4 System Level Aspects for Single Cell Scenarios

4.1 Efficient Analysis of OFDM Channels

N. Grip, Luleå University of Technology, Sweden
G. E. Pfander, Jacobs University Bremen, Germany

4.1.1 Introduction

Narrowband finite lifelength systems such as wireless communications can be well modelled by smooth and compactly supported spreading functions. We show how to exploit this fact to derive a fast algorithm for computing the matrix representation of such operators with respect to well time-frequency localized Gabor bases (such as pulse shaped OFDM bases). Hereby we use a minimum of approximations, simplifications, and assumptions on the channel.

The derived algorithm and software can be used, for example, for comparing how different system settings and pulse shapes affect the diagonalization properties of an OFDM system acting on a given channel.

4.1.2 The channel matrix G

A *Gabor* (or *Weyl-Heisenberg*) system with *window* g and *lattice constants* a and b is the sequence $(g_{q,r})_{q,r\in\mathbb{Z}^d}$ of translated and modulated functions

$$g_{q,r} \stackrel{\text{def}}{=} T_{ra}M_{qb}g = \mathrm{e}^{\mathrm{i}2\pi\langle qb, x-ra\rangle}g(x-ra).$$

For OFDM communications applications, information is stored in the coefficients of the transmitted signal $s = \sum_{q,r\in\mathbb{Z}} c_{q,r}g_{q,r}$. In order to guarantee that the coefficients can be recovered from s in a numerically stable way, s and its coefficients should be equivalent in the sense that for some nonzero and finite A, B independent of s, $A\|s\|^2 \leq \|c\|^2 \leq B\|s\|^2$ with $\|c\|^2 \stackrel{\text{def}}{=} \sum_{q,r}|c_{q,r}|^2$ and $\|s\|^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}}|s(t)|^2\,\mathrm{d}t$. This means that the sequence of functions $(g_{q,r})_{q,r\in\mathbb{Z}}$ is a *Riesz basis* for the function space $L^2(\mathbb{R})$ of square integrable functions (or an *orthonormal basis* in the special case $A = B = 1$). This guarantees the existence of a *dual basis* $(\widetilde{g}_{q,r})$ that also is a Gabor basis. Such bases are also called *biorthogonal*, or, in the special case $\widetilde{g} = g$, *orthonormal*.

In communications applications, s is sent through a channel with linear channel operator H and the receiver typically tries to reconstruct the transmitted coefficients

$c_{q,r} = \langle s, \widetilde{g}_{q,r} \rangle$ from the received signal Hs using some (possibly other) Gabor Riesz basis $(\gamma_{q',r'})$. A series expansion in this basis gives

$$\begin{aligned} Hs &= \sum_{q',r' \in \mathbb{Z}} \langle Hs, \gamma_{q',r'} \rangle \, \widetilde{\gamma}_{q',r'} = \sum_{q',r' \in \mathbb{Z}} \left\langle H \sum_{q,r \in \mathbb{Z}} c_{q,r} g_{q,r} \, , \gamma_{q',r'} \right\rangle \widetilde{\gamma}_{q',r'} \\ &= \sum_{q',r' \in \mathbb{Z}} \left(\sum_{q,r \in \mathbb{Z}} c_{q,r} \, \langle H g_{q,r}, \gamma_{q',r'} \rangle \right) \widetilde{\gamma}_{q',r'}, = \sum_{q',r' \in \mathbb{Z}} (Gc)_{q',r'} \widetilde{\gamma}_{q',r'}, \end{aligned}$$

where G is the coefficient mapping $(c_{q,r})_{q,r} \mapsto \left(\sum_{q,r \in \mathbb{Z}} c_{q,r} \, \langle H g_{q,r}, \gamma_{q',r'} \rangle \right)_{q',r'}$ with biinfinite matrix representation

$$\mathrm{G}_{q',r';q,r} = \langle H g_{q,r}, \gamma_{q',r'} \rangle \, ,$$

and with indices (q', r') and (q, r) for rows and columns respectively. The matrix elements are usually called *intercarrier interference* (ICI) for $p = p'$ and $q \neq q'$. Similarly, the matrix elements are called *intersymbol interference* (ISI) when $p \neq p'$. Recovering the transmitted coefficients corresponds to inverting G, which is unreasonably time-consuming unless g and γ can be chosen so that G is diagonal or at least has fast off-diagonal decay.

