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Some remarks

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Summary

Constructive design of Gabor frame windows is rare, and most results come from the one-dimensional case. The connection between the geometry of fundamental domains of lattices and Gabor systems was explored first in a series of papers by Han and Wang [HW01], [HW04]. We build upon these results to construct Gabor frames with smooth and compactly supported window functions in higher dimensions. For this purpose we study pairs of lattices with equal density allowing compact and star-shaped fundamental domains. Concrete examples are provided and the results are extended to other special class of lattices. In addition, we make observations on the intricate behavior of Gabor systems with multivariate Gaussian windows.

Contents

1 Introduction

Various applications from signal processing and communications engineering involve time-frequency representations of functions and distributions. Gabor systems provide a useful basic structure for constructing discrete representations of signals as a sum of a translated and modulated copies of a *window function g*. A *Gabor system* we shall denote by $(g, \Lambda) = \{T_x M_\omega : (x, \omega) \in \Lambda\}$ where T_x is a translation $(T_x f)(y) = f(y - x)$, $x \in \mathbb{R}^d$, and M_ω a modulation $(M_\omega f)(y) = e^{2\pi i \langle \omega, y \rangle} f(y)$, $\omega \in \mathbb{R}^d$. The set of translation and modulation parameters is often chosen to be a full-rank lattice in $\mathbb{R}^d$. An expression of the type

$$f = \sum_{x,\omega \in \Lambda} c(x, \omega) T_x M_\omega g, \tag{1}$$

is useful if the sum is (unconditionallty) convergent, and if the computation of the coefficients and storing the necessary information is fast and stable. Gabor systems considered here are frames, and they therefore provide these properties.

Criteria Gabor systems to be frames have been considered in many studies. The most famous one is the density condition, which states that a Gabor system whose lattice has density less than one, is not a Gabor frame for $L^2(\mathbb{R})$ [Dau92], [Jan94]. In the case of a one-dimensional Gaussian γ_1 the lattice density being larger than one is sufficient for the system to be a frame. A characterization of irregular Gabor systems based on the one-dimensional Gaussian window has also been established. In fact, the Gabor system (γ_1, Λ) is a frame for $L^2(\mathbb{R})$ if and only if the density of Λ is greater than 1 [Lyu92], [SW92].

Constructing Gabor frames for lattices Λ of higher dimension ($\Lambda \subset \mathbb{R}^{2d}, d \geq 2$) is a much harder task. For lattices generated by a diagonal matrix, an explicit construction of a window function g is not difficult [DGM86]. Moreover, for any $\Lambda = \alpha A \mathbb{Z}^{2d}, A$ - symplectic, $\alpha < 1$, it is easy to construct a Gabor frame for $L^2(\mathbb{R}^d)$ by taking the image of the Gabor frame $(g, \alpha \mathbb{Z}^{2d})$ under a metaplectic transform [Grö01].

Recently, the existence of Gabor frames for arbitrary lattices has been proven. There exists a function $g \in L^2(\mathbb{R}^d)$, such that (g, Λ) is a frame in $L^2(\mathbb{R}^d)$, if and only if the density of Λ is greater or equal to 1. In particular, g can be chosen so that (g, Λ) is a Parseval frame for $L^2(\mathbb{R}^d)$ [Bek04]. In spite of its importance, Bekka proves existence only and his work reveals nothing more about the window besides membership in $L^2(\mathbb{R}^d)$.

The intricate structure of Gabor systems based on characteristic functions on the unit interval is studied in [Jan03]. We add an interesting example of a Gabor system based on a characteristic function in $\mathbb{R}^2$ in Section 4.4.

Another recent study constructs Gabor frames for $L^2(\mathbb{R}^{2d}), d \geq 2$ for separable lattices $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$. It shows that any separable lattice $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ with $D(\Lambda) = 1$ admits a Gabor orthonormal basis, as well as that any separable lattice Λ with $D(\Lambda) > 1$ admits a Gabor frame for $L^2(\mathbb{R}^d)$. However, the Gabor windows are constructed as characteristic functions on sets that are fundamental domains for pairs of lattices in $\mathbb{R}^d$ [HW01], [HW04]. The fundamental domains may well be unbounded, so the constructed Gabor window decays neither in time nor in frequency.

This technical report aims to provide some examples and to give results on constructing Gabor frames (g, Λ) in $L^2(\mathbb{R}^2)$ with window $g \in C_c^\infty(\mathbb{R}^2)$ for separable lattices Λ. Our search for window functions, which are smooth and compactly supported, is motivated by the following consideration. Whenever, g is compactly supported, the Gabor coefficients $\langle f, T_x M_\omega g\rangle$ provides a good information about the size of f near time x. However, if g is discontinuous, the coefficients $\langle \widehat{f}, M_{-x} T_\omega \widehat{g}\rangle$ do not provide information about localization in frequency because of the poor decay of $\widehat{g}$ (due to Heisenberg's uncertainty principle). Whenever bandwidth limitations are imposed (for instance, in radio communications the available bandwidth is portioned between users in order to avoid signal interference), poor frequency localization is a problem as well. However, a well-known result [Kat76] states that smoothness in the time-domain implies fast decay in the frequency-domain, guaranteeing better time-frequency localization of the function f.

Expansions similar to (1) would be very easy to work with, if (g, Λ) were an orthonormal basis (ONB). However, Gabor ONBs (and more generally, Riesz bases) are severely restricted by a fact known as the Amalgam Balian-Low theorem [BHW98], [GHHK03], [Grö01], [CP06], [BCM03]. This theorem states that a Gabor orthonormal basis is not possible even under weak assumptions about the time-frequency localization of the window function. In other words, if we require both $g, \widehat{g}$ to have (a) fast decay (implying good time-frequency localization), and (b) non-redundancy, uniqueness and unconditional convergence of (1), then (g, Λ) can span at most a subspace of $L^2(\mathbb{R}^d)$. Under the requirement that every $f \in L^2(\mathbb{R}^d)$ has an unconditionally convergent expansion 1 and both $g, \widehat{g}$ to have fast decay, then (g, Λ) can be at most a frame but not a Riesz basis, so we forego uniqueness of the expansion coefficients. The Amalgam Balian-Low theorem essentially states that (a) non-redundancy with completeness and convergence and (b) good time-frequency localization are mutually incompatible in the context of Gabor systems.

The structure of this technical report is as follows. Section 2 provides a background on lattices, fundamental domains of lattices and some aspects of Fourier analysis related to translational tiling. Furthermore, we recall some notions from functional analysis, such as Bessel sequences, Riesz bases and frames, with a focus on basic properties of Gabor frames and Riesz bases.

Section 3 outlines the geometric approach for demonstrating existence of certain Gabor windows for certain lattices. In particular, we give an overview of the approach in [HW01], [HW04] for constructing Gabor orthonormal bases and frames for separable lattices in $\mathbb{R}^{2d}$. This result states that whenever $A, B \in GL(d, \mathbb{Q})$, with $|\det A \cdot \det B| = 1$ then there exists a window function for the lattice

$A\mathbb{Z}^d \times B\mathbb{Z}^d$, creating a Gabor ONB for $L^2(\mathbb{R}^d)$ [HW01], [HW04]. The existence of an ONB for $L^2(\mathbb{R}^d)$ is an essential ingredient in proving existence of a Gabor frame for $L^2(\mathbb{R}^d)$. However, the constructed window functions are always (discontinuous) characteristic functions.

In Section 4 we discuss a "hands-on" approach, involving an explicit construction of smooth window functions for certain class of separable lattices. We observe that an extra property of a fundamental domains for lattices (star-shapedness) allows the construction of Gabor frames with smooth windows (Theorem 4.2). Furthermore, these explicit constructions can be extended to lattices generated by lower-block diagonal matrices with additional properties (Proposition 4.4). As an illustration we provide some examples of pairs of lattices in $\mathbb{R}^4$ which admit a common star-shaped fundamental domains in Section 4, namely Theorem 4.8 and following.

In Section 5 we consider some examples of Gabor systems with Gaussian windows in dimensions greater than 2. These examples demonstrate that the behavior of Gaussians in higher-dimensional Gabor systems does not depend on the density of the relevant lattice in a straightforward way. We discuss there a potential alternative criterion, called symplectic capacity of a lattice.

2 Theoretical background

2.1 Basic notation

This section provides a basic review of terminology and notation used in this paper. $C^\infty(\mathbb{R}^d)$ denotes smooth (arbitrarily often differentiable), complex-valued functions on $\mathbb{R}^d$, $C_0(\mathbb{R}^d)$ the continuous functions vanishing at infinity, $C_c(\mathbb{R}^d)$ the compactly supported continuous functions. $L^1(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$ are the standard Banach spaces of integrable, resp. square-integrable functions. $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class of rapidly decreasing functions on $\mathbb{R}^d$, in other words,

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta f(x)| < \infty : \quad \text{for all multi-indices } \alpha, \beta\}.$$

Its dual space $\mathcal{S}'(\mathbb{R}^d)$, or the space of tempered distributions, is the space of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$.

We now present the three most important unitary operators in time-frequency analysis on $\mathbb{R}^d$. Using the notation for the frequency domain $\widehat{\mathbb{R}}^d$ as the domain dual to the time domain $\mathbb{R}^d$, we have $\widehat{\mathbb{R}}^d \simeq \mathbb{R}^d$ as topological groups [Kat76]. A *translation* or a *time shift* is the operator $(T_x f)(t) = f(t-x)$, $x \in \mathbb{R}^d$, and a *modulation* or a *frequency shift* is the operator $(M_\omega f)(t) = e^{2\pi i \langle \omega, t\rangle} f(t)$, $\omega \in \widehat{\mathbb{R}}^d$. A *time-frequency shift* is

$$(\pi(\lambda)f)(t) = (M_\omega T_x f)(t) = e^{2\pi i \langle \omega, t\rangle} f(t-x)$$

for $\lambda = (x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

2.2 Fourier transform and short-time Fourier transform

For our purposes the Fourier transform will be defined with the standard normalization. The *Fourier transform* is the map $\mathcal{F} : L^1(\mathbb{R}^d) \to C_0(\widehat{\mathbb{R}}^d)$, mapping f to $\widehat{f} = \mathcal{F}f$, which is given by

$$\widehat{f}(\omega) = \int_{\mathbb{R}^d} f(y) e^{-2\pi i \langle \omega, y \rangle} dy.$$

While the Fourier transform is defined for integrable functions, it can be naturally extended to a unitary operator on $L^2(\mathbb{R}^d)$ (a result known as Plancherel-Parseval theorem) [Kat76]. The Fourier transform is an isomorphism of the Schwarz class $\mathcal{S}(\mathbb{R}^d)$ to itself [Kat76]. It 'intertwines' the translation and modulation operators in the following way: $\mathcal{F}T_x f = M_{-x}\mathcal{F}f$ and $\mathcal{F}M_\omega f = T_\omega \mathcal{F}f$.

A key tool in Gabor analysis is the short-time Fourier transform (STFT), also called 'continuous Gabor transform' or 'windowed Fourier transform'. The short-time Fourier transform is defined by

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega t} \overline{g(t-x)} dt = \langle f, M_\omega T_x g \rangle = \langle \widehat{f}, T_\omega M_{-x} \widehat{g} \rangle \qquad (2)$$

The STFT is a time-frequency representation of f because it carries information simultaneously from the time and frequency domains. For example, (2) states that if g (respectively $\hat{g}$) is well-localized, then $|V_g f(x, \omega)|$ can only be large if a significant amount of the energy of f is concentrated around x (and, if a significant amount of the frequency content of f is near ω). For a set Λ, the collection $\{V_g f(\lambda), \lambda \in \Lambda\}$ is called the set of Gabor coefficients of f.

The short-time Fourier transform $V_g f$, just like the Fourier transform, completely determines f as shown by the following well-known inversion formula. All $f \in L^2(\mathbb{R}^d)$ can be reconstructed from its STFT via the equality

$$f = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^{2d}} V_g f(x, \omega) M_\omega T_x \gamma \, d\omega dx, \qquad (3)$$

for all $g, \gamma \in L^2(\mathbb{R}^d)$ with $\langle g, \gamma \rangle \neq 0$. The integral on the right-hand side is vector-valued. The equality (3), therefore, is understood in a weak sense [Grö01].

2.3 Lattices and fundamental domains

Next we recall some definitions about lattices. A *lattice* in $\mathbb{R}^{2d}$ is a discrete subgroup of the additive group $\mathbb{R}^{2d}$, i.e. $\Lambda = A\mathbb{Z}^{2d}$. In our discussion Λ will always be a *full-rank lattice* in $\mathbb{R}^{2d}$ ($\det A \neq 0$). In line with the notation for the dual group of $\mathbb{R}^d$, we will sometimes write the lattice as a subset of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. The *dual lattice* $\Lambda^\perp$ of Λ is defined as

$$\Lambda^\perp = \{\lambda \in \mathbb{R}^{2d} : \langle \lambda, \mu \rangle \in \mathbb{Z}, \forall \mu \in \Lambda\}.$$

The *adjoint lattice* Λ° of Λ is defined as

$$\Lambda^\circ = \{\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d : \pi(\lambda)\pi(\mu) = \pi(\mu)\pi(\lambda), \forall \mu \in \Lambda\}.$$

Note that for a lattice $\Lambda = A\mathbb{Z}^{2d}$, the dual $\Lambda^\perp = A^{-T}\mathbb{Z}^{2d}$. A lattice is *separable* if it can be represented as $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$. The adjoint lattice for a separable lattice is $\Lambda^\circ = B^{-T}\mathbb{Z}^d \times A^{-T}\mathbb{Z}^d$.

