INFINITE DIMENSIONAL RESTRICTED INVERTIBILITY

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Abstract. The 1987 *Bourgain-Tzafriri Restricted Invertibility Theorem* is one of the most celebrated theorems in analysis. At the time of their work, the authors raised the question of a possible infinite dimensional version of the theorem. In this paper, we will give a quite general definition of restricted invertibility for operators on infinite dimensional Hilbert spaces based on the notion of *density* from frame theory. We then prove that localized Bessel systems have large subsets which are Riesz basic sequences. As a consequence, we prove the strongest possible form of the infinite dimensional restricted invertibility theorem for ℓ_1-localized operators and for Gabor frames with generating function in the Feichtinger Algebra. For our calculations, we introduce a new notion of *density* which has serious advantages over the standard form because it is independent of index maps - and hence has much broader application. We then show that in the setting of the restricted invertibility theorem, this new density becomes equivalent to the standard density.

Keywords. Restricted invertibility; density, localization, Hilbert space frames, Gabor analysis.
AMS MSC (2000). 42C15, 46C05, 46C07.

1. Introduction

In 1987, Bourgain and Tzafriri proved one of the most celebrated and useful theorems in analysis [5]: *The Bourgain-Tzafriri Restricted Invertibility Theorem.* The form we give now can be found in Casazza [6], Vershynin [19] (where the restriction that the norms of the vectors Te_i equal one - or even are bounded below - is removed), and Vershynin [20, 21] (also see Casazza and Tremain [11]).

Theorem 1.1 (**Restricted Invertibility Theorem**). *There exists a function* $\mathbf{c} : (0,1) \longrightarrow (0,1)$ *so that for every* $n \in \mathbb{N}$ *and every linear operator* $T : \ell_2^n \to \ell_2^n$ *with* $\|Te_i\| = 1$ *for* $i = 1, 2, \cdots, n$ *and* $\{e_i\}_{i=1}^n$ *an orthonormal basis for* ℓ_2^n*, there is a subset* $J_\epsilon \subseteq \{1, 2, \cdots, n\}$ *satisfying*

$$\text{(1)} \qquad \frac{|J_\epsilon|}{n} \geq \frac{(1-\epsilon)}{\|T\|^2},$$

Date: May 6, 2009.
The first author was supported by NSF DMS 0704216 and thanks the American Institute of Math for their continued support.

(2) For all $\{b_j\}_{j\in J_\epsilon}\in \ell_2(J_\epsilon)$ *we have*

$$\|\sum_{j\in J_\epsilon} b_j Te_j\|^2 \geq \mathbf{c}(\epsilon)\sum_{j\in J_\epsilon}|b_j|^2.$$

Throughout this paper, $\|\cdot\|$ represents the Hilbert space norm on vectors and the operator norm for operators acting on Hilbert spaces.

In our proofs we will need a minor extension of Theorem 1.1 which is stated and proved in the appendix (see Theorem 6.1). It is easily seen that (1) is best possible in Theorem 1.1. Letting $Te_{2i}=e_i=Te_{2i-1}$ for $i=1,2,\cdots,n$ in ℓ_2^{2n}, we see that $1/\|T\|^2$ is necessary. In [8] it is shown that the class of equal norm Parseval frames $\{f_i\}_{i=1}^{2n}$ in ℓ_2^n are not *2-pavable*. In the current setting, this says that Theorem 1.1 (1) fails if $\epsilon=0$.

In their paper [5], Bourgain and Tzafriri raised the question of a possible infinite dimensional version of their theorem. They then gave a weakened version of this for the special case of families of exponentials. Vershynin [21] proves an infinite dimensional restricted invertibility theorem for restrictions of exponentials to subsets of the torus.

In this paper, we will use the notion of *density* from frame theory to give a precise definition for infinite dimensional restricted invertibility. We then prove a very general theorem on restricted invertibility for classes of Bessel systems which are ℓ_1-localized with respect to frames. As a consequence, we obtain the general restricted invertibility theorem for ℓ_1-localized operators on arbitrary Hilbert spaces. We apply our general results to prove the restricted invertibility theorem for Gabor systems with generator in the Feichtinger algebra as well as for systems of Gabor molecules in the Feichtinger algebra.

Standard density theory requires an *index map* (see Section 2.) This can be problematic in some applications. So we will introduce a new notion of density which is independent of index maps and as a consequence should have much broader application in the field. We will then show that in the presence of *localization*, this form of density becomes equivalent to the standard form.

The notion of localization with respect to an orthonormal basis is not usable in Gabor theory due to the Balian-Low Theorem [13]. This is why we have to move from *rectangular* coordinate systems to *overcomplete* coordinate systems. This leads us to introduce a new concept of *relative density*, because there, the overcompleteness of the coordinate system factors out.

The paper is organized as follows. Section 2 contains the notation, the first form of *density* and the statements of the fundamental results in the paper. Section 3 is a detailed discussion of *localization* with a number of examples. Here, we also introduce our new notion of *density* which has the major advantage that it is independent of index maps. We then show its relationship to the standard *density* and show that in the setting of ℓ_2-localized frames, the two forms of density are the same. We also restate our main results using the second notion of density. Section 4 contains the proof of the main results on restricted invertibility. Section 5 addresses the restricted invertibility theorem for Gabor systems and Section 6 is an appendix containing some intermediate results used in this paper.

2. Notation and statement of results

Hilbert space frame theory has traditionally been used in signal processing (see [13]) but recently has also had a significant impact on problems in pure mathematics, applied mathematics and engineering. (See, for example, [7, 9, 10, 12, 17] and their references.)

Definition 2.1. *A family of vectors $\{f_i\}_{i\in I}$ in a Hilbert space $\mathbb{H}$ is called a* **frame** *for $\mathbb{H}$ if there are constants $0 < A \leq B < \infty$ (called* **lower and upper frame bounds** *respectively) if*

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad \textit{for all } f \in \mathbb{H}.$$

If we only have the right hand side inequality, we call the family a **Bessel sequence** with **Bessel bound** B. If we can choose $A = B$ in Definition 2.1, then we say the frame is **tight** with tight frame bound A. If $A = B = 1$, it is a **Parseval frame**. The **analysis operator** $T : \mathbb{H} \to \ell_2(I)$ of the frame $\{f_i\}_{i\in I}$ is defined by

$$T(f) = \sum_{i\in I} \langle f, f_i\rangle e_i,$$

where $\{e_i\}_{i\in I}$ is the unit vector basis of $\ell_2(I)$. The adjoint of T is the **synthesis operator** given by

$$T^*(e_i) = f_i, \quad \text{for all } i \in I.$$

The **frame operator** is the positive, self-adjoint, invertible operator $S : \mathbb{H} \to \mathbb{H}$ where $S = T^*T$. That is, for all $f \in \mathbb{H}$,

$$S(f) = \sum_{i\in I} \langle f, f_i\rangle f_i.$$

Reconstruction of $f \in \mathbb{H}$ comes from

$$f = \sum_{i\in I} \langle f, f_i\rangle S^{-1}(f_i).$$

The family $\{S^{-1}(f_i)\}_{i\in I}$ is also a frame for $\mathbb{H}$ called the **dual frame** of $\{f_i\}_{i\in I}$.

A family of vectors $\{f_i\}_{i\in I}$ in $\mathbb{H}$ is called a **Riesz sequence** with **Riesz bounds** $0 < A \leq B < \infty$ if for all families of scalars $\{a_i\}_{i\in I}$ we have

$$A\sum_{i\in I} |a_i|^2 \leq \|\sum_{i\in I} a_i f_i\|^2 \leq B\sum_{i\in I} |a_i|^2.$$

We will use the notion of *density* from frame theory to give the correct formulation of restricted invertibility for infinite dimensional Hilbert spaces. In the following section we will define the previously mentioned new notion of density which does not require an index map and then show that for ℓ_2-localized frames, the two notions of density are equivalent, a result which is interesting in itself.

Over the last few years, a considerable amount of work has been done on *density theory.* We refer the reader to [1, 2, 3, 4] for the latest developments. The common notions on density involve countable point sets in σ-finite discrete measure spaces. We follow this approach and, throughout the paper, I will denote a countable index set and G will denote a finitely generated Abelian group $G = \mathbb{Z}^{d_1} \times \mathbb{Z}_{N_1} \times \cdot \times \mathbb{Z}_{N_{d_2}}$ with $d_1, d_2 \in \mathbb{N}$ and $\mathbb{Z}_N = \{0, 1, 2, \cdots, N-1\}$ being the cyclic group of order N.

Definition 2.2. *Let I be a set and $a : I \longrightarrow G$ (called a* **localization map***). For $J \subseteq I$, the* **lower and upper density** *of J with respect to a are given, respectively, by*

$$D^-(a; J) = \liminf_{R\to\infty} \inf_{k\in G} \frac{|a^{-1}(B_R(k)) \cap J|}{|B_R(0)|}, \tag{2.1}$$

$$D^+(a; J) = \limsup_{R\to\infty} \sup_{k\in G} \frac{|a^{-1}(B_R(k)) \cap J|}{|B_R(0)|}, \tag{2.2}$$

where $|\cdot|$ denotes the cardinality of the set and

$$B_R(k) = \{g \in G\,;\, \|g - k\|_\infty = \max_{1\le j\le d_1+d_2} |g(j) - k(j)| \le R\}$$

is the box of radius R and center k in G. Note that $|B_R(k)| = |B_R(k')|$ for all $k, k' \in G$ and $R > 0$.

If $D^-(a; J) = D^+(a; J)$, then we say that J is of **uniform density** *and write $D(a; J) = D^-(a; J) = D^+(a; J)$.*

Remark 2.3. In the case that $I = G$ and $a = id$, we write the lower and upper density as $D^-(J), D^+(J)$ and these are called the **Beurling densities** of J [1, 2, 3, 13].

The dependence of $D^-(a; J)$ and $D^-(a; J)/D^-(a; I)$ on a is illustrated in the following example.

Example 2.4. Let $I = G = \mathbb{Z}$ and $J = 2\mathbb{Z}$.

(1) For $a = \mathrm{id}$, we have $D^-(a; J)/D^-(a; I) = \frac{1/2}{1} = \frac{1}{2}$.
(2) For $a = 2\mathrm{id}$ we have $D^-(a; J)/D^-(a; I) = \frac{1/4}{1/2} = \frac{1}{2}$.
(3) For a bijective with even numbers mapping bijectively to $\mathbb{Z} \setminus 4\mathbb{Z}$ and odd numbers to $4\mathbb{Z}$, we have $D^-(a; J)/D^-(a; I) = \frac{3/4}{1} = \frac{3}{4}$.

Nonetheless, this dependence on a will not introduce ambiguity when combined with standard localization notions from frame theory (see, for example, [1, 2, 3, 4]).

Definition 2.5. *Let $p = 1$ or $p = 2$. Let $a : I \longrightarrow G$, and let $\mathcal{G} = \{g_k : k \in G\}$ be a frame for $\mathbb{H}$ and $\mathcal{F} = \{f_i\}_{i\in I} \subseteq \mathbb{H}$. We say that $(\mathcal{F}, a, \mathcal{G})$ is* **ℓ_p-localized** *if there exists $r \in \ell_p(G)$ with $|\langle f_i, g_{k'}\rangle| \le r(k)$ whenever $a(i) - k' = k$. Also, $\mathcal{G} = \{g_k : k \in G\}$ is* **ℓ_p-self-localized** *if $(\mathcal{G}, \mathrm{id}, \mathcal{G})$ is ℓ_p-localized.*

The operator $T : \mathbb{H}' \longrightarrow \mathbb{H}$ is **ℓ_p-localized** *if there exists an orthonormal basis $\mathcal{E}$ of $\mathbb{H}'$ indexed by I, a frame $\mathcal{G}$ of $\mathbb{H}$ indexed by the finitely generated Abelian group G, and a map $a : I \longrightarrow G$ so that so that $(T(\mathcal{E}), a, \mathcal{G})$ is ℓ_p-localized.*

As discussed in detail in Section 3, given $\mathcal{F}$ and $\mathcal{G}$, $D^-(a; J)$ and $D^+(a; J)$ do not depend on the choice of a as long as $(\mathcal{F}, a, \mathcal{G})$ is ℓ_2-localized.

