Sampling of stochastic operators

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Abstract—We develop sampling methodology aimed at determining stochastic operators that satisfy a support size restriction on the autocorrelation of the operators stochastic spreading function. The data that we use to reconstruct the operator (or, in some cases only the autocorrelation of the spreading function) is based on the response of the unknown operator to a known, deterministic test signal.

Index Terms—autocorrelation,delta train, Gabor frames, Haar property, operator sampling, scattering function, spreading function, stochastic operators, time-frequency analysis

I. INTRODUCTION

IN wireless and wired communication, in radar detection, and in signal processing it is usually assumed that a signal is passed through a filter, whose parameters have to be determined from the output. Commonly, such systems are modeled with a time-variant linear operator acting on a space of signals. For narrowband signals, we can model the effects of Doppler shifts and multi-path propagation as the sum of few time-frequency shifts that are applied to the sent signal. In general, the channel consists of a continuum of time-frequency scatterers: the channel is formally represented by an operator with a superposition integral

$$(Hf)(x) = \iint \eta(t,\nu) \, M_\nu T_t \, f(x) \, \mathrm{d}t \, \mathrm{d}\nu, \qquad (1)$$

where T_t is a *time-shift* by t, that is, $T_t f(x) = f(x-t)$, $t \in \mathbb{R}$, M_ν is a *frequency shift* or *modulation* given by $M_\nu f(x) = e^{2\pi i \nu x} f(x)$, $\nu \in \mathbb{R}$. Taking Fourier transforms, it follows that $\widehat{M_\nu f}(\xi) = \hat{f}(\xi - \nu) = T_\nu \widehat{f(\xi)}$ for all $\xi \in \mathbb{R}$. The function $\eta(t,\nu)$ is called a *(Doppler-delay) spreading function* of H. To *identify* the operator means to determine the spreading function η of H from the response $Hf(x)$ of the operator to a given sounding signal $f(x)$. The not necessarily rectangular support of the spreading function is known as the *occupancy pattern*, and its area as *spread* of the operator H. The fundamental restriction for the spread to be less than one (rather than the area of the *bounding box* of the occupancy pattern) has been shown to be necessary and sufficient for the identifiability of channels. This extension of the class of "underspread", hence identifiable, channels is particularly of interest in the field of sonar communication [1] and in the multiple input-single output channel settings. Acoustic channels possess larger spreads than radar and wireless channels. This is due to the speed of sound being magnitudes

lower than that of electromagnetic waves. This leads to time delays up to several seconds and Doppler spreads in the tens of Hertz for high-frequency channels [2]. Another type of channels with large values for the area of the occupancy pattern are multiple input – single output channels; they combine several deterministic spreading functions into one channel, thus covering a larger region of the time-frequency plane [3]. In recent work, a sampling theory for operators has been developed [4, 5]. It establishes identifiability by providing concrete reconstruction formulas for operators that satisfy the above mentioned constraints. In this paper, we develop operator sampling in the stochastic setting. Taking into account the random nature of the real-world communication environments, we model such channels with stochastic time-variant operators [6, 7, 8, 9]. In this setting, the spreading function $\boldsymbol{\eta}(t,\nu)$ of the operator $\boldsymbol{H}$ in (1) is a random process[†] indexed by (t,ν) that is to be recovered from the output $\boldsymbol{H}f(x)$. For the purposes of this paper, it would be enough to think of $\boldsymbol{\eta}(t,\nu;\omega)$ as a (t,ν)-indexed family of random variables in a common probability space Ω. The fine points of the underlying mathematical analysis will be indicated in few side remarks. We elaborate these points in [10] using the theory of modulation spaces[‡]. There, we rigorously define and prove functional analytic identification results for operators with spreading functions belonging to the class of generalized random processes.

A. Operator sampling theory in the historical perspective

The progress of operator identification theory largely follows the evolution of the related theory of function sampling. The two major directions of generalization are the introduction of stochasticity (stationary and general) and the removal of the requirement for the "bandlimitation" to be rectangular. To simplify the orientation, we summarize this development. For example, building up on the classical Shannon-Nyquist sampling theorem, a corresponding result for bandlimited stationary stochastic processes was proven in a more general form in [13]. We cite it here in the form it is given in a classic book of Papoulis.

Theorem 1 ([18, page 378]). *If a stationary process $\boldsymbol{x}(t)$ is bandlimited, that is, its power spectral density $S(\omega) = \mathcal{F}_{\tau \to \omega} \mathbb{E}\{\boldsymbol{x}(t)\overline{\boldsymbol{x}(t+\tau)}\}$ is integrable and $\operatorname{supp} S(\omega) \subset [-\Omega/2, \Omega/2]$, then we can recover $\boldsymbol{x}(t)$ in the mean-square*

[†]Here, and in remainder of the paper, random functions and operators are denoted by boldface characters, $\overline{x}$ is a complex conjugate of x, x^* conjugate transpose, and $\mathbb{E}$ expectation.

[‡]To accommodate delta functions and other generalized functions, we let $\boldsymbol{\eta}(t,\nu)$ to be a mapping from the dual $M^\infty(\mathbb{R}^2)$ of the Feichtinger algebra $M^1(\mathbb{R}^2)$ into the Hilbert space of zero-mean random variables $\mathrm{RV}(\Omega)$.

			rectangular	non-rectangular
Sampling of	functions	Deterministic	Shannon [11]	Kluvánek [12]
		Stochastic stationary	Lloyd [13]	Lloyd [13]
		Stochastic	Lee [14]	Lee [14]
	operators	Deterministic	Kozek and Pfander [15]	Pfander and Walnut [16], Pfander [5]
		Stochastic stationary	Oktay, Pfander, and Zheltov [17]	Theorem 12
		Stochastic	Theorem 7	Theorem 11

TABLE I: Development of function and operator sampling.

sense from the samples taken at rate $T = \Omega^{-1}$. *In fact,*

$$\boldsymbol{x}(t) = \underset{N \to \infty}{\text{l.i.m.}} \sum_{n=-N}^{N} \boldsymbol{x}(nT) \frac{\sin \pi \Omega(t - nT)}{\pi \Omega(t - nT)}.$$

The requirement of stationarity and the bandlimitation of the spectrum to the symmetric interval were later relaxed [13, 19, 14]. In 1963, Kailath realized that for a deterministic time-variant channel to be identifiable, it is sufficient that the product ΩT of the maximum time delay T and maximum Doppler spread Ω is not greater than 1 [7]. Since, channels were called *underspread* whenever $\Omega T < 1$, and *overspread* if $\Omega T > 1$ [20]. The insight of Kailath has been generalized and made precise by Kozek and Pfander [15]. Following in Kailath footsteps, the seminal paper of Bello lays the groundwork for channel sampling and characterization tools and vocabulary [8]. In the sequel [9] Bello further argues that it is not the product ΩT that matters for identification of a deterministic time-variant channel, but rather the *spread*, or the area of what he calls an *occupancy pattern*, that is, the not necessarily rectangular support of the spreading function supp η. In particular, Bello's conjecture has been put into a rigorous mathematical framework and was proven using tools from Gabor analysis in [16]. Also, the ideas developed there were used to estimate the capacity of the channels with non-rectangular spread [21]. A brief comment of Kailath in [7] suggests sufficiency of $\Omega T \leq 1$ for the identification of the stationary stochastic channels as well as deterministic, and Bello treats this question as a side matter, more interested in developing the estimator for $\eta(t, \nu)$ when the output has been contaminated by additive noise. Just like the final results of Lee were predated by the work of Lloyd [13], who gave a sampling theorem for wide-sense stationary stochastic processes with non-rectangular support, most research on identification of stochastic operators has been done within the assumptions of a simpler stationary model: the channel has the property of *wide-sense stationarity with uncorrelated scattering* (WSSUS), that is, the autocorrelation function of $\eta(t, \nu)$ has the form

$$R_{\boldsymbol{\eta}}(t, \nu; t', \nu') = \mathbb{E}\big\{\boldsymbol{\eta}(t, \nu) \overline{\boldsymbol{\eta}(t', \nu')}\big\}$$
$$= \delta(t - t') \, \delta(\nu - \nu') \, C_{\boldsymbol{\eta}}(t, \nu).$$

In other words, taps at different delays are uncorrelated and stationary. The function $C_{\boldsymbol{\eta}}(t, \nu)$ is known as the *scattering*

function of $\boldsymbol{H}$. It completely characterizes the second-order statistics of $\boldsymbol{\eta}$ and represents the power spectral density of the transfer function of the channel. This means that the scattering function represents the expected behavior of the operator. Two common types of methods to identify the scattering function are deconvolution and direct measurement methods [22, 23, 24]. In [17], we used the WSSUS assumption and offered an alternative method for the reconstruction of the scattering function by sounding the channel with a weighted delta train, using Zak transform methods. In this paper, we pursue this approach further in order to address a more general problem of stochastic spreading function reconstruction and stochastic operator sampling and identification. It turns out that the key criterion for identifiability is the 4-dimensional volume of the support of the autocorrelation of the spreading function. Hence, our object of analysis will be the class $\text{StOPW}(M)$ of operators with M indicating the support of the autocorrelation of the spreading function.

B. Overview of the paper

Stochastic or not, it is easy to see that the deterministic reconstruction formula discussed in Section II-A below allows recovery of a *sample spreading function* from the echo whenever its support has area less than one [16]. However, on its own, a sample will provide little information on the average behavior. Secondly, it is possible that the 4D volume of the support of the 4D autocorrelation function $R_{\boldsymbol{\eta}}(t, \nu; t', \nu')$ is less than one, while some samples $\boldsymbol{\eta}(t, \nu; \omega)$ have 2D area greater than one. Our contributions are as follows. In Section II-C we consider the case of the 4D region supporting $R_{\boldsymbol{\eta}}(t, \nu; t', \nu')$ representable as a tensor product of a 2D region S with itself. In this scenario, we prove that it is possible to recover the stochastic spreading function of the channel in the mean-square sense provided that the support S of the stochastic spreading function supp $\boldsymbol{\eta}(t, \nu)$ occupies a region of area less than one. We provide an explicit reconstruction formula (10) from the echo of a specially constructed weighted delta train sounding signal. This case will include as a special case the deterministic operators as their autocorrelation functions satisfy $R_{\boldsymbol{\eta}}(t, \nu; t', \nu') = \eta(t, \nu) \overline{\eta(t', \nu')}$. In Section II-D the case of an arbitrary support region $M = \text{supp}\, R_{\boldsymbol{\eta}}(t, \nu; t', \nu')$ is considered. We give a criterion for identifiability of the family $\text{StOPW}(M)$ and discover regions of 4D volume less

than one that are nonetheless unidentifiable by our methods. Section III discusses connections of the operator identification theory to finite frame theory. In the theory of deterministic operator identification, which is mentioned above as a particular case of the stochastic operator with a degenerate probability distribution, the proofs pivot on the so-called *Haar property* of Gabor frames. A finite-dimensional Gabor frame is defined as

$$G = \{M^n T^k c\}_{k,n=0}^{L-1}, \tag{2}$$

where the finite-dimensional translation operators T^k and modulation operators M^n operating on a vector $c \in \mathbb{C}^L$ are given by

$$T^k c[p] = c[p-k] \quad \text{and} \quad M^n c[p] = e^{2\pi i n p / L} c[p]. \tag{3}$$

A frame has the *Haar property* whenever its elements are in general linear position, that is, any subset of L elements is linearly independent. A finite-dimensional Gabor frame G generated by window c as defined in (2), has the Haar property for almost every choice of the window c [25]. Application of our methods to the stochastic case spawns a more peculiar Gabor frame on $\mathbb{Z}_L \times \mathbb{Z}_L$, the properties of which differ in the key respect of linear independence of its subsets. In particular, such a frame has small subsets that are linearly dependent for any choice of window c. We will show that the support patterns of the the autocorrelation functions of the operators are in one-to-one correspondence with the column subsets of such Gabor frames. We completely classify autocorrelation patterns for $L = 2$ and analyze several phenomena that emerge for general L.