An important notion is that of a fundamental domain (a tiling set) for a lattice. A *fundamental domain* Ω for a lattice Λ in $\mathbb{R}^d$ is a measurable set in $\mathbb{R}^d$ with the following properties

- $(\Omega + \lambda_1) \cap (\Omega + \lambda_2)$ is a null set for $\lambda_1 \neq \lambda_2$ from Λ. (4)
- $\mathbb{R}^d = \bigcup_{\lambda \in \Lambda} (\Omega + \lambda)$. (5)

An alternative formulation is that Ω is a *tiling set* for Λ, or that Ω tiles Λ. An condition equivalent to (4) and (5) is to require that the equality

$$\sum_{\lambda \in \Lambda} \chi_\Omega(x - \lambda) = 1 \tag{6}$$

holds for almost all $x \in \mathbb{R}^d$. Condition (4) on its own is equivalent to the inequality

$$\sum_{\lambda \in \Lambda} \chi_\Omega(x - \lambda) \leq 1, \quad \text{for almost all } x.$$

Furthermore, 6 can be converted int a general statement about tiling with a function f, in other words, f *tiles* by Λ if

$$\sum_{\lambda \in \Lambda} f(x - \lambda) = 1.$$

Condition (4) on its own is equivalent to the inequality

$$\sum_{\lambda \in \Lambda} \chi_\Omega(x - \lambda) \leq 1, \quad \text{for almost all } x.$$

If only (4) is satisfied, then Ω is a *packing set* for Λ, or that Ω packs Λ. Furthermore, if (4) and $m(\Omega) = \operatorname{vol}\Lambda$ hold simultaneously, then Ω is a fundamental domain for Λ.

The *volume* of the lattice $\Lambda = A\mathbb{Z}^d$, $\operatorname{vol}\Lambda = m(\mathbb{R}^d/\Lambda) = |\det A|$, and the *density* of $\Lambda, d(\Lambda) = (\operatorname{vol}\Lambda)^{-1}$. Clearly, $\operatorname{vol}\Lambda$ equals the area of a fundamental domain for Λ.

Next, we turn our attention to *symplectic lattices*, which are parametrized by symplectic matrices. These matrices are a special subclass of volume-preserving matrices $\mathrm{SO}(2d, \mathbb{R})$. Let $\lambda = (x, \omega)$, $\lambda' = (x', \omega') \in \mathbb{R}^{2d}$. The bilinear form

$[\lambda, \lambda'] = x'\omega - x\omega'$ is called symplectic. A matrix $M \in \mathrm{GL}(2d, \mathbb{R})$ is *symplectic* if $[M\lambda, M\lambda'] = [\lambda, \lambda']$. The symplectic matrices form a group $\mathrm{Sp}(d)$, a proper subgroup of $\mathrm{SO}(d)$ [Fol89].

A characterization of a symplectic matrix can be expressed through relations between its blocks: a block matrix $M = \begin{pmatrix} A & C \\ D & B \end{pmatrix}$ is *symplectic* if and only if $AD^T = A^T D$, $BC^T = B^T C$ and $A^T B - D^T C = I$ [Fol89], [Grö01]. It is a well-known fact [Fol89] that $\mathrm{Sp}(d)$ is generated by the matrices

$$\begin{pmatrix} B & 0 \\ 0 & B^* \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}, \quad \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where B, C are such that $\det B \neq 0, C = C^T$ [Fol89].

To every symplectic matrix M there is an associated metaplectic operator $\mu(M)$, which is unitary on $L^2(\mathbb{R}^d)$. The metaplectic operators associated to the canonical generators are dilations D_B, multiplication by the chirp $e^{-\pi i \langle t, Ct\rangle}$ and the Fourier transform respectively. Hence, every metaplectic operator is a composition of these three operators, up to a unit factor [Fol89], [Grö01].

2.4 Translational tiling and Fourier analysis

Fundamental domains can be studied using methods from Fourier analysis. A deep result from this field states that a bounded measurable set Ω of unit measure in $\mathbb{R}^d$ is a tiling set for a lattice if it possesses a spectrum (an ONB of exponentials for $L^2(\Omega)$) [Fug74].

First, we recall some relations between measures on lattices. For a lattice Λ we define the σ-finite measure

$$\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$$

While δ_Λ is infinite on $\mathbb{R}^d$ ($\delta_\Lambda(\mathbb{R}^d) = \infty$), it is finite on any compact subset of $\mathbb{R}^d$. Because Λ is a discrete subgroup of $\mathbb{R}^d$, δ_Λ is of bounded variation on any compact subset of $\mathbb{R}^d$, and thus can be interpreted as a tempered distribution in $\mathcal{S}'(\mathbb{R}^d)$ [Bag92].

We use the Poisson summation formula [Kat76],[Grö01]:

$$\sum_{\lambda \in \Lambda} f(x + \lambda) = \frac{1}{\operatorname{vol} \Lambda} \sum_{\lambda \in \Lambda^\perp} \widehat{f}(\lambda) e^{2\pi i \langle \lambda, x\rangle} \tag{7}$$

to derive a formula for the Fourier transform of the distribution δ_Λ. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then

$$\langle \widehat{\phi}, \widehat{\delta_\Lambda}\rangle = \langle \phi, \delta_\Lambda\rangle = \sum_{\lambda \in \Lambda} \phi(\lambda) = \frac{1}{\operatorname{vol} \Lambda} \sum_{\tilde{\lambda} \in \Lambda^\perp} \widehat{\phi}(\tilde{\lambda}) = \frac{1}{\operatorname{vol} \Lambda} \langle \widehat{\phi}, \delta_{\Lambda^\perp}\rangle.$$

Therefore, as tempered distributions

$$\widehat{\delta_\Lambda} = \frac{1}{\operatorname{vol}\Lambda}\delta_{\Lambda^\perp}. \tag{8}$$

The following lemma presents a criterion for a set Ω to be a fundamental domain for Λ in terms of the zero set of a function. We denote the set of zeros of a function f by $\mathcal{N}(f)$.

Lemma 2.1 *Let Ω be a measurable set in $\mathbb{R}^d$, Λ a lattice in $\mathbb{R}^d$ such that $m(\Omega) = \operatorname{vol}\Lambda$. Assume $\widehat{\chi_\Omega} \in C^\infty(\mathbb{R}^d)$. The following statements are equivalent:*

1. *Ω is a fundamental domain for Λ.*
2. *$\widehat{\chi_\Omega}$ vanishes on $\Lambda^\perp\backslash\{0\}$.*
3. *$\operatorname{supp}\widehat{\delta_\Lambda} \subseteq \{0\} \cup \mathcal{N}(\widehat{\chi_\Omega})$.*

Proof. Formally speaking, (6) is equivalent to

$$\chi_\Omega * \delta_\Lambda(t) = 1.$$

Taking Fourier transforms of both sides (as distributions) we obtain $\widehat{\chi_\Omega} \cdot \widehat{\delta_\Lambda} = \delta_0$. Since the right side is Dirac's delta, the zeros of $\widehat{\chi_\Omega}$ have to eliminate all point masses of $\widehat{\delta_\Lambda}$ except that at 0, in other words,

$$\operatorname{supp}\widehat{\delta_\Lambda} \subseteq \{0\} \cup \mathcal{N}(\widehat{\chi_\Omega}).$$

Taking into account (8), this condition is equivalent to saying $\widehat{\chi_\Omega}$ vanishes on $\Lambda^\perp\backslash\{0\}$. Hence, *2.* $\Leftrightarrow$ *3.*

We follow the approach from [Kol04]. Assume $\sum_{\lambda\in\Lambda}\chi_\Omega(x-\lambda) = 1$ almost everywhere. We have to determine $\operatorname{supp}\widehat{\delta_\Lambda}$. From the definition of support for a tempered distribution we must show that $\langle \phi, \widehat{\delta_\Lambda}\rangle = 0$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ which are supported in the complement of (the closed set) $\{0\} \cup \mathcal{N}(\widehat{\chi}_\Omega)$. Let us denote in the following $f^*(x) = \overline{f(-x)}$.

According to the convolution formula for the Fourier transform for functions in $L^1(\mathbb{R}^d)$.

$$\mathcal{F}((\widehat{\chi}_\Omega)^* \cdot h)(\xi) = \chi_\Omega^* * \widehat{h}(\xi) \tag{9}$$

Since χ_Ω is a nonnegative function, we have $(\widehat{\chi}_\Omega)^* = \widehat{\chi}_\Omega$. So every $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp}\phi \subseteq (\{0\} \cup \mathcal{N}(\widehat{\chi}_\Omega))^c$ can be represented as $(\widehat{\chi}_\Omega)^* \cdot \psi$, where

$$\psi(s) = \begin{cases} \frac{\phi(s)}{(\widehat{\chi}_\Omega)^*(s)}, & s \notin \{0\} \cup \mathcal{N}(\widehat{\chi}_\Omega), \\ 0, & s \in \{0\} \cup \mathcal{N}(\widehat{f}). \end{cases}$$

is also in $\mathcal{S}(\mathbb{R}^d)$, and ψ vanishes on $(\{0\} \cup \mathcal{N}(\widehat{\chi}_\Omega))^c$. We can now apply the Fourier calculus for tempered distributions to $\langle \phi, \widehat{\delta_\Lambda}\rangle$ and derive the following.

$$\begin{aligned} \langle \phi, \widehat{\delta_\Lambda}\rangle &= \langle (\widehat{\chi}_\Omega)^* \cdot \psi, \widehat{\delta_\Lambda}\rangle \\ &= \langle \mathcal{F}^{-1}((\widehat{\chi}_\Omega)^* \cdot \psi), \delta_\Lambda\rangle \end{aligned}$$

We write out this inner product as a sum of inner products with the individual Dirac deltas, and use (9) to simplify it:

$$
\begin{aligned}
&= \sum_{\lambda\in\Lambda} \langle \chi_\Omega^* * \widehat{\psi}, \delta_\lambda \rangle \\
&= \sum_{\lambda\in\Lambda} \chi_\Omega^* * \widehat{\psi}(\lambda) \\
&= \sum_{\lambda\in\Lambda} \int_{\mathbb{R}^d} \chi_\Omega(t-\lambda)\widehat{\psi}(t)\, dt
\end{aligned}
$$

due to $\chi_\Omega^*(\lambda - t) = \chi_\Omega(t-\lambda)$ (χ_Ω is positive). Next, Fubini's theorem allows us to exchange sum and integral and substitute (6):

$$
\langle \phi, \widehat{\delta_\Lambda} \rangle = \int_{\mathbb{R}^d} \sum_{\lambda\in\Lambda} \chi_\Omega(t-\lambda)\widehat{\psi}(t)\, dt = \int \widehat{\psi}(t)\, dt = \psi(0) = 0
$$

Thus $\operatorname{supp}\widehat{\delta_\Lambda} \subseteq \{0\} \cup \mathcal{N}(\widehat{\chi_\Omega})$, proving implication *1.* $\Rightarrow$ *3.*

To prove *3.* $\Rightarrow$ *1.*, we have to show that $h(t) = \sum_{\lambda\in\Lambda} \chi_\Omega(t-\lambda)$ is a constant under condition *3.* In other words, we must prove there exists a constant C such that

$$
\int h\phi = C \int \phi, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d).
$$

For $\phi \in \mathcal{S}(\mathbb{R}^d)$, we compute

$$
\int h\widehat{\phi} = \int_{\mathbb{R}^d} \sum_{\lambda\in\Lambda} \chi_\Omega(s-\lambda)\widehat{\phi}(s)\, ds
$$

after a change of variables $t = s - \lambda$, and exchanging sum and integral

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} \chi_\Omega(t) \sum_{\lambda\in\Lambda} \widehat{\phi}(t+\lambda)\, dt \\
&= \int_{\mathbb{R}^d} \chi_\Omega(t) \langle T_{-t}\widehat{\phi}, \delta_\Lambda \rangle\, dt
\end{aligned}
$$

We again apply the formula for the Fourier transform for $\mathcal{S}(\mathbb{R}^d)$ to get for the inner product

$$
= \int_{\mathbb{R}^d} \chi_\Omega(t) \langle M_t\phi, \widehat{\delta_\Lambda} \rangle\, dt
$$

By (8), $\widehat{\delta_\Lambda}$ is a σ-finite measure, whence

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} \chi_\Omega(t) \int_{\mathbb{R}^d} \phi(\lambda) e^{2\pi i \langle t, \lambda\rangle}\, d\widehat{\delta_\Lambda}(\lambda)\, dt \\
&= \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \chi_\Omega(t) e^{2\pi i \langle t, \lambda\rangle} \phi(\lambda)\, d\widehat{\delta_\Lambda}(\lambda)\, dt
\end{aligned}
$$

Integrating in t gives the Fourier transform of χ_Ω:

$$= \int_{\mathbb{R}^d} \widehat{\chi_\Omega}(-\lambda)\phi(\lambda)\, d\widehat{\delta_\Lambda}(\lambda)$$

Now we plug in the condition on the support of $\widehat{\delta_\Lambda}$ and obtain

$$= \widehat{\delta_\Lambda}(0)\widehat{\chi_\Omega}(0)\phi(0) = C\phi(0),$$

for some constant C. However,

$$\widehat{\chi_\Omega}(0) = \int \chi_\Omega(t)\, dt = m(\Omega) = \operatorname{vol}\Lambda,$$

and (8) implies that $\widehat{\delta_\Lambda}(0) = \frac{1}{\operatorname{vol}\Lambda}$. Thus,

$$\int h\widehat{\phi} = \phi(0) = \int \widehat{\phi},$$

implying that $h(x)$ is constant 1 for almost all x. $\square$

A direct and important consequence of Lemma 2.1 is the existence of a spectrum (an ONB of pure frequencies) for fundamental domains.