We can now state the main results of the paper. The first is the frame theoretic form of restricted invertibility.

Theorem 2.6. *Let* $\mathbf{c}$ *be the function provided in Theorem 6.1. Let* $\mathcal{F} = \{f_i\}_{i\in I}$, $\|f_i\| \geq u > 0$ *for all* $i \in I$, *be a Bessel system with Bessel bound* B *in a Hilbert space* $\mathbb{H}$. *Let* G *be a finitely generated Abelian group and assume either*

(A) $\mathcal{G} = \{g_k : k \in G\}$ *is a Riesz basis for* $\mathbb{H}$ *with Riesz bounds* A, B,
or
(B) $\mathcal{G} = \{g_k : k \in G\}$ *is a frame for* $\mathbb{H}$ *with* ℓ_1*- self-localized dual frame* $\widetilde{\mathcal{G}} = \{\widetilde{g}_k : k \in G\}$.

Let $a : I \to G$ *be a localization map with* $0 < D^-(a; I) \leq D^+(a; I) < \infty$. *If* $(\mathcal{F}, a, \mathcal{G})$ *is* ℓ_1*-localized, then for every* $\epsilon > 0$ *and* $\delta > 0$ *there is a subset* $J = J_{\epsilon\delta} \subseteq I$ *of uniform density satisfying*

(1) $$\frac{D(a; J)}{D^-(a; I)} \geq \frac{(1-\epsilon)u^2}{B},$$

(2) For all scalars $\{b_j\}_{j\in J}$ *we have*

$$\|\sum_{j\in J} b_j f_j\|^2 \geq \mathbf{c}(\epsilon)(1-\delta)\frac{A}{B}\, u^2 \sum_{j\in J} |b_j|^2$$

with $A = B$ *in the case of (B).*

A special case of Theorem 2.6 is the restricted invertibility theorem (as envisioned by Bourgain and Tzafriri) for ℓ_1-localized operators on infinite dimensional Hilbert spaces. In fact, for an orthonormal basis $\mathcal{E} = \{e_i\}_{i\in I}$ in $\mathbb{H}'$ and a bounded operator $T : \mathbb{H}' \longrightarrow \mathbb{H}$, $\{Te_i\}_{i\in I}$ is Bessel with optimal Bessel bound $\|T\|^2$.

The reader may substitute $\mathbb{Z}$ or even $\mathbb{N}$ for the finitely generated Abelian group $G = \mathbb{Z}^d \times H$, H finite Abelian, in the theorem below.

Theorem 2.7 (Infinite Dimensional Restricted Invertibility Theorem). *Let* $\{e_k\}_{k\in G}$ *and* $\mathcal{G} = \{g_k\}_{k\in G}$ *be orthonormal bases for a Hilbert space* $\mathbb{H}$, $T : \mathbb{H} \to \mathbb{H}$ *be a bounded linear operator satisfying* $\|Te_k\| = 1$ *for all* $k \in G$ *and* $\mathcal{F} = T(\mathcal{G})$. *Let* $a : G \to G$ *be a one to one map and assume that* $(\mathcal{F}, a, \mathcal{G})$ *is* ℓ_1*-localized. Then for all* $\epsilon, \delta > 0$, *there is a subset* $J = J_{\epsilon\delta} \subseteq G$ *of uniform density so that (with* $\mathbf{c}$ *being the function provided in Theorem 6.1),*

(1) $D(a; J) \geq \dfrac{1-\epsilon}{\|T\|^2}$,

(2) For all $\{b_j\}_{j\in J} \in \ell_2(J)$ *we have*

$$\|\sum_{j\in J} b_j Te_j\|^2 \geq \mathbf{c}(\epsilon)(1-\delta)\sum_{j\in J} |b_j|^2.$$

Theorem 2.7 is best possible in the sense that the theorem fails in general if $\epsilon = 0$ in (1). This follows easily from the corresponding finite dimensional result discussed after Theorem 1.1.

The density concepts outlined above were developed in part to obtain sophisticated results on the density of Gabor frames for $L^2(\mathbb{R}^d)$ [2, 3, 4, 13].

For $\lambda = (x, \omega) \in \mathbb{R}^{2d}$ we define **modulation by** ω M_ω and **translation by** x T_x on $L^2(\mathbb{R}^d)$ by

$$M_\omega(\varphi)(\cdot) = e^{2\pi i \omega \cdot}\varphi(\cdot), \quad T_x(\varphi)(\cdot) = \varphi(\cdot - x), \quad \varphi \in L^2(\mathbb{R}^d)$$

For $\varphi \in L^2(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ discrete, we consider the set $(\varphi, \Lambda) = \{\pi(\lambda)\varphi\}_{\lambda \in \Lambda} \subseteq L^2(\mathbb{R}^d)$ where $\pi(\lambda)\varphi = \pi(x, \omega)\varphi = M_\omega T_x \varphi$, $\lambda = (x, \omega) \in \mathbb{R}^{2d}$. The set (φ, Λ) is called **Gabor system** with generating function φ, and if (φ, Λ) is a frame for $L^2(\mathbb{R}^d)$, then we call (φ, Λ) a **Gabor frame**

The **Feichtinger algebra** is given by

$$S_0(\mathbb{R}^d) = \big\{ f \in L^2(\mathbb{R}^d) : \ \ \langle f, \pi(\cdot) g_0 \rangle \in L^1(\mathbb{R}^{2d}) \big\},$$

with φ_0 being a Gaussian [13]. Theorem 5.1 in Section 5 is Theorem 2.6 applied to time–frequency molecules. In terms of Gabor frames and the lower Beurling density $D^-(\Lambda)$ (respectively uniform Beurling density $D(\Lambda)$), if $\Lambda \subseteq \mathbb{R}^{2d}$, it reduces to the following result.

Theorem 2.8. *Let $\epsilon, \delta > 0$. Let $\varphi \in S_0(\mathbb{R})$ and let the Gabor system (φ, Λ) have Bessel bound $B < \infty$. Then exists a set $\Lambda_{\epsilon\delta} \subseteq \Lambda$, of uniform density, so that*

(1) $$\frac{D(\Lambda_{\epsilon\delta})}{D^-(\Lambda)} \geq \frac{(1-\epsilon)}{B}\|\varphi\|^2,$$

(2) For all $\{b_\lambda\}_{\lambda \in \Lambda} \in \ell_2(\Lambda)$,

$$\|\sum_{\lambda \in \Lambda_{\epsilon\delta}} b_\lambda \pi(\lambda)\varphi\|^2 \geq \mathbf{c}(\epsilon)(1-\delta)\|\varphi\| \sum_{\lambda \in \Lambda_{\epsilon\delta}} |b_\lambda|^2$$

Note that Theorem 2.8 (2) states that $(\varphi, \Lambda_{\epsilon\delta})$ is a Riesz sequence with lower Riesz bound $\mathbf{c}(\epsilon)(1-\delta)\|\varphi\|$. That is, the lower Riesz bound of $(\varphi, \Lambda_{\epsilon\delta})$ depends only on ϵ, δ, and $\|g\|$, but not on any geometric properties of Λ or other specifics of g. Certainly, such properties of g and Λ affect the Bessel bound of (φ, Λ) and therefore (1) in Theorem 2.8. Moreover, note that if (φ, Λ) is a tight frame, then $D^-(\Lambda) = \frac{B}{\|\varphi\|^2}$, and (1) in Theorem 2.8 becomes simply [3]

$$D(\Lambda_{\epsilon\delta}) \geq (1-\epsilon).$$

Balan, Casazza, and Landau [4] introduced some of the tools used here to resolve an old problem in frame theory: *What is the correct quantitative measure for redundancy for infinite dimensional Hilbert spaces?* In [4], the following complementary result to Theorem 2.8 is obtained.

Theorem 2.9. *Let $\varphi \in S_0(\mathbb{R})$ and let (φ, Λ) be a Gabor frame. Then exists a set $\Lambda_\epsilon \subseteq \Lambda$ so that $(\varphi, \Lambda_\epsilon)$ is still a frame, while*

$$D^+(\Lambda_\epsilon) \leq 1 + \epsilon.$$

To prove results as Theorem 2.9 one has to maintain completeness while removing large subsets from frames. The challenge when proving Theorem 2.6 is to obtain a given lower Riesz bound while choosing as many elements as possible from a Bessel system.

3. Relative density and restricted invertibility

Definitions 2.2 and 2.5 are based on the work of Balan, Casazza, Heil, Landau [1, 2, 3, 4] (see also Gröchenig [14]). They lead to a density concept of subsets of $\mathcal{F}$ when $(\mathcal{F}, a, \mathcal{G})$ is ℓ_1-localized. The definition of density of $\mathcal{F}' \subseteq \mathcal{F} = \{f_i\}_{i \in I}$ relies on the localization map $a : I \longrightarrow G$, as does the left hand side of (1) in Theorem 2.6, while the right hand side of (1) in Theorem 2.6 does not depend on a. In fact, as mentioned briefly in Section 2, in combination with localized function systems though, $D^-(a; J)$ becomes independent of a. This fact is well illustrated in the following example.

Example 3.1. Let $\mathcal{E} = \mathcal{G} = \{g_k\}_{k \in \mathbb{Z}}$ be an orthonormal basis of $\mathbb{H}$. Let $T : \mathbb{H} \longrightarrow \mathbb{H}$ be defined by $Tg_k = g_{[\frac{k}{2}]}$. Let $\mathcal{F} = T\mathcal{G}$ and $a : \mathbb{Z} \mapsto \mathbb{Z}$ be so that

$$r(k) \geq |\langle f_{k''}, g_{a(k'')-k} \rangle| = |\langle g_{[\frac{k''}{2}]}, g_{a(k'')-k} \rangle| = \delta([\tfrac{k''}{2}] - a(k'') + k), \quad k \in \mathbb{Z}.$$

Clearly, $r \in \ell_2(\mathbb{Z})$ then implies $[\frac{k''}{2}] - a(k'')$, $k'' \in \mathbb{Z}$, is bounded. Given a_1, a_2 with $[\frac{k''}{2}] - a_1(k'')$, $k'' \in \mathbb{Z}$, and $[\frac{k''}{2}] - a_2(k'')$, $k'' \in \mathbb{Z}$, bounded, then $a_1(k'') - a_2(k'')$, $k'' \in \mathbb{Z}$, bounded, and, clearly $D^-(a_1; J) = D^-(a_2; J)$ for all subsets $J \subseteq \mathbb{Z}$. (See Proposition 3.4 for a detailed argument.).

In general, for a family of functions $\mathcal{F}$ and a reference system $\mathcal{G}$, each element $f \in \mathcal{F}$ is naturally placed within $\mathcal{G}$ as the coefficient sequence $\{\langle f, g_k \rangle\}_k$ decays away from its *center of mass* as $\|k\|_\infty \to \infty$ by virtue of $\{\langle f, g_k \rangle\} \in \ell_2(G)$. The function family $\mathcal{F}$ being ℓ_2–localized with respect to $\mathcal{G}$ simply means that the decay behavior of $\{\langle f, g_k \rangle\}_k$ away from its *center of mass* is independent of $f \in \mathcal{F}$.

As each $f \in \mathcal{F}$ is local within the *coordinate system* $\mathcal{G}$, an explicit location map $a : I \longrightarrow G$ is not needed. Localization and density of $\mathcal{F}$ with respect to $\mathcal{G}$ are fully determined by $\mathcal{G}$. To address this, we give a definition of localization and density which is independent of an explicit index set map $a : I \longrightarrow G$.