II. OPERATOR SAMPLING

A. Deterministic operators

Equivalently to (1), any operator H acting on one-variable signals can be represented with its symbols,

1) its *time-varying impulse response* $h(x,t) = \int \eta(t,\nu)\, e^{-2\pi i \nu(x-t)}\, d\nu$, then
$$(Hf)(x) = \int h(x,t) f(x-t)\, dt,$$

2) its *kernel* $\kappa(x,t) = h(x, x-t) = \int \eta(x-t, \nu) e^{-2\pi i \nu t} d\nu$, then
$$(Hf)(x) = \int \kappa(x,t) f(t)\, dt,$$

3) and its *Kohn-Nirenberg symbol* $\sigma(x,\xi) = \mathcal{F}_{t\to\xi} h(x,t)$, then
$$(Hf)(x) = \int \sigma(x,\xi) \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi.$$

All these symbols of H can be transformed into each other using partial Fourier transforms normalized by $\mathcal{F}f(\xi) = \hat{f}(\xi) = \int \boldsymbol{f}(t)\, e^{-2\pi i \xi t}\, dt$ and area-preserving shears, such as $\mathcal{I}_2 f(x,t) = f(x+t, x)$ [26]. In particular, the spreading function and the Kohn-Nirenberg symbol are related through

$$\sigma(x,\xi) = \mathcal{F}_s \eta(t,\nu),$$

where the *symplectic Fourier transform* is given by

$$(\mathcal{F}_s \eta)(x,\xi) = \iint \eta(t,\nu)\, e^{-2\pi i (\nu x - \xi t)}\, dt\, d\nu.$$

In the following, it will sometimes be advantageous to present the results with h, κ, or σ instead of η and we will not hesitate to do so. However, we formulate our results primarily with η due to its particularly simple relationship to the short-time Fourier transform. On the Schwartz dual space $\mathcal{S}'$ of tempered distributions, we define the *short-time Fourier transform* to be

$$V_\varphi f(t,\nu) = \langle f, M_\nu T_t \varphi \rangle$$

for any φ in the Schwartz space $\mathcal{S}$. Then for all $f, \varphi \in \mathcal{S}$ we have the useful equality

$$\langle Hf, \varphi \rangle = \langle \eta, V_f \varphi \rangle.$$

The inner product $\langle \cdot, \cdot \rangle$ is taken to be conjugate linear in the second variable. We will say that H belongs to an *operator Paley-Wiener space* if the support of the spreading function $\eta(t,\nu) = \mathcal{F}_s \sigma(x,\xi)$ is contained in a subset of the time-frequency plane, usually taken to be compact,

$$\mathrm{OPW}(S) = \{H : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R}),$$
$$\sigma \in L^2(\mathbb{R}^2), \quad \mathrm{supp}\, \mathcal{F}_s \sigma \subseteq S\}.$$

Colloquially, we refer to H as *"bandlimited"* to S. The connection of the operator sampling theory to the more established function sampling is best observed on the following theorem for Hilbert-Schmidt operators with the spreading function supported on a rectangle in the time-frequency plane. Note that if H is a multiplication operator with a bandlimited multiplier, Theorem 2 reduces to the classic Shannon sampling theorem.

Theorem 2. *[5, Theorem 1.2] For $H : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ Hilbert-Schmidt (that is, $\eta(t,\nu) \in L^2(\mathbb{R}^2)$) such that $\mathrm{supp}\, \eta(t,\nu) \subseteq [0,T) \times [-\Omega/2, \Omega/2]$ and $T\Omega \leq 1$ we have*

$$\left\| H \sum_{k\in\mathbb{Z}} \delta_{kT} \right\|_{L^2} = T \|\eta\|_{L^2},$$

and H can be reconstructed by

$$\kappa(x+t, x) = \sum_{n\in\mathbb{Z}} (H\mathrm{m})(t + nT) \frac{\sin \pi T(x-n)}{\pi T(x-n)},$$

with convergence in $L^2(\mathbb{R}^2)$.

Remark. Functional analytic arguments show that whenever $\eta(t,\nu)$ is compactly supported (a physically reasonable assumption as discussed in [15]) it is possible to extend the domain of H from L^2 to the whole of M^∞.[†] Here, the family of *modulation spaces* M^p is defined via the finiteness of the norms $\|f\|_{M^p} = \|V_g f\|_{L^p} < \infty$, where L^p is the space of functions whose p-th power is integrable [26]. Note that M^∞ is the analytic dual space of bounded linear functionals on the Feichtinger algebra $M^1 = \mathcal{S}_0 \supset \mathcal{S}$. It is therefore well-defined as a linear functional on $W(A(\mathbb{R}^{2d}), \ell^\infty)$. This allows us to use certain tempered distributions, in particular, the delta trains $\mathrm{m}_c = \sum_{k\in\mathbb{Z}} c_k \delta_{kT}$ as input for H. A reader unfamiliar with modulation spaces should ignore the functional analytic subtleties and think of functions in $L^2(\mathbb{R}) \subset M^\infty$.

[†]In fact, Pfander and Walnut show that if $\eta(t,\nu) \in M^\infty$ and $\eta(t,\nu)$ has compact distributional support, then it also belongs to a Wiener amalgam space $W(\mathcal{F}L^\infty(\mathbb{R}^{2d}), \ell^1)$ [27].

A deeper result provides guarantees for recovery of the operator whose spreading function is supported on a set of arbitrary shape, not just a rectangle, as long as the area of the support is less than 1. Additionally, it is possible to replace $\sin x/x$ with smooth functions with better decay through the use of partitions of unity generated by continuous functions. For $a, \varepsilon > 0$ we say $r(t)$ generates an (a, ε)-*partition of unity* $\{r(t + ak)\}_{k \in \mathbb{Z}}$ whenever

$$r(t) = 0 \text{ if } t \notin (-\varepsilon, a + \varepsilon) \quad \text{and} \quad \sum_{k \in \mathbb{Z}} r(t + ak) = 1.$$

Definition 3. The set S is (a, b, Λ)-*rectified* if it can be covered by $L = \frac{1}{ab}$ translations $\Lambda = \{(k_j, n_j)\}_{j=0}^{L-1}$ of the rectangle $\square = [0, a) \times [0, b)$ along the lattice $a\mathbb{Z} \times b\mathbb{Z}$:

$$S \subset \bigcup_{j=0}^{L-1} \square + (ak_j, bn_j).$$

Lemma 4. *For any compact set S of measure $\mu(S) < 1$ contained within $[0, T) \times [-\Omega/2, \Omega/2]$ there exist $a, b > 0$ and Λ such that S is (a, b, Λ)-rectified, $L = \frac{1}{ab}$ is prime, $a < \frac{1}{T}$ and $b < \frac{1}{\Omega}$.*

Proof: This is a standard result from a theory of Jordan domains. A proof can be found in [28]. Any compact set with Jordan content less than one can be covered with L rectangles of cumulative area $L \cdot \frac{1}{L} = 1$, and it is easy to see that we do not lose generality requiring a, b to be small enough to satisfy $a < 1/T$, $b < 1/\Omega$ and L prime. ∎

Theorem 5. *[16, Theorem 1.7], [4] Let $S \subset [0, T) \times [-\Omega/2, \Omega/2] \subset \mathbb{R}^2$ be a compact set with measure $\mu(S) < 1$, such that S is (a, b, Λ)-rectified as in Definition 3 (with ΩT not necessarily smaller than 1). Then there a vector $c \in \mathbb{C}^L$, and the identifier signal*

$$\text{Ш}_c = \sum_n c_{n \bmod L} \delta_{n/L},$$

such that for any $H \in \text{OPW}(S)$

$$h(x + t, t) = aL \sum_{j=0}^{L-1} \sum_{q \in \mathbb{Z}} a_{j,q} (H\text{Ш}_c)(x - a(k_j + q))$$
$$\times r(x - ak_j)\varphi(t + a(k_j + q)) e^{2\pi i b n_j t}$$

unconditionally in $L^2(\mathbb{R}^2)$. Here, the coefficients $a_{j,k}$ are uniquely determined by the choice of $\{c_n\}$, and $r(t)$, $\varphi(t)$ are any functions such that $r(t - ak)$ and $\widehat{\varphi}(\gamma - bn)$ are (a, ε)- (resp., (b, ε)-) partitions of unity in time (resp., frequency) domains, with $\varepsilon > 0$ dependent on S.

Since we can always rectify a compact S with measure $\mu(S) < 1$, Theorem 5 holds for a wider class of regions, namely, of all regions S whose so-called Jordan outer content is less than one [16]

B. Stochastic operator Paley-Wiener spaces

Let H be a stochastic operator with integral representation

$$Hf = \iint \eta(t, \nu) \, M_\nu T_t \, f \, \mathrm{d}t \, \mathrm{d}\nu,$$

and stochastic *spreading function* $\eta(t, \nu)$ a zero-mean random process such that $\eta(t, \nu; \omega) \in L^2(\mathbb{R}^2)$ for all $\omega \in \Omega$. If the autocorrelation of the spreading function given by

$$R_{\eta}(t, \nu; t', \nu') = \mathbb{E}\{\eta(t, \nu) \overline{\eta(t', \nu')}\}$$

is supported on a closed set $M \subseteq \mathbb{R}^4$, we call H a *stochastic Paley-Wiener operator bandlimited* to M, or $H \in \text{StOPW}(M)$. In this section, we always assume M already rectified, as done in the deterministic case. The symmetries of the autocorrelation function require a symmetric rectification, defined below. However, it will not cause any confusion that whenever we speak of a rectified 4D region, we always mean a symmetrically rectified one.

Definition 6. The set M is (a, b, Λ)-*symmetrically rectified* if it can be covered by the translations $\Lambda = \{\lambda_j = (k_j, n_k, k'_j, n'_j)\}_{j=0}^{L^2-1}$ of the prototype parallelepiped $\square^2 = [0, a) \times [0, b) \times [0, a) \times [0, b)$ along the lattice $(a\mathbb{Z} \times b\mathbb{Z})^2$:

$$M \subset \bigcup_{j=0}^{L^2-1} \square^2 + (ak_j, bn_j, ak'_j, bn'_j)$$

such that the 4D volume of $\square^2$ is small: $ab = \frac{1}{L}$, L is prime, and the right-hand side is a symmetric set.

It is easy to see that the above requirements imply that the volume of M satisfies $\mu(M) \leq 1$, and conversely, it can be shown with little work that any symmetric compact set with Jordan content less than one can be rectified with a sufficiently large L [28]. This restriction on the area is crucial for operator identification. Before giving results for general operators whose autocorrelation functions are supported on arbitrary sets $M \subset \mathbb{R}^4$, we look at the sets of a special kind, where $M = S \times S$. A general case will be considered in Section II-D below.