Theorem 2.2 ([Fug74]) *Let Ω be a bounded open measurable set in $\mathbb{R}^d$, and Λ a lattice in $\mathbb{R}^d$ with $\operatorname{vol}\Lambda = m(\Omega)$. If Ω is a fundamental domain for Λ, then the normalized exponentials $\{(m(\Omega)^{-\frac{1}{2}}e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda^\perp\}$ form an orthonormal basis for $L^2(\Omega)$. Furthermore, if $\operatorname{vol}\Lambda = m(\Omega) = 1$, and $\{(m(\Omega)^{-\frac{1}{2}}e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda^\perp\}$ form an orthonormal basis for $L^2(\Omega)$, then Ω is a fundamental domain for Λ.*

Proof. We observe first of all that

$$\langle e^{2\pi i\langle x\cdot\rangle}, e^{2\pi i\langle\lambda,\cdot\rangle}\rangle = \int_\Omega e^{2\pi i\langle x-\lambda,y\rangle}dy = \widehat{\chi_\Omega}(x-\lambda). \tag{10}$$

If Ω is a fundamental domain, the equality (10) and statement 2. from Lemma 2.1 imply that the exponentials $\{e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda^\perp\}$ are orthogonal as elements of $L^2(\Omega)$.

Assume now $\operatorname{vol}\Lambda = m(\Omega) = 1$ and $\{m(\Omega)^{-\frac{1}{2}}e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda^\perp\}$ is an ONB for $L^2(\Omega)$. Then it is enough to check that we have

$$\begin{aligned}\sum_{\lambda\in\Lambda^\perp} |\widehat{\chi_\Omega}(x-\lambda)|^2 &= \sum_{\lambda\in\Lambda^\perp} |\langle e^{2\pi i\langle x,\cdot\rangle}, e^{2\pi i\langle\lambda,\cdot\rangle}\rangle|^2 \\ &= \|e^{2\pi i\langle x,\cdot\rangle}\|^2_{L^2(\Omega)} \\ &= 1\end{aligned} \tag{11}$$

Therefore, by (11), $|\widehat{\chi_\Omega}|^2$ is a tiling for $\Lambda^\perp$ (compare (6),[Kol04]). The remark after Lemma 2.1 implies that the Fourier transform of $|\widehat{\chi_\Omega}|^2$, which is precisely $\chi_\Omega * \chi_\Omega^*$, vanishes on $\Lambda\backslash\{0\}$. But the support of the convolution $\chi_\Omega * \chi_\Omega^*$ is $\Omega - \Omega$, so in fact $(\Omega-\Omega)\cap\Lambda = \{0\}$, equivalent to $\Omega \cap (\Omega + \Lambda\backslash\{0\}) = \varnothing$. Because by assumption $\operatorname{vol}\Lambda = m(\Omega) = 1$, Ω is a fundamental domain for Λ. $\square$

2.5 Frames, Riesz bases, Bessel sequences

This is a brief overview of some general properties of Bessel sequences, Riesz bases, and frames for a separable Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$.

A *Bessel sequence* in $\mathcal{H}$ with bound b is a family of functions $\{f_j\}_{j\in\mathbb{N}}$ such that for all $f \in \mathcal{H}$,

$$\sum_{j\in\mathbb{N}} |\langle f, f_j\rangle|^2 \leq b\|f\|^2. \tag{12}$$

A sequence $\{f_j\} \subset \mathcal{H}$ is a *Riesz sequence* if and only if there exist constants $a, b > 0$ such that for all finitely supported sequences of scalars $\{c_j\}_{j\in\mathbb{N}}$,

$$a\sum_{j\in\mathbb{N}} |c_j|^2 \leq \Big\|\sum_{j\in\mathbb{N}} c_j f_j\Big\|^2 \leq b\sum_{j\in\mathbb{N}} |c_j|^2 \tag{13}$$

A *Riesz basis* for $\mathcal{H}$ is a Riesz sequence whose linear span is complete in $\mathcal{H}$.

A sequence $\mathcal{F} = \{f_j\}_{j\in\mathbb{N}}$ is a *frame* for $\mathcal{H}$ if there exist $0 < a \leq b$ such that for all $f \in \mathcal{H}$,

$$a\|f\|^2 \leq \sum_{j\in\mathbb{N}} |\langle f, f_j\rangle|^2 \leq b\|f\|^2\,. \tag{14}$$

A *frame sequence* is a frame for the closure of its linear span.

The constants $0 < a \leq b$ are called *lower* and *upper frame bound* respectively. A frame is called *tight* if we can choose $a = b$. If $a = b = 1$, the frame is called a *Parseval tight frame.*

The linear map associated to a sequence $\mathcal{F}$

$$S_{\mathcal{F}} : \mathcal{H} \to \mathcal{H}, \quad S_{\mathcal{F}} : f \mapsto \sum_{j\in\mathbb{N}} \langle f, f_j\rangle f_j.$$

is called a *frame operator.* By definition $S_{\mathcal{F}}$ is self-adjoint, but if $\mathcal{F}$ is a frame for $\mathcal{H}$ then $S_{\mathcal{F}}$ is positive, invertible and bounded (details can be found for example in [Chr03]).

Proposition 2.3 *For every frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{N}}$ there exists a dual frame $\tilde{\mathcal{F}} = \{g_j\}_{j\in\mathbb{N}}$ such that every $f \in \mathcal{H}$ has expansion*

$$f = \sum_{j\in\mathbb{N}} \langle f, g_j\rangle f_j = \sum_{j\in\mathbb{N}} \langle f, f_j\rangle g_j.$$

2.6 Gabor frames and Gabor Riesz sequences

This section summarizes the most important definitions and properties from the theory of Gabor frames and Gabor Riesz basic sequences.

Definition 2.4 *Let $\Lambda \subset \mathbb{R}^{2d}$ be a discrete set. A Gabor system (g, Λ) for $L^2(\mathbb{R}^d)$ is the set of all time-frequency shifts of the window function g by $\lambda = (x, \omega) \in \Lambda$, i.e.*

$$(g, \Lambda) := \{g_\lambda : \lambda \in \Lambda\},$$

for $g_\lambda(t) = \pi(\lambda)g(t) = T_x M_\omega g = g(t-x)e^{2\pi i \langle \omega, t\rangle}$

We outline the basic definitions:

- A Gabor system (g, Λ) is a *Riesz basis sequence* if there exist constants $0 < a \leq b$ such that for all $\boldsymbol{c} \in \ell^2(\Lambda)$,

$$a\|\boldsymbol{c}\|_{\ell^2}^2 \leq \|\sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda) g\|_2^2 \leq b\|\boldsymbol{c}\|_{\ell^2}^2 \,. \tag{15}$$

- A *Gabor Riesz basis* is a Riesz basis for $L^2(\mathbb{R}^d)$ if it is also complete in $L^2(\mathbb{R}^d)$.

- A Gabor system (g, Λ) is a *frame* for $L^2(\mathbb{R}^d)$ with frame bounds $0 < a \leq b$ if such that for all $f \in L^2(\mathbb{R}^d)$,

$$a\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda) g\rangle|^2 \leq b\|f\|^2 \,. \tag{16}$$

- A *Gabor frame sequence* is a frame for the L^2-closure of its linear span.

- A Gabor *Bessel sequence* is a sequence for which (14) holds with $a = 0, b > 0$.

The operator

$$S_{(g,\Lambda)} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d); \quad S_{(g,\Lambda)} : f \mapsto \sum V_g f(\lambda)\, \pi(\lambda) g$$

is called a *Gabor frame operator.* It is a positive, bounded, invertible and self-adjoint operator if (g, Λ) is a frame for $L^2(\mathbb{R}^d)$.

For the rest of the paper, Λ will always denote a regular lattice, parametrized by a matrix from $\mathrm{GL}(2d, \mathbb{R})$. Such lattices provide a large set of properties of the respective Gabor frames. The frame operator $S_{g,\Lambda}$ commutes with the time-frequency shifts $\{\pi(\lambda), \lambda \in \Lambda\}$ [Chr03]. This property of the frame operator underlies the fundamental observation that the dual frame of a Gabor frame on a regular lattice has the structure of a Gabor frame with the same lattice. Gabor frames posses therefore a very useful reconstruction formula:

$$f = \sum_{\lambda \in \Lambda} V_g f(\lambda)\, \pi(\lambda)\gamma = \sum_{\lambda \in \Lambda} V_\gamma f(\lambda)\, \pi(\lambda) g, \tag{17}$$

where γ is the (canonical) dual window. For a detailed discussion of further properties of Gabor frames, their duals and the Gabor frame operator we refer to [FK98], [FZ98], [Chr03], [Grö01].

The following very important duality principle relates properties of two Gabor systems on two time-frequency lattices.

Theorem 2.5 (Ron-Shen duality principle) *Let $g \in L^2(\mathbb{R}^d)$, and Λ a full rank lattice. Then (g, Λ) is a frame for $L^2(\mathbb{R}^d)$ if and only if (g, Λ°) is a Riesz sequence.*

This result will be used frequently in Section 3 and 4.

The separability of Λ is crucial for the following statements. If $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$, $S_{g,A\mathbb{Z}^d \times B\mathbb{Z}^d}$ can be represented as a sesquilinear form on $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to \mathbb{R}$: $(f, h) \to \langle S_{g,\gamma} f, h\rangle$. We define the bi-infinite cross-ambiguity Gramian matrix

$$\begin{aligned} &\mathbf{G}(x) = (G_{ij}(x))_{i,j\in\mathbb{Z}^d} : \\ &G_{ij}(x) = (\det B)^{-1} \sum_{k\in\mathbb{Z}^d} \overline{g(x - B^{-T}j - Ak)} g(x - B^{-T}i - Ak). \end{aligned} \tag{18}$$

This matrix is studied in [Wal92], [RS97]. The proof of the statement uses a double application of the Poisson summation formula (7) for $A\mathbb{Z}^d$ and $B\mathbb{Z}^d$. Thus, the Gabor frame operator has the following matrix representation:

Proposition 2.6 (Walnut representation) *Let $g, \gamma \in W(\mathbb{R}^d)$. Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a full-rank lattice in $\mathbb{R}^{2d}$. For $f, h \in L^2(\mathbb{R}^d)$, define the sequences*

$$\mathbf{f}(x) := \{f(x - B^{-T}i) : i \in \mathbb{Z}^d\}, \quad \mathbf{h}(x) := \{h(x - B^{-T}j) : j \in \mathbb{Z}^d\}.$$

Then for all $f, h \in L^2(\mathbb{R}^d)$, the following holds:

$$\langle S_{g,A\mathbb{Z}^d\times B\mathbb{Z}^d} f, h\rangle = \int_{B^{-T}\mathbb{T}^d} \langle \mathbf{G}(x)\mathbf{f}(x), \mathbf{h}(x)\rangle dx. \tag{19}$$

The following proposition characterizes the boundedness and stability of the operator S_g in terms of the matrix $\mathbf{G}(x)$.

Proposition 2.7 *Let $g \in L^2(\mathbb{R}^d)$, and $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a full-rank lattice in $\mathbb{R}^{2d}$. Then*

- S_g is a bounded operator on $L^2(\mathbb{R}^d)$ if and only if there exists $b > 0$ such that $\mathbf{G}(x) \leq bI_{\ell^2}$ for almost all $x \in \mathbb{R}^d$.
- S_g is an invertible operator on $L^2(\mathbb{R}^d)$ if and only if there exists $a > 0$ such that $\mathbf{G}(x) \geq aI_{\ell^2}$ for almost all $x \in \mathbb{R}^d$.

The following theorem shows that symplectic transformations of the lattice leave the Gabor frame property 'invariant'.

Theorem 2.8 *Let Λ be a full rank lattice in $\mathbb{R}^{2d}$ and $M \in \mathrm{Sp}(d)$. Then the following are equivalent:*

1. *There exists a $g \in L^2(\mathbb{R}^d)$ such that (g, Λ) is a Gabor frame for $L^2(\mathbb{R}^d)$.*
2. *There exists a $\tilde{g} \in L^2(\mathbb{R}^d)$ such that $(\tilde{g}, M\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$.*

Remark: The window $\tilde{g} = \mu(M)g$, where $\mu(M)$ is the metaplectic operator associated to M.

Furthermore, Theorem 2.8 remains valid if we replace L^2 by $\mathcal{S}, M^1$ because $\tilde{g}$ is the image of g under a metaplectic operator [Grö01]. In short, this result states that the spanning properties of the Gabor system (g, Λ) are carried onto the Gabor system $(\tilde{g}, M\Lambda)$, since the latter set is the image of the former under a unitary map [Grö01].