Definition 3.2. *Let $p = 1$ or $p = 2$. The set $\mathcal{F} \subseteq \mathbb{H}$ is ℓ_p-**localized** with respect to $\mathcal{G} = \{g_k\}_{k \in G}$ if there exists a sequence $r \in \ell_p(G)$ so that for each $f \in \mathcal{F}$ there is a $k \in G$ with $\langle f, g_n \rangle \leq r(n-k)$ for all $n \in G$.*

*The operator $T : \mathbb{H}' \longrightarrow \mathbb{H}$ is ℓ_p-**localized** if there exists an orthonormal basis $\mathcal{E}$ of $\mathbb{H}'$ and a frame $\mathcal{G}$ of $\mathbb{H}$ so that $T(\mathcal{E})$ is ℓ_p-localized with respect to $\mathcal{G}$.*

Note, that any diagonalizable operator, for example, a compact normal operator on a separable Hilbert space is ℓ_1-localized.

Definition 3.3. *The* **lower density** *and* **upper density** *of $\mathcal{F}$ with respect to $\mathcal{G}$ are given, respectively, by*

$$D^-(\mathcal{F};\mathcal{G}) = \liminf_{R\to\infty} \inf_{k\in G} \frac{\sum_{f\in\mathcal{F}} a_f \sum_{n\in B_R(k)} |\langle f, g_{n-k}\rangle|^2}{|B_R(0)|}, \tag{3.3}$$

$$D^+(\mathcal{F};\mathcal{G}) = \limsup_{R\to\infty} \sup_{k\in G} \frac{\sum_{f\in\mathcal{F}} a_f \sum_{n\in B_R(k)} |\langle f, g_{n-k}\rangle|^2}{|B_R(0)|}, \tag{3.4}$$

where $a_f = \big(\sum_{n\in G} |\langle f, g_n\rangle|^2\big)^{-1}$, $f \in \mathcal{F}$. If $D^-(\mathcal{F};\mathcal{G}) = D^+(\mathcal{F};\mathcal{G})$, then $\mathcal{F}$ has **uniform density** *$D(\mathcal{F};\mathcal{G}) = D^-(\mathcal{F};\mathcal{G}) = D^+(\mathcal{F};\mathcal{G})$ with respect to $\mathcal{G}$.*

Note that if $\mathcal{G}$ is a tight frame with upper and lower frame bound A, then $a_f = (A\|f\|^2)^{-1}$ for $f \in \mathcal{F}$. The following four propositions describe the relationship between Definitions 2.2 and 2.5 and Definitions 3.2 and 3.3

Proposition 3.4. *Let $p = 1$ or $p = 2$. If $(\mathcal{F}, a, \mathcal{G})$ is ℓ_p-localized, $\|f_i\| \geq u > 0$ for all $i \in I$ and $\mathcal{G}$ is a frame, then $(\mathcal{F}, b, \mathcal{G})$ is ℓ_p-localized if and only if $a - b : I \longrightarrow G$ is bounded.*

Proof. If $a - b$ bounded, then clearly $(\mathcal{F}, b, \mathcal{G})$ is ℓ_p-localized.

To see the converse, let us assume that $a - b$ is not bounded while $(\mathcal{F}, a, \mathcal{G})$ and $(\mathcal{F}, b, \mathcal{G})$ are ℓ_p-localized. Choose $r \in \ell_p(G)$ with $|\langle f_i, g_k\rangle| \leq r(a(i) - k), r(b(i) - k)$ for all $i \in I, k \in G$. Observe that

$$\sum_{k\in G} \min\{r(k), r(k-n)\}^2 \longrightarrow 0 \quad \text{as} \quad \|n\|_\infty \to \infty.$$

Let A be the lower frame bound of $\mathcal{G}$. Choose M so that $\sum_{k\in G} \min\{r(k), r(k - n)\}^2 \leq \frac{1}{2}Au^2$ for all n with $\|n\|_\infty \geq M$ and choose i with $\|a(i) - b(i)\|_\infty \geq M$. Then

$$\begin{aligned} 0 < Au^2 &\leq A\|f_i\|^2 \leq \sum_{k\in G} |\langle f_i, g_k\rangle|^2 \leq \sum_{k\in G} \min\{r(a(i) - k), r(b(i) - k)\}^2 \\ &= \sum_{k\in G} \min\{r(k), r(k - (a(i) - b(i))\}^2 \leq \tfrac{1}{2}Au^2, \end{aligned}$$

a contradiction. □

Proposition 3.5. *If $a - b$ is bounded, then $D^-(J, a) = D^-(J, b)$ and $D^+(J, a) = D^+(J, b)$ for all $J \subseteq I$.*

Proof. Let $\|a(i) - b(i)\|_\infty \leq M$ for all $i \in I$ and choose $J \subseteq I$.

Clearly,

$$D^-(b; J) = \liminf_{R\to\infty} \inf_{k\in G} \frac{|b^{-1}(B_R(k)) \cap J|}{|B_R(0)|} = \liminf_{R\to\infty} \inf_{k\in G} \frac{|b^{-1}(B_{R+M}(k)) \cap J|}{|B_R(0)|}. \tag{3.5}$$

Choose $k_m \in G$, $R_m \in \mathbb{R}^+$ with

$$D^-(b; J) = \lim_{m\to\infty} \frac{|b^{-1}(B_{R_m+M}(k_m)) \cap J|}{|B_R(0)|}$$

Now observe that due to the boundedness of $a - b$, we have $a(j) \in B_R(k)$ implies $b(j) \in B_{R+M}(k)$ and we conclude that

$$|a^{-1}(B_{R_m}(k_m) \cap J| \leq |b^{-1}(B_{R_m+M}(k_m) \cap J|$$

and so

$$\begin{aligned} D^-(a;J) &= \liminf_{R\to\infty} \inf_{k\in G} \frac{|a^{-1}(B_R(k)) \cap J|}{|B_R(0)|} \leq \liminf_{n\to\infty} \frac{|a^{-1}(B_{R_m}(k_m)) \cap J|}{|B_{R_m}(0)|} \\ &\leq \lim_{n\to\infty} \frac{|b^{-1}(B_{R_m+M}(k_m)) \cap J|}{|B_{R_m}(0)|} = D^-(b;J). \end{aligned}$$

The inequalities $D^-(a;J) \geq D^-(b;J)$, $D^+(a;J) \leq D^+(b;J)$ and $D^+(a;J) \geq D^+(b;J)$ follow similarly. $\square$

Proposition 3.6. *Let $\mathcal{F}$ be Bessel with $\|f\| \geq u > 0$ and $\mathcal{G}$ be a frame. If $(\mathcal{F}, a, \mathcal{G})$ is ℓ_2-localized, then $D^+(a;I) < \infty$.*

Proof. Let $B_\mathcal{F}$ be a Bessel bound of $\mathcal{F}$ and $A_\mathcal{G}, B_\mathcal{G}$ be frame bounds of $\mathcal{G}$. Choose $r \in \ell_2(G)$ with $|\langle f_i, g_n\rangle| \leq r(a(i)-n)$ for $i \in I, n \in G$, and M with $\sum_{n\notin B_M(0)} r(n)^2 \leq \frac{1}{2}A_\mathcal{G}u^2$. Suppose $D^+(a;I) = \infty$. Then exists for each $m \in \mathbb{N}$ an element $k_m \in G$ with $|a^{-1}(k_m)| \geq m$. We compute

$$\begin{aligned} B_\mathcal{F}B_\mathcal{G}|B_M(0)| &\geq B_\mathcal{F} \sum_{n\in B_M(k_m)} \|g_n\|^2 \geq \sum_{i\in I} \sum_{n\in B_M(k_m)} |\langle f_i, g_n\rangle|^2 \\ &\geq \sum_{i\in a^{-1}(k_m)} \Big(\sum_{n\in G} |\langle f_i, g_n\rangle|^2 - \sum_{n\notin B_M(k_m)} |\langle f_i, g_n\rangle|^2 \Big) \\ &\geq \sum_{i\in a^{-1}(k_m)} \Big(A_\mathcal{G}\|f_i\|^2 - \sum_{n\notin B_M(k_m)} r(k_m-n)^2 \Big) \\ &\geq m\Big(A_\mathcal{G}u^2 - \tfrac{1}{2}A_\mathcal{G}u^2\Big) \geq \tfrac{1}{2}mA_\mathcal{G}u^2. \end{aligned}$$

As the left hand side above is finite and independent of m while the right hand side grows linearly with m, we have reached a contradiction. $\square$

Proposition 3.7. *Let $\mathcal{G}$ be a frame and $(\mathcal{F}, a, \mathcal{G})$ be ℓ_2-localized where $\|f_i\| \geq u > 0$, $i \in I$. Then for any $J \subseteq I$, $\mathcal{F}_J = \{f_j\}_{j\in J}$, we have $D^-(a;J) \leq D^-(\mathcal{F}_J;\mathcal{G})$ and $D^+(a;J) = D^+(\mathcal{F}_J;\mathcal{G})$. If, moreover, $\mathcal{F}$ is Bessel, then $D^-(a;J) = D^-(\mathcal{F}_J;\mathcal{G})$.*

Proof. Let $r \in \ell_2(G)$ be given with $|\langle f_i, g_n\rangle| \leq r(a(i)-n))$ for all $i \in I$, $n \in G$. Let A be the lower frame bound of $\mathcal{G}$. Then for all $i \in I$,

$$0 < u^2A \leq \sum_{n\in G} |\langle f_i, g_n\rangle|^2 \leq \sum_{n\in G} r(a(i)-n)^2 = \|r\|^2,$$

so $\|r\|^{-2} \leq a_{f_i} \leq u^{-2}A^{-1}$. For $\epsilon > 0$ choose M so that

$$u^{-2}A^{-1} \sum_{n\notin B_M(0)} r(n)^2 < \epsilon.$$

For all $i \in I$ this implies

$$1 - a_{f_i} \sum_{n \in B_M(a(i))} |\langle f_i, g_n \rangle|^2 = a_{f_i} \sum_{n \notin B_M(a(i))} |\langle f_i, g_n \rangle|^2 < \epsilon.$$

Let $J \subseteq I$. For any k and $R > M$ we have

$$\begin{aligned}(1-\epsilon)|a^{-1}(B_{R-M}(k)) \cap J| &= \sum_{j \in J, a(j) \in B_{R-M}(k)} (1-\epsilon) \\ &\leq \sum_{j \in J, a(j) \in B_{R-M}(k)} a_{f_j} \sum_{n \in B_R(k)} |\langle f_j, g_n \rangle|^2 \\ &\leq \sum_{j \in J} a_{f_j} \sum_{n \in B_R(k)} |\langle f_j, g_n \rangle|^2 .\end{aligned}$$

Equation (3.5) implies then $D^-(a;J) \leq D^-(\mathcal{F}_J;\mathcal{G})$ and $D^+(a;J) \leq D^+(\mathcal{F}_J;\mathcal{G})$.

Note that if $D^+(a;J) = \infty$, then $D^+(a;J) = D^+(\mathcal{F}_J;\mathcal{G})$ follows from $D^+(a;J) \leq D^+(\mathcal{F}_J;\mathcal{G})$.