C. Sampling of stochastic operators supported on $S \times S$

Under the special circumstance that M can be represented as a product $S \times S$ of some set S in $\mathbb{R}^2$, or rather, if we assume M to be (a, b, Λ)−symmetrically rectified, $\Lambda = \Gamma \times \Gamma$ for some $\Gamma \subset \mathbb{Z} \times \mathbb{Z}$ such that $|\Gamma| = L$, the operator (via its spreading function) can be reconstructed directly from the output of the weighted delta train input with an explicit formula similar to the one presented by Pfander and Walnut [29]. We define the *non-normalized Zak transform* $\mathcal{Z} : L^2(\mathbb{R}) \mapsto L^2([0, aL] \times [0, b))$ as

$$\mathcal{Z}_{aL} f(x, \nu) = \sum_{n \in \mathbb{Z}} f(x - anL) \, e^{2\pi i aLn\nu}.$$

Theorem 7. *Let $H \in \text{StOPW}(S \times S)$ such that the compact set $S \subset \mathbb{R}^2$ has measure $\mu(S) < 1$. Then there exist L prime, $a, b > 0$ with $ab = \frac{1}{L}$, a complex vector $c \in \mathbb{C}^L$ and a sequence $\{\alpha_{jp}\}_{j,p=1}^L$ depending only on c such that we can reconstruct H from its response to an L−periodic c−weighted*

delta train $\text{III}_c = \sum_{k\in\mathbb{Z}} c_{k \ (\mathrm{mod}\ L)}\,\delta_{ak}$ *via*

$$\boldsymbol{\eta}(t,\nu) \overset{m.s.}{=} aL \sum_{j=0}^{L-1}\sum_{p=0}^{L-1} \alpha_{jp}\, e^{-2\pi i a(\nu-bn_j)(p+k_j)}\,\widetilde{\chi}_\square(t,\nu)$$
$$\times\,(\mathcal{Z}_{aL}\boldsymbol{H}\text{III}_c)\,(t+a(p-k_j),\nu-bn_j),$$

where the translations $\widetilde{\chi}_\square(t+ak,\nu+bn)$ generate an (a,b,ε)-partition of unity in the time-frequency domain.

This means that it is always possible to find c such that any two operators from $\mathrm{StOPW}(S\times S)$ can be distinguished from their response to the pilot signal $\text{III}_c = \sum_{k\in\mathbb{Z}} c_{k\ (\mathrm{mod}\ L)}\,\delta_{ak}$.

Proof: Let $a,b>0$, L – prime with $ab=\frac{1}{L}$ be fixed numbers to be chosen later. This choice will depend on the support of $\boldsymbol{\eta}(t,\nu)$. Let $c\in\mathbb{C}^L$; indices of c should always be understood modulo L.

$$\boldsymbol{H}\text{III}_c(x) = \iint \boldsymbol{\eta}(t,\gamma)\,e^{2\pi i\gamma x}\,\text{III}_c(x-t)\,\mathrm{d}t\,\mathrm{d}\gamma$$
$$= \iint \boldsymbol{\eta}(t,\gamma)\,e^{2\pi i\gamma x}\sum_{k\in\mathbb{Z}}c_k\delta(x-t-ak)\,\mathrm{d}t\,\mathrm{d}\gamma$$
$$= \int \sum_{k\in\mathbb{Z}}c_k\,\boldsymbol{\eta}(x-ak,\gamma)\,e^{2\pi i\gamma x}\,\mathrm{d}\gamma$$
$$= \sum_{k\in\mathbb{Z}}c_{k+p}\int \boldsymbol{\eta}(x-a(k+p),\gamma)\,e^{2\pi i\gamma x}\,\mathrm{d}\gamma,$$

with all the equalities holding ω-surely, and the last equality being true for arbitrary $p\in\mathbb{Z}$. Applying Zak transform to both sides, we get

$$\mathcal{Z}_{aL}(\boldsymbol{H}\text{III}_c)(x,\nu)$$
$$= \sum_{n\in\mathbb{Z}}\boldsymbol{H}\text{III}_c(x-anL)\,e^{2\pi ianL\nu}$$
$$= a\sum_{k,n\in\mathbb{Z}}c_{k+p}\int\boldsymbol{\eta}\big(x-a(k+p+nL),\gamma\big)e^{2\pi i(anL\nu+\gamma(x-anL))}\mathrm{d}\gamma$$

substitute $k = k+nL$

$$= \sum_{k\in\mathbb{Z}}c_{k+p}\int\boldsymbol{\eta}\big(x-a(k+p),\gamma\big)e^{2\pi i\gamma x}\sum_{n\in\mathbb{Z}}e^{2\pi ianL(\nu-\gamma)}\,\mathrm{d}\gamma$$

by Poisson summation formula $\sum e^{2\pi inqx}=q^{-1}\sum\delta(qx-n)$ with $q=aL={1}/{b}$

$$= \sum_{k\in\mathbb{Z}}c_{k+p}\int\boldsymbol{\eta}\big(x-a(k+p),\gamma\big)e^{2\pi i\gamma x}\frac{1}{aL}\sum_{n\in\mathbb{Z}}\delta\big(aL(\nu-\gamma)-n\big)\mathrm{d}\gamma$$
$$= \sum_{k\in\mathbb{Z}}c_{k+p}\int\boldsymbol{\eta}\big(x-a(k+p),\gamma\big)e^{2\pi i\gamma x}\,b\sum_{n\in\mathbb{Z}}\delta(\nu-\gamma-bn)\,\mathrm{d}\gamma$$

carry out integration in γ and set $n=-n,k=-k$

$$= b\sum_{k,n\in\mathbb{Z}}c_{p-k}\int\boldsymbol{\eta}(x-a(p-k),\nu+bn)\,e^{2\pi i(\nu+bn)x}.$$

Substitute $t=x-ap$ and observe in the exponent $(\nu+bn)x = \nu(t+ap)+bnt+\frac{n}{L}p$:

$$\boldsymbol{\mathcal{Z}}_p(t,\nu) = b^{-1}\,e^{-2\pi i\nu(t+ap)}\,\mathcal{Z}_{aL}\boldsymbol{H}\text{III}_c(t+ap,\nu)$$
$$= \sum_{k,n\in\mathbb{Z}}c_{p-k}\,e^{2\pi ipn/L}\,\underbrace{\boldsymbol{\eta}(t+ak,\nu+bn)\,e^{2\pi ibnt}}_{\boldsymbol{\eta}_{k,n}(t,\nu)}.$$
$$(4)$$

For all $t,\nu\in\square$ and all $\omega\in\Omega$ we then have a mixing matrix equation

$$\boldsymbol{\mathcal{Z}}(t,\nu) = \sum_{k,n\in\mathbb{Z}}M^nT^kc\,\boldsymbol{\eta}_{k,n}(t,\nu), \qquad (5)$$

where $\boldsymbol{\mathcal{Z}}(t,\nu)=[\boldsymbol{\mathcal{Z}}_p(t,\nu)]_{p=0}^{L-1}$ is a vector-valued function on $(t,\nu)\in\square$. By Lemma 4, the set S can be $(a,b,\Gamma)-$rectified, that is, there exists a collection of indices $\Gamma=\{(k_j,n_j)\}_{j=0}^{L-1}$ such that $\boldsymbol{\eta}_{k,n}(t,\nu)\equiv 0$ on $(t,\nu)\in\square$ for any $(k,n)\notin\Gamma$. Therefore, by Lemma 8 below, the infinite sums in (5) can be trimmed to finite sums to obtain

$$\boldsymbol{\mathcal{Z}}(t,\nu) = \sum_{k,n\in\mathbb{Z}}M^nT^kc\,\boldsymbol{\eta}_{k,n}(t,\nu)$$
$$\overset{m.s.}{=} \sum_{(k,n)\in\Gamma}M^nT^kc\,\boldsymbol{\eta}_{k,n}(t,\nu)$$
$$= G\big|_\Gamma\,\boldsymbol{\eta}_\Gamma(t,\nu), \qquad (6)$$

where G is the $L\times L^2$ Gabor matrix of all time-frequency shifts of $c\in\mathbb{C}^L$ given by (2), $G\big|_\Gamma$ is the submatrix of G corresponding to columns indexed by $\Gamma \bmod L$, and

$$\boldsymbol{\eta}_\Gamma(t,\nu) = [\boldsymbol{\eta}_{k_j,n_j}(t,\nu)]_{(k_j,n_j)\in\Gamma}$$

is a column vector of nonzero patches of $\boldsymbol{\eta}(t,\nu)$ defined by (4). Without loss of generality, no two indices in Γ correspond to the same column of G, that is, $(k,n)\neq(k',n')\bmod L\in\Gamma$ for all $(k,n)\neq(k',n')\in\Gamma$. (Such collisions can always be avoided by choosing a different rectification (a',b',Γ') such that the entire support set S is within $[0,a'L')\times[0,b'L')$, which can always be achieved using Lemma 4. By assumption, $|\Gamma|\leq L$, therefore, $G\big|_\Gamma$ is invertible for some complex vector $c\in\mathbb{C}^L$.[†] Denote

$$A = \big(G\big|_\Gamma\big)^{-1} = [\alpha_{jp}]_{j,p=0}^{L-1}$$

the inverse of $G\big|_\Gamma$. The solution to (6) can now be found, giving

$$\boldsymbol{\eta}_{k_j,n_j}(t,\nu) \overset{m.s.}{=} \big(G\big|_\Gamma\big)^{-1}\boldsymbol{\mathcal{Z}}(t,\nu)$$
$$= aL\,e^{-2\pi i\nu t}\sum_{p=0}^{L-1}\alpha_{jp}\,e^{-2\pi i\nu ap}\mathcal{Z}_{aL}\boldsymbol{H}\text{III}(t+ap,\nu).$$

We can now combine the patches $\boldsymbol{\eta}_{k,n}(t,\nu)$ into the whole spreading function $\boldsymbol{\eta}(t,\nu)$. In the mean-square sense we have

$$\boldsymbol{\eta}(t,\nu) \overset{m.s.}{=} \sum_{j=0}^{L-1}\boldsymbol{\eta}(t,\nu)\widetilde{\chi}_\square(t-ak_j,\nu-bn_j)$$

[†]In fact, the set of $c\in\mathbb{C}^L$ such that *every* $L\times L$ submatrix of G is invertible is a dense open subset of $\mathbb{C}^L$ [25]. Furthermore, all entries of c can be additionally chosen to have absolute value one.

$$\overset{m.s.}{=} \sum_{j=0}^{L-1} \boldsymbol{\eta}_{k_j,n_j}(t-ak_j, \nu-bn_j)\, e^{-2\pi i b n_j(t-ak_j)}$$
$$\times \widetilde{\chi}_\square(t-ak_j, \nu-bn_j)$$
$$\overset{m.s.}{=} aL \sum_{j=0}^{L-1}\sum_{p=0}^{L-1} \alpha_{jp}\, e^{-2\pi i(\nu(t+a(p-k_j))-bn_j ap)}$$
$$\times \widetilde{\chi}_\square(t,\nu) \cdot \mathcal{Z}_{aL}\boldsymbol{H}\mathrm{m}_c(t+a(p-k_j), \nu-bn_j). \qquad \blacksquare$$

Lemma 8. *Let $\boldsymbol{\eta}(t,\nu)$ be such that $\operatorname{supp} R_{\boldsymbol{\eta}}(t,\nu;t',\nu') \subset S \times S$ with $S \subset \mathbb{R}^2$ that is (a,b,Γ)-rectified. Then for all $(t,\nu) \in [0,a) \times [0,b)$,*

$$\sum_{k=0}^{L-1} \sum_{l,n \in \mathbb{Z}} a_{k,n,l}(t,\nu)\boldsymbol{\eta}(t-a(lL+k), \nu+bn)$$
$$\overset{m.s.}{=} \sum_{\gamma \in \Gamma} a_\gamma(t,\nu)\boldsymbol{\eta}_\gamma(t,\nu),$$

where $\boldsymbol{\eta}_\gamma(t,\nu) = \boldsymbol{\eta}(t-a(lL+k), \nu+bn)$.