We call H *time-invariant* if it commutes with the time-shift operator $T_{t_0} f(t) = f(t - t_0)$ for any t_0, that is, if $T_{t_0} H = H T_{t_0}$. Linear and time-invariant H are convolution operators, for which it is well-known that the family of complex exponentials $\mathrm{e}^{i2\pi\xi t}$ are "eigenfunctions" in the sense that for the restriction of such functions to an interval $[0, L]$, that is, $s(\cdot) = \mathrm{e}^{i2\pi\langle\xi,\cdot\rangle} \chi_{[0,L]}(\cdot)$, there is some complex scalar λ_ξ such that if h lives on $[0, L_h]$, then $Hs = \lambda_\xi s$ in the interval $[L_h, L]$. Thus G is easily diagonalized by using Gabor windows $g = \chi_{[0,L]}$, $\gamma = \chi_{[L_h,L]}$ and lattice constants such that the resulting Gabor systems $(g_{k,l})$ and $(\gamma_{k,l})$ are biorthogonal bases. This trick is used in wireline communications, where the smaller support of γ is obtained by removing a guard interval (often called *cyclic prefix*) from g. See, for example, [4, Section 2.3] for more details and further references.

In *wireless* communications, due to reflections on different structures in the environment, the transmitted signal reaches the receiver via a possibly infinite number of different wave propagation paths. Because of the highly time varying nature of this setup of paths and the corresponding channel operator, we can at most hope for approximate diagonalization of the channel operator. In fact, two different time-varying operators do in general not commute, so both cannot be diagonalized with the same choice of bases. Thus, diagonalization is usually only possible in the following sense: Typically, $(H g_{q,r})$ is a finite and linearly independent sequence, and thus a Riesz basis with some dual basis $\left(\widetilde{H g_{q,r}}\right)$, so for true diagonalization of G, we would have to set $\gamma_{q',r'} = \widetilde{H g_{q,r}}$, but then $\gamma_{q',r'}$ would typically not be a Gabor basis or have any other simple structure that enables efficient computation of all $\gamma_{q',r'}$ and all the diagonal elements $\langle H g_{q,r}, \gamma_{q',r'} \rangle$. Hence, for computational complexity to meet practical restrictions we have to settle for "almost dual" Gabor bases $(g_{q,r})$ and $(\gamma_{q',r'})$, such as the Gabor bases proposed in [7]. We are primarily interested in bases that are good candidates for providing low intersymbol and interchannel interference (ISI and ICI). As proposed in [7], we expect excellent joint time-frequency concentration of g and γ to be the most important requirement for achieving that goal.

For such g and γ we propose a fast algorithm for computing G in Section 4.1.4, based on a channel operator model described in Section 4.1.3. Our model is deterministic, so a typical example use is in coverage predictions for radio network planning [1, Section 3.1.3]. The algorithm computes the ISI and ICI dependence on, for example, pulse shaping and threshold choices from input data. It depends on describing a particular channel, that we assume to be known, for example, from measurements or computed from ray tracing, finite element or finite difference methods (described with more references in [1]). Moreover, the performance of a communication system is usually evaluated by means of extensive Monte-Carlo simulations [1], which also might be a potential future application where fast algorithms are required.

4.1.3 Common channel operator models

The channel operator H maps an input signal s to a weighted superposition of time and frequency shifts of s:

$$Hs(\cdot) = \int_{K\times[A,\infty)} S_H(\nu,t)\,\mathrm{e}^{i2\pi\nu(t-t_0)} s(\,\cdot\, - t)\,\mathrm{d}(\nu,t)\,, \qquad K \text{ compact.}$$

This standard model is usually formulated for so-called *Hilbert–Schmidt operators* with the *spreading function* S_H in the space L^2 of square integrable functions (e.g., in [8,9]) or for S_H in some subspace of the tempered distributions S' (e.g., in [10,12]). The weakest such assumption is that $S_H \in S'$, which restricts the input signal s to be a Schwartz class function.