To obtain $D^+(a;J) \geq D^+(\mathcal{F}_J;\mathcal{G})$ if $D^+(a;I) < \infty$ and $D^-(a;J) \geq D^-(\mathcal{F}_J;\mathcal{G})$ if $\mathcal{F}$ is Bessel, we may assume $D^+(a;J) < \infty$ (see Proposition 3.6). Then there exists $K \in \mathbb{N}$ with $|a^{-1}(k)| \leq K$ for all $k \in G$. For $\epsilon > 0$ and M sufficiently large, we have for $n \in B_R(k)$, $k \in G$,

$$\begin{aligned}\sum_{j \in J, a(j) \notin B_{R+M}(k)} |\langle f_j, g_n \rangle|^2 &\leq \sum_{j \in J, a(j) \notin B_{R+M}(k)} r(a(j)-n)^2 \\ &\leq K \sum_{m \notin B_{R+M}(k)} r(m-n)^2 \leq \epsilon.\end{aligned}$$

We conclude for $k \in G$ and R large that

$$\begin{aligned}\sum_{j \in J} a_{f_i} \sum_{n \in B_R(k)} |\langle f_j, g_n \rangle|^2 &\leq \sum_{j \in J, a(j) \in B_{R+M}} a_{f_i} \sum_{n \in B_R(k)} |\langle f_j, g_n \rangle|^2 \\ &\qquad + \sum_{j \in J, a(j) \notin B_{R+M}} a_{f_i} \sum_{n \in B_R(k)} |\langle f_j, g_n \rangle|^2 \\ &\leq |a^{-1}(B_{R+M}(k)) \cap J| \\ &\qquad + \sum_{n \in B_R(k)} \sum_{j \in J, a(j) \notin B_{R+M}} a_{f_i} |\langle f_j, g_n \rangle|^2 \\ &\leq |a^{-1}(B_{R+M}(k)) \cap J| + u^{-2} A^{-1} B_R(0) \epsilon,\end{aligned}$$

and $D^-(a;J) \geq D^-(\mathcal{F}_J;\mathcal{G})$, $D^+(a;J) \geq D^+(\mathcal{F}_J;\mathcal{G})$ follows. □

The following example illustrates the role of the Bessel bound of $\mathcal{F}$ to achieve $D^-(a;J) = D^-(\mathcal{F}_J;\mathcal{G})$.

Example 3.8. Let $\mathcal{G} = \{e_k\}_{k \in \mathbb{Z}}$ be an orthonormal basis, and let the members of $\mathcal{F}$ be given by $f_i = f = \sum_{m \in \mathbb{Z}}^{\infty} 2^{-|m|} e_m$, $i \in \mathbb{Z}$. For $a : \mathbb{Z} \longrightarrow \mathbb{Z}$, $i \mapsto 0$, we have

$(\mathcal{F}, a, \mathcal{G})$ is ℓ_1-localized, $D^-(a;\mathbb{Z}) = 0$, but

$$\begin{aligned} D^-(\mathcal{F};\mathcal{G}) &= \liminf_{R\to\infty} \inf_{k\in\mathbb{Z}} \frac{\sum_{f\in\mathcal{F}} \sum_{n\in B_R(k)} |\langle f, e_{n-k}\rangle|^2}{|B_R(0)|} \\ &= \liminf_{R\to\infty} \inf_{k\in\mathbb{Z}} \frac{\sum_{i\in\mathbb{Z}} \sum_{n\in B_R(k)} 2^{-|n-k|}}{|B_R(0)|} \\ &= \liminf_{R\to\infty} \inf_{k\in\mathbb{Z}} \infty = \infty. \end{aligned}$$

Definition 3.9. *Let $\mathcal{F}$ be ℓ_1-localized with respect to $\mathcal{G}$ with $0 < D^-(\mathcal{F};\mathcal{G}) \le D^+(\mathcal{F};\mathcal{G}) < \infty$. The* **relative lower density**, *respectively* **relative upper density** *of $\mathcal{F}' \subseteq \mathcal{F}$ is*

$$R^-(\mathcal{F}', \mathcal{F};\mathcal{G}) = \frac{D^-(\mathcal{F}';\mathcal{G})}{D^+(\mathcal{F};\mathcal{G})}, \tag{3.6}$$

$$R^+(\mathcal{F}', \mathcal{F};\mathcal{G}) = \frac{D^+(\mathcal{F}';\mathcal{G})}{D^-(\mathcal{F};\mathcal{G})}. \tag{3.7}$$

If $R^-(\mathcal{F}', \mathcal{F};\mathcal{G}) = R^+(\mathcal{F}', \mathcal{F};\mathcal{G})$, then we say that $\mathcal{F}'$ has **uniform relative density** *$R(\mathcal{F}', \mathcal{F};\mathcal{G}) = R^+(\mathcal{F}', \mathcal{F};\mathcal{G})$ in $\mathcal{F}$.*

Examples 3.12 and 3.13 below illustrate the interaction of density and localization. We are now ready to restate the main result of the paper.

Theorem 3.10. *Let $\mathbf{c}$ be the function provided in Theorem 6.1. Let $\mathcal{F} \subseteq \mathbb{H}$ be ℓ_1-localized with respect to the frame $\mathcal{G}$ and assume that $\|f\| \ge u$ for all $f \in \mathcal{F}$ and $\mathcal{F}$ is Bessel with Bessel bound $B_{\mathcal{F}}$. Assume either*

(A) $\mathcal{G} = \{g_k : k \in G\}$ is a Riesz basis for $\mathbb{H}$ with Riesz bounds $A_{\mathcal{G}}, B_{\mathcal{G}}$,

or

(B) $\mathcal{G} = \{g_k : k \in G\}$ is a frame for $\mathbb{H}$ with ℓ_1- self-localized dual frame $\widetilde{\mathcal{G}} = \{\widetilde{g}_k : k \in G\}$.

If $(\mathcal{F};\mathcal{G})$ is ℓ_1-localized with $0 < D^-(\mathcal{F};\mathcal{G}) \le D^+(\mathcal{F};\mathcal{G}) < \infty$. Then for every $\epsilon > 0$ and $\delta > 0$ there is a subset $\mathcal{F}_{\epsilon\delta} \subseteq \mathcal{F}$ of uniform density with

(1) $$R^+(\mathcal{F}_{\epsilon\delta}, \mathcal{F};\mathcal{G}) = \frac{D(\mathcal{F}_{\epsilon\delta};\mathcal{G})}{D^-(\mathcal{F};\mathcal{G})} \ge \frac{(1-\epsilon)u^2}{B_{\mathcal{F}}},$$

(2) $\mathcal{F}_{\epsilon\delta}$ is a Riesz sequence with Riesz bounds

(A) $\mathbf{c}(\epsilon)(1-\delta)\dfrac{A_{\mathcal{G}}}{B_{\mathcal{G}}}\, u^2,\ B_{\mathcal{F}}$.

(B) $\mathbf{c}(\epsilon)(1-\delta)\, u^2,\ B_{\mathcal{F}}$.

Proof. Note that for $\mathcal{F} = \{f_i\}_{i\in I}$, $J_{\epsilon\delta} \subseteq I$, and $\mathcal{F}_{\epsilon\delta} = \{f_j\}_{j\in J_{\epsilon\delta}}$, we have in general

$$\frac{D(a; J_{\epsilon\delta})}{D^-(a; I)} \geq \frac{D(\mathcal{F}_{J_{\epsilon\delta}}; \mathcal{G})}{D^-(\mathcal{F}; \mathcal{G})} = R^+(\mathcal{F}_{\epsilon\delta}, \mathcal{F}; \mathcal{G}).$$

But under the given assumptions, Proposition 3.7 implies equality above, and, hence, Theorem 3.10 is a restatement of Theorem 2.6. □

Theorem 3.10 can be rephrased in terms of ℓ_1-localized operators. Again, given an ℓ_1-localized operator $T : \mathbb{H}' \longrightarrow \mathbb{H}$ and respective orthonormal basis $\mathcal{E}$ of $\mathbb{H}'$ and a frame $\mathcal{G}$ of $\mathbb{H}$ with $\mathcal{F} = T(\mathcal{E})$ being ℓ_1-localized with respect to $\mathcal{G}$, then Theorem 3.10 holds verbatim with the Bessel bound $B_{\mathcal{F}}$ being replaced with $\|T\|^2$.

Theorem 3.11 (**General Infinite Dimensional Restricted Invertibility Theorem**)**.** *Let* $\mathbf{c}$ *be the function provided in Theorem 6.1. Let* $\mathcal{E}$ *and* $\mathcal{G} = \{g_k\}_{k\in G}$ *be orthonormal bases for an Hilbert space* $\mathbb{H}$*, let* $T : \mathbb{H} \to \mathbb{H}$ *be a bounded linear operator satisfying* $\|Te\| = 1$*, for all* $e \in \mathcal{E}$*. Assume that* $T\mathcal{E}$ *is* ℓ_1*-localized with respect to* $\{g_k\}_{k\in G}$*. Then for all* $\epsilon, \delta > 0$*, there is a subfamily* $\mathcal{E}_{\epsilon\delta} \subseteq \mathcal{E}$ *of uniform density with*

(1) $R^+(T\mathcal{E}_{\epsilon\delta}, T\mathcal{E}; \mathcal{G}) \geq \dfrac{1-\epsilon}{\|T\|^2}$*, and*

(2) $T\mathcal{E}_{\epsilon\delta}$ *is a Riesz system with Riesz bounds* $\mathbf{c}(\epsilon)(1-\delta)$, $\|T\|^2$.

We close this section with two examples displaying the interaction of density, localization, and Theorems 3.10 respectively 3.11.

Example 3.12. In the following, we shall consider as reference system for $\mathbb{H}$

- an orthonormal basis $\mathcal{G} = \{g_n\}_{n\in\mathbb{Z}}$ of $\mathbb{H}$;
- the system $\mathcal{G}' = \{g'_n\}$ given by $g'_{2n} = g'_{2n+1} = g_n$, $n \in \mathbb{Z}$, that is, $\mathcal{G}'$ consists of two intertwined copies of $\mathcal{G}$;
- the system $\mathcal{G}''$ given by the sequence

$$\cdots, e_{-7}, e_{-2}, e_{-5}, e_{-3}, e_{-1}, e_0, e_1, e_3, e_5, e_2, e_7, e_9, e_{11}, e_4, e_{13}, e_{15}, e_{17}, e_6, e_{19}, \cdots$$

.

Moreover, we shall consider the operators $T_1, T_2, T_3 : \mathbb{H} \longrightarrow \mathbb{H}$ given by

- $T_1 e_n = e_n$, $n \in \mathbb{Z}$, that is, $T_1 = \mathrm{Id}$;
- $T_2 e_n = e_{2[\frac{n}{2}]}$, $n \in \mathbb{Z}$;
- $T_3 e_0 = e_1$, $T_3 e_n = e_n$ for $n \in \mathbb{Z} \setminus \{0\}$;

Clearly, $\|T_1\| = 1$ and $\|T_2\| = \|T_3\| = \sqrt{2}$. Note that the right hand side in Theorem 3.11 (1) is $(1-\epsilon)$ for T_1 and $(1-\epsilon)/2$ for T_2, T_3. Set $\mathcal{F} = \{e_n\}_{n\in\mathbb{Z}}$, $\mathcal{F}_{\text{even}} = \{e_{2n}\}_{n\in\mathbb{Z}}$, and observe that they form orthonormal bases for their closed linear span. Hence, we could choose $\mathcal{F}_{\epsilon\delta} = \mathcal{F} \subseteq \mathcal{F}$ in case of T_1, and $\mathcal{F}_{\epsilon\delta} = \mathcal{F}_{\text{even}}$ in case of T_1, T_2. We now discuss strengths and shortcomings of Theorem 3.11 when using as reference systems $\mathcal{G}$, $\mathcal{G}'$, and $\mathcal{G}''$.

(1) Clearly, $\mathcal{F}$ is ℓ_1–localized with respect to $\mathcal{G}$. Moreover $D^-(\mathcal{F};\mathcal{G}) = D(\mathcal{F};\mathcal{G}) = 1$, $D(\mathcal{F}_{\text{even}};\mathcal{G}) = \frac{1}{2}$, and
$$R(\mathcal{F},\mathcal{F};\mathcal{G}) = D(\mathcal{F};\mathcal{G})/D(\mathcal{F};\mathcal{G}) = 1,$$
$$R(\mathcal{F}_{\epsilon\delta},\mathcal{F};\mathcal{G}) = D(\mathcal{F}_{\text{even}};\mathcal{G})/D(\mathcal{F};\mathcal{G}) = \frac{1}{2}.$$
So $\mathcal{F}$, $\mathcal{F}_{\text{even}}$ satisfy the conclusions of (1) in Theorem 3.11 for T_1 respectively T_2 and T_3.