Proof: Denote $Q = \mathbb{Z}/L\mathbb{Z} \times L\mathbb{Z} \times \mathbb{Z}$ the set of all indices on the left hand side.

$$\mathbb{E}\left\{|\sum_{\gamma \in Q} a_\gamma \boldsymbol{\eta}_\gamma - \sum_{\gamma \in \Gamma} a_\gamma \boldsymbol{\eta}_\gamma|^2\right\} = \sum_{\gamma_1,\gamma_2 \in Q \setminus \Gamma} a_{\gamma_1}\overline{a_{\gamma_2}}\, \mathbb{E}\left\{\boldsymbol{\eta}_{\gamma_1}\overline{\boldsymbol{\eta}_{\gamma_2}}\right\}.$$

The rectification of $R_{\boldsymbol{\eta}}$ guarantees that

$$\mathbb{E}\left\{\boldsymbol{\eta}_\gamma(t_1,\nu_1)\overline{\boldsymbol{\eta}_{\gamma_2}(t_2,\nu_2)}\right\} = 0$$

whenever either γ_1 or γ_2 are not in Γ. We conclude

$$\sum_{\gamma \in Q} a_\gamma \boldsymbol{\eta}_\gamma(t,\nu) \overset{m.s.}{=} \sum_{\gamma \in \Gamma} a_\gamma \boldsymbol{\eta}_\gamma(t,\nu). \qquad \blacksquare$$

Remark. Since the autocorrelation functions of WSSUS operators have the special form

$$R_{\boldsymbol{\eta}}(t,\nu;t',\nu') = \delta(t-t')\,\delta(\nu-\nu')\,C_{\boldsymbol{\eta}}(t,\nu),$$

Theorem 7 is applicable to WSSUS operators whenever the area of the support of the scattering function (also known as the *spread* of the operator) satisfies $\mu(\operatorname{supp} C_{\boldsymbol{\eta}}(t,\nu)) < 1$. However, it is intuitively clear that it is excessive to cover a 4D diagonal (a set of measure zero in $\mathbb{R}^4$) with a single 4D parallelepiped of volume one. Indeed, covering it with a string of small parallelepipeds (as in Figure 2) is enough for identification and allows recovery of $C_{\boldsymbol{\eta}}$ for $\operatorname{supp} C_{\boldsymbol{\eta}}$ bounded, of arbitrary size. This comes as a corollary from Theorem 12 and Theorem 15

D. Stochastic operators with non-tensored support

In this section we proceed to the most general case of stochastic operators. Consider an operator $\boldsymbol{H}$ with a spreading function $\boldsymbol{\eta}(t,\nu)$ such that the autocorrelation function $R_{\boldsymbol{\eta}}(t,\nu;t',\nu')$ is supported on some arbitrary bounded set M in $\mathbb{R}^4$. We will see that in general it is no longer possible to reconstruct $\boldsymbol{\eta}(t,\nu)$ itself. For instance, this happens because with nonzero probability some samples $\boldsymbol{\eta}(t,\nu;\omega)$ may have

spread larger than one, violating the necessary condition of the deterministic Theorem 5. Nevertheless, one may hope to recover $R_{\boldsymbol{\eta}}(t,\nu;t',\nu')$ from the autocorrelation of the received information $R_{\mathbf{f}}(t,t')$. To prove Theorem 11 below, we will need to solve equations of the form $Y = GXG^*$ with both X, Y positive semi-definite.[†] To this end, we need a standard technique from linear algebra. Let *vectorization* $\operatorname{vec} : \mathbb{C}^{n \times n} \mapsto \mathbb{C}^{n^2}$ be the linear isomorphism between the space of matrices $\mathbb{C}^{n \times n}$ and the space of column vectors $\mathbb{C}^{n^2}$ given by stacking of the columns

$$(\operatorname{vec} A)_i = A_{i \bmod n, \lfloor i/n \rfloor}, \quad i = 0, \ldots, n^2-1.$$

The following identity relating vectorization and the Kronecker product of matrices is well known [30, 31]:

$$\operatorname{vec}(AXB^t) = (A \otimes B)\operatorname{vec} X. \qquad (7)$$

For an arbitrary matrix A, the set $\Lambda_A = \{(\lambda,\lambda') : A_{\lambda,\lambda'} \neq 0\}$ is called *the support set* of A. From the properties of positive semi-definite matrices it is easy to see that

$$(\lambda,\lambda') \in \Lambda \text{ implies } \{(\lambda',\lambda),(\lambda,\lambda),(\lambda',\lambda')\} \subseteq \Lambda. \quad (8)$$

With some abuse of terminology, we call sets that may appear as support sets of positive semi-definite matrices, and hence satisfy (8), *positive semi-definite patterns* or, for short, *spd patterns*.

Lemma 9. *Let Λ be a fixed finite spd pattern, $\Lambda \subseteq \{(0,0),\ldots,(n-1,n-1)\}$, and let $G \in \mathbb{C}^{n \times m}$. The following are equivalent:*

(i) For each positive semi-definite $Y \in \mathbb{C}^{m \times m}$, there exists a unique $X \in \mathbb{C}^{n \times n}$ such that a) X is positive semi-definite, b) $\operatorname{supp} X = \Lambda$, and c) $Y = GXG^*$.

(ii) If a Hermitian matrix $N \in \mathbb{C}^{n \times n}$ with $\operatorname{supp} N \subseteq \Lambda$ solves the homogeneous equation $0 = GNG^*$, then $N = 0$.

(iii) The matrix $\overline{G} \otimes G\big|_\Lambda$ has a left inverse (is full rank).

Proof: (i) $\Rightarrow$ (ii) By contraposition, let there be $0 \neq N \in \mathbb{C}^{n \times n}$ such that $GNG^* = 0$; let E be the diagonal matrix whose λth diagonal entry is one if $(\lambda,\lambda) \in \Lambda$ and zero else. Then there exists a positive real number C (for example, Gershgorin circle theorem guarantees $C = \|N\|_1 = \sup_{1 \leq i \leq m} \sum_{j=1}^m |N_{ij}| < \infty$ will be enough) such that both $CE + N$ and CE are positive semi-definite, and $G(CE+N)G^* = G(CE)G^* + GNG^* = G(CE)G^*$, thus violating the uniqueness of X in (i), as both $\operatorname{supp}(CE+N)$ and $\operatorname{supp} N \subseteq \Lambda$. (ii) $\Rightarrow$ (i) Suppose, that there exist $X_1 \neq X_2$, both positive semi-definite, supported on Λ, such that $GX_1G^* = GX_2G^*$. Then $N = X_1 - X_2 \neq 0$ is necessarily Hermitian, $\operatorname{supp} N \subseteq \Lambda$, and $GNG^* = 0$, a contradiction. (iii) $\Rightarrow$ (ii) Let an Hermitian matrix N be supported on Λ and be a solution to $0 = GNG^*$. Applying vectorization to both sides of $0 = GNG^*$, we get by (7)

$$\operatorname{vec} 0 = (\overline{G} \otimes G)\operatorname{vec} N, \qquad (9)$$

[†]A a Hermitian matrix $X \in \mathbb{C}^{n \times n}$ is *positive semi-definite* if for any $a \in \mathbb{C}^n$, $a^* X a \geq 0$.

If $\overline{G} \otimes G\big|_\Lambda$ is invertible, (9) implies $N = 0$. (ii) $\Rightarrow$ (iii) Let $A \in \mathbb{C}^{m \times m}$ be an arbitrary square matrix supported on Λ such that $\overline{G} \otimes G\big|_\Lambda \operatorname{vec} A = 0$, that is, $GAG^* = 0$. The *Cartesian decomposition* of A is given by $A = H_1 + iH_2$ with both H_1, H_2 Hermitian, defined by

$$H_1 = \frac{1}{2}(A + A^*), \quad H_2 = \frac{1}{2i}(A - A^*).$$

It is easy to see that both $\operatorname{supp} H_1, \operatorname{supp} H_2 \subseteq \Lambda$, and $GH_1G^* = GH_2G^* = 0$, since $GA^*G^* = (GAG^*)^* = 0$. Therefore, by (ii), $H_1 = H_2 = 0$, and $A = 0$. It follows that $\operatorname{vec} A = 0$. Since $\operatorname{vec} A$ is arbitrary, it follows that the columns of $\overline{G} \otimes G\big|_\Lambda$ are linearly independent, that is, $\overline{G} \otimes G\big|_\Lambda$ is left invertible. ∎

Definition 10. We would say that a spd support set Λ is a *permissible pattern* if some $c \in \mathbb{C}^L$ generates a Gabor frame $G = [M^n T^k c]_{k,n=0}^{L-1}$ such that the equivalent conditions of Lemma 9 are satisfied. If an spd pattern is not permissible, it will be called *defective*.

This designation reflects the emergence of these patterns as those which permit the sampling of operators with delta trains.

Theorem 11. *Let $H \in \operatorname{StOPW}(M)$ such that M is properly (a, b, Λ)-rectified for some $a, b > 0$, $|\Lambda| = L^2$, and $ab = 1/L$. If some $c \in \mathbb{C}^L$ generates $G = [M^n T^k c]_{k,n=0}^{L-1}$ such that the submatrix $(\overline{G} \otimes G)|_\Lambda$ is (left) invertible, then we can reconstruct the autocorrelation of the spreading function $\mathbb{E}\{\eta \overline{\eta}\}$ from the autocorrelation $R_{\mathbf{f}} = \mathbb{E}\{\mathbf{f}\,\overline{\mathbf{f}}\}$ of the response $\mathbf{f} = H \mathrm{III}_c$ of H to the L−periodic c−weighted delta train $\mathrm{III}_c = \sum_{k \in \mathbb{Z}} c_{k \bmod L} \delta_{ak}$ with the reconstruction formula*

$$R_{\boldsymbol{\eta}}(t, \nu; t', \nu')$$
$$= b^{-2} \sum_{j=0}^{L^2-1} \sum_{p,p'=0}^{L-1} \alpha_{j,p,p'}\, e^{2\pi i a b (n_j p - n'_j p')}$$
$$\times \sum_{m,m' \in \mathbb{Z}} e^{-2\pi i a (\nu(k_j + mL + p) - \nu'(k'_j + m'L + p'))}$$
$$\times R_{\mathbf{f}}(t - a(k_j + mL - p),\, t' - a(k'_j + m'L - p')). \tag{10}$$

Proof: Again, as in the deterministic case, we start with the support already rectified in the sense of Definition 6. We proceed in the same manner as in the direct product case until the mixing formula (5), which guarantees that for $p = 0, \ldots, L - 1$ for all (t, ν) within the base rectangle $\square = [0, a) \times [0, b)$

$$\boldsymbol{\mathcal{Z}}(t, \nu) = \sum_{k,n \in \mathbb{Z}} M^n T^k c\, \boldsymbol{\eta}_{k,n}(t, \nu),$$

Taking autocorrelation on both sides, we get for all $p, p' = 0, \ldots, L - 1$, and $(t, \nu), (t', \nu') \in \square$,

$$\mathbb{E}\{\boldsymbol{\mathcal{Z}}_p(t, \nu) \overline{\boldsymbol{\mathcal{Z}}_{p'}(t', \nu')}\}$$
$$= \mathbb{E}\Big\{\sum_{k,n \in \mathbb{Z}} (M^n T^k c)_p \boldsymbol{\eta}_{k,n}(t, \nu) \sum_{k',n' \in \mathbb{Z}} \overline{(M^{n'} T^{k'} c)_{p'} \boldsymbol{\eta}_{k',n'}(t', \nu')}\Big\}$$
$$= \sum_{k,n,k',n' \in \mathbb{Z}} (M^n T^k c)_p \overline{(M^{n'} T^{k'} c)_{p'}} \mathbb{E}\{\boldsymbol{\eta}_{k,n}(t, \nu) \overline{\boldsymbol{\eta}_{k',n'}(t', \nu')}\}.$$

Let $\operatorname{supp} R_{\boldsymbol{\eta}}(t, \nu; t', \nu')$ be covered by L^2 4D parallelepipeds indexed by $\Lambda = \{(k, n, k', n')\}$, that is,

$$R_{\boldsymbol{\eta}}(t, \nu; t', \nu') =$$
$$\sum_{(k,n,k',n') \in \Lambda} \mathbb{E}\{\boldsymbol{\eta}_{k,n}(t - ak, \nu - bn) \overline{\boldsymbol{\eta}_{k',n'}(t' - ak', \nu' - bn')}\}.$$