2.7 Frames of exponentials

The following propositions list properties of families of exponential functions.

Proposition 2.9 *Let I and J be two bounded open sets in $\mathbb{R}^d$, and Λ be a discrete set in $\mathbb{R}^d$. Then the family $\{e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda\}$ is a Bessel sequence for $L^2(I)$ if and only if it is a Bessel sequence for $L^2(J)$.*

Proof. Assume $\{e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda\}$ is a Bessel sequence for $L^2(I)$ with bound b. We shall show the following:

1. $\{e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda\}$ is a Bessel sequence for $L^2(I')$, for all $I' \subseteq I$.
2. $\{e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda\}$ is a Bessel sequence for $L^2(I + y)$, for all $y \in \mathbb{R}^d$.
3. $\{e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda\}$ is a Bessel sequence for $L^2(I \cup I + y)$, for all $y \in \mathbb{R}^d$.

By covering J with a finite number of translates of I it will follow that $\{e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda\}$ is a Bessel sequence for $L^2(J)$.

To show point 1. we note that we can consider $L^2(I')$ as a subspace of $L^2(I)$, so for any $f \in L^2(I')$,

$$\sum_{\lambda\in\Lambda} |\langle f, e^{2\pi i\langle\lambda,\cdot\rangle}\rangle|^2 \leq b\|f\|^2_{L^2(I)} = b\|f\|^2_{L^2(I')}.$$

Furthermore, $\|f\|_{L^2(I+y)} = \|T_y f\|_{L^2(I)}$, so for $f \in L^2(I + y)$,

$$\begin{aligned}
\sum_{\lambda\in\Lambda} |\langle f, e^{2\pi i\langle\lambda,\cdot\rangle}\rangle|^2 &= \sum_{\lambda\in\Lambda} |\langle T_y f, e^{2\pi i\langle\lambda,\cdot-y\rangle}\rangle|^2 \\
&= \sum_{\lambda\in\Lambda} |e^{2\pi i\langle\lambda,y\rangle}\langle T_y f, e^{2\pi i\langle\lambda,\cdot\rangle}\rangle|^2 \\
&= \sum_{\lambda\in\Lambda} |\langle T_y f, e^{2\pi i\langle\lambda,\cdot\rangle}\rangle|^2 \\
&\leq b\|T_y f\|^2_{L^2(I)} = b\|f\|^2_{L^2(I+y)},
\end{aligned}$$

proving point 2.

Last, without loss of generality we assume $I \cap I + y = \varnothing$, as otherwise we have $I \cup I + y = (I \cap (I + y)^c) \sqcup I + y$, and as $I \cap (I + y)^c \subseteq I$, we can apply point 1. We have for $f \in L^2(I \cup I + y)$, $\|f\|^2_{L^2(I\cup I+y)} = \|P_I f\|^2_{L^2(I)} + \|P_{I+y} f\|^2_{L^2(I+y)}$, hence

$$\begin{aligned}
\sum_{\lambda\in\Lambda} |\langle f, e^{2\pi i\langle\lambda,\cdot\rangle}\rangle|^2 &= \sum_{\lambda\in\Lambda} |\langle P_I f + P_{I+y} f, e^{2\pi i\langle\lambda,\cdot\rangle}\rangle|^2 \\
&\leq 2\sum_{\lambda\in\Lambda} |\langle P_I f, e^{2\pi i\langle\lambda,\cdot\rangle}\rangle|^2 + 2\sum_{\lambda\in\Lambda} |\langle P_{I+y} f, e^{2\pi i\langle\lambda,\cdot\rangle}\rangle|^2 \\
&\leq 2b\|P_I f\|^2_{L^2(I)} + 2b\|P_{I+y} f\|^2_{L^2(I+y)} \\
&= 2b\|f\|^2_{L^2(I\cup I+y)},
\end{aligned}$$

whence point 3. follows. ☐

Proposition 2.10 *Let I be a bounded open set in $\mathbb{R}^d$, $J \subset I$ and Λ be a discrete set in $\mathbb{R}^d$. Then the family $\{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ is a Riesz sequence in $L^2(I)$ if it is a Riesz sequence in $L^2(J)$.*

Proof. Let $\{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ is a Riesz sequence in $L^2(J)$. Let $\boldsymbol{c} = \{c_\lambda\}$ be a finitely supported sequence. Then

$$\|\sum_\lambda c_\lambda e^{2\pi i\langle\lambda,\cdot\rangle}\|^2_{L^2(I)} \geq \|\sum_\lambda c_\lambda e^{2\pi i\langle\lambda,\cdot\rangle}\|^2_{L^2(J)} \geq a\sum_{\lambda\in\Lambda} |c_\lambda|^2 = a\|\boldsymbol{c}\|^2_{\ell^2}.$$

Furthermore, there exist a finite number N of translates of J which cover I, so

$$\|\sum_\lambda c_\lambda e^{2\pi i\langle\lambda,\cdot\rangle}\|^2_{L^2(I)} \leq N\|\sum_\lambda c_\lambda e^{2\pi i\langle\lambda,\cdot\rangle}\|^2_{L^2(J)} \leq Nb\sum_\lambda |c_\lambda|^2 = Nb\|\boldsymbol{c}\|^2_{\ell^2}..$$

Thus $\{e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda\}$ is a Riesz sequence in $L^2(I)$. ☐

The following statement shows that working with exponentials requires more care.

Proposition 2.11 *Let Λ be a lattice in $\mathbb{R}^d$, and Ω be a bounded measurable set in $\mathbb{R}^d$ with $m(\Omega) < \operatorname{vol}\Lambda$. If $(\Omega + \lambda) \cap \Omega$ is null for all $\lambda \in \Lambda \setminus \{0\}$ then the family $\{e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda^\perp\}$ is a tight frame for $L^2(\Omega)$.*

Proof. Let us assume that the intersection $(\Omega + \lambda) \cap \Omega$ is null for all $\lambda \in \Lambda \setminus \{0\}$. Suppose first that $\Omega \subseteq \Delta$, where Δ is a fundamental domain for Λ. We can consider $L^2(\Omega)$ as a subspace of $L^2(\Delta)$ with the embedding $f \mapsto \chi_\Omega f$. By Theorem 2.2 the family $\{e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda^\perp\}$ is an orthonormal basis for $L^2(\Delta)$. Then for $f \in L^2(\Delta)$,

$$\sum_{\lambda\in\Lambda^\perp} |\langle f, e^{2\pi i\langle\lambda,\cdot\rangle}\rangle|^2 = \|f\|^2_{L^2(\Delta)}$$

This holds in particular for $f \in L^2(\Omega)$. Hence $\{e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda^\perp\}$ is a Parseval tight frame for $L^2(\Omega)$. In general we have that the closure of Ω is compact and can be covered by finitely many compact fundamental domains Δ of Λ. Without loss of generality we take Δ to be the canonical domain $\mathbb{R}^d/\Lambda$. Now the set of exponentials $\{e^{2\pi i\langle\lambda,\cdot\rangle} : \lambda \in \Lambda^\perp\}$ constitutes a tight frame for the union of these fundamental domains. Then applying Proposition 2.10, we obtain the desired result. ☐

3 Existence of L^2-windows

In this section we present the known results about separable lattices, their extensions to lattices spanned by block-triangular matrices and at the end we make some observations on tensor frames.

An entry point to the geometric approach to constructing Gabor frames is the following basic observation: a fundamental domains Ω for lattice Λ in $\mathbb{R}^d$ allows a spectrum, i.e. an orthonormal basis for $L^2(\Omega)$ consisting of pure frequencies [Fug74], [IKT03], [KM06], in our notation Theorem 2.2. Whenever Ω is a fundamental domain for a lattice Λ, the translates of Ω along the lattice points are all disjoint up to a set of zero measure. Theorem 3.2 combines these two observations to construct a Gabor ONB for $L^2(\mathbb{R}^d)$. This method can be extended to construct Gabor frames for $L^2(\mathbb{R}^d)$ for special lattices as shown in Theorem 3.3.

3.1 Gabor frames for separable lattices

In this section we review the literature on Gabor frames, with windows being characteristic functions, for separable lattices [HW01],[HW04]. In the one-dimensional case, we refer to [Jan03] for a discussion of Gabor systems with windows characteristic functions. At first we consider a separable lattice $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$. The central result in [HW01] about fundamental domains is the following

Theorem 3.1 ([HW01]) *Let $A\mathbb{Z}^d$ and $B\mathbb{Z}^d$ be two full-rank lattices in $\mathbb{R}^d$, such that* $\det A = \det B$. *Then there exists a measurable set Ω, which is a fundamental domain for both $A\mathbb{Z}^d$ and $B\mathbb{Z}^d$. If $\det B \geq \det A$, then there exists a measurable set Ω, which is a fundamental domain for $A\mathbb{Z}^d$ and $(\Omega + \lambda_1) \cap (\Omega + \lambda_2)$ is a null set for any $\lambda_1 \neq \lambda_2$ from $B\mathbb{Z}^d$.*

This theorem is used in [HW04] to show the existence of Gabor orthonormal bases and Gabor frames on a separable lattice $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ as illustrated in Theorem 3.2. The Gabor window function is a characteristic function.

Theorem 3.2 ([HW04]) *Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a lattice in $\mathbb{R}^{2d}$ with $D(\Lambda) = 1$. There exists $g \in L^2(\mathbb{R}^d)$ such that (g, Λ) is an orthonormal basis for $L^2(\mathbb{R}^d)$.*

Proof. The assumption $D(\Lambda) = 1$ implies $|\det A| = |\det B^{-T}|$. By Theorem 3.1 the lattices $A\mathbb{Z}^d$ and $B^{-T}\mathbb{Z}^d$ have a common fundamental domain, which we denote by Ω.

We consider the Gabor system (χ_Ω, Λ). For $k_1 \neq k_2$ in $\mathbb{Z}^d$, we have

$$\langle T_{Ak_1}\chi_\Omega, T_{Ak_2}\chi_\Omega \rangle = 0, \tag{20}$$

because the supports of the two functions are disjoint. Theorem 2.2 shows that the family $\{e^{2\pi i \langle Bl, y\rangle} : l \in \mathbb{Z}^n\}$ forms an orthogonal basis for $L^2(\Omega)$. To obtain an orthonormal basis, it is enough to normalize the L^2-norm of χ_Ω by setting as a window function

$$g = \frac{\chi_\Omega}{\sqrt{m(\Omega)}} = \frac{\chi_\Omega}{\sqrt{|\det A|}}.$$

The union of all those functions $M_{Bl}T_{Ak}g$ for all $k \in \mathbb{Z}^n$ is (g, Λ). Now if we denote the projections of $f \in L^2(\mathbb{R}^d)$ onto $L^2(\Omega + Ak)$ by $P_k f = \chi_{\Omega+Ak} \cdot f$, we have by (20),

$$\|f\|_2^2 = \sum_{k\in\mathbb{Z}^d} \|P_k f\|_2^2$$

For all $k \in \mathbb{Z}^d$ Theorem 2.2 implies that

$$\|P_k f\|_2^2 = \sum_{l\in\mathbb{Z}^d} |\langle e^{2\pi i\langle Bl,y\rangle}, P_k f\rangle|^2 = \sum_{l\in\mathbb{Z}^d} |\langle T_{Ak}M_{Bl}g, f\rangle|^2.$$

Therefore,

$$\|f\|_2^2 = \sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} |\langle T_{Ak}M_{Bl}g, f\rangle|^2$$

which implies that (g, Λ) is an orthonormal basis for $L^2(\mathbb{R}^d)$ (since the L^2-operator norm of $T_{Ak}M_{Bl}$ is 1). $\square$

Theorem 3.3 ([HW04]) *Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a lattice in $\mathbb{R}^{2d}$ with $D(\Lambda) > 1$. There exists $g \in L^2(\mathbb{R}^d)$ such that (g, Λ) is a Gabor frame for $L^2(\mathbb{R}^d)$.*

Proof. The density condition $D(\Lambda) > 1$ implies $|\det A| < |\det B^{-T}|$. By Theorem 3.1 there exists a measurable set Ω which is a fundamental domain for $A\mathbb{Z}^d$ and such that the set $\{\Omega + B^{-T}k\} \cap \{\Omega + B^{-T}k'\}$ is null for $k \neq k' \in \mathbb{Z}^d$.

We apply Theorem 2.5 to the system $(\chi_\Omega, B^{-T}\mathbb{Z}^d \times A^{-T}\mathbb{Z}^d)$. We claim that this is a Riesz sequence. Let $\boldsymbol{c} = \{c_{k,l}\}$ be a finitely supported sequence in $\mathbb{C}$.

$$\left\|\sum_{k,l} c_{k,l}T_{B^{-T}k}M_{A^{-T}l}\chi_\Omega\right\|_2^2 = \left\langle \sum_{k,l} c_{k,l}T_{B^{-T}k}M_{A^{-T}l}\chi_\Omega, \sum_{k',l'} c_{k',l'}T_{B^{-T}k'}M_{A^{-T}l'}\chi_\Omega \right\rangle \tag{21}$$

Theorem 2.2 implies that $\{M_{A^{-T}l} : l \in \mathbb{Z}^d\}$ form an orthonormal basis for $L^2(\Omega)$ because Ω is a fundamental domain for $A\mathbb{Z}^d$. On the other hand, the set $\{\Omega + B^{-T}k\} \cap \{\Omega + B^{-T}k'\}$ is null for $k \neq k' \in \mathbb{Z}^d$, so

$$\langle T_{B^{-T}k}M_{A^{-T}l}\chi_\Omega,\, T_{B^{-T}k'}M_{A^{-T}l'}\chi_\Omega\rangle = \delta_{k,k'}\delta_{l,l'}m(\Omega)$$

Hence, the right-hand side in (21) equals nothing but $m(\Omega)\sum_{k,l}|c_{k,l}|^2 = m(\Omega)\|\boldsymbol{c}\|_{\ell^2}^2$. According to (15), $(\chi_\Omega, B^{-T}\mathbb{Z}^d \times A^{-T}\mathbb{Z}^d)$ is a Riesz orthogonal basis for its closed linear span, hence the Gabor system $(\chi_\Omega, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is a frame for $L^2(\mathbb{R}^d)$ (which follows from the Ron-Shen duality Theorem 2.5). $\square$

Next we give another result about spectra.