(2) $\mathcal{F}$ is ℓ_1–localized with respect to $\mathcal{G}'$. We have $D^-(\mathcal{F};\mathcal{G}') = D(\mathcal{F};\mathcal{G}') = \frac{1}{2}$ and $D(\mathcal{F}_{\text{even}};\mathcal{G}') = \frac{1}{4}$, so again $R(\mathcal{F},\mathcal{F};\mathcal{G}') = D(\mathcal{F};\mathcal{G}')/D(\mathcal{F};\mathcal{G}') = 1$ and $R(\mathcal{F}_{\epsilon\delta},\mathcal{F};\mathcal{G}') = D(\mathcal{F}_{\text{even}};\mathcal{G}')/D(\mathcal{F};\mathcal{G}') = \frac{1}{2}$ for T_2, T_3, satisfying Theorem 3.11 (1) for T_1 respectively T_2 and T_3.

(3) $\mathcal{F}$ is also ℓ_1–localized with respect to $\mathcal{G}''$. Now, $D^-(\mathcal{F};\mathcal{G}'') = D(\mathcal{F};\mathcal{G}'') = 1$, but $D(\mathcal{F}_{\text{even}};\mathcal{G}'') = \frac{1}{4}$, consequently, $R(\mathcal{F},\mathcal{F};\mathcal{G}'') = D(\mathcal{F};\mathcal{G}'')/D(\mathcal{F};\mathcal{G}'') = 1$, so Theorem 3.11 (1) for T_1 is satisfied, but as
$$R(\mathcal{F}_{\epsilon\delta},\mathcal{F};\mathcal{G}'') = D(\mathcal{F}_{\text{even}};\mathcal{G}'')/D(\mathcal{F};\mathcal{G}'') = \frac{1}{4},$$
so $\mathcal{F}_{\epsilon\delta} = \mathcal{F}_{\text{even}}$ is not a valid choice satisfying Theorem 3.11 (1) for T_2, T_3. Theorem 3.11 guarantees for any $\epsilon, \delta > 0$ the existence of a Riesz sequence $\mathcal{F}_{\epsilon\delta}$ with $R(\mathcal{F}_{\epsilon\delta},\mathcal{F};\mathcal{G}'') = D(\mathcal{F}_{\epsilon\delta};\mathcal{G}'')/D(\mathcal{F};\mathcal{G}'') \geq (1-\epsilon)\frac{1}{2}$, and clearly, in the case of T_2 and T_3, we may choose $\mathcal{F}_{\epsilon\delta} = \mathcal{F}_{\text{odd}} = \mathcal{F}\setminus\mathcal{F}_{\text{even}}$. Then we have the seemingly better result $R(\mathcal{F}_{\epsilon\delta} = \mathcal{F}_{\text{odd}},\mathcal{F};\mathcal{G}'') = D(\mathcal{F}_{\text{odd}};\mathcal{G}'')/D(\mathcal{F};\mathcal{G}'') = \frac{3}{4} > (1-\epsilon)\frac{1}{2}$.

(4) Note that regardless of how we adjust $\mathcal{G}$, we will not be guaranteed a Riesz system as large as the optimal one for T_3, namely $\mathcal{F}_{\epsilon\delta} = \mathcal{F}\setminus\{Te_0\}$. Clearly, this shortcoming is shared by the finite dimensional version of Bourgain-Tzafriri.

The following example illustrates that the possible choices of index set G of $\mathcal{G}$ is strongly influenced by $\mathcal{F}$ in Theorem 3.10 respectively T in Theorem 3.11.

Example 3.13. Consider the operator $T_4 : \mathbb{H} \longrightarrow \mathbb{H}$ given by $T_4 e_n = e_n + e_{2n}$, $n \in \mathbb{Z}$. We have $\|T_4 e_n\| \geq u = \sqrt{2}$ for $n \in \mathbb{Z}$ and
$$\|T_4(\sum c_n e_n)\| = \|\sum c_n e_n + \sum_n c_n e_{2n}\| \leq \|\sum c_n e_n\| + \|\sum_n c_n e_{2n}\| = 2\|\sum c_n e_n\|.$$
As $\|T_4 e_0\| = \|2e_0\| = 2$, we have $\|T_4\| = 2$. Note that also
$$\|T_4(N^{-\frac{1}{2}}\sum_{n=1}^{N} e_{2^n})\| = N^{-\frac{1}{2}}\|e_2 + 2e_4 + 2e_8 + \cdots + 2e_{2^{N-1}} + e_{2^N}\| = \sqrt{\tfrac{4(N-2)+2}{N}} \to 2$$
as $N \to \infty$.

The right hand side in Theorem 3.10 (1) is $(1-\epsilon)/2$ for T_4, and the orthogonal family $\mathcal{F}_{\epsilon\delta} = T_4\{e_{2n+1}\}_{n\in\mathbb{Z}}$ satisfies the conclusions of Theorem 3.10 (2). But $T_4(\mathcal{E})$ is not ℓ_1-localized with respect to $\mathcal{G}$ whenever $\mathcal{G}$ is a linear ordering of $\mathcal{E} = \{e_n\}_{n\in\mathbb{Z}}$. To see this, presume that $T_4\mathcal{E}$ is ℓ_1-localized with respect to $\mathcal{G} = \{g_n = e_{\sigma(n)}\}_{n\in\mathbb{Z}}$

where σ is a permutation on $\mathbb{Z}$. Let $r \in \ell_1(\mathbb{Z})$ be the respective bounding sequence and choose N so that $r(k) < 1$ for $|k| \geq N$. Now, for some $k_{2N} \in \mathbb{Z}$, we have

$$\delta_{2N,\sigma(n)} + \delta_{4N,\sigma(n)} = \langle Te_{2N}, g_n \rangle \leq r(k_{2N} - \sigma(n)), \quad n \in \mathbb{Z}.$$

Inserting $n_1 = \sigma^{-1}(2N)$ respectively $n_2 = \sigma^{-1}(4N)$, we obtain $1 \leq r(k_{2N} - 2N)$ respectively $1 \leq r(k_{2N} - 4N)$, and, by choice of N, $|k_{2N} - 2N|, |k_{2N} - 4N| < N$, leading to the contradiction $2N < 2N$.

As an alternative to linear orders on $\mathcal{E}$, consider the following as reference system $\mathcal{G}_{\mathbb{Z}^2}$

$$\begin{array}{ccccccccccccc}
 & & & & & & \vdots & & & & & & \\
 & e_{-208} & e_{-104} & e_{-52} & e_{-26} & e_{-13} & e_{13} & e_{26} & e_{52} & e_{104} & e_{208} & e_{416} & \\
 & e_{-144} & e_{-72} & e_{-36} & e_{-18} & e_{-9} & e_{9} & e_{18} & e_{36} & e_{72} & e_{144} & e_{288} & \\
 & e_{-80} & e_{-40} & e_{-20} & e_{-10} & e_{-5} & e_{5} & e_{10} & e_{20} & e_{40} & e_{80} & e_{160} & \\
\cdots & e_{-16} & e_{-8} & e_{-4} & e_{-2} & e_{-1} & e_{0} & e_{1} & e_{2} & e_{4} & e_{8} & e_{16} & \cdots \\
 & e_{-48} & e_{-24} & e_{-12} & e_{-6} & e_{-3} & e_{3} & e_{6} & e_{12} & e_{24} & e_{48} & e_{96} & \\
 & e_{-112} & e_{-56} & e_{-28} & e_{-14} & e_{-7} & e_{7} & e_{14} & e_{28} & e_{56} & e_{112} & e_{224} & \\
 & e_{-176} & e_{-88} & e_{-44} & e_{-22} & e_{-11} & e_{11} & e_{22} & e_{44} & e_{88} & e_{176} & e_{352} & \\
 & e_{-240} & e_{-120} & e_{-60} & e_{-30} & e_{-15} & e_{15} & e_{30} & e_{60} & e_{120} & e_{240} & e_{480} & \\
 & & & & & & \vdots & & & & & &
\end{array}.$$

Clearly, T_4 is ℓ_1-localized with respect to $\mathcal{G}_{\mathbb{Z}^2}$. In fact, we can choose $r = \delta_{(0,0)} + \delta_{(0,1)} \in \ell_1(\mathbb{Z}^2)$.

Theorem 3.10 guarantees for $\delta, \epsilon > 0$ the existence of a Riesz sequence $\mathcal{F}_{\epsilon\delta}$ with $R(\mathcal{F}_{\epsilon\delta}, \mathcal{F}; \mathcal{G}_{\mathbb{Z}^2}) = D(\mathcal{F}_{\epsilon\delta}; \mathcal{G}_{\mathbb{Z}^2})/D(\mathcal{F}_{\epsilon\delta}; \mathcal{G}_{\mathbb{Z}^2}) \geq (1-\epsilon)/2$. We have $D(\mathcal{F}; \mathcal{G}_{\mathbb{Z}^2}) = 1$, but for the natural choice $\mathcal{F}_{\epsilon\delta} = T_4\{e_{2n+1}\}_{n\in\mathbb{Z}}$, we have $D(\mathcal{F}_{\epsilon\delta}; \mathcal{G}_{\mathbb{Z}^2}) = 0$. For $\mathcal{F}_{\epsilon\delta} = T_4\{e_{2^{2k}(2n+1)}\}_{n\in\mathbb{Z}, k\in\mathbb{N}_0}$, we have $D(\mathcal{F}_{\epsilon\delta}; \mathcal{G}_{\mathbb{Z}^2}) = \frac{1}{2}$, therefore satisfying the conclusions of Theorem 3.10.

For completeness sake, note that $T_4(\mathcal{E})$ itself is not a Riesz sequence. To see this, observe that

$$\|\sum_{n=1}^{N}(-1)^n T_4 e_{2^n}\|^2 = \|\sum_{n=1}^{N}(-1)^n(e_{2^n} + e_{2^{n+1}})\|^2 = \|-e_1 + (-1)^N e_{2^{N+1}}\|^2 = 2$$

while $\sum_{n=1}^{N} |(-1)^n|^2 = N$.

4. Proof of Theorem 2.6

Note that the generality assumed here, namely that G is any finitely generated Abelian group, is quite useful in practice as the group is often given by the structure of the problem at hand. For example, in time–frequency analysis, the group $G = \mathbb{Z}^{2d}$ is generally used when considering single window Gabor systems. If we consider multi-window Gabor systems, then an index set $\mathbb{Z}^{2d} \times H$ with H being a finite group is natural. (Also, see Example 3.13 for the dependence of G on T and $\mathcal{F}$.) The following proposition will allow us to consider in our proofs only localization with respect to $\mathcal{G}$ with $G = \mathbb{Z}^d$.

Proposition 4.1. *Let H be a finite Abelian group of order N and $G = \mathbb{Z}^d \times H$. Choose a bijection $u : \{0, 1, \cdots, N-1\} \longrightarrow H$ and*

$$U : \mathbb{Z}^d \longrightarrow G, \quad (k_1, \cdots, k_d) \mapsto (k_1, \cdots, k_{d-1}, \lfloor k_d/N \rfloor, u(k_d \mod N)).$$

For $a : I \longrightarrow G$ set $b = U^{-1} \circ a : I \longrightarrow \mathbb{Z}^d$. Then

1. *$D^-(b; J) = D^-(a; J)$ and $D^+(b; J) = D^+(a; J)$ for all $J \subseteq I$.*
2. *$(\mathcal{F}, a, \mathcal{G})$ is ℓ_1-localized if and only if $(\mathcal{F}, b, \mathcal{G}')$ is ℓ_1-localized where $\mathcal{G}' = \{g_{U(k)}\}_{k \in \mathbb{Z}^d}$.*

Proof. First, observe that for all $P \in \mathbb{N}$ we have

$$D^-(a; J) = \liminf_{R\to\infty} \inf_{k\in G} \frac{|a^{-1}(B_R(k)) \cap J|}{|B_R(0)|} = \liminf_{M\to\infty} \inf_{k\in G} \frac{|a^{-1}(B_{MP}(k)) \cap J|}{|B_{MP}(0)|}$$

where $M \to \infty$, $M \in \mathbb{N}$.