Without loss of generality, all (k, n) and (k', n') are distinct modulo L.[†] For brevity, let $\mathcal{T}_j$ be the translation

$$\mathcal{T}_j f(t, \nu; t', \nu') = f(t - ak_j, \nu - bn_j, t' - ak'_j, \nu' - bn'_j).$$

Let us denote $Y_{p,p'} = \mathbb{E}\{\boldsymbol{\mathcal{Z}}_p \boldsymbol{\mathcal{Z}}_{p'}^*\}$ and $X_{k,n,k',n'} = \mathbb{E}\{\boldsymbol{\eta}_{k,n} \boldsymbol{\eta}_{k',n'}^*\}$, where $*$ stands for conjugate transpose. As covariance matrices of zero-mean random vectors, both $X_{k,n,k',n'}$ and $Y_{p,p'}$ are positive semi-definite, and conversely, every positive semi-definite matrix is a covariance matrix for some random vector [18]. With this notation, we write for each $(t, \nu), (t', \nu') \in \square$ a deterministic matrix equation

$$Y = GXG^*, \tag{11}$$

where $G = [M^n T^k c]$ is a Gabor matrix as in (2). To have a a unique positive semi-definite solution X of the underdetermined system of equations (11) with an arbitrary positive semi-definite Y, by Lemma 9 it is necessary and sufficient that $\overline{G} \otimes G\big|_\Lambda$ is left invertible. Let A denote its left inverse,

$$A = [\alpha_{(k,n),(p,p')}]_{k,n,p,p'=0}^{L-1} \text{ such that } A(\overline{G} \otimes G\big|_\Lambda) = \operatorname{Id}.$$

Such A may exist only if $|\Lambda| \leq L^2$. The autocorrelation of the channel spreading function can be recovered from X by observing (starting with the definition (4) of $\boldsymbol{\eta}_{k,n}$),

$$R_{\boldsymbol{\eta}}(t, \nu; t', \nu') = \mathbb{E}\{\boldsymbol{\eta}(t, \nu) \overline{\boldsymbol{\eta}(t', \nu')}\}$$
$$= \sum_{j=0}^{L^2-1} \mathcal{T}_j X(t, \nu; t', \nu')_{k_j, n_j, k'_j, n'_j}\, e^{-2\pi i a(\nu k_j - \nu' k'_j)}$$
$$= \sum_{j=0}^{L^2-1} (A^{-1} \operatorname{vec} \mathcal{T}_j Y(t, \nu; t', \nu')_{k_j, n_j, k'_j, n'_j}\, e^{-2\pi i a(\nu k_j - \nu' k'_j)}$$
$$= \sum_{j=0}^{L^2-1} \sum_{p,p'=0}^{L-1} \alpha_{j,p,p'} \mathcal{T}_j Y_{p,p'}(t, \nu; t', \nu')\, e^{-2\pi i a(\nu k_j - \nu' k'_j)}, \tag{12}$$

where we construct Y from the autocorrelation of the received signal $\mathbf{f}(t) = H \mathrm{III}_c(t)$ via

$$Y_{p,p'}(t, \nu; t', \nu') = \mathbb{E}\{\boldsymbol{\mathcal{Z}}_p(t, \nu) \boldsymbol{\mathcal{Z}}_{p'}^*(t', \nu')\}$$
$$= b^{-2} e^{-2\pi i a(\nu p - \nu' p')} \mathbb{E}\{\mathcal{Z}_{aL}\mathbf{f}(t + ap, \nu) \overline{\mathcal{Z}_{aL}\mathbf{f}(t' + ap', \nu')}\}$$
$$= b^{-2} e^{-2\pi i a(\nu p - \nu' p')} \sum_{m,m' \in \mathbb{Z}} e^{-2\pi i b^{-1}(m\nu - m'\nu')}$$
$$\times \mathbb{E}\{\mathbf{f}(t + a(p - mL)) \overline{\mathbf{f}(t' + a(p' - m'L))}\}.$$

[†]As in the deterministic case, the parameters a, b, L can always be chosen in such a way that $\operatorname{supp} R_{\boldsymbol{\eta}}(t, \nu; t', \nu') \subseteq [0, aL) \times [0, bL)$ up to a translation.

Translating and simplifying, we get

$$
\begin{aligned}
Y_{p,p'}&(t - ak_j, \nu - bn_j; t' - ak'_j, \nu' - bn'_j) \\
&= b^{-2}\, e^{2\pi i a b(n_j p - n'_j p')} \sum_{m,m' \in \mathbb{Z}} e^{-2\pi i a((p+mL)\nu - (p'+m'L)\nu')} \\
&\quad \times R_{\mathbf{f}}(t - a(k_j + mL - p), t' - a(k'_j + m'L - p)).
\end{aligned}
$$

Upon substitution of the above back into (12), we get the desired reconstruction formula (10). This completes the proof of Theorem 11. ∎

If the support of $R_{\boldsymbol{\eta}}$ can be factored into a tensor square of some set of area less than one on the time-frequency plane, then Theorem 11 guarantees reconstruction of $R_{\boldsymbol{\eta}}$ from the autocorrelation $R_{\mathbf{f}}$ of the received measurement $\boldsymbol{H}\text{Ш}_c$, which is a weaker result than Theorem 7. To wit, assume that $R_{\boldsymbol{\eta}}$ is supported on a product set $S \times S$ such that $S \times S$ is (a, b, Λ)-rectified, and that the area of S is less than one. It follows that Λ itself can be represented as a direct product of some index set, say, $\Lambda = \Gamma \times \Gamma$ such that S is (a, b, Γ)-rectified. For some vector c the submatrix of its time-frequency shifts G_Γ is left invertible [16, 25]. Then $(\overline{G} \otimes G)|_\Lambda = \overline{G_\Gamma} \otimes G_\Gamma$ is also left invertible, and $R_{\boldsymbol{\eta}}(t, \nu; t', \nu')$ can be recovered from $R_{\mathbf{f}}(t, t')$. The strength of Theorem 11 is that it allows to exploit information about more complicated dependencies between scatterers manifested in the geometry of the support of the autocorrelation function $R_{\boldsymbol{\eta}}(t, \nu; t', \nu')$.
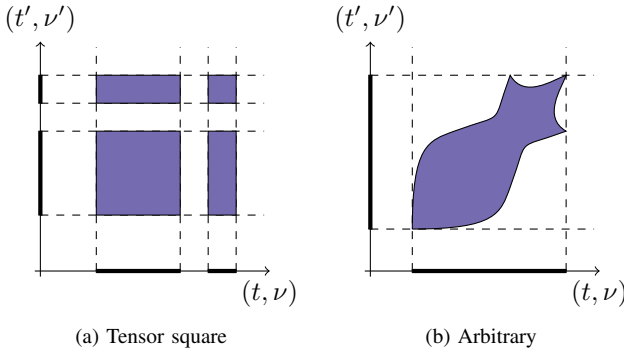

(a) Tensor square

(b) Arbitrary

Fig. 1: Different types of support sets of autocorrelation functions.

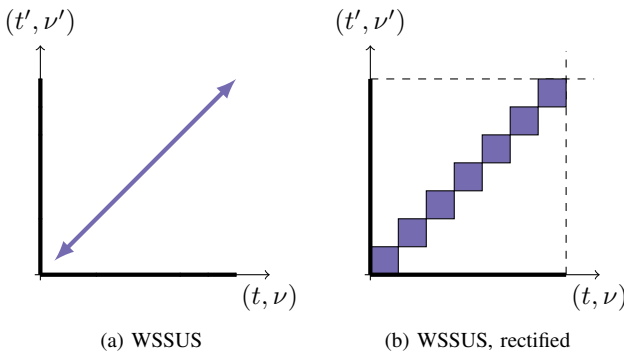
(a) WSSUS

(b) WSSUS, rectified

Fig. 2: Different types of support sets of autocorrelation functions, WSSUS case.

For example, consider the important case of the WSSUS operators, already mentioned in the Lemma 8 after Theorem 7. Let $\boldsymbol{H}^{\text{big}}$ be some WSSUS operator whose scattering function $C_{\boldsymbol{\eta}}(t, \nu)$ is supported on a 2D rectangle $S^{\text{big}} = [0, 1) \times [0, L)$. This rectangle S^{big} of area greater than one can always be rectified with a collection Λ^{big} of L^2 translations of a box $[0, 1) \times [0, \frac{1}{L})$. From the WSSUS assumption we know that $R_{\boldsymbol{\eta}}(t, \nu; t', \nu') = 0$ whenever (t, ν) and (t', ν') are in different boxes, therefore, the 4D set $\text{supp}\, R_{\boldsymbol{\eta}}(t, \nu; t', \nu')$ is properly $(1, \frac{1}{L}, \Lambda^{\text{big}})$-rectifiable, as shown in Figure 2b. It follows that in the sense of Theorem 11, $\boldsymbol{H}^{\text{big}}$ is identifiable whenever the diagonal set Λ^{big} is a permissible pattern. Since, according to Theorem 15, this is always the case with any such diagonal set, we obtain the following corollary of Theorem 11.

Theorem 12. *Let $\boldsymbol{H} \in \text{StOPW}(S \times S)$ a WSSUS operator with $S = \text{supp}\, C(t, \nu)$ a compact set of arbitrary finite measure $\mu(S) < \infty$. Then there exist L prime, $L > \mu(S)$, $a, b > 0$ with $ab = 1/L$, a complex vector $c \in \mathbb{C}^L$ and a sequence $\{\alpha_{jp}\}_{j,p=1}^L$ depending only on c such that we can reconstruct the scattering function $C(t, \nu)$ of $\boldsymbol{H}$ from the autocorrelation of its response to an $L-$periodic $c-$weighted delta train $\text{Ш}_c = \sum_{k \in \mathbb{Z}} c_{k \bmod L} \delta_{ak}$ using* (13).