Corollary 3.4 *Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a lattice in $\mathbb{R}^{2d}$ such that $D(\Lambda) \geq 1$. If Ω is bounded open measurable set in $\mathbb{R}^d$, which is a fundamental domain for $A\mathbb{Z}^d$ and $\Omega \cap \{\Omega + B^{-T}l\}$ is null for $l \neq 0 \in \mathbb{Z}^d$, then the family $\{M_{Bl}\chi_\Omega : l \in \mathbb{Z}^d\}$ is complete in $L^2(\Omega)$.*

Proof. Suppose for contradiction that there exists $h \in L^2(\Omega), h \neq 0$ such that $\langle h, M_{Bl}\chi_\Omega \rangle = 0$ for all $l \in \mathbb{Z}^d$. Then for all $k \in \mathbb{Z}^d$,

$$\langle h, T_{Ak} e^{2\pi i \langle Bl, y \rangle} \chi_\Omega \rangle = 0$$

The proof of Theorem 3.3 then would imply that there exist $a, b > 0$ such that

$$a\|h\|_2^2 \leq \sum_k \sum_l |\langle h, T_{Ak} e^{2\pi i \langle Bl, y \rangle} \chi_\Omega \rangle|^2 \leq b\|h\|_2^2,$$

requiring $\|h\|_2 = 0$. □

In the following we combine the theory of the matrix form of the Gabor frame operator and the results on the geometric constructions. We have the following "no-go" result.

Proposition 3.5 *Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d, A, B \in \mathrm{GL}(d, \mathbb{R})$ with $D(\Lambda) > 1$. Let Ω be a fundamental domain for $A\mathbb{Z}^d$ and a packing for $B^{-T}\mathbb{Z}^d$. If $g \in C(\mathbb{R}^d)$ is supported on Ω, then the Gabor system (g, Λ) is not a frame for $L^2(\mathbb{R}^d)$.*

Proof. Let $g \in L^2(\mathbb{R}^d)$, $\operatorname{supp} g \subseteq \Omega$, generate a Gabor frame $(g, A\mathbb{Z}^d \times B\mathbb{Z}^d)$. We analyze the structure of the associated cross-ambiguity matrix $\mathbf{G}(x)$. If $j \neq i$, then

$$\operatorname{supp} g(x - B^{-T}j - Ak)g(x - B^{-T}i - Ak) \subseteq Ak + [(\Omega + B^{-T}j) \cap (\Omega + B^{-T}i)].$$

Since $(\Omega + B^{-T}j) \cap (\Omega + B^{-T}i)$ is null, then for almost all x, $G_{ji}(x) = 0$, whenever $j \neq i$. Thus the matrix $\mathbf{G}(x)$ (18) is diagonal for almost all x. Consider the matrix entry

$$G_{00}(x) = \sum_{k \in \mathbb{Z}^d} |g(x - Ak)|^2.$$

Since $\operatorname{supp} g = \Omega$, we have only one nonzero term in the summation (namely, the one with $k = 0$), yielding $G_{00}(x) = |g(x)|^2$. By a substituon of the the sequence $\mathbf{c} = \{\delta_{n,0}\}_{n \in \mathbb{Z}^d}$ in the Ron-Shen criterion [Grö01] we see that $a \leq \langle \mathbf{G}(x)\mathbf{c}, \mathbf{c} \rangle \leq b$, because S_g is a bounded and invertible operator on $L^2(\mathbb{R}^d)$. Therefore, the same properties must be transferred to the matrix $\mathbf{G}(x)$ for almost every x. But then

$$\langle \mathbf{G}(x)\mathbf{c}, \mathbf{c} \rangle = G_{00}(x) = |g(x)|^2,$$

which in turn implies

$$a \leq |g(x)|^2 \leq b.$$

Hence g cannot be continuous on $\mathbb{R}^d$. □

Proposition 3.5 implies that results of the type [HW04] do not provide windows with good time-frequency localization. However, we have the following weaker result

Proposition 3.6 *Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ with $D(\Lambda) > 1$. Let Ω be a fundamental domain for $A\mathbb{Z}^d$ and packing for $B^{-T}\mathbb{Z}^d$. Let $g \in C(\mathbb{R}^d), \operatorname{supp} g = \Omega$, that is, $g \neq 0$ almost everywhere on Ω, then (g, Λ) is complete in $L^2(\mathbb{R}^d)$.*

Proof. Let $f \in L^2(\mathbb{R}^d)$. Denote by f_k the restriction of f to $\Omega + Ak$, $k \in \mathbb{Z}^d$. f_k belongs to $L^2(\Omega)$ and can be identified with a Ω-periodic function on $\mathbb{R}^d$. Then

$$\|f\|_2^2 = \sum_{k \in \mathbb{Z}^d} \|f_k\|_2^2$$

Suppose there exists $f \in L^2(\mathbb{R}^d)$ such that

$$\langle f, M_{Bl} T_{Ak} g \rangle = 0, \forall k, l \in \mathbb{Z}^d. \tag{22}$$

However, because $\operatorname{supp} g = \Omega$, for a fixed $k \in \mathbb{Z}^d$, (22) is the Fourier transform of $f_k \cdot T_{Ak} g$ evaluated at Bl. The Fourier expansion expansion of $f_k \cdot T_{Ak} g$ implies that $f_k \cdot T_{Ak} g$ is identically 0 almost everywhere. Because g does not vanish on a subset of Ω of positive measure, $f_k = 0$ almost everywhere for all k. Therefore $f = 0$ almost everywhere. $\square$

We note also that whenever the window function is a characteristic function supported on a (union of) fundamental domains Ω, the matrix $\mathbf{G}(x)$ given by (18) is independent of the shape of Ω.

3.2 Gabor frames for lattices parametrized by block-triangular matrices

In this section we review results from [HW04] about geometric construction of Gabor windows for lattices generated by block-triangular matrices. At first, we consider the lattice $\Lambda \in \mathbb{R}^{2d}$ given by

$$\Lambda = \begin{pmatrix} A & 0 \\ D & B \end{pmatrix} \mathbb{Z}^d,$$

where A, B are full-rank matrices. The idea is to transform this lattice into a separable lattice by multiplying Λ by the matrix

$$T = \begin{pmatrix} I & 0 \\ -DA^{-1} & I \end{pmatrix}$$

which produces the separable lattice

$$T\Lambda = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mathbb{Z}^d \tag{23}$$

Proposition 3.7 ([HW04]) *Let $\Lambda = \begin{pmatrix} A & 0 \\ D & B \end{pmatrix} \mathbb{Z}^{2d}$ be a lattice in $\mathbb{R}^{2d}$ with $A, B \in \mathrm{GL}(d, \mathbb{R})$ and $D(\Lambda) > 1$. Then there exists a function $g \in L^2(\mathbb{R}^d)$ such that (g, Λ) is a Gabor frame for $L^2(\mathbb{R}^d)$.*

Proof. We denote by Ω the set which is tiling for $A\mathbb{Z}^d$ and packing for $B^{-T}\mathbb{Z}^d$, and compute for $\lambda = (x, \omega) \in \Lambda$, the action of the time-frequency shift $\pi(T\lambda)$ on $L^2(\mathbb{R}^d)$, with T defined in (23)

$$
\begin{aligned}
\pi(T\lambda)h =&\pi(x, -DA^{-1}x + \omega)h \\
=&e^{2\pi i\langle -DA^{-1}x+\omega, t-x\rangle}h(t-x) \\
=&M_{AD^{-1}x}e^{2\pi i\langle DA^{-1}x,x\rangle}e^{2\pi i\langle \omega,t\rangle}e^{2\pi i\langle \omega,-x\rangle}h(t-x) \\
=&M_{AD^{-1}x}e^{-2\pi i\langle DA^{-1}t,t\rangle}e^{2\pi i\langle \omega,t\rangle}e^{2\pi i\langle \omega,-x\rangle}e^{2\pi i\langle DA^{-1}t,t\rangle}\times \\
&e^{2\pi i\langle DA^{-1}t,-x\rangle}e^{2\pi i\langle -DA^{-1}x,t\rangle}e^{2\pi i\langle DA^{-1}x,x\rangle}h(t-x) \\
=&M_{AD^{-1}x}e^{\pi i\langle DA^{-1}t,t\rangle}e^{2\pi i\langle w,t-x\rangle}e^{2\pi i\langle DA^{-1}(t-x),t-x\rangle}h(t-x) \\
=&M_{AD^{-1}x}UT_xM_\omega U^{-1}h \\
=&M_{AD^{-1}x}U\pi(\lambda)U^{-1}h
\end{aligned}
\tag{24}
$$

where U is the unitary operator (chirp)

$$
U : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad (Uh)(t) = e^{-2\pi i\langle DA^{-1}t,t\rangle}h(t). \tag{25}
$$

Since $\Omega + Ak$ are disjoint up to a null set for different $k \in \mathbb{Z}^d$, it is clear that for all $\phi \in L^2(\mathbb{R}^d)$

$$
\|\phi\|_2^2 = \sum_{k\in\mathbb{Z}^d} \|P_{Ak}\phi\|_2^2,
$$

where $P_k\phi = \chi_{\Omega+Ak}\phi$. The Gabor system $(\chi_\Omega, T\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ by Theorem 3.3, with bounds, say, $a \le b$. For k fixed, we estimate

$$
a\|P_k\phi\|_2^2 \le \sum_{l\in\mathbb{Z}^d} |\langle \phi, \pi(Ak, Bl)\chi_\Omega|^2 \le b\|P_k\phi\|_2^2. \tag{26}
$$

Let $g = U^{-1}\chi_\Omega$, where U is the operator defined by (25). Let us denote the x-coordinate of $\lambda \in \Lambda$ by x_λ. Rearranging (24), for $f \in L^2(\mathbb{R}^d)$ we calculate

$$
\begin{aligned}
\sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 &= \sum_{\lambda\in\Lambda} |\langle f, M_{AD^{-1}x_\lambda}U^{-1}\pi(T\lambda)\chi_\Omega\rangle|^2 \\
&= \sum_{\lambda\in\Lambda} |\langle M_{AD^{-1}x_\lambda}Uf, \pi(T\lambda)\chi_\Omega\rangle|^2,
\end{aligned}
$$

because $M_{AD^{-1}x_\lambda}$ and U are just phase factors. Now due to choice of T, for all $\lambda \in \Lambda$, there exist $k, l \in \mathbb{Z}^d$ such that $T\lambda = (Ak, Bl)^T$.

$$
\sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 = \sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} |\langle M_{AD^{-1}Ak}Uf, \pi(Ak, Bl)\chi_\Omega\rangle|^2 \tag{27}
$$

Let k be fixed for the moment.

We apply the inequality (26) to $\phi = M_{AD^{-1}Ak}Uf$, and substitute into (27).

$$\begin{aligned} a\|P_k e^{2\pi i\langle DA^{-1}y,Ak\rangle}Uf\|_2^2 &\le \sum_{l\in\mathbb{Z}^d} |\langle M_{AD^{-1}Ak}Uf, \pi(Ak,Bl)\chi_\Omega\rangle|^2 \\ &\le b\|P_k M_{AD^{-1}Ak}Uf\|_2^2, \end{aligned}$$

which is equivalent to

$$\begin{aligned} a\|P_kUf\|_2^2 &\le \sum_{l\in\mathbb{Z}^d} |\langle M_{AD^{-1}Ak}Uf, \pi(Ak,Bl)\chi_\Omega\rangle|^2 \\ &\le b\|P_kUf\|_2^2 \end{aligned}$$

because a modulation leaves the L^2-norm unchanged. As we sum over all $k \in \mathbb{Z}^d$, and then use (27), we obtain

$$\begin{aligned} a\sum_{k\in\mathbb{Z}^d}\|P_kUf\|_2^2 &\le \sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} |\langle M_{AD^{-1}Ak}Uf, \pi(Ak,Bl)\chi_\Omega\rangle|^2 \\ &\le b\sum_{k\in\mathbb{Z}^d}\|P_kUf\|_2^2 \quad \text{and} \\ a\|f\|_2^2 = a\|Uf\|_2^2 &\le \sum_{\lambda\in\Lambda}|\langle f,\pi(\lambda)g\rangle|^2 \le b\|Uf\|_2^2 = b\|f\|_2^2 \end{aligned}$$

Hence (g,Λ) is a Gabor frame for $L^2(\mathbb{R}^d)$. □

Note: g is discontinuous because it is the multiplication of a characteristic function χ_Ω with a chirp given by U^{-1}.