Alternatively, one can assume s to be in the Wiener amalgam space $W(A, l^1) = S_0(\mathbb{R}^d)$ $W(A, l^1) = S_0$ (Feichtinger algebra) S_0 (Feichtinger algebra) (also named the Feichtinger algebra), which consists of all continuous $f\colon \mathbb{R}^d \to \mathbb{C}$ for which

$$\sum_{n\in\mathbb{Z}^d} \|(f(\cdot)\psi(\cdot - n))\hat{}\,\|_1 < \infty, \qquad \|g\|_1 \stackrel{\text{def}}{=} \int_{\mathbb{R}} |g(x)|\,\mathrm{d}x$$

for some compactly supported[1] ψ with integrable Fourier transform $\widehat{\psi}$ and satisfying $\sum_{n\in\mathbb{Z}^d} \psi(x-n) = 1$. We write S_0' for the space of linear bounded functionals on S_0. S_0 is also a so-called modulation space, described at more depth and with notation $S_0 = M^{1,1} = M^1$ and $S_0' = M^{\infty,\infty} = M^\infty$ in [3, 6].

Since the space $S_0'(\mathbb{R}\times\mathbb{R})$ includes Dirac delta distributions, this model includes important idealized borderline cases such as the following:

Line-of-sight path transmission: $S_H = a\delta_{\nu_0,t_0}$, a Dirac distribution at (ν_0, t_0) representing a time- and Doppler-shift with attenuation a.

Time-invariant systems: $h(x,t) = h(0,t)$ and $S_H(\nu,t) = h(t)\delta_0(\nu)$.

[1] A function is said to have *compact support* if it vanishes outside some finite length interval.

Moreover, S_0' excludes derivatives of Dirac distributions, which can be used to avoid complex-valued Hs with no physical meaning [11, Sec. 3.1.1]. Further, S_0 is the smallest Banach space of test functions with some useful properties like invariance under time-frequency shifts [6, p. 253], thus allowing for time-frequency analysis on its dual S_0' which is, in that particular sense, the largest possible Banach space of tempered distributions that is useful for time-frequency analysis. One more motivation for considering spreading functions in S_0' is that Hilbert–Schmidt operators are compact, hence, they exclude invertible operators, such as the example $S_H = a\delta_{\nu_0,t_0}$ above, and small perturbations of invertible operators, which are useful in the theory of radar identification and in some mobile communication applications. For results using a Banach space setup, see for example [9, 12].

Nevertheless, for narrowband finite lifelength channels such as those typical for radio communications, all analysis can be restricted to the time window and frequency band of interest. We show in [5] that the full system behaviour within this time-frequency window can be modelled with an infinitely many times differentiable spreading function $S_H(\nu, t)$ that vanishes for frequencies ν outside some finite interval and which has subexponential decay as a function of t. That a function f has *subexponential decay* means that for $0 < \varepsilon < 1$ there is some $C_\varepsilon > 0$ such that

$$|f(x)| \leq C_\varepsilon \mathrm{e}^{-|x|^{1-\varepsilon}} \qquad \text{for all } x \in \mathbb{R}.$$

Hence we can with negligible errors also do a smooth cutoff to a compactly supported and infinitely many times differentiable spreading function. A big advantage of this Hilbert–Schmidt model is that Fourier analysis can be applied without the need of deviating into distribution theory.