Proof: For an arbitrary ε, it is always possible to find $a, b > 0, \frac{1}{ab} > \sqrt{\mu(S) + \varepsilon}$ such that the support of the scattering function S is covered by N translations of the rectangle $[0, a) \times [0, b)$ (as usual, indexed by Λ), such that $\mu(S) \le Nab \le \mu(S) + \varepsilon$. Without loss of generality, all $(k, n) \in \Lambda$ are distinct modulo $L = \frac{1}{ab}$. We cover the 4D diagonal of the set $S \times S$ with N 4D parallelepipeds of volume $1/(ab)^2$, so the total volume of the cover is $\frac{N}{L^2} \le 1$. By Theorem 11, there exists $c \in \mathbb{C}^L$ such that we can reconstruct the autocorrelation of the spreading function from the echo $\boldsymbol{f} = \boldsymbol{H}\text{Ш}_c$ using a version of (10) that takes advantage of the stationarity of $\boldsymbol{H}$ and periodicity of Ш_c:

$$
\begin{aligned}
C(t, \nu) = b^{-2} \sum_{j=0}^{L^2-1} \sum_{p,p'=0}^{L-1} \sum_{m \in \mathbb{Z}} \alpha_{j,p,p'}\, e^{2\pi i a[(p-p')(\nu+bn_j) - \nu m L]} \\
\times R_{\mathbf{f}}(t - a(k_j - p),\, t - a(k_j + m'L - p')).
\end{aligned}
$$
(13)

∎

III. PERMISSIBLE PATTERNS

The preceding discussion shows that it is ultimately important to study which spd patterns Λ are permissible, that is, for which sets Λ there exists $c \in \mathbb{C}^L$ such that $\overline{G} \otimes G\big|_\Lambda$ is full rank. Consider the spreading function $\boldsymbol{\eta}(t, \nu)$ of $\boldsymbol{H}$ given by

$$
\boldsymbol{\eta}(t, \nu) = \sum_{k=0}^{L-1} \sum_{n=0}^{L-1} \boldsymbol{\eta}_{k,n}(t - ka, \nu - nb).
$$

with $\boldsymbol{\eta}_{k,n}(t, \nu) \in L^2(\mathbb{R}^2)$ supported on $[0, a) \times [0, b)$. Due to the specifics of the underlying time-frequency analysis, it will be convenient to index the matrices with the finite index set $\mathcal{J}$ given by

$$
\mathcal{J} = \{(0, 0), (0, 1), \ldots, (0, L-1), (1, 0), \ldots, (L-1, L-1)\}.
$$
(14)

We call the *autocorrelation pattern* of $\boldsymbol{\eta}$ the indicator matrix $\mathrm{ACP} \in \{0,1\}^{\mathcal{J} \times \mathcal{J}}$

$$\mathrm{ACP}(\lambda, \lambda') = \begin{cases} 1, & \text{if for some } (t,\nu), (t',\nu') \in \square, \\ & \mathbb{E}\{\boldsymbol{\eta}_\lambda(t,\nu) \overline{\boldsymbol{\eta}_{\lambda'}(t',\nu')}\} \neq 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda = (k,n)$, $\lambda' = (k',n')$, $\lambda, \lambda' \in \mathcal{J}$. Clearly, ACP must be symmetric, moreover, just as in (8),

$$\mathrm{ACP}(\lambda, \lambda') = 1 \quad \text{implies}$$
$$\mathrm{ACP}(\lambda', \lambda) = \mathrm{ACP}(\lambda, \lambda) = \mathrm{ACP}(\lambda', \lambda') = 1, \qquad (15)$$

and conversely, for any matrix $M \in \mathbb{C}^{\mathcal{J} \times \mathcal{J}}$ that satisfies (15), there exists an operator with the spreading function $\boldsymbol{\eta}_M(t, \nu; t', \nu')$ whose autocorrelation pattern is M. We visualize autocorrelation patterns with diagrams such as Figure 3, where shaded boxes correspond to nonzero correlation between patches. Lemma 9 indicates that the problem at hand is linked to the Haar property of Gabor systems. We explore this connection in more detail in Section III-C. The Haar property of a classical finite-dimensional Gabor frame reads that any L columns of a generic Gabor matrix G are in general linear position if L is prime. Therefore, any deterministic spreading function supported on L boxes of area $1/L$ on a time-frequency plane can be identified. In other words, any set $\Gamma \subset \mathcal{J}$ such that $|\Gamma| = L$ indexes a subset of columns of G that is linearly independent for almost every seed vector c. This contrasts with the stochastic setting, where we will see that there exist plenty of spd patterns that correspond to submatrices of $\overline{G} \otimes G$ which are rank-deficient for every choice of c. Generally, $\overline{G} \otimes G$ can be viewed as a Gabor system on the non-cyclic group $\mathbb{Z}_L \times \mathbb{Z}_L \cong \mathcal{J}$, generated by the window $\overline{c} \otimes c$,

$$\overline{G} \otimes G = \{ \pi(k,n,k',n')(\overline{c} \otimes c) : (k,n), (k',n') \in \mathcal{J},$$
$$\pi(k,n,k',n') = \overline{M^n T^k c} \otimes M^{n'} T^{k'} c \}.$$

The Haar property of $\overline{G} \otimes G$ would require all subsets of $\overline{G} \otimes G$ of order L^2 to be linearly independent, which is not the case for any prime number $L \geq 2$. However, the positive-definiteness of the autocorrelation function demands the autocorrelation pattern to satisfy property (15). This precludes certain subsets of $\overline{G} \otimes G$ from being tested for linear dependence. Below we show that this restriction implies that for $L = 2$ every spd pattern is permissible; not so for $L \geq 3$.

A. Case $L = 2$

On Figures 3 and 4 we show all possible spd patterns for $L = 2$. In addition, the last pattern on Figure 4b, corresponds to a linear dependent set of elements of $\overline{G} \otimes G$. However, due to lack of symmetry, such a set cannot appear as a support set for the autocorrelation of the spreading function of the stochastic operator. We mention it here to highlight the connection of the stochastic operator identification theory with the analysis of Gabor frames on general abelian groups, for this example, $\mathbb{Z}_2 \times \mathbb{Z}_2$. Further properties of such frames, including this pattern, can be found in [32, 33]. Interestingly, the only admissible patterns on $L = 2$ belong to one of the
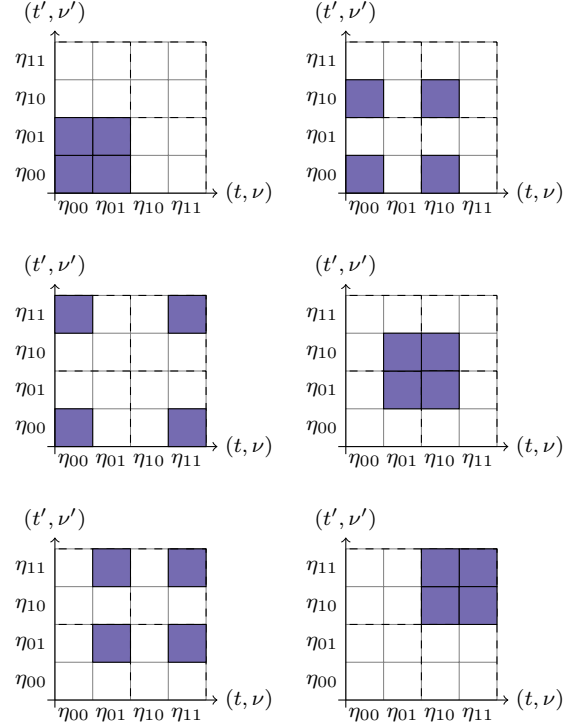

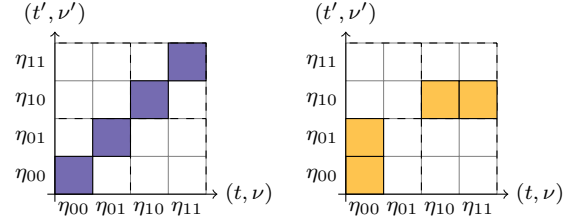Fig. 3: Tensor rank-1 autocorrelation patterns for $L = 2$. Any rank-1 tensor pattern is permissible by (16).

Fig. 4: Other autocorrelation patterns for $L = 2$.

two types, shown on Figure 3 and Figure 4a. The first type contains sets that can be represented as rank-1 products[†] of sets of order L in $\mathcal{J}$, that is, there exists a set Γ or order $|\Gamma| = L$ such that $\Lambda = \Gamma \times \Gamma$. Such product patterns inherit the linear independence properties of their factors, since the Haar property of G guarantees

$$\mathrm{rank}\left(\overline{G} \otimes G\big|_{\Gamma \times \Gamma}\right) = (\mathrm{rank}\, G_\Gamma)^2 = L^2. \qquad (16)$$

The second type, all boxes on the diagonal, of maximal possible tensor rank, is also a permissible pattern for $L = 2$, as we observe the determinant of the matrix

$$\det\left(\overline{G} \otimes G\big|_{\mathrm{diag}}\right) = 4(|c_0|^4 - |c_1|^4)(c_0^2\, \overline{c_1}^2 - c_1^2\, \overline{c_0}^2) \neq 0$$

or almost all choices of the generating atom $c = [c_0\ c_1]^t$. This concludes the analysis of the $L = 2$ case with a happy ending:

Proposition 13. *Let a function $R_{\boldsymbol{\eta}}(t, \nu; t', \nu')$ be supported on a set of measure less than or equal to one covered with 4 boxes*

[†]We say that an element $b \in S \otimes S$ has *tensor rank* k if k is the smallest number with the property that there exist $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k \in S$ such that $\Lambda = a_1 \otimes b_1 + \cdots + a_k \otimes b_k$.

$\square^2 + (k_j a, n_j b, k'_j a, n'_j b)$ *for some* $\{(k_j, n_j, k'_j, n'_j) \in \mathcal{J} \times \mathcal{J}\}_{j=0}^3$, *and some* $\square^2 = [0, a) \times [0, b) \times [0, a) \times [0, b)$ *such that* $ab = {}^1\!/{}_2$ *and all* $(k, n), (k', n')$ *distinct modulo 2. Let* $\boldsymbol{\eta}(t, \nu)$ *be some function with* $R_{\boldsymbol{\eta}}(t, \nu; t', \nu')$ *as its autocorrelation function, and* $\boldsymbol{H}$ *a stochastic channel with* η *its spreading function. Then we can recover* $R_{\boldsymbol{\eta}}(t, \nu; t', \nu')$ *from the echo of* $\boldsymbol{H}$ *to the sounding signal*

$$\text{III}_c(t) = c_0 \sum_m \delta(t - a - 2am) + c_1 \sum_m \delta(t - 2am)$$

for almost all $c = [c_0 \ c_1]^t \in \mathbb{C}^2$, $|c_0| = |c_1| = 1$.

B. Case $L > 2$

Classification of all autocorrelation patterns with $L = 3$ already presents some challenges. For once, there are over 5000 possible spd patterns

$$1 + \binom{9}{7}\binom{7}{2} + \binom{9}{5}\left(2\binom{5}{4} + 3\binom{5}{3}\right) = 5796.$$

Here, 1 corresponds to the diagonal case, plus there are $\binom{9}{7}$ choices of 7 boxes on the diagonal, any two of which can be correlated corresponding to the $\binom{7}{2}$ factor, etc.[†] The problem of classifying all possible arrangements of boxes becomes intractable very quickly with L. In Section III-C, we show that whenever some Λ is a permissible pattern for almost all c, there is a related (via the permutations of the Gabor elements) family of Λ^σ that will be permissible patterns for almost all c. Such observations simplify the classification problem, although they do not suffice to give a complete classification. Secondly, alas, unlike $L = 2$, some spd patterns correspond to rank-deficient subsets of $\overline{G} \otimes G$. That is, there are sets that satisfy (8) but are not permissible, for example, see Figure 5. It turns out that there are structural reasons for patterns on Figure 6 and Figure 5 to be permissible (respectively, defective). With little effort it can be seen that patterns with tensor rank 2 (as on Figure 5) can never be permissible, owing solely to the properties of the tensor product space and not to the specifics of the underlying Gabor structure. In fact, for any L, permissible patterns cannot contain two complete large sets of pairwise tensor products. (Here, such two sets are $\{\boldsymbol{\eta}_{00}, \boldsymbol{\eta}_{01}\}$ and $\{\boldsymbol{\eta}_{02}, \boldsymbol{\eta}_{10}\}$, and in this context, *large* means that $\#\{\boldsymbol{\eta}_{00}, \boldsymbol{\eta}_{01}\} + \#\{\boldsymbol{\eta}_{02}, \boldsymbol{\eta}_{10}\} = 2 + 2 > 3$.)