Proposition 3.8 ([HW04]) *Let* $\Lambda = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\mathbb{Z}^{2d}$ *be a lattice in* $\mathbb{R}^{2d}$ *with* $D(\Lambda) > 1$. *Then there exists a function* $g \in L^2(\mathbb{R}^d)$ *such that* (g,Λ) *is a Gabor frame for* $L^2(\mathbb{R}^d)$.

Proof. The upper block-triangular case can be reduced to the lower block-triangular case via a Fourier transform. We consider again a Gabor frame (g,Λ'), where

$$\Lambda' = \begin{pmatrix} -B & 0 \\ C & A \end{pmatrix}\mathbb{Z}^{2d}$$

A, B are full-rank $d\times d$-matrices, and g is the window function from Proposition 3.7. We take the Fourier transform of the elements (g,Λ')

$$\mathcal{F}\pi(\lambda')g = \mathcal{F}M_{Ck+Al}T_{-Bk}g = T_{Ck+Al}M_{Bk}\widehat{g},$$

which are nothing but the time-frequency shifts of g over a lattice

$$\Lambda = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\mathbb{Z}^{2d}$$

that is generated by an upper block-triangular matrix. Because the Fourier transform $\mathcal{F}$ is unitary on $L^2(\mathbb{R}^d)$, the Gabor system $(\widehat{g},\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$. □

Proposition 3.9 ([HW04]) *Let* $\Lambda = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \mathbb{Z}^{2d}$ *be a full-rank matrix in* $\mathbb{R}^{2d}$ *such that* $D(\Lambda) \geq 1$ *and either* DA^{-1} *or* CB^{-1} *is symmetric. There exists* $g \in L^2(\mathbb{R}^d)$ *such that* (g, Λ) *is a Gabor frame for* $L^2(\mathbb{R}^d)$.

Proof. Depending on whether DA^{-1} or CB^{-1} is symmetric, we choose a matrix

$$T = \begin{pmatrix} I & 0 \\ -DA^{-1} & I \end{pmatrix} \quad \text{or} \quad T = \begin{pmatrix} I & -CB^{-1} \\ 0 & I \end{pmatrix}$$

so that $T\Lambda$ is a lower or upper block-triangular lattice. Since the symmetry of a real-valued matrix M implies that $M = M^T$, the matrix T is in both cases symplectic. By Propositions 3.7 and 3.8 and Theorem 2.8, there exists $g \in L^2(\mathbb{R}^d)$ such that (g, Λ) is a frame. $\square$

Proposition 3.10 *Let* $\Lambda = M\mathbb{Z}^{2d}$ *be a lattice in* $\mathbb{R}^{2d}$ *with* $M \in \mathrm{Sp(d)}$. *There does not exist* $g \in \mathcal{S}(\mathbb{R}^d)$ *such that* (g, Λ) *is a Gabor frame for* $L^2(\mathbb{R}^d)$.

Proof. We argue by contradiction. Suppose that there exists a $g \in \mathcal{S}(\mathbb{R}^d)$ such that (g, Λ) is a Gabor frame for $L^2(\mathbb{R}^d)$. By Theorem 2.8, there exists a $\tilde{g} \in \mathcal{S}(\mathbb{R}^d)$ such that $(\tilde{g}, \mathbb{Z}^d)$ is a frame for $L^2(\mathbb{R}^d)$. By the Ron-Shen duality principle, $(\tilde{g}, \mathbb{Z}^d)$ is a Riesz sequence too, for $L^2(\mathbb{R}^d)$. But then the amalgam Balian-Low theorem [GHHK03], [BHW98], [Grö01] holds, yielding a contradiction to our assumption. $\square$

4 Construction of smooth windows

We consider a separable lattice $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$, with $D(\Lambda) > 1$, where A and B are $d \times d$-matrices of full rank. In view of Proposition 3.5 we have to look for window functions in $C_c(\mathbb{R}^d)$ whose support extends beyond the fundamental domain Ω of $A\mathbb{Z}^d$. We shall construct a smooth window function by using a smoothened characteristic function to obtain results similar to Theorem 3.2 and Theorem 3.3.

To simplify our computations we shall obtain statements about Riesz basic sequences based on the adjoint lattice, $\Lambda^\circ = B^{-T}\mathbb{Z}^d \times A^{-T}\mathbb{Z}^d$, for which automatically $D(\Lambda^\circ) < 1$. We will construct a smooth window g such that (g, Λ°) is a Riesz sequence. The existence of a frame (g, Λ) for $L^2(\mathbb{R}^d)$ will be then deduced from Theorem 2.5. In this section for two sets $X, Y \in \mathbb{R}^d$ we shall denote by $X + Y$ the set $\{x + y, x \in X, y \in Y\}$.

4.1 Existence of smooth windows supported on star-shaped fundamental domains

Our goal is to construct a fundamental domain Ω' for $B^{-T}\mathbb{Z}^d$ with the following property: there exists a proper subset Ω of Ω' with $\Omega + B(0, \epsilon) \subset \Omega'$ for some

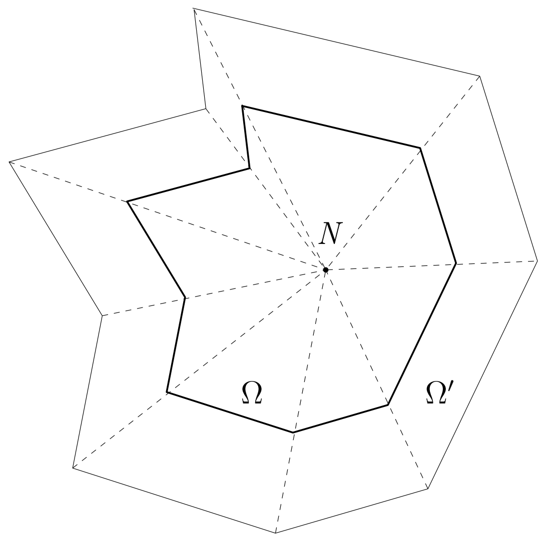

Figure 1: Ω is the scaled image of Ω' under dilation with centre O.

$\epsilon > 0$, which is a fundamental domain for $A\mathbb{Z}^d$ (Figure 1). Due to the fact that $m(\Omega') = |\det B^{-T}| > |\det A| = m(\Omega)$, such sets Ω and Ω' could theoretically exist. In addition as long as the set Ω' is bounded, we can apply the result on perturbed frames [CC97] to a mollified characteristic function.

This leads to the question: *Which lattices allow fundamental domains Ω and Ω' with such properties?* We note that a sufficient condition for our purposes is star-shapedness of the tiling set, stated as

Lemma 4.1 *Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a lattice with $D(\Lambda) > 1$. Let $\tilde{A} = \operatorname{vol}\Lambda^{-\frac{1}{d}}A$. If the lattices $\tilde{A}\mathbb{Z}^d$ and $B^{-T}\mathbb{Z}^d$ have a common fundamental domain Ω' which is star-shaped and compact, then there exists $\epsilon > 0$ and a fundamental domain Ω for $A\mathbb{Z}^d$ such that $\Omega + B(0,\epsilon) \subset \Omega'$.*

Proof. Under the notation

$$\tilde{A} = D(\Lambda)^{\frac{1}{d}}A,$$

we obtain a scaling of the lattice $A\mathbb{Z}^d$ to $\tilde{A}\mathbb{Z}^d$. Furthermore, $|\det\tilde{A}\det B| = 1$. By Theorem 3.1 there exists a measurable set Ω' which is a common fundamental domain for $\tilde{A}\mathbb{Z}^d$ and $B^{-T}\mathbb{Z}^d$. We claim that there exists a fundamental domain Ω for $A\mathbb{Z}^d$ such that $\Omega \subset \Omega'$. For a star-shaped set Ω', there exists a point $N \in \Omega'$ such that for all points $Q \in \Omega'$ the segment $\overrightarrow{NQ}$ is contained entirely within Ω'. We apply a dilation with center N and coefficient $D(\Lambda^\circ)^{-\frac{1}{d}}$ to Ω' and obtain a set Ω which is similar to Ω' and moreover, $\Omega \cap \Omega' = \Omega$ (as illustrated in Figure 1). In addition, there is a δ-neighborhood Ω_δ of Ω (for δ sufficiently small), contained inside Ω'. We claim that Ω is a fundamental domain for the lattice $A\mathbb{Z}^d$.

Using Lemma 2.1 and a change of variables $x = N + yD(\Lambda^\circ)^{\frac{1}{d}}$, we derive

$$\begin{aligned}
\widehat{\chi_\Omega}(\xi) &= \int_\Omega e^{-2\pi i\langle \xi, x\rangle} dx \\
&= \frac{1}{D(\Lambda^\circ)} \int_{T_N\Omega'} e^{-2\pi i\langle \xi, N + yD(\Lambda^\circ)^{\frac{1}{d}}\rangle} dy \\
&= \frac{1}{D(\Lambda^\circ)} \int_{\Omega'} e^{-2\pi i\langle D(\Lambda^\circ)^{\frac{1}{d}}\xi, y\rangle} dy \\
&= \frac{1}{D(\Lambda^\circ)} \cdot \widehat{\chi_{\Omega'}}\left(\xi D(\Lambda^\circ)^{-\frac{1}{d}}\right)
\end{aligned}$$

Since $\widehat{\chi_{\Omega'}}$ vanishes on $\tilde{A}^{-T}\mathbb{Z}^d\backslash\{0\}$, $\widehat{\chi_\Omega}$ vanishes on $D(\Lambda^\circ)^{-\frac{1}{d}}\tilde{A}^{-T}\mathbb{Z}^d\backslash\{0\} = A^{-T}\mathbb{Z}^d\backslash\{0\}$. Lemma 2.1 implies that Ω is a fundamental domain for $A\mathbb{Z}^d$. □

Theorem 4.2 *Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a lattice in $\mathbb{R}^{2d}$ with $D(\Lambda) > 1$. Let $\tilde{A} = D(\Lambda)^{\frac{1}{d}}A$. If the lattices $B^{-T}\mathbb{Z}^d$ and $\tilde{A}\mathbb{Z}^d$ have a common compact star-shaped fundamental domain, there exists $g \in C_c^\infty(\mathbb{R}^d)$ such that (g, Λ) is a frame for $L^2(\mathbb{R}^d)$.*

Proof. We denote the common star-shaped fundamental domain of the lattices $B^{-T}\mathbb{Z}^d$ and $\tilde{A}\mathbb{Z}^d$ by Ω'. The condition on its shape allows us to construct a compact fundamental domain Ω for $A\mathbb{Z}^d$ such that $\Omega \subset \Omega'$.

Let $g \in C_c^\infty(\mathbb{R}^d)$ be such that $g(x) = 1$ for $x \in \Omega$, $g(x) = 0$ for $x \notin \Omega'$, $|g| \leq 1$ elsewhere. We know from the proof of Theorem 3.3 that $(\chi_\Omega, \Lambda^\circ)$ is a Riesz sequence in $L^2(\mathbb{R}^d)$.

Just as Ω is bounded set, so is Ω'. Therefore it is covered by finitely many copies of Ω. Hence for any finitely supported sequence $\mathbf{c} = (c_k)$,

$$\begin{aligned}
\|\sum_{k\in\mathbb{Z}^d} c_k M_{A^{-T}k} g\|^2_{L^2(\Omega')} &= \||g(\cdot)| \cdot |\sum_{k\in\mathbb{Z}^d} c_k e^{2\pi i\langle A^{-T}k, \cdot\rangle}|\|^2_{L^2(\Omega')} \\
&\leq \|\sum_{k\in\mathbb{Z}^d} c_k e^{2\pi i\langle A^{-T}k, \cdot\rangle}\|^2_{L^2(\Omega')},
\end{aligned} \tag{28}$$

because $|g| \leq 1$ on Ω'. The family $\{M_{A^{-T}k}\chi_\Omega : k \in \mathbb{Z}^d\}$ forms a tight frame for $L^2(\Omega)$ (Theorem 2.2). Applying now Proposition 2.9, we see that there exists $C > 0$ such that (28) is bounded by $C\|\mathbf{c}\|_{\ell^2}$ for all sequences $\mathbf{c}$. This implies that $\{M_{A^{-T}k}g : k \in \mathbb{Z}^d\}$ forms a Bessel sequence for $L^2(\Omega')$. Since the translates of $\operatorname{supp} g \subseteq \Omega'$ are disjoint up to a null set by translation in $B^{-T}\mathbb{Z}^d$, we can derive easily that (g, Λ°) is a Bessel sequence. We follow the same argument as in the proofs of Theorem 3.3 and Theorem 3.2 - namely for all finitely supported sequences $\mathbf{c} = (c_{k,n})$:

$$\begin{aligned}
\|\sum_{k,n} c_{k,n} T_{B^*n} M_{A^*k} g\|^2_{L^2(\Omega')} &= \sum_n \|\sum_k c_{k,n} M_{A^*k} g\|^2_{L^2(\Omega')} \\
&\leq C \sum_{k,l} |c_{k,l}|^2 = C\|\mathbf{c}\|^2_{\ell^2},
\end{aligned} \tag{29}$$

Equation (29) holds since the set $\{\Omega' + B^{-T}n_1\} \cap \{\Omega' + B^{-T}n_2\}$ is null whenever $n_1 \neq n_2 \in \mathbb{Z}^d$.