4.1.4 Computing the channel matrix G

For $\epsilon > 0$ we define the *ϵ-essential support* of a bounded function $f : \mathbb{R} \to \mathbb{C}$ to be the closure of the set $\{x : |f(x)| \geq \epsilon \cdot \operatorname{ess\,sup}_x |f(x)|\}$. For communications applications with Q carrier frequencies, at least Q samples of every received symbol are needed in the receiver. Thus a hasty and naive approach to computing the matrix elements could start with a $Q \times Q$ matrix representation of H for computing the samples of $Hg_{q,r}$. If up to R neighbouring transmission symbols have overlapping ϵ-essential support, then we need to compute $(RQ)^2$ matrix elements $\langle Hg_{q,r}, \gamma_{q',r'}\rangle$, which, with this approach, would require $R^2 \cdot \mathcal{O}(Q^5)$ arithmetic operations with Q typically being at least of the size 256–1024 in radio communications, and with $R = 4$ for $\epsilon = 10^{-6}$ and the optimally well-localized Gaussian windows that we have used for example applications described in [5]. This is a quite demanding task, so therefore more efficient formulas and algorithms were derived in [5] for the Hilbert–Schmidt channel models described in last section. With notation $I_{C,B} \stackrel{\text{def}}{=} \left[C - \frac{B}{2}, C + \frac{B}{2}\right]$, the resulting model is based on the following assumptions about supports and index sets for the involved functions:

$$\begin{aligned}
\operatorname{supp}\widehat{g} \subseteq & I_{\Omega_c,\Omega}, \qquad T_g \stackrel{\text{def}}{=} \tfrac{1}{\Omega}, \qquad T_\gamma \stackrel{\text{def}}{=} \tfrac{1}{\Omega+\omega}, \\
\operatorname{supp} S_H \subseteq & I_{\omega_c,\omega} \times I_{C,L}, \qquad \operatorname{supp}\widehat{Hg} \subseteq \operatorname{supp}\widehat{\gamma} \subseteq I_{\Omega_c+\omega_c,\Omega+\omega}, \\
\mathcal{K},\mathcal{M} \subset & \mathbb{Z}^d, \qquad |\mathcal{K}| < \infty, \qquad |\mathcal{M}| < \infty \qquad \text{and} \\
g(mT_g) = & \gamma(kT_\gamma) = (Hg)(kT_\gamma) = 0 \quad \text{for} \quad k \in \mathbb{Z}^d \setminus \mathcal{K} \text{ and } m \in \mathbb{Z}^d \setminus \mathcal{M}.
\end{aligned}$$

The analysis takes place in an interval $I_{C_0+t_0,L_0}$ containing the support of all perturbed basis functions $Hg_{q,r}$. We refer to [5] for details, but in short, the algorithm is based on a smooth truncation of $\widehat{S_H(\nu,\cdot)}$ to a band of width $1/T''$ containing the full transmission frequency band, in which $S_H(\nu,\cdot)$ can be fully represented by sample values $S_{n,p}$, from which the spreading function S_H^q experienced by the functions $(g_{q,r})_r$ can be computed:

$$\widehat{S_H^q(\cdot,t)}(t_0) = |\omega_0 T''|\, \chi_{I_{C_0,L_0}}(t-t_0) \sum_{p\in\mathcal{P}} \mathrm{e}^{\mathrm{i}2\pi\Omega_{c,q}(t-pT'')} \operatorname{sinc}_\Omega(t-pT'') \sum_{n\in\mathcal{N}} S_{n,p} \mathrm{e}^{\mathrm{i}2\pi\frac{t-t_0-pT''}{L_0}}, \tag{4.2}$$

with $\Omega_{c,q}$ being the centerpoint of the support of $\widehat{g_{q,r}}$ and $\operatorname{sinc}_\Omega(x) \stackrel{\text{def}}{=} \frac{\sin(\pi\Omega x)}{\pi x}$ extended continuously to $\mathbb{R}$. Using (4.2), we can compute the samples $(Hg_{q,r})(kT_\gamma) = |T_g| \sum_{m\in\mathbb{Z}^d} f(mT_g)\, (S_H^q(\cdot,kT_\gamma - mT_g))\hat{}(-mT_g)$ and finally the matrix element $\langle Hg_{q,r}, \gamma_{q',r'}\rangle$ using the formula

$$\langle u,v\rangle_{L^2(\mathbb{R}^d)} = |T| \sum_{k\in\mathcal{I}_u} u(kT)\overline{v_{\text{bpf}}(kT)}$$

for functions with supports

$$\operatorname{supp}\widehat{u} \subseteq I_{C_u,B}, \quad \operatorname{supp}\widehat{v} \subseteq I_{C_v,B}, \quad I_{C_{uv},B_{uv}} \stackrel{\text{def}}{=} I_{C_u,B} \cap I_{C_v,B} \neq \emptyset, \qquad T = \tfrac{1}{B}$$

and with v_{bpf} being defined by its Fourier transform $\widehat{v_{\text{bpf}}}(\xi) \stackrel{\text{def}}{=} \widehat{v}(\xi)\chi_{I_{C_{uv},B_{uv}}}(\xi)$.