Proposition 14. *Let* $G \in \mathbb{C}^{n \times m}$ *be an arbitrary matrix, and* G_1, G_2 *its submatrices comprising columns of* G *indexed, respectively, by* Γ_1 *and* Γ_2 *such that* $\Gamma_1 \cap \Gamma_2 = \varnothing$. *Denote* $\Lambda = (\Gamma_1 \times \Gamma_1) \cup (\Gamma_2 \times \Gamma_2)$. *If* $|\Gamma_1| + |\Gamma_2| > \operatorname{rank} G$, *then the set of elements in* $\overline{G} \otimes G\big|_\Lambda$ *is linearly dependent.*

Proof: Since $|\Gamma_1| + |\Gamma_2| > \operatorname{rank} G$, it follows that there exists $[b_1 \ b_2]^t \in \ker[G_1 | G_2] \neq 0$, that is,

$$G_1 b_1 + G_2 b_2 = 0.$$

[†]The case of even number of boxes on the diagonal is always counted within the larger odd case; by symmetry considerations, every correlation between boxes adds two boxes on the field).
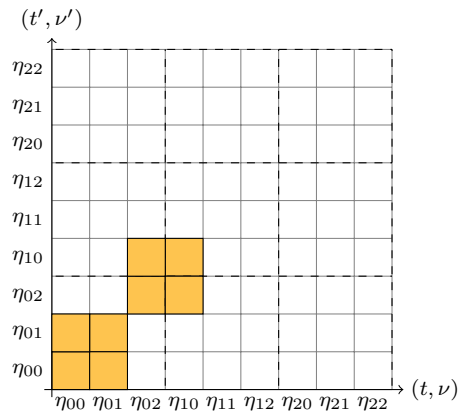

Fig. 5: According to Proposition 14, any rank-2 pattern is not a permissible pattern.

Now consider $B_1 = \overline{b_1} \otimes b_1$, $B_2 = \overline{b_2} \otimes b_2$. Both are Hermitian, and

$$\begin{aligned}
&G_1 B_1 G_1^* - G_2 B_2 G_2^* \\
&= (G_1 b_1)(G_1 b_1)^* - (G_2 b_2)(G_2 b_2)^* \\
&= (G_1 b_1)(G_1 b_1)^* - (-G_1 b_1)(-G_1 b_1)^* = 0.
\end{aligned}$$

This means that we have found a Hermitian matrix

$$N = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

supported on Λ such that $GNG^* = 0$ and, hence, $\overline{G} \otimes G\big|_\Lambda$ is linearly dependent. It follows that Λ is not a permissible pattern. ∎

On the other hand, the patterns with maximal tensor rank (that is, diagonal patterns) will *always* be permissible. This corresponds to the important case of WSSUS operators. We give here an elementary proof for finite dimensions. (See also [34, Theorem 3.1] for a discussion of this in the realm of Hilbert-Schmidt operators in infinite dimensions.)

Theorem 15. *Denote* $\pi(k, n) = M^n T^k$. *Let* $G = \{\pi(k, n)c\}_{k,n=0}^{L-1}$ *be a Gabor frame generated by* c. *Then for almost all* $c \in \mathbb{C}^L$, *the set* $\overline{G} \otimes G\big|_{diag}$ *of all tensor products of each Gabor element with itself*

$$\overline{G} \otimes G\big|_{diag} = \{\overline{\pi(k, n)c} \otimes \pi(k, n)c\}_{k,n=0}^{L-1}$$

is linearly independent, that is, for almost all $c \in \mathbb{C}^L$ *the diagonal set*

$$\Lambda_{diag} = \{(k, n), (k, n)\}_{k,n=0}^{L-1}$$

is a permissible pattern.

Proof: It is a straightforward observation that the linear independence of the set $\overline{G} \otimes G\big|_{\text{diag}} \in \mathbb{C}^{\mathcal{J} \times \mathcal{J}}$ is equivalent to the linear independence of the family of the corresponding finite-dimensional Hilbert-Schmidt operators $P_{k,n} : \mathbb{C}^L \mapsto \mathbb{C}^L$ given by

$$P_{k,n} x = \langle x, \pi(k, n)c \rangle \pi(k, n)c.$$

Clearly, $P_{k,n} = (M^n T^k) P_{0,0} (M^n T^k)^*$. From the spreading function representation of $H : \mathbb{C}^L \mapsto \mathbb{C}^L$,

$$H = \sum_{p,q=0}^{N-1} \eta[p,q] M^p T^q$$

it is easy to see with the help of the commutation relation $T^k M^n = e^{-2\pi i k n} M^n T^k$ that for any H we have

$$(M^n T^k) H (M^n T^k)^* = \sum_{p,q=0}^{L-1} (M^{(k,n)} \eta[p,q]) M^p T^q,$$

where $M^{(k,n)} \eta[p,q] = e^{2\pi i (qk - pn)} \eta[p,q]$. In particular, the spreading functions of $P_{k,n}$ and $P_{0,0}$ satisfy $\eta_{k,n} = M^{(k,n)} \eta_{0,0}$. Therefore, the linear independence of the family of operators $\{P_{k,n}\}_{k,n=0}^{L-1} \subseteq \mathrm{HS}(\mathbb{C}^L)$ is equivalent to the linear independence of the family $\{M^{(k,n)} \eta_{0,0}\} \subseteq \mathbb{C}^{L^2}$. In finite dimensions, the short time Fourier transform of c with respect to itself takes form

$$V_c c[p,q] = \langle c, M^p T^q c \rangle = \sum_r e^{-2\pi i p r / L} c[r] \overline{c}[r-q].$$

Here and in what follows, the summation is over $\mathbb{Z}_L$ and all the indices of vectors $c, x \in \mathbb{C}^L$ are taken modulo L. We compute

$$P_{0,0} x[m]$$
$$= \langle x, c \rangle c[m] = \sum_q x[q] \overline{c}[q] c[m]$$
$$= \sum_q x[m-q] \overline{c[m-q]} c[m]$$
$$= \sum_q x[m-q] \sum_r \delta[r-m] \overline{c[r-q]} c[r]$$
$$= \sum_q x[m-q] \sum_r \left(\frac{1}{L} \sum_p e^{2\pi i p (r-m)/L} \right) \overline{c}[r-q] c[r]$$
$$= \sum_p \sum_q \left(\frac{1}{L} \sum_r e^{-2\pi i p r / L} c[r] \overline{c}[r-q] \right) e^{2\pi i m p / L} x[m-q]$$
$$= \sum_p \sum_q \left(\frac{1}{L} V_c c[p,q] \right) M^p T^q x[m],$$

which proves that $\eta_{0,0}[p,q] = \frac{1}{L} V_c c[p,q]$. We claim that for almost every $c \in \mathbb{C}^L$, $V_c c$ has full support. For a fixed pair $p, q \in \mathbb{Z}_L \times \mathbb{Z}_L$, the set of solutions $Z_{p,q} = \{c \in \mathbb{C}^L : V_c c[p,q] = 0\}$ is a manifold of zero measure in $\mathbb{C}^L$. Hence, $V_c c$ has full support almost everywhere, namely whenever $c \in \mathbb{C}^L \setminus \bigcup_{p,q} Z_{p,q}$. The proof is complete by observing that whenever $\eta_{0,0}$ has full support, the family $\{M^{(k,n)} \eta_{0,0}\}_{(k,n) \in \mathbb{Z}_L \times \mathbb{Z}_L}$ is linearly independent in $\mathbb{C}^{L^2}$. ∎

Another source of deficiency that begins to appear only with $L \geq 4$ is illustrated on Figure 7.

Proposition 16. *Let the* height *(or, equivalently, the width) of the pattern Λ exceed L, or, formally, for some $\lambda_0 \in \Lambda$,*

$$\#\{\lambda' : (\lambda_0, \lambda') \in \Lambda\} > L.$$

Then Λ is not the permissible pattern. The matrix $\overline{G} \otimes G\big|_\Lambda$ is singular for all $c \in \mathbb{C}^L$.
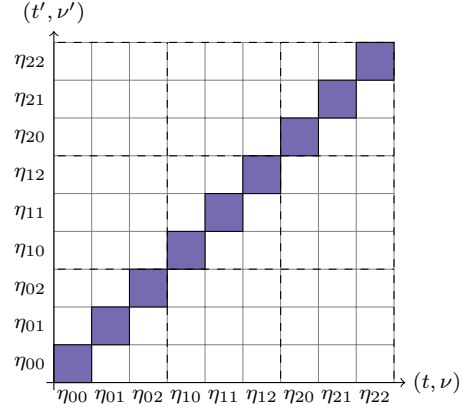

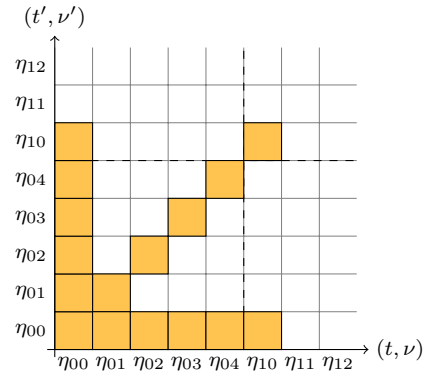Fig. 6: By Theorem 15, all diagonal patterns are permissible for almost all c.

Fig. 7: Defective autocorrelation pattern on $L = 5$. Total area $16/25$ leads to a 16×25 matrix of rank 13.

Proof: Let $\Gamma = \{\lambda' : (\lambda_0, \lambda') \in \Lambda\}$. Then $|\Gamma| > L$, therefore, the set $G\big|_\Gamma$ is linear dependent. Hence, we can find a nontrivial vector $a \in \ker G\big|_\Gamma$. We now have $(\overline{G} \otimes G) a = \overline{G} \otimes G a = 0$. That is, we have found a nontrivial linear combination $\sum_{i \in \Gamma} a_i g_i = 0$, and

$$\sum_{i \in \Gamma} \overline{a_i g_i} \otimes g_1 = \sum_{i \in \Gamma} a_i (\overline{g_i} \otimes g_1) = 0.$$

The theorem immediately follows from Lemma 9 by observing that whenever there exists a non-trivial A supported on Λ, not skew-Hermitian and such that $GA = 0$, (here, $A = [a\ 0]$), then a non-trivial Hermitian matrix $A + A^*$ satisfies $G(A + A^*) G^* = 0$. ∎

The existence of defective spd sets shows that one must be careful when trying to reconstruct the autocorrelation of the spreading function by sampling an operator with the delta train. Luckily, the task of weeding out defective patterns is uncoupled from the sampling procedure. All defective patterns Λ^{bad} can be discovered numerically and inexpensively by testing a Gabor matrix generated by a random vector c. It is easy to see that the rank deficiency of $\overline{G} \otimes G\big|_\Lambda$ will (with probability one) indicate whether Λ is good or bad.

Remark. Theorem 11 gives a procedure to recover $\mathbb{E}\{\eta \overline{\eta}\}$ from the autocorrelation $\mathbb{E}\{f \overline{f}\}$ of the response $f = H \text{Ⅲ}_c$ of H to an L−periodic c−weighted delta train $\text{Ⅲ}_c = $

$\sum_{k\in\mathbb{Z}} c_{k \bmod L}\, \delta_{ak}$. The result is based on finding a permissible rectification of the set M. If some rectification satisfies the symmetry conditions for autocorrelations, but is not permissible, one can seek alternative (possibly finer) rectifications with the hope that one of them is permissible. However, Propositions 14 and 16 indicate that for some sets M there are no permissible rectifications, hence, Theorem 11 does not apply to $\mathrm{StOPW}(M)$. Indeed, if for some (t_0,ν_0), we have $\mu(\mathbb{E}\{\boldsymbol{\eta}(t_0,\nu_0)\,\boldsymbol{\eta}(t',\nu')\}) > 1$, then every rectification will have the defect discussed in Proposition 16, and our procedure to recover $R_{\boldsymbol{\eta}}(t,\nu;t',\nu')$ is not applicable. Similarly, if for two sets S_1, S_2 we have $\mu(S_1) + \mu(S_2) > 1$ and $(S_1 \times S_1)\cup(S_2 \times S_2) \subseteq M$. Consider M be (a,b,Λ)-symmetrically rectified, and S_1' be the induced (a,b,Γ_1)-rectification of S_1, and S_2' be the induced (a,b,Γ_2)-rectification of $S_2 \setminus S_1'$. (Note that Γ_2 is not empty unless $S_2 \subset S_1'$, but then $\mu(S_1') > 1$, and the rectification is again defective by Proposition 16). Clearly, $\Gamma_1 \cap \Gamma_2 = \varnothing$, $(\Gamma_1 \times \Gamma_1) \cup (\Gamma_2 \times \Gamma_2) \subset \Lambda$, and $\mu(S_1') + \mu(S_2') \geq \mu(S_1) + \mu(S_2) > 1$, and Proposition 14 is applies. This observation does not imply that there exist symmetric M with 4D volume less than one and the property that no test signal g allows to recover $\mathbb{E}\{\boldsymbol{\eta}\,\overline{\boldsymbol{\eta}}\}$ from the autocorrelation $\mathbb{E}\{Hg\,\overline{Hg}\}$. In fact, recovery may be possible using a different type of a pilot signal, for example, a non-periodic delta train.