Furthermore, since $g|_\Omega = \chi_\Omega$, we see that

$$\begin{aligned}\|\sum_{k\in\mathbb{Z}^d} c_k M_{A^{-T}k} g\|^2_{L^2(\Omega')} &\geq \|\sum_{k\in\mathbb{Z}^d} c_k M_{A^{-T}k} g\|^2_{L^2(\Omega)} \\ &= \|\sum_{k\in\mathbb{Z}^d} c_k M_{A^{-T}k} \chi_\Omega\|^2_{L^2(\Omega)}\end{aligned}$$

Thus as in (29) we show that

$$\|\sum_{k,n} c_{k,n} T_{B^{-T}n} M_{A^{-T}k} g\|^2_{L^2(\Omega')} \geq C' \|\mathbf{c}\|_{\ell^2}$$

for some constant $C' > 0$. Then the Gabor system (g, Λ°) is also a Riesz basic sequence. By Theorem 2.5, (g, Λ) is a frame for $L^2(\mathbb{R}^d)$. $\square$

In fact the following more general fact is also true:

Corollary 4.3 *Let $\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a lattice in $\mathbb{R}^{2d}$ such that $D(\Lambda) > 1$. Let $\Omega \subset \mathbb{R}^d$ and $\epsilon > 0$ be such that Ω is a bounded fundamental domain for $A\mathbb{Z}^d$ and $\Omega + B(0,\epsilon)$ tiles $B^{-T}\mathbb{Z}^d$. Then there exists a compactly supported and smooth function g such that (g, Λ) is a Gabor frame for $L^2(\mathbb{R}^d)$.*

The result from Theorem 4.2 can be extended to a lattices generated by lower-block triangular matrices.

Proposition 4.4 *Let $\Lambda = \begin{pmatrix} A & 0 \\ D & B \end{pmatrix} \mathbb{Z}^{2d}$ be such that $D(\Lambda)^{\frac{1}{d}} A\mathbb{Z}^d$ and $B^{-T}\mathbb{Z}^d$ have a common fundamental domain Ω' which is star-shaped and compact. Then there exists $g \in C_c^\infty(\mathbb{R}^d)$ such that $(g, \Lambda$ is a frame for $L^2(\mathbb{R}^d)$.*

Proof. We consider the lattice

$$\Lambda = \begin{pmatrix} A & 0 \\ D & B \end{pmatrix} \mathbb{Z}^{2d}.$$

Let

$$T = \begin{pmatrix} I & 0 \\ -DA^{-1} & I \end{pmatrix}.$$

Then we note as in the proof of Proposition 3.7 that $T\Lambda = A\mathbb{Z}^d \times B\mathbb{Z}^d$ is separable and fulfils the conditions of Lemma 4.1.

Then Theorem 4.8 assures that there exists $\tilde{g} \in C_c^\infty(\mathbb{R}^d)$ such that $(\tilde{g}, T\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. From the details of the proof of Proposition 3.7 we conclude that (g, Λ) is a Gabor frame for $L^2(\mathbb{R}^d)$, where $g(t) = e^{2\pi i \langle DA^{-1}t, t\rangle} \tilde{g}(t)$. Obviously $g \in C_c^\infty(\mathbb{R}^d)$. $\square$

4.2 Examples of smooth windows in 2-D

In this section we illustrate the statement given in Proposition 4.1 and Theorem 4.2. For a given pair of matrices in $\mathrm{GL}(2,\mathbb{R})$, we construct explicitly a common convex fundamental domain.

Proposition 4.5 *Let* $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\tilde{A} = \begin{pmatrix} \frac{m}{n} & 0 \\ 0 & \frac{n}{m} \end{pmatrix}$ *where* m, n *are co-prime integers. There exists a common convex fundamental domain for* $B^{-T}\mathbb{Z}^2 = \mathbb{Z}^2$ *and* $\tilde{A}\mathbb{Z}^2$*.*

Proof. The two-dimensional torus is $\mathbb{T}^2 = [0,1) \times [0,1)$. Consider the parallelogram

$$\Omega = \begin{pmatrix} \frac{1}{n} & m \\ 0 & n \end{pmatrix} \mathbb{T}^2.$$

We claim that Ω is a common fundamental domain for the lattices $\mathbb{Z}^2$ and $\begin{pmatrix} \frac{m}{n} & 0 \\ 0 & \frac{n}{m} \end{pmatrix} \mathbb{Z}^2$.

Suppose that there exists $(k,l)^T \neq \overrightarrow{0} \in \mathbb{Z}^2$ such that $\left\{\Omega + \begin{pmatrix} k \\ l \end{pmatrix}\right\} \cap \Omega \neq \varnothing$. Then there exist points $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathbb{T}^2$ such that

$$\frac{\alpha_1}{n} + m\beta_1 + k = \frac{\alpha_2}{n} + m\beta_2 \quad \text{and} \quad n\beta_1 + l = n\beta_2$$

Therefore,

$$\beta_2 - \beta_1 = \frac{l}{n},$$

which implies that $\alpha_1 - \alpha_2 = ml - kn$ must be an integer. Since $0 \leq \alpha_1, \alpha_2 < 1$, we have necessarily $\alpha_1 = \alpha_2$, and also that

$$\beta_2 - \beta_1 = \frac{k}{m}.$$

Since $\gcd(m,n) = 1$ and $0 \leq \beta_2 - \beta_1 < 1$, this is possible only if $k = l = 0$. Thus $(\Omega + \mathbb{Z}^2 \backslash \{0\}) \cap \Omega = \varnothing$. As $m(\Omega) = 1$, Ω is a fundamental domain for $\mathbb{Z}^2$.

We apply the same argument to the lattice $\begin{pmatrix} \frac{m}{n} & 0 \\ 0 & \frac{n}{m} \end{pmatrix} \mathbb{Z}^2$. Suppose that there exists $\begin{pmatrix} k \\ l \end{pmatrix} \neq \overrightarrow{0} \in \mathbb{Z}^2$ such that $\{\Omega + \begin{pmatrix} \frac{m}{n} & 0 \\ 0 & \frac{n}{m} \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix}\} \cap \Omega \neq \varnothing$. Then there exist points $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathbb{T}^2$ such that

$$\frac{\alpha_1}{n} + m\beta_1 + \frac{km}{n} = \frac{\alpha_2}{n} + m\beta_2 \quad \text{and} \quad n\beta_1 + \frac{ln}{m} = n\beta_2$$

The equality

$$\beta_2 - \beta_1 = \frac{l}{m},$$

again leads to the conclusion that $\alpha_2 - \alpha_1 = mk - ln$ is an integer. Because $0 \le \alpha_1, \alpha_2 < 1$, we have $\alpha_1 = \alpha_2$, and

$$\beta_2 - \beta_1 = \frac{k}{n}.$$

Again, since $\gcd(m,n) = 1$ and $0 \le \beta_2 - \beta_1 < 1$, this is possible only if $k = l = 0$. Thus $(\Omega + \mathbb{Z}^2 \backslash \{0\}) \cap \Omega = \varnothing$. Because $m(\Omega) = 1$, Ω is a fundamental domain also for $\begin{pmatrix} \frac{m}{n} & 0 \\ 0 & \frac{n}{m} \end{pmatrix} \mathbb{Z}^2$. □

Example 4.6 *The lattices* $\mathbb{Z}^2$, $\begin{pmatrix} \frac{m}{n} & 0 \\ 0 & \frac{n}{m} \end{pmatrix} \mathbb{Z}^2$ *and* $\begin{pmatrix} \frac{1}{n} & m \\ 0 & n \end{pmatrix} \mathbb{Z}^2$, *where* $\gcd(m,n) = 1$, *have a common convex fundamental domain.*

Example 4.7 *The lattices* $\mathbb{Z}^2$, $\begin{pmatrix} \frac{n}{m} & 0 \\ 0 & \frac{m}{n} \end{pmatrix} \mathbb{Z}^2$ *and* $\begin{pmatrix} n & 0 \\ m & \frac{1}{n} \end{pmatrix} \mathbb{Z}^2$, *where* $\gcd(m,n) = 1$, *have a common convex fundamental domain.*

Theorem 4.8 *Let $m, n \in \mathbb{Z}$ be relatively prime. Let $\Lambda = A\mathbb{Z}^2 \times B\mathbb{Z}^2$ be a lattice in $\mathbb{R}^4$. Whenever $B^T A$ is of the form*

1. kI, $|k| \le 1$;
2. $\begin{pmatrix} m^2 k & 0 \\ 0 & n^2 k \end{pmatrix}$, *where* $|k| < (mn)^{-1}$;
3. $\begin{pmatrix} k & mnk \\ 0 & n^2 k \end{pmatrix}$, *where* $|k| < n^{-1}$; *or*
4. $\begin{pmatrix} n^2 k & 0 \\ mnk & k \end{pmatrix}$, *where* $|k| < n^{-1}$,

there exists a function $g \in C_c^\infty(\mathbb{R}^2)$ such that (g, Λ) is a Gabor frame for $L^2(\mathbb{R}^2)$.

Proof. We have

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \underbrace{\begin{pmatrix} B^{-T} & 0 \\ 0 & B \end{pmatrix}}_{M} \begin{pmatrix} B^T A & 0 \\ 0 & I \end{pmatrix}$$

which shows that $\Lambda = M((B^T A)\mathbb{Z}^2 \times \mathbb{Z}^2)$, and M is symplectic. Since $\det B^T A \le 1$, we can rescale $B^T A \mathbb{Z}^2$ to make its density 1 as in Lemma 4.1. Then $\widetilde{B^T A}\mathbb{Z}^2$ is respectively of the form

$$\mathbb{Z}^2, \begin{pmatrix} \frac{m}{n} & 0 \\ 0 & \frac{n}{m} \end{pmatrix} \mathbb{Z}^2, \begin{pmatrix} \frac{1}{n} & m \\ 0 & n \end{pmatrix} \mathbb{Z}^2, \begin{pmatrix} n & 0 \\ m & \frac{1}{n} \end{pmatrix} \mathbb{Z}^2$$

Examples 4.6 and 4.7 assure the existence of a common convex fundamental domain for $(B^T A)\mathbb{Z}^2$ and $\mathbb{Z}^2$ accordingly. Theorem 4.2 ensures that there exists a smooth and compactly-supported function g' such that $(g', (B^T A)\mathbb{Z}^2 \times \mathbb{Z}^2)$ is a Gabor frame for $L^2(\mathbb{R}^2)$. The matrix M is symplectic, and its associated unitary operator U from Theorem 2.8 is the dilation

$$(Uh)(x) = (\det B)^{-\frac{1}{2}} h(B^{-1}x),$$

see also [Fol89]. Hence, $g = U^* g' \in C_c^\infty(\mathbb{R}^2)$ and (g, Λ) is a Gabor frame for $L^2(\mathbb{R}^2)$. $\square$

Using Proposition 3.7, we can extend the result from Theorem 4.8 to a larger class of 4×4-block-matrices.

Proposition 4.9 *Let* $\Lambda = \begin{pmatrix} A & 0 \\ D & B \end{pmatrix} \mathbb{Z}^4$ *be a lattice in* $\mathbb{R}^4$ *with* $A, B \in \mathrm{GL}(2, \mathbb{R})$ *such that* $B^T A$ *is of the form given in Theorem 4.8. Then there exists a function* $g \in C_c^\infty(\mathbb{R}^2)$ *such that* (g, Λ) *is a Gabor frame for* $L^2(\mathbb{R}^2)$.

Proof. We transform the lattice Λ into a separable lattice $A\mathbb{Z}^2 \times B\mathbb{Z}^2$ by multiplying it by

$$T = \begin{pmatrix} I & 0 \\ -DA^{-1} & I \end{pmatrix}$$

Then Theorem 4.8 assures that there exists $\tilde{g} \in C_c^\infty(\mathbb{R}^2)$ such that $(\tilde{g}, T\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^2)$. The proof of Proposition 3.7 states that (g, Λ) is a Gabor frame for $L^2(\mathbb{R}^2)$, where $g(t) = e^{2\pi i \langle DA^{-1}t, t\rangle} \tilde{g}(t)$. Obviously $g \in C_c^\infty(\mathbb{R}^2)$. $\square$

4.3 A pair of lattices which does not allow a common star-shaped fundamental domain

This construction of windows in C_c^∞ from Sections 4.1 and 4.2 strongly relied on the existence of a common compact and star-shaped fundamental domain for a given lattice pair (Λ_1, Λ_2). Here we provide an example of lattice pair (Λ_1, Λ_2) such that Λ_1, Λ_2 do not have a compact fundamental domain with these properties. This example illustrates the limitations of the method described in Theorem 4.2.

Proposition 4.10 *There exist pairs of lattices in* $\mathbb{R}^2$*, which do not allow a common star-shaped fundamental domain.*

Proof. Assume that there exists a compact star-shaped set Ω' serving as a common fundamental domain for the lattices

$$\Lambda_1 = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix} \mathbb{Z}^2, \quad \Lambda_2 = \mathbb{Z}^2.$$

Then we would be able for (small enough) $\epsilon > 0$ to locate $\Omega \subset \Omega + B_0(\epsilon) \subset \Omega'$ which is a scaled copy of Ω' by $\frac{1}{\sqrt{2}}$ and thus a fundamental domain for $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbb{Z}^d$ (a consequence of Lemma 4.1 - since $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbb{Z}^2 = D_{\frac{1}{\sqrt{2}}} \Lambda_1$).

