbb{Z}} |h_H(t, t+nN+\alpha_j)|^2 dt \\
&=\quad \sum_{j=1}^{MN} \int_0^N \sum_{n\in\mathbb{Z}} |H(\sum_{k\in\mathbb{Z}} \delta_{kN+\alpha_j})(t+nN+\alpha_j)|^2 dt \\
&=\quad \sum_{j=1}^{MN} \|H(\sum_{k\in\mathbb{Z}} \delta_{kN+\alpha_j})\|_{L^2}^2,
\end{aligned}$$

that is, $\{\sum_{k\in\mathbb{Z}} \delta_{kN+\alpha_j}\}_{j=1}^{MN}$ identifies $OPW([0,N]\times[-\frac{M}{2},\frac{M}{2}])$. $\square$

Remark 5.5. Note that as in the multichannel sampling theory for functions, Theorem 5.4 can be applied to reduce the rate of impulse transmission. For example, as $OPW([0,1]\times[-\frac{1}{2},\frac{1}{2}]) \subseteq OPW([0,N]\times[-\frac{1}{2},\frac{1}{2}])$, we can identify any operator in $OPW([0,1]\times[-\frac{1}{2},\frac{1}{2}])$ by its action on the tempered distributions $\sum_{n\in\mathbb{Z}} \delta_{nN+j\alpha}$, $0 \le \alpha \le 1, j=0,\ldots,N-1$, each of which has impulse rate $1/N$.

Another important multi-channel sampling concept can be applied to operator sampling, namely, derivative sampling. We apply a multi-channel sampling formula consisting of samples of f and its $MN-1$ derivatives [18, 19].

Theorem 5.6. *For $M, N \in \mathbb{N}$, we have*

$$\|H\|_{HS}^2 \asymp \sum_{j=0}^{MN-1} \Big\| \sum_{r=0}^{j} \binom{j}{r} (-1)^r \big(H\sum_k \delta_{kN}^{(r)}\big)^{(j-r)} \Big\|^2,\ H \in OPW([0,N]\times[-\tfrac{M}{2},\tfrac{M}{2}]).$$

Here, $f^{(r)}$ denotes the r-th derivative of f in the distributional sense.

Proof. If we apply multi-channel derivative sampling theorem to $h_H(t,\cdot) \in PW([-\frac{M}{2},\frac{M}{2}])$, then we obtain

$$h_H(t,x) = \sum_{j=0}^{MN-1} \sum_{n\in\mathbb{Z}} \frac{\partial^j}{\partial x^j} h_H(t,x)|_{x=t+nN}\, \varphi_j(x-t-nN),\ x\in\mathbb{R},\ a.e.\ t\in[0,N],$$

and

$$\|h_H(t,\cdot)\|^2 \asymp \sum_{j=0}^{MN-1} \sum_{n\in\mathbb{Z}} \big|\frac{\partial^j}{\partial x^j} h_H(t,x)|_{x=t+nN}\big|^2, \ a.e. \ t\in[0,N],$$

where $\{\varphi_j(x-t-nN)\}_{j=0,n\in\mathbb{Z}}^{MN-1}$ is a Riesz basis for $PW([-\frac{M}{2},\frac{M}{2}])$ for each fixed $t\in[0,N]$. Let H_j be the operator defined through the j-th derivative of $h_H(t,\cdot)$, that is,

$$H_j f(x) = \int \frac{\partial^j}{\partial x^j} h_H(t,x) f(x-t) dt \quad a.e.$$

Since $\operatorname{supp} \frac{\partial^j}{\partial x^j} h_H(\cdot,x) \subseteq \operatorname{supp} h_H(\cdot,x)$, we have

$$\chi_{[0,N]}(t)(H_j \sum_{k\in\mathbb{Z}} \delta_{kN})(t+nN) = \frac{\partial^j}{\partial x^j} h_H(t,x)|_{x=t+nN},$$

so that

$$\begin{aligned}
\|h_H\|^2 &= \int_0^N \|h_H(t,\cdot)\|^2 dt \asymp \int_0^N \sum_{j=0}^{MN-1} \sum_{n\in\mathbb{Z}} |(H_j \sum_{k\in\mathbb{Z}} \delta_{kN})(t+nN)|^2 dt \\
&= \sum_{j=0}^{MN-1} \|H_j \sum_{k\in\mathbb{Z}} \delta_{kN}\|^2.
\end{aligned}$$