C. Equivalence classes of permissible patterns

Let $S^{\mathcal{J}}$ be the *symmetry group* of all permutations on the group $\mathcal{J}$ of order L^2 defined in (14).

Definition 17. We call two spd patterns Λ and Λ^σ on the grid $\mathcal{J} \times \mathcal{J}$ *homologous* whenever there exists a permutation $\sigma \in S^{\mathcal{J}}$ such that

$$(\lambda, \lambda') \in \Lambda \quad \Leftrightarrow \quad (\sigma(\lambda), \sigma(\lambda')) \in \Lambda^\sigma.$$

Two homologous patterns Λ and Λ^σ are *equivalent* if for almost all choices of the window $c \in \mathbb{C}^L$ generating G,

$$\mathrm{rank}\,\overline{G} \otimes G\big|_\Lambda = \mathrm{rank}\,\overline{G} \otimes G\big|_{\Lambda^\sigma}.$$

For example, the patterns on Figure 8 are all homologous, with the first two provably equivalent, according to Proposition 18 below. Numerical evidence shows that linear independence of the subsets of the tensored Gabor frame $\overline{G} \otimes G$ is invariant under homology in the sense of Definition 17, for example, all the patterns on Figure 8 are permissible, as tested on a large number of randomly chosen c. Also, for $L = 3, 5$, and 7, entire orbits of patterns $\{\Lambda^\sigma\}_{\sigma\in S^{\mathcal{J}}}$ are permissible. Additional indirect support to the hypothesis that any two patterns that are homologous with respect to the permutation of the Gabor atoms have the same linear independence status for almost all c comes from the equivalence of all patterns for the case $L = 2$ (see Figure 3) and the universality of Theorems 14, 15 and 16 with respect to the permutations of the Gabor atoms. We will describe the permutations $\sigma \in S^{\mathcal{J}}$ such that Λ and Λ^σ are provably equivalent for any Λ for almost all $c \in \mathbb{C}^L$. Clearly, all such permutations form a subgroup of $S^{\mathcal{J}}$. It remains an open question whether this subgroup is
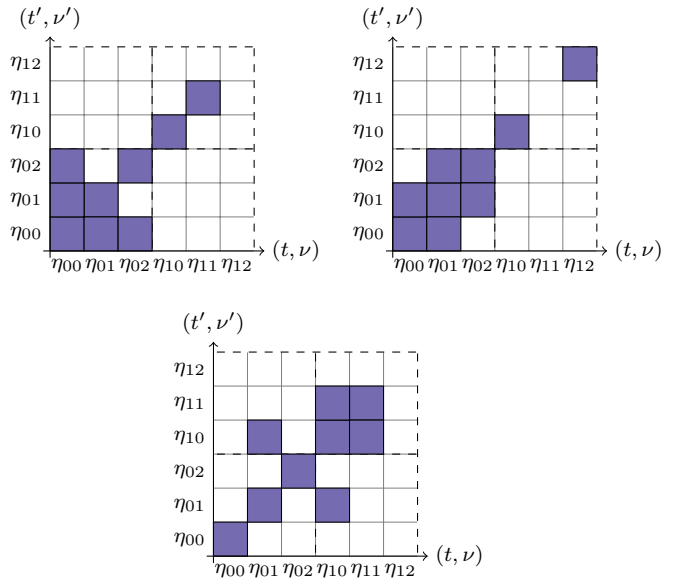

Fig. 8: Three homologous patterns for $L = 3$.

the whole group $S^{\mathcal{J}}$. For sake of notational simplicity, let us identify the atoms of a Gabor frame G generated by a window $c \in \mathbb{C}^L$ with the elements of the torus $\mathbb{Z}_L \times \mathbb{Z}_L$. The (k,n)th atom $g = M^n T^k c = \pi(k,n)\,c$ would correspond to $(k,n) \in \mathbb{Z}_L \times \mathbb{Z}_L$, where finite-dimensional operators M and T are defined in (3), and $\pi(k,n) = M^n T^k$.

Proposition 18. *Let A denote the group of permutations $\sigma \in S^{\mathcal{J}}$ such that Λ and Λ^σ are equivalent for all spd patterns $\Lambda \subseteq \mathcal{J}$ for all $c \in \mathbb{C}^L$, except maybe for a set of measure zero.*

a) *translations of the torus $\sigma_a(k,n) = (k+q, n+p) \bmod L$ belong to A for all $p, q \in \mathbb{Z}$.*

b) *reflection of the torus $\sigma_b(k,n) = (-n, k) \bmod L$ belongs to A.*

c) *rotation of the torus $\sigma_c(k,n) = (k, -n) \bmod L$ belongs to A.*

For an illustration of these transformations in the case of $\mathbb{Z}_4 \times \mathbb{Z}_4$, see Figures 9 and 10.

Proof: Note that the linear independence of the subset G_Λ of elements of G is invariant under scaling the atoms by scalars p_λ, complex conjugation, and any invertible linear transformation T. Let $G = [M^n T^k c]_{k,n=0}^{L-1}$. First, we construct several Gabor frames closely related to G.

a) It is easy to see that the Gabor frame G_a constructed from the atom $c_a = \varphi_a(c) = M^p T^q c$ is a column permutation of G up to constant multiples of columns by $C = e^{2\pi i p k}$. Indeed,

$$\begin{aligned}
\pi(k,n)\,c_a &= M^n T^k (M^p T^q c)\\
&= M^n\, e^{2\pi i p k} M^p\, T^k\, T^q c\\
&= e^{2\pi i p k} M^{n+p} T^{k+q}\, c\\
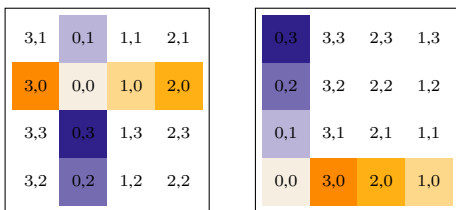&= C_a \pi(k+q, n+p) c.
\end{aligned}$$

Fig. 9: Actions of the time-frequency shift (a) and conjugation (b) are translation and horizontal reflection, respectively.

b) Similarly, constructing the Gabor frame G_b with the Gabor window being the Fourier transform of the atom $c_b = \varphi_b(c) = \mathcal{F}c$ produces a Fourier image of the original Gabor matrix. This holds since the Fourier transform commutes with time-frequency shifts in the following sense:

$$\begin{aligned}
\pi(k,n)c_b &= M^n T^k \mathcal{F}c \\
&= \mathcal{F}(T^{-n} M^k c) \\
&= \mathcal{F}(\,e^{2\pi i k n} M^k T^{-n} c) \\
&= C_b \mathcal{F} \pi(-n, k)c.
\end{aligned}$$

The Fourier transform is a unitary operation which acts on the Gabor matrix from the left, hence it does not change the linear independence of the columns, scaling of columns by the constant C_b notwithstanding. In other words, G_b is a frame that is *unitarily equivalent* to a particular permutation of G, namely, it corresponds to rotating the torus, see Figure 10 [35, 36]. Curiously, the vertical flip operation $\mathcal{I}c_j = c_{-j}$ also creates an equivalent pattern. This can be understood by observing $\mathcal{I} = \mathcal{F}^2$. This corresponds to the reflection of both rows and columns of the torus $\mathbb{Z}_L \times \mathbb{Z}_L$.
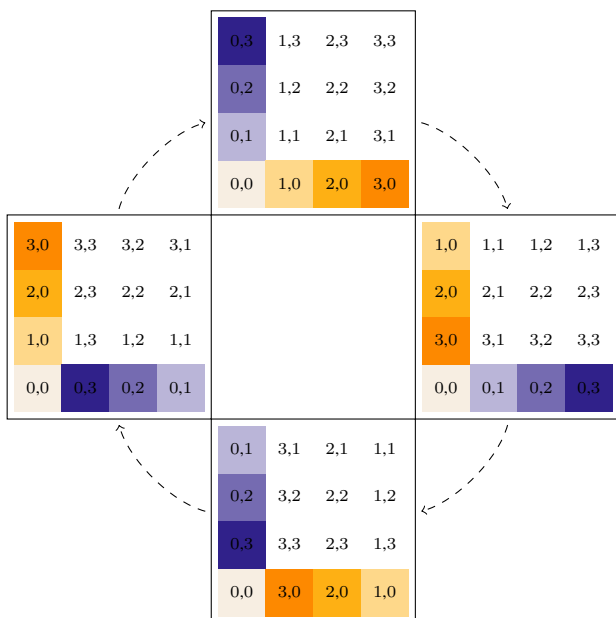

Fig. 10: Action of the Fourier transform is $90°$ rotation.

c) Another operation that preserves linear independence is conjugation. Let the Gabor frame G_c be generated by the window $c_c = \varphi_c(c) = \bar{c}$. Then $\overline{G_c}$ contains all the atoms of

G in the order that corresponds to the horizontal reflection of the torus group on itself. Note that taking conjugation does not destroy the linear independence of the vectors.

Now let Λ be any spd pattern. From the above constructions of the Gabor frames $G_\alpha, \alpha = a, b, c$, it is evident that for all generating atoms $c_\alpha = \varphi_\alpha(c) \in \mathbb{C}^L$, $\mathrm{rank} \left. \overline{G_\alpha} \otimes G_\alpha \right|_{\Lambda^{\sigma_\alpha}} = \mathrm{rank} \left. \overline{G} \otimes G \right|_{\Lambda}$. For any $\alpha = a, b, c$ the transformation $\varphi_\alpha : c \mapsto c_\alpha$ is a bijective mapping $C^L \mapsto \mathbb{C}^L$, therefore, if Λ is permissible for some c, then Λ^σ is also permissible (for some other c), and moreover, if Λ is permissible for all $c \in \mathbb{C}^L$ except a zero set Z, then Λ^σ is also permissible for all $c \in \mathbb{C}^L$ except a possibly different zero set Z^σ. And conversely, if Λ is defective, then Λ^σ is also defective. Since there are finitely many permutations σ, the union of all bad sets $\bigcup_{\sigma \in S^{\mathcal{J}}} Z^\sigma$ has measure zero, and it follows that Λ and Λ^σ are equivalent. ∎

IV. CONCLUSION

We show that a large class of stochastic operators can be reconstructed from the stochastic process resulting from the application of the operator to an appropriately designed deterministic test signal. Our results are based on combining recently developed sampling results for deterministic band limited operators [15, 16, 5] and the sampling theory of stochastic processes [14, 18]. The classes of stochastic operators considered here are characterized by support constraints on the autocorrelation of the stochastic spreading function of the operator. Our results do not necessitate the frequently used WSSUS condition, nor the so-called underspread condition on operators. Moreover, we show that in some cases when the operator cannot be fully determined from its action on a test signal, still, the second order statistics of the spreading function, that is, its autocorrelation can be determined from the operator output. While in the deterministic case the possibility of operator reconstruction only depends on the support size of the spreading function, we show that in the stochastic case geometric considerations play a fundamental role for reconstruction.

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