Note that for $R = MN$, $M \in \mathbb{N}$,

$$\begin{aligned} \left|b^{-1}\big(B_R^{Z^d}(k)\big) \cap I\right| &= \left|a^{-1} \circ U\big(B_{MN}^{Z^d}(k)\big) \cap I\right| \\ &= \left|a^{-1}\big(B_{MN}^{Z^{d-1}}(k_1, \cdots, k_{d-1}) \times B_M^{\mathbb{Z}}(\lfloor \tfrac{k_d}{N} \rfloor) \times H\big) \cap I\right|. \end{aligned}$$

Now, compare

$$\begin{aligned} D^-(b; J) &= \liminf_{M\to\infty} \inf_{k\in\mathbb{Z}^d} \frac{\left|b^{-1}\big(B_{MN}^{\mathbb{Z}^d}(k)\big) \cap I\right|}{|B_{MN}^{\mathbb{Z}^d}(0)|} \\ &= \liminf_{M\to\infty} \inf_{k\in\mathbb{Z}^d} \frac{\left|a^{-1}\big(B_{MN}^{\mathbb{Z}^{d-1}}(k_1, \cdots, k_{d-1}) \times B_M^{\mathbb{Z}}(\lfloor \frac{k_d}{N} \rfloor) \times H\big) \cap I\right|}{(2MN+1)^d} \end{aligned}$$

and

$$\begin{aligned} D^-(a; J) &= \liminf_{M\to\infty} \inf_{(k,h)\in\mathbb{Z}^d\times H} \frac{\left|a^{-1}\big(B_{MN}^{Z^d\times H}(n, h)\big) \cap I\right|}{|B_{MN}^{\mathbb{Z}^d\times H}(0)|} \\ &= \liminf_{M\to\infty} \inf_{k\in\mathbb{Z}^d} \frac{\left|a^{-1}\big(B_{MN}^{\mathbb{Z}^{d-1}}(k_1, \cdots, k_{d-1}) \times B_{MN}^{\mathbb{Z}}(k_d) \times H\big) \cap I\right|}{N(2MN+1)^d}. \end{aligned}$$

As the sets $B_R(k) = RB_1(0) + k$ in Definition 2.2 can be replaced by sets of the form $D_R(k) = RD + k$ if D is a compact set of measure 1 and 0 measure boundary (Lemma 4 in [16]), we conclude that $D^-(b; J) = D^-(a; J)$, and, similarly $D^+(b; J) = D^+(a; J)$.

The second assertion is obvious. □

Lemma 4.2. *Let $(\mathcal{F} = \{f_i\}_{i\in I}, a, \mathcal{G} = \{g_k\}_{k\in G})$ be ℓ_1-localized with $D^+(a; I) < \infty$. For $M^R : \ell_2(G) \mapsto \ell_2(I)$ given by $(M^R)_{i,k} = \langle f_i, g_k \rangle$ if $\|a(i) - k\|_\infty > R$, and $(M^R)_{i,k} = 0$ otherwise, we have*

$$\lim_{R\to\infty} \|M^R\| = 0.$$

Proof. As $(\mathcal{F} = \{f_i\}_{i\in I}, a, \mathcal{G} = \{g_k\}_{k\in G})$ is ℓ_1-localized, there exists $r \in \ell_1(G)$ with

$$r(k) \geq |\langle f_i, g_{k'} \rangle| \quad \text{if} \quad a(i) - k' = k.$$

Hence,

$$\begin{aligned}\sup_{i\in I}\sum_{k\in G}|(M^R)_{i,k}| &= \sup_{i\in I}\sum_{k':\|a(i)-k'\|_\infty>R}|\langle f_i,g_{k'}\rangle| \le \sup_{i\in I}\sum_{k':\|a(i)-k'\|_\infty>R} r(a(i)-k')\\ &\le \sup_{i\in I}\sum_{k:\|k\|_\infty>R} r(k) =: \Delta_r(R).\end{aligned}$$

Similarly, setting $K=\max_{k\in G}|a^{-1}(k)|$ (it is finite since $D^+(a;I)<\infty$) we obtain

$$\begin{aligned}\sup_{k\in G}\sum_{i\in I}|(M^R)_{i,k}| &= \sup_{k'\in G}\sum_{i:\|a(i)-k'\|_\infty>R}|\langle f_i,g_{k'}\rangle| \le \sup_{k'\in G}\sum_{i:\|a(i)-k'\|_\infty>R} r(a(i)-k')\\ &\le \sup_{k'\in G} K\sum_{k:\|k\|_\infty>R} r(k) = K\Delta_r(R).\end{aligned}$$

The result now follows from Schur's criterion [15, 18] since $\Delta_r(R)\longrightarrow 0$. $\square$

The following lemma is similar to Lemma 3.6 of [4].

Lemma 4.3. *Let $\mathcal{G}=\{g_k\}_{k\in\mathbb{Z}^d}$ be a frame for $\mathbb{H}$ with dual frame $\widetilde{\mathcal{G}}=\{\widetilde{g}_k\}_{k\in\mathbb{Z}^d}$ and let $a:I\to G$ be a localization map of finite upper density so that the Bessel system $(\{f_i\}_{i\in I},a,\mathcal{G})$ is ℓ_1-localized. For $R>0$ set*

$$f_{iR}=\sum_{n\in\mathbb{Z}^d:\,\|a(i)-n\|_\infty<R}\langle f_i,g_n\rangle\widetilde{g}_n$$

and set

$$L_I:\mathbb{H}\longrightarrow\ell_2(I),\ \ h\mapsto\{\langle h,f_i\rangle\}\ \text{ and }\ L_{IR}:\mathbb{H}\longrightarrow\ell_2(I),\ \ h\mapsto\{\langle h,f_{iR}\rangle\}.$$

Then

$$\lim_{R\to\infty}\|L_I-L_{IR}\|=0.$$

Proof. For $h\in\mathbb{H}$, we compute

$$\begin{aligned}\|(L_I-L_{IR})h\|_{\ell_2}^2 &= \sum_{i\in I}|\langle h,f_i\rangle-\langle h,f_{iR}\rangle|^2 = \sum_{i\in I}|\langle h,f_i-f_{iR}\rangle|^2\\ &= \sum_{i\in I}|\langle h,\sum_{\|a(i)-n\|_\infty>R}\langle f_i,g_n\rangle\widetilde{g}_n\rangle|^2 = \sum_{i\in I}|\sum_{\|a(i)-n\|_\infty>R}\langle f_i,g_n\rangle\langle h,\widetilde{g}_n\rangle|^2\\ &= \|M^R\{\langle h,\widetilde{g}_n\rangle\}_{n\in\mathbb{Z}^d}\|^2,\end{aligned}$$

with M^R given by $M_{i,n}=\langle f_i,g_n\rangle$ if $\|a(i)-n\|_\infty>R$ and $M_{i,n}=0$ otherwise. Since $(\{f_i\}_{i\in I},a,\mathcal{G})$ is ℓ_1-localized, we can apply Lemma 4.2 and obtain $\|M^R\|\longrightarrow 0$ as $R\to\infty$. The result now follows from the boundedness of the map $h\mapsto\{\langle h,\widetilde{g}_n\rangle\}_n$. $\square$

4.1. **Proof of Theorem 2.6, assuming (A).** Fix $\epsilon, \delta > 0$. As $\mathbf{c}$ given in Theorem 6.1 is positive and continuous, we can choose $\epsilon' > 0$ with $\epsilon' < \epsilon$ and

$$\mathbf{c}(\epsilon)(1-\delta) \le \mathbf{c}(\epsilon')(1-\tfrac{\delta}{2}).$$

Choose $\alpha > 0$ satisfying $\alpha \le \frac{\delta}{8}$ and

$$(1-\epsilon')\frac{(1-\alpha)^2}{(1+\alpha)^2} \ge (1-\epsilon).$$

Recall that for $R \in \mathbb{N}$,

$$D^-(a;I) = \liminf_{R\to\infty} \inf_{k\in\mathbb{Z}^d} \frac{\left|a^{-1}(B_R(k)) \cap I\right|}{\left|B_R(0)\right|} = \liminf_{R\to\infty} \inf_{k\in\mathbb{Z}^d} \frac{\left|a^{-1}(B_R(k)) \cap I\right|}{(2R+1)^d}$$

where $B_R(k) = \{k' : \|k-k'\|_0 \le R\}$. Hence, we may choose $P > 0$ such that for all $R \ge P$, $k \in \mathbb{Z}^d$, we have

$$\left|a^{-1}(B_R(k)) \cap I\right| \ge (1-\alpha)\, D^-(a;I)\, (2R+1)^d\,.$$

Let $\{\widetilde{g}_n\}_{n\in G}$ be the dual basis of $\{g_n\}_{n\in G}$. For any $R > 0$, set

$$f_{iR} = \sum_{n\in\mathbb{Z}^d\colon \|a(i)-n\|_\infty < R} \langle f_i, g_n\rangle \widetilde{g}_n.$$

For

$$L_I : \mathbb{H} \longrightarrow \ell_2(I),\ h \mapsto \{\langle h, f_i\rangle\} \text{ and } L_{IR} : \mathbb{H} \longrightarrow \ell_2(I),\ h \mapsto \{\langle h, f_{iR}\rangle\},$$

Lemma 4.3 implies that there is $Q > 0$ with the property that for all $R \ge Q$ we have

$$\|L_I - L_{IR}\| \le \min\Big\{\alpha u, \alpha\|T\|, \Big(\frac{A\delta\, \mathbf{c}(\epsilon')u^2}{8B}\Big)^{\frac{1}{2}}\Big\}. \tag{4.8}$$

Also, since

$$D^+(a;I) = \limsup_{R\to\infty} \sup_{k\in\mathbb{Z}^d} \frac{\left|a^{-1}(B_R(k)) \cap I\right|}{\left|B_R(k)\right|} < \infty,$$

we can pick $K > 0$ with $|a^{-1}(k)| < K$, for all $k \in \mathbb{Z}^d$.

By possibly increasing P and Q, we can assume $P > Q$ and

$$\begin{aligned} K \quad & \big((2P+1)^d - (2(P-Q)+1)^d\big) \\ & \le \alpha\, u^2\, \frac{(1-\epsilon')(1-\alpha)}{(1+\alpha)^2} \frac{D^-(a;I)(2P+1)^d}{\|T\|^2} \end{aligned} \tag{4.9}$$

Set $\mathcal{F}_k = \{f_i :\ a(i) \in B_P(k)\}$ and correspondingly $\mathcal{F}_{kQ} = \{f_{iQ} :\ a(i) \in B_P(k)\}$. Equation (4.8) implies

$$\|f_i - f_{iQ}\| = \|L_I^*\{\delta_i\} - L_{IQ}^*\{\delta_i\}\| = \|(L_I - L_{IQ})^*\{\delta_i\}\| \le \|L_I - L_{IQ}\| \le \alpha u,$$

and, therefore, $\|f_{iQ}\| \ge (1-\alpha)u$. Similarly, we conclude for $T_Q : e_i \mapsto f_{iQ}$, that $\|T - T_Q\| < \alpha\|T\|$ and for $h = \sum a_i e_i \in \mathbb{H}$,

$$\begin{aligned} \|(T-T_Q)h\| &= \|\sum a_i (T-T_Q)e_i\| = \|\sum a_i(f_i - f_{iQ})\| \\ &= \|(L_I - L_{IQ})^*\{a_i\}\| \le \alpha\|T\|\|\{a_i\}\| = \alpha\|T\|\|h\|. \end{aligned}$$

Applying Theorem 6.1 to the finite sets $\mathcal{F}_{kQ}$ with cardinality

$$n \geq (1-\alpha) D^{-}(a;I)\,(2P+1)^d$$

and ϵ', we obtain Riesz sequences $\mathcal{F}'_{kQ} \subseteq \mathcal{F}_{kQ}$ with

$$\begin{aligned} |\mathcal{F}'_{kQ}| &\geq (1-\epsilon')u^2\,(1-\alpha)\,D^{-}(a;I)\,(2P+1)^d/\|T_Q\|^2 \\ &\geq \frac{(1-\epsilon')(1-\alpha)}{(1+\alpha)^2}\,u^2\,D^{-}(a;I)\,(2P+1)^d/\|T\|^2 \end{aligned} \tag{4.10}$$

and lower Riesz bounds $\mathbf{c}(\epsilon')(1-\alpha)^2u^2$.