As explained in [5], this way the full matrix G can be computed in $R^2 \cdot \mathcal{O}(M^2 \cdot Q^2)$ arithmetic operations with $M \stackrel{\text{def}}{=} |\mathcal{M}|$, which can be compared to the $R^2 \cdot \mathcal{O}(Q^5)$ operations of the more naive and straightforward matrix computation approach described above.

Bibliography

[1] Ernst Bonek, Henrik Asplund, Conor Brennan, Christian Bergljung, Peter Cullen, Dirk Didascalou, Patrick C.F. Eggers, José Fernandes, Cristophe Grangeat, Ralf Heddergott, Peter Karlsson, Ralf Kattenbach, Mikael B. Knudsen, Preben E. Mogensen, Andreas F. Molisch, Bo Olsson, Jörg Pamp, Gert Frølund Pedersen, I. Pedersen, Martin Steinbauer, martin Weckerle, and Thomas Zwick. *Antennas and Propagation*, chapter 3, pages 77–306. John Wiley & Sons, 2001.

[2] B. Delyon and A. Juditsky. On minimax wavelet estimators. *Appl. Comput. Harmon. Anal.*, 3(3):215–228, 1996.

[3] Hans G. Feichtinger and Georg Zimmermann. A Banach space of test functions for Gabor analysis. In Hans G. Feichtinger and Thomas Strohmer, editors, *Gabor Analysis and Algorithms*, chapter 3, pages 123–170. Birkhäuser, Boston, MA, USA, 1998.

[4] Niklas Grip. *Wavelet and Gabor Frames and Bases: Approximation, Sampling and Applications.* Doctoral thesis 2002:49, Luleå University of Technology, SE-971 87 Luleå, 2002. WWW: `http://www.sm.luth.se/~grip/Research/publications.php`.

[5] Niklas Grip and Götz Pfander. A discrete model for the efficient analysis of time-varying narrowband communication channels. *Multidim. Syst. Sign. Process.*, 19(1):3–40, March 2008. WWW: `http://dx.doi.org/10.1007/s11045-007-0032-1`.

[6] Karlheinz Gröchenig. *Foundations of Time-Frequency Analysis.* Birkhäuser, 2000.

[7] G. Matz, D. Schafhuber, K. Gröchenig, M. Hartmann, and F. Hlawatsch. Analysis, optimization, and implementation of low-interference wireless multicarrier systems. *preprint (submitted to IEEE Transactions on Wireless Communications)*, 2005.

[8] Gerald Matz and Franz Hlawatsch. Time-frequency transfer function calculus (symbolic calculus) of linear time-varying systems (linear operators) based on a generalized underspread theory. *J. Math. Phys.*, 39(8):4041–4070, August 1998. (Special issue on Wavelet and Time-Frequency Analysis.) WWW: `http://www.lss.supelec.fr/perso/matz_gerald/GM_jrnl.html`.

[9] Götz E. Pfander and David F. Walnut. Measurement of time-variant linear channels. *IEEE Trans. Inform. Theory*, 52(11):4808–4820, November 2006. WWW: `http://ieeexplore.ieee.org/iel5/18/36107/01715527.pdf`.

[10] Götz E. Pfander and David F. Walnut. Operator identification and Feichtinger's algebra. *Sampl. Theory Signal Image Process*, 5(2):183–200, May 2006. WWW: `http://www.math.iu-bremen.de/pfander/operatoridentfei.pdf`.

[11] Scott Rickard. *Time-frequency and time-scale representations of doubly spread channels.* Ph.D. dissertation, Princeton University, November 2003.

[12] Thomas Strohmer. Pseudodifferential operators and Banach algebras in mobile communications. *Appl. Comput. Harmon. Anal.*, 20(2):237–249, March 2006. WWW: `http://www.math.ucdavis.edu/~strohmer`.