Observing that $H_j f(x) = \sum_{r=0}^j \binom{j}{r}(-1)^r \big(H f^{(r)}\big)^{(j-r)}$ by Leibniz's rule completes the proof. $\qquad\square$

We conclude this section by presenting two explicit multi-channel reconstruction formulas for operators.

Example 5.7. Consider the Riesz bases $\{e^{-2\pi i(t+2n)\nu}\}_{n\in\mathbb{Z}} \cup \{-2\pi i\nu\, e^{-2\pi i(t+2n)\nu}\}_{n\in\mathbb{Z}}$, $t\in\mathbb{R}$, of $L^2[-1/2,1/2]$, as well as their Riesz basis duals given by $\{2(1-2|\nu|)\, e^{-2\pi i(t+2n)\nu}\}_{n\in\mathbb{Z}} \cup \{\frac{2i}{\pi} \operatorname{sgn}(\nu)\, e^{-2\pi i(t+2n)\nu}\}_{n\in\mathbb{Z}}$ [9]. We obtain

$$\begin{aligned}
\eta_H(t,\nu) &= \sum_n \langle \eta_H(t,\cdot), e^{-2\pi i(t+2n)\cdot}\rangle 2(1-2|\nu|)\, e^{-2\pi i(t+2n)\nu} \\
&+ \sum_n \langle \eta_H(t,\cdot), -2\pi i\cdot e^{-2\pi i(t+2n)\cdot}\rangle \frac{2i}{\pi} \operatorname{sgn}(\nu)\, e^{-2\pi i(t+2n)\nu}, \ \nu\in[-\tfrac{1}{2},\tfrac{1}{2}].
\end{aligned}$$

Taking an inverse Fourier transform with respect to the variable ν, we have

$$h_H(t,x) = \sum_n h_H(t,t+2n) S_n(t,x) + \frac{\partial}{\partial x} h_H(t,x)|_{x=t+2n} T_n(t,x), \ x\in\mathbb{R}, \ t\in[0,2].$$

where

$$S_n(t,x) = \operatorname{sinc}^2 \tfrac{1}{2}(x-t-2n)$$

and

$$T_n(t,x) = \tfrac{2}{\pi} \operatorname{sinc}\tfrac{1}{2}(x-t-2n) \sin\tfrac{\pi}{2}(x-t-2n).$$

We give a second reconstruction formula for the operator class $OPW([0,2]\times[-\frac{1}{2},\frac{1}{2}])$.

Example 5.8. Let $H\in OPW([0,2]\times[-\frac{1}{2},\frac{1}{2}])$. Then for $0<\alpha<1$,

$$h_H(t,x) = \sum_{n\in\mathbb{Z}}(H\sum_{k\in\mathbb{Z}}\delta_{2k})(t+2n) S_1(x-t-2n) + (H\sum_{k\in\mathbb{Z}}\delta_{2k+\alpha})(t+2n+\alpha) S_2(x-t-2n-\alpha),$$

where the reconstruction functions are

$$S_1(x) = \frac{2}{e^{\pi i \alpha} - 1} \mathcal{F}^{-1} \left(e^{\pi i \alpha} \chi_{[-\frac{1}{2}, 0)}(\nu) - \chi_{[0, \frac{1}{2}]}(\nu) \right)(x)$$

and

$$S_2(x) = \frac{2}{e^{\pi i \alpha} - 1} \mathcal{F}^{-1} \left(-\chi_{[-\frac{1}{2}, 0)}(\nu) + e^{\pi i \alpha} \chi_{[0, \frac{1}{2}]}(\nu) \right)(x).$$

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