We further reduce $\mathcal{F}'_{kQ} \subseteq \mathcal{F}_{kQ}$ by setting

$$F''_{kQ} = F'_{kQ} \cap a^{-1}(B_{P-Q}(k)). \tag{4.11}$$

Now,

$$\begin{aligned} |\mathcal{F}''_{kQ}| &\geq \frac{(1-\epsilon')(1-\alpha)}{(1+\alpha)^2}\,u^2\,D^{-}(a;I)\,(2P+1)^d/\|T\|^2 \\ &\quad - K\big((2P+1)^d - (2(P-Q)+1)^d\big) \\ &\geq \frac{(1-\epsilon')(1-\alpha)}{(1+\alpha)^2}\,u^2\,D^{-}(a;I)\,(2P+1)^d/\|T\|^2 \\ &\quad - \alpha\frac{(1-\epsilon')(1-\alpha)}{(1+\alpha)^2}\,u^2\,D^{-}(a;I)(2P+1)^d/\|T\| \\ &\geq \frac{(1-\epsilon')(1-\alpha)^2}{(1+\alpha)^2}\,u^2\,D^{-}(a;I)\,(2P+1)^d/\|T\|^2. \end{aligned} \tag{4.12}$$

Claim 1: If $J_k = \{j \in I : f_{jR} \in \mathcal{F}''_{kQ}\}$, $J = \bigcup_{k\in(2P+1)\mathbb{Z}^d} J_k$ and

$$\mathcal{F}_Q(J) = \bigcup_{k\in(2P+1)\mathbb{Z}^d} \mathcal{F}''_{kQ},$$

then $\mathcal{F}_Q(J)$ is a Riesz sequence with lower Riesz bound $\frac{A}{B}\mathbf{c}(\epsilon')(1-\alpha)^2u^2$.

Proof of Claim 1. To see this, consider $\widetilde{\mathcal{G}}_k = \{\widetilde{g}_{k'} :\ \|k'-k\|_\infty < P\}$ which are disjoint subsets of $\mathcal{G}$. Furthermore, (4.11) ensures that for $k \in (2P+1)\mathbb{Z}^d$, the set $\mathcal{F}''_{kQ}$ is a Riesz basis sequence in $\operatorname{span}\mathcal{G}_k$, where the lower Riesz constant $\mathbf{c}(\epsilon')(1-\alpha)^2u^2$ is given by Theorem 6.1 and does not depend on k or P. For $\{a_j\}_{j\in J} \in \ell_2(J)$ we have, using Lemma 6.2,

$$\begin{aligned} \Big\|\sum a_j f_{jQ}\Big\|^2 &= \Big\|\sum_{k\in(2P+1)\mathbb{Z}^d}\sum_{j\in J_k} a_j f_{jQ}\Big\|^2 \geq \frac{1/B}{1/A}\sum_{k\in(2P+1)\mathbb{Z}^d}\|\sum_{j\in J_k} a_j f_{jQ}\|^2 \\ &\geq \frac{A}{B}\sum_{k\in(2P+1)\mathbb{Z}^d}\mathbf{c}(\epsilon')(1-\alpha)^2u^2\|\{a_j\}_{j\in J_k}\|^2 \\ &= \frac{A\mathbf{c}(\epsilon')}{B}(1-\alpha)^2u^2\|\{a_j\}_{j\in J}\|^2. \end{aligned}$$

We conclude that $\mathcal{F}_Q(J)$ is a Riesz sequence with lower Riesz bound

$$\frac{A}{B}\mathbf{c}(\epsilon')(1-\alpha)^2u^2,$$

so Claim 1 is shown.

It remains to show that we can replace $\mathcal{F}_Q(J)$ by $\mathcal{F}(J)=\{f_j,\ j\in J\}=\{f_i,\ f_{iQ}\in \mathcal{F}_Q(J)\}$, while controlling the lower Riesz bound. For $\{a_j\}_{j\in J}$ we have

$$\begin{aligned}
\|\sum a_jf_j\| &\geq \|\sum a_jf_{jQ}\|-\|\sum a_j(f_j-f_{jQ})\|\\
&\geq \Big(\frac{A}{B}\mathbf{c}(\epsilon')\Big)^{\frac12}(1-\alpha)u\|\{a_j\}\|-\|(L_I-L_{IQ})^*\{a_i\}\|\\
&\geq \Big(\frac{A}{B}\mathbf{c}(\epsilon')(1-\alpha)^2u^2\Big)^{\frac12}\ \|\{a_j\}\|-\Big(\frac{A\delta\,\mathbf{c}(\epsilon')u^2}{8B}\Big)^{\frac12}\|\{a_i\}\|\\
&\geq \Big(\frac{A}{B}\mathbf{c}(\epsilon')u^2((1-\alpha)^2-\frac{\delta}{8})\Big)^{\frac12}\|\{a_i\}\|\\
&\geq \Big(\frac{A}{B}\mathbf{c}(\epsilon')u^2((1-2\alpha-\frac{\delta}{8})\Big)^{\frac12}\|\{a_i\}\|\\
&\geq \Big(\frac{A}{B}\mathbf{c}(\epsilon')u^2((1-\frac{\delta}{4}-\frac{\delta}{4})\Big)^{\frac12}\|\{a_i\}\|\geq\Big((1-\delta)\mathbf{c}(\epsilon)\ u^2\ \frac{A}{B}\Big)^{\frac12}\|\{a_i\}\|.
\end{aligned}$$

Clearly,

$$\begin{aligned}
D(a;J) &= D^-(a;J)\\
&\geq \frac{(1-\epsilon')(1-\alpha)^2}{(1+\alpha)^2}\,u^2\,\frac{D^-(a;I)}{\|T\|^2}\geq(1-\epsilon)\,u^2\frac{D^-(a;I)}{\|T\|^2}.
\end{aligned}$$

□

4.2. **Proof of Theorem 2.6, assuming (B).** The only arguments in the proof of Theorem 2.6, assuming (A), that require adjustments are Claim 1 and the subsequent computations.

Let $K,P,\epsilon,\epsilon',\alpha$ be given as in the proof of Theorem 3.10, assuming (A). Choose Q as in (4.8) with $\frac{A}{B}$ replaced by 1. Let $r'\in\ell_1(\mathbb{Z}^d)$ with $|\langle\widetilde{g}_n,\widetilde{g}_{n'}\rangle|\leq r'(n-n')$. Let B' be the optimal Bessel bound of $\{g_n\}$. Choose $R'>0$ so that

$$\Delta_{r'}(R')\frac{\|T\|^2B'}{\mathbf{c}(\epsilon')u^2}K^2(2Q+1)^{2d}<\frac{\delta}{8}.$$

Set $W=2P+R'$. Similarly to (4.9), we increase P so that

$$K\big(W^d-(2(P-Q)+1)^d\big)\leq\alpha u^2\frac{(1-\epsilon')(1-\alpha)}{(1+\alpha)^2}D^-(a;I)(2P+1)^d/\|T\|,$$

while maintaining $W=2P+R'$.

Define J and J_k, $k\in\mathbb{Z}^d$ as done in the proof of Theorem 3.10, assuming (A). Let $x_k=\sum_{j\in J_k}a_jf_{jQ}$, $k\in\mathbb{Z}^d$, and $x=\sum x_k$.

Then

$$\begin{aligned}
\sum_k \sum_{k'\neq k} & \langle x_k, x_{k'}\rangle \\
&= \sum_k \sum_{k'\neq k} \sum_{j\in J_k} \sum_{j'\in J_{k'}} \sum_{n:\|a(j)-n\|\leq Q} \sum_{n':\|a(j')-n'\|\leq Q} a_j \overline{a}_{j'} \langle f_j, g_n\rangle \overline{\langle f_{j'}, g_{n'}\rangle} \langle \widetilde{g}_n, \widetilde{g}_{n'}\rangle \\
&= \sum_k \sum_{k'\neq k} \sum_{j\in J_k} \sum_{j'\in J_{k'}} \sum_{n:\|a(j)-n\|\leq Q} \sum_{n':\|a(j')-n'\|\leq Q} a_j \overline{a}_{j'} \langle f_j, g_n\rangle \overline{\langle f_{j'}, g_{n'}\rangle} M^W_{n,n'} \\
&= \sum_n \sum_{n'} \sum_{j\in J:\|a(j)-n\|\leq Q} \sum_{j'\in J:\|a(j')-n'\|\leq Q} a_j \overline{a}_{j'} \langle f_j, g_n\rangle \overline{\langle f_{j'}, g_{n'}\rangle} M^{R'}_{n,n'} \\
&= \langle S, M^{R'} S\rangle,
\end{aligned}$$

where

$$S = \Big\{ \sum_{j\in J:\|a(j)-n\|\leq Q} a_j \langle f_j, g_n\rangle \Big\}_n.$$

We now compute the norm of S.

$$\begin{aligned}
\|S\|^2_{\ell_2(\mathbb{Z}^d)} &= \sum_n \Big| \sum_{j\in J:\|a(j)-n\|\leq Q} a_j \langle f_j, g_n\rangle \Big|^2 \leq \sum_n \Big| \sum_{j\in J:\|a(j)-n\|\leq Q} |a_j| \|f_j\| \|g_n\| \Big|^2 \\
&\leq \|T\|^2 B' \sum_n \Big| \sum_{j\in J:\|a(j)-n\|\leq Q} |a_j| \Big|^2 \\
&\leq \|T\|^2 B' K(2Q+1)^d \sum_n \sum_{j\in J:\|a(j)-n\|\leq Q} |a_j|^2 \\
&\leq \|T\|^2 B' K(2Q+1)^d \sum_j \sum_{n:\|a(j)-n\|\leq Q} |a_j|^2 \\
&\leq \|T\|^2 B' K(2Q+1)^{2d} \sum_j |a_j|^2 \\
&\leq \|T\|^2 B' K(2Q+1)^{2d} \sum_k \sum_{j\in J_k} |a_j|^2 \\
&\leq \|T\|^2 B' K(2Q+1)^{2d} \sum_k \frac{1}{\mathbf{c}(\epsilon')u^2} \|x_k\|^2 \\
&= \frac{\|T\|^2 B'}{\mathbf{c}(\epsilon')u^2} K(2Q+1)^{2d} \sum_k \|x_k\|^2.
\end{aligned}$$

Here, we used that for each n at most $K(2Q+1)^d$ indices j satisfy $\|a(j)-n\|_\infty \leq Q$, and, for each j there are at most $(2Q+1)^d$ indices n with $\|a(j)-n\|_\infty \leq Q$. Recall that B' is the Bessel bound of $\{g_n\}$ which therefore bounds $\{\|g_n\|\}_n$.

We conclude that for $x_k = \sum_{j\in J_k} a_j f_{jQ}$, $k \in \mathbb{Z}^d$,

$$\begin{aligned}
\big|\sum_{k\neq k'}\langle x_k, x_{k'}\rangle\big| &\leq \|S\|\,\|M^{R'}\|\,\|S\| \\
&\leq \frac{\|T\|^2 B'}{\mathbf{c}(\epsilon')u^2} K(2Q+1)^{2d}\big(\sum_k \|x_k\|^2\big) K\Delta_{r'}(R') \\
&\leq \frac{\delta}{8}\sum_k \|x_k\|^2.
\end{aligned}$$

Now,

$$\begin{aligned}
\|\sum_{j\in J} a_j f_{jQ}\|^2 &= \|\sum x_k\|^2 = \sum_k\sum_{k'}\langle x_k, x_{k'}\rangle \\
&= \sum_k \|x_k\|^2 + \sum_k\sum_{k'\neq k}\langle x_k, x_{k'}\rangle \geq \sum_k \|x_k\|^2 - \sum_k\sum_{k'\neq k}|\langle x_k, x_{k'}\rangle| \\
&\geq \sum_k \|x_k\|^2 - \frac{\delta}{8}\sum_k \|x_k\|^2 \geq \mathbf{c}(\epsilon')u^2(1-\alpha)^2(1-\frac{\delta}{8})\sum_{j\in J}|a_j|^2
\end{aligned}$$

For $\mathcal{F}(J) = \{f_j,\ j\in J\} = \{f_i,\ f_{iQ} \in j \in \mathcal{F}_Q(J)\}$ and $\{a_j\} \in \ell_2(J)$ we compute

$$\begin{aligned}
\|\sum a_j f_j\| &\geq \|\sum a_j f_{jQ}\| - \|\sum a_j(f_j - f_{jQ})\| \\
&\geq \Big(\mathbf{c}(\epsilon')u^2(1-\alpha)^2(1-\tfrac{\delta}{8})\Big)^{\frac{1}{2}}\|\{a_j\}\| - \|(L_I - L_{IQ})^*\{a_j\}\| \\
&\geq \Big(\mathbf{c}(\epsilon')u^2(1-\alpha)^2(1-\tfrac{\delta}{8}) - \frac{\delta\mathbf{c}(\epsilon')u^2}{8}\Big)^{\frac{1}{2}}\|\{a_j\}\| \\
&\geq \mathbf{c}(\epsilon')^{\frac{1}{2}}\Big((1-\frac{\delta}{8})^3 - \tfrac{\delta}{8}\Big)^{\frac{1}{2}} u\|\{a_j\}\| \\
&\geq \mathbf{c}(\epsilon')^{\frac{1}{2}}\Big(1 - 3\tfrac{\delta}{8} - \tfrac{\delta}{8}\Big)^{\frac{1}{2}} u\|\{a_j\}\| \\
&\geq \mathbf{c}(\epsilon')^{\frac{1}{2}}(1-\tfrac{\delta}{2})^{\frac{1}{2}} u\|\{a_j\}\| \geq \Big((1-\delta)\mathbf{c}(\epsilon)u^2\Big)^{\frac{1}{2}}\|\{a_j\}\|.
\end{aligned}$$

□

5. Gabor molecules and the Proof of Theorem 2.8

Similarly to the notion **Gabor system** $(\varphi;\Lambda)$ in Section 2, we define a **Gabor multi-system** $(\varphi^1, \varphi^2, \cdots, \varphi^n; \Lambda^1, \Lambda^2, \cdots, \Lambda^n)$ generated by n functions and n sets of time frequency shifts as the union of the corresponding Gabor systems

$$(\varphi^1;\Lambda^1) \cup (\varphi^2;\Lambda^2) \cup \cdots \cup (\varphi^n;\Lambda^n).$$

Recall that the **short-time Fourier transform** of a tempered distribution $f \in S'(\mathbb{R}^d)$ with respect to a Gaussian window function $g_0 \in S(\mathbb{R}^d)$ is

$$V_{g_0}f(x,\omega) = \langle f, \pi(x,\omega)g_0\rangle = \langle f, M_\omega T_x g_0\rangle, \quad \text{for} \quad \lambda = (x,w) \in \mathbb{R}^{2d}.$$

A **system of Gabor molecules** $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ associated to an enveloping function $\Gamma:\mathbb{R}^{2d}\to\mathbb{R}$ and a set of time frequency shifts $\Lambda\subseteq\mathbb{R}^{2d}$ consists of elements whose short-time Fourier transform have a common envelope of concentration:

$$|V_{g_0}\varphi_{x,\omega}(y,\xi)|\leq\Gamma(y-x,\xi-\omega),\ \text{for all } \lambda=(x,\omega)\in\Lambda,\ (y,\xi)\in\mathbb{R}^{2d}.$$

For $1\leq p\leq\infty$, the **modulation space** $M^p(\mathbb{R}^d)$ consists of all tempered distributions $f\in S'(\mathbb{R}^d)$ such that

$$\|f\|_{M^p}=\|V_{g_0}f\|_{L^p}=\left(\int\int_{\mathbb{R}^{2d}}|\langle f,M_\omega T_x g_0\rangle|^p dx\ dw\right)^{1/p}<\infty \tag{5.13}$$

with the usual adjustment for $p=\infty$. It is known [13] that M^p is a Banach space for all $1\leq p\leq\infty$, and any non-zero function $g\in M^1$ can be substituted for the Gaussian g_0 in (5.13) to define an equivalent norm for M^p. It is known (see [3] Theorem 8 (a)) that in case (φ,Λ) is a frame, $\varphi\in S_0(\mathbb{R}^d)$, then (φ,Λ) is ℓ_1–self-localized.

Theorem 2.8 is a special case of the following, more general result.

Theorem 5.1. *Let $\epsilon,\delta>0$. Let $\{g_\lambda\}_{\lambda\in\Lambda}\subseteq S_0(\mathbb{R}^d)$ be a set of ℓ_1–self-localized Gabor molecules with $\|g_\lambda\|\geq u$ and Bessel bound $B<\infty$. Then exists a set $\Lambda_{\epsilon\delta}\subseteq\Lambda$ so that*

(1) $$\frac{D(\Lambda_{\epsilon\delta})}{D^-(\Lambda)}\geq\frac{(1-\epsilon)}{B}u^2$$

(2) $\{g_\lambda\}_{\lambda\in\Lambda_{\epsilon\delta}}$ *is a Riesz sequence with lower Riesz bound* $\mathbf{c}(\epsilon)(1-\delta)u^2$.

Proof. Set

$$a:\Lambda\longrightarrow\mathbb{Z}^{2d},\quad \lambda\mapsto\arg\min_{n\in\mathbb{Z}^{2d}}\|\lambda-\tfrac{1}{2}n\|_\infty.$$

Now, $D^-(a,\mathbb{Z}^{2d})=2^{-2d}D^-(\Lambda)$. Choose $g\in S_0(\mathbb{R}^d)$ with $\mathcal{G}=(g,\frac{1}{2}\mathbb{Z}^{2d})=\{\pi(\frac{1}{2}n)g\}$ being a tight frame. As $g\in S_0(\mathbb{R}^d)$, we have $(g,\frac{1}{2}\mathbb{Z}^{2d})$ is ℓ_1-self-localized and $(\{\varphi_\lambda\}_\lambda,a,(g,\frac{1}{2}\mathbb{Z}^{2d}))$ is ℓ_1-localized [3].

A direct application of Theorem 3.10, assuming (B), guarantees for each $\epsilon,\delta>0$ the existence of $\Lambda_{\epsilon\delta}\subseteq\Lambda$ with $\{\varphi_\lambda\}_{\lambda\in\Lambda_{\epsilon\delta}}$ is a Riesz sequence and

$$D^-(\Lambda_\epsilon)=2^{2d}D(a;\Lambda_\epsilon)\geq 2^{2d}\frac{(1-\epsilon)}{B}D^-(a;I)=\frac{(1-\epsilon)}{B}D^-(\Lambda). \tag{5.14}$$

□

6. Appendix

We will need a minor extension of Theorem 1.1. Its proof is based on the formulation of Casazza [6] and Vershynin [19].

Theorem 6.1 (**Restricted Invertibility Theorem**). *There exists a continuous and monotone function $\mathbf{c}:(0,1)\longrightarrow(0,1)$ so that for every $n\in\mathbb{N}$ and every linear operator $T:\ell_2^n\to\ell_2^n$ with $\|Te_i\|\geq u$ for $i=1,2,\cdots,n$ and $\{e_i\}_{i=1}^n$ an orthonormal basis for ℓ_2^n, there is a subset $J_\epsilon\subseteq\{1,2,\cdots,n\}$ satisfying*

(1) $\quad \dfrac{|J_\epsilon|}{n} \geq \dfrac{(1-\epsilon)u^2}{\|T\|^2}$, *and*

(2) $\quad \|\sum_{j\in J_\epsilon} b_j Te_j\|^2 \geq \mathbf{c}(\epsilon)\, u^2 \sum_{j\in J_\epsilon} |b_j|^2, \quad \{b_j\}_{j\in J} \in \ell_2(J_\epsilon).$

Proof. Theorem 1.1 does not assert continuity of $\mathbf{c}$. Due to the defining property of $\mathbf{c}$, we can choose $\mathbf{c}$ monotone, and replacing $\mathbf{c}$ with $\mathbf{c}_\zeta$, $\zeta > 0$, with $\mathbf{c}_\zeta(\epsilon) = \int_{\min\{0,\epsilon-\zeta\}}^{\epsilon} \mathbf{c}(\epsilon')\, d\epsilon'$ ensures continuity of $\mathbf{c}$.

Next, we want to replace the traditional assumption $\|Te_i\| = 1$ with $\|Te_i\| \geq u$. Given T, define an operator S by

$$Se_i = \frac{Te_i}{\|Te_i\|}, \quad i = 1, 2, \cdots, n.$$

Now,

$$\Big\|\sum_{i=1}^n a_i Se_i\Big\| \;=\; \Big\|\sum_{i=1}^n \frac{a_i}{\|Te_i\|} Te_i\Big\| \leq \|T\| \left(\sum_{i=1}^n \left|\frac{a_i}{\|Te_i\|}\right|^2\right)^{1/2} \leq \frac{\|T\|}{u}\left(\sum_{i=1}^n |a_i|^2\right)^{1/2}.$$

Hence,

$$\|S\| \leq \frac{\|T\|}{u}.$$

Applying Theorem 1.1 to the operator S with $\|Se_i\| = 1$, $i = 1, \cdots, n$, we obtain $J \subseteq \{1, \cdots, n\}$ with

(1) $\quad |J| \geq \dfrac{(1-\epsilon)u^2}{\|T\|^2} n,$

(2) $\quad \Big\|\sum_{j\in J} b_j Te_j\Big\|^2 = \Big\|\sum_{j\in J} b_j \|Te_j\| Se_j\Big\|^2 \geq \mathbf{c}(\epsilon)^2 \sum_{j\in J} |b_j|^2 \|Te_j\|^2 \geq \mathbf{c}(\epsilon)^2 u^2 \sum_{j\in J} |b_j|^2.$

□

We will also need a simple inequality for Riesz sequences.

Lemma 6.2. *Let $\{f_i\}_{i\in I}$ be a Riesz basis sequence with bounds A, B. Then for any partition $\{I_j\}_{j\in J}$ of I we have for all scalars $\{a_i\}_{i\in I}$,*

$$\frac{A}{B}\sum_{j\in J} \|\sum_{i\in I_j} a_i f_i\|^2 \leq \|\sum_{i\in I} a_i f_i\|^2 \leq \frac{B}{A} \sum_{j\in J} \|\sum_{i\in I_j} a_i f_i\|^2.$$

Proof.

$$\begin{aligned} \frac{A}{B}\sum_{j\in J} \|\sum_{i\in I_j} a_i f_i\|^2 &\leq \frac{A}{B} B \sum_{j\in J}\sum_{i\in I_j} |a_i|^2 = A\sum_{i\in I} |a_i|^2 \leq \|\sum_{i\in I} a_i f_i\|^2 \\ &\leq B\sum_{i\in I}|a_i|^2 = B\sum_{j\in J}\sum_{i\in I_j}|a_i|^2 \leq \frac{B}{A}\sum_{j\in J}\|\sum_{i\in I_j} a_i f_i\|^2 \end{aligned}$$

□

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