Furthermore, because Ω' tiles Λ_2, translates of Ω by $\Lambda_2 = \mathbb{Z}^2$ do not have a common side and are never adjacent (due to the fact that each one is a scaled copy of Ω' contained entirely in Ω').

Now we consider the lattices

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbb{Z}^2, \quad \mathbb{Z}^2.$$

We see that the set $\Omega = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbb{T}^2$ is a fundamental domain for $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbb{Z}^2$ and forms a packing for $\mathbb{Z}^2$. This set, however, has the drawback that there is no "free space" under horizontal translations by vectors of the form $(n, 0)^T, n \in \mathbb{Z}$. We may hope to find another tiling set for $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbb{Z}^2$, $\tilde{\Omega}$, such that there exists $\epsilon > 0$ such that $\tilde{\Omega} + B_0(\epsilon)$ packs $\mathbb{Z}^2$. We shall show that this is impossible.

Let Ω be a general fundamental domain for $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbb{Z}^2$. Consider its corona, which is the collection of all adjacent (i.e. having at least one point in common) translates of Ω under vectors $\vec{v} \in \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbb{Z}^2$. We have $\mathbb{Z}^2 \subset \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbb{Z}^2$. If no element from the corona of Ω is a $\mathbb{Z}^2$-translate of Ω, they must all result from vectors $\vec{v} = (n, m + \frac{1}{2})^T, m, n \in \mathbb{Z}$. But then if we take two adjacent translates $T_{\vec{v}_1}\Omega, T_{\vec{v}_2}\Omega$ from the corona, then $\vec{v}_1 - \vec{v}_2 \in \mathbb{Z}^2$. Because the arrangement of the tiles is invariant under translation by vectors from the set $\mathbb{Z}^2 = \Lambda_2$, then $T_{\vec{v}_1 - \vec{v}_2}\Omega$ is an adjacent to Ω, hence it is in the corona of Ω, which is contradiction. Hence a Λ_2-translate of Ω adjacent to Ω always exists. For every $\epsilon > 0$, no set Ω' containing $\Omega + B_0(\epsilon)$ as a proper subset can form a packing for $\mathbb{Z}^2$. Ω'

However, this is a contradiction coming from the assumption that Λ_1, Λ_2 have a common star-shaped fundamental domain. $\square$

4.4 "Janssen's tie" in 2-D

The intricate structure of Gabor systems based on characteristic functions on the unit interval is studied in [Jan03]. In two dimensions the behavior of characteristic functions is even more intricate, as the following examples show:

The pair of lattices considered in Proposition 4.10 have the following interesting property - for a characteristic function on a bounded domain not the size of a domain but it spacial orientation determine the Gabor window property.

Example 4.11 *Let* $\Lambda = A\mathbb{Z}^2 \times \mathbb{Z}^2$*, where* $A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$*. Let* Ω *be a bounded fundamental domain for* $A\mathbb{Z}^2$*, and* $g = \chi_\Omega + T_{j_0}\chi_\Omega, j_0 \in \mathbb{Z}^2, j_0 \neq \vec{0}$*. Then the Gabor system* (g, Λ) *is not a frame for* $L^2(\mathbb{R}^2)$*.*

Proof. As we have seen before, the entries of the cross-ambiguity Gramian matrix defined in (18) for $g = \chi_{\mathcal{B}}$ are given by

$$G_{i,j}(x) = \sum_{k \in \mathbb{Z}^2} \overline{g(x - j - Ak)} g(x - i - Ak). \tag{30}$$

Due to the planar arrangement of the support of g, we see that $G_{i,i}(x) = 2$ for almost all x, while $G_{i,j}(x) = 1$ for $i - j = \pm j_0$ and $G_{i,j} = 0$ for all other i, j. It is easy to see that these coefficients do not depend on the shape of Ω, whence $\mathbf{G}(x)$ is independent of the shape of Ω.

Thus we may at first consider the simplest case, namely $\Omega = A\mathbb{T}^2, j_0 = (1, 0)^T$, which is a $2 \times \frac{1}{2}$-rectangle with sides parallel to the axes in $\mathbb{R}^2$. In this case we know that $g = \chi_{[0,2)} \otimes \chi_{[0,\frac{1}{2})}$. Then $(g, A\mathbb{Z}^2 \times \mathbb{Z}^2)$ is essentially the tensor Gabor system of $(\chi_{[0,2)}, \mathbb{Z} \times \mathbb{Z})$ and $(\chi_{[0,\frac{1}{2})}, \frac{1}{2}\mathbb{Z} \times \mathbb{Z})$. But we know that $(\chi_{[0,2)}, \mathbb{Z} \times \mathbb{Z})$ is not a Gabor frame for $L^2(\mathbb{R}^2)$ (this result is stated as Proposition 3.3.2.1. in [Jan03]), because the lower frame bound is 0. Hence, the tensor system is not a Gabor frame for $L^2(\mathbb{R}^2)$. In return, the cross-ambiguity matrix $\mathbf{G}(x)$ of (g, Λ) is not stable for almost every x

For another j_0 we shall also prove that $\mathbf{G}(x)$ is not stable. We fix a natural number $N > 0$, and consider $f = \sum_{s=0}^{N-1} (-1)^s T_{s \cdot j_0} \chi_\Omega$. Clearly, $\|f\|_2^2 = Nm(\Omega)$. We compute $\langle S_g f, f \rangle$ according to Proposition 2.6.

$$\begin{aligned}
\langle S_g f, f \rangle &= \langle \mathbf{G}(x)\mathbf{f}(x), \mathbf{f}(x) \rangle \\
&= \int_{\mathbb{T}^2} \langle \mathbf{G}(x)\mathbf{f}(x), \mathbf{h}(x) \rangle dx \\
&= \sum_{i,j \in \mathbb{Z}^2} \int_{\mathbb{T}^2} G_{i,j}(x) T_i f(x) \overline{T_j f(x)} dx \\
&= \sum_{i \in \mathbb{Z}^2} \int_{\mathbb{T}^2} 2|T_i f(x)|^2 dx + \sum_{(i,j): i-j = \pm j_0} \int_{\mathbb{T}^2} T_i f(x) \overline{T_j f(x)} dx \\
&= 2 \int_{\mathbb{R}^2} |f(x)|^2 dx + \int_{\mathbb{R}^2} f(x) \overline{T_{j_0} f(x)} dx + \int_{\mathbb{R}^2} f(x) \overline{T_{-j_0} f(x)} dx \\
&= 2\langle f, f \rangle + \langle f, T_{j_0} f \rangle + \langle T_{j_0} f, f \rangle \\
&= \|f + T_{j_0} f\|_2^2,
\end{aligned} \tag{31}$$

after we have applied the periodization trick several times in the computations (31).

But due to the choice of f, $\|f + T_{j_0} f\|_2^2 = 2m(\Omega)$ because $f + T_{j_0} f$ is a telescoping sum, and $\operatorname{supp} f + T_{j_0} f = \Omega \cup T_{Nj_0}\Omega$. This implies that for this choice of f, $\langle S_g f, f \rangle$ remains constant, whereas $\|f\|_2$ can vary (because of N). Hence no lower frame bound exists for the Gabor family (g, Λ). Thus (g, Λ) is not a frame for $L^2(\mathbb{R}^2)$. $\square$

Example 4.12 *Let* $\Lambda = A\mathbb{Z}^2 \times \mathbb{Z}^2$*, where* $A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. *Let* Ω *be a compact fundamental domain for* $A\mathbb{Z}^2$*, and* $g = \chi_\Omega \cup T_{j_0}\chi_\Omega$*, where* $j_0 \in A\mathbb{Z}^2 \setminus \mathbb{Z}^2$*. Then* (g, Λ) *is a tight frame for* $L^2(\mathbb{R}^2)$*.*

Proof. We consider again the cross-ambiguity Gramian matrix defined in (18) for g, whose coefficients are given by

$$G_{i,j}(x) = \sum_{k \in \mathbb{Z}^2} \overline{g(x - j - Ak)} g(x - i - Ak). \tag{32}$$

We note that unless $i = j$, $\operatorname{supp} T_{i+Ak}g$ and $\operatorname{supp} T_{j+Ak}g$ are disjoint because $g = \chi_\Omega \cup T_{j_0}\chi_\Omega$, where $j_0 \in A\mathbb{Z}^2 \setminus \mathbb{Z}^2$ and Ω is a fundamental domain for $A\mathbb{Z}^2$. Therefore, the matrix $\mathbf{G}(x)$ is diagonal. Then we obtain that the diagonal entry

$$G_{i,i}(x) = \sum_{k \in \mathbb{Z}^2} |g(x - Ak - i)|^2 = G_{0,0}(x) = 2,$$

for almost all x. Therefore, Proposition 2.7 guarantees that $(\chi_{\mathcal{B}}, \Lambda)$ is a tight frame for $L^2(\mathbb{R}^2)$. $\square$

5 Multivariate Gaussian Gabor frames

In this section we study several examples of multivariate Gabor Gaussian systems. We denote by γ_n the standard n-dimensional Gaussian function. Clearly,

$$\gamma_n = \bigotimes_{n \text{ times}} \gamma_1.$$

A big goal is to give a criterion on lattices $\Lambda \subset \mathbb{R}^{2d}$ which determines whether (γ_n, Λ) is a Gabor frame for $L^2(\mathbb{R}^d)$. For $d = 1$ the problem is solved.

If $d = 1$, the problem is solved. Its solution depends on the concept of density $d(\Lambda)$ of a lattice Λ. In fact, the following theorem has been proven in [SW92]:

Theorem 5.1 *The family* (γ_1, Λ)*, for a discrete set* $\Lambda \subseteq \mathbb{R}^2$*, is with respect to* $L^2(\mathbb{R})$

- *a frame if* $D(\Lambda) > 1$*;*
- *complete but not a frame if* $D(\Lambda) = 1$*;*
- *a Riesz sequence if* $D(\Lambda) < 1$*.*

In higher dimensions a similar characterization is not trivial, and only very little is known. In higher dimensions, pretty much all that is known is that the above result does not hold in full. What remains true though is

Theorem 5.2 *The family (γ_d, Λ), where $\Lambda \subseteq \mathbb{R}^d$, is with respect to $L^2(\mathbb{R}^d)$*

- *not a frame if $D(\Lambda) = 1$;*
- *not complete if $D(\Lambda) > 1$;*

We provide some examples, and discuss an alternative criterion, called symplectic capacity. First we recall some theory which is applicable to the study of Gabor frames.

5.1 Tensor frames

The easiest way to create frames for spaces of functions in higher dimensions is to take tensor products. In this section we present some results on tensor products of Gabor frames. For n lattices $\Lambda_1, \ldots, \Lambda_n$ of the same dimension, we set $\odot_{i=1}^n \Lambda_i = \{(x_1, \ldots, x_n) \times (\omega_1, \ldots, \omega_n) : (x_i, \omega_i) \in \Lambda_i\}$.

Lemma 5.3 *Let (g_1, Λ_1) and (g_2, Λ_2) be frames for $L^2(\mathbb{R}^d)$. Then $(g_1 \otimes g_2, \Lambda_1 \odot \Lambda_2)$ is a frame for $L^2(\mathbb{R}^{2d})$.*

Proof. Let $f = f(x, y) \in L^2(\mathbb{R}^{2d}), \lambda \in \Lambda_1, \mu \in \Lambda_2$. Then we have

$$\pi(\lambda, \mu)(g_1 \otimes g_2)(x, y) = \pi(\lambda)g_1(x) \cdot \pi(\mu)g_2(y).$$

We compute

$$\begin{aligned}
&\sum_{\lambda,\mu} |\langle f, \pi(\lambda, \mu)(g_1 \otimes g_2)\rangle|^2 \\
=& \sum_{(\lambda,\mu)\in\Lambda_1\times\Lambda_2} \left| \iint f(x, y)\, \overline{\pi(\lambda)g_1(x)\pi(\mu)g_2(y)}\, dxdy \right|^2 \\
=& \sum_{(\lambda,\mu)\in\Lambda_1\times\Lambda_2} \left| \int \langle F_y, \pi(\lambda)g_1\rangle \overline{\pi(\mu)g_2(y)}\, dy \right|^2, \qquad (33)
\end{aligned}$$

where we set $F_y(x) = f(x, y)$ and

$$\langle F_y, \pi(\lambda)g_1\rangle = \int f(x, y)\overline{\pi(\lambda)g_1(x)}\, dx.$$

The function $f \in L^2(\mathbb{R}^{2d}, d(x, y))$, which implies that for almost every y, $F_y(x) = f(x, y) \in L^2(\mathbb{R}^d, dx)$. This is due to the Fubini theorem (the function $|f|^2 \in L^1(\mathbb{R}^{2d}, d(x, y))$, hence $|F_y|^2 \in L^1(\mathbb{R}^{2d}, dx)$ for almost every y, hence for almost every y, $F_y \in L^2(\mathbb{R}^d, dx)$.Then applying the frame inequality (g_1, Λ_1) to F_y, we obtain that for some $a, b > 0$ and almost every $y \in \mathbb{R}^d$

$$\begin{aligned} a\|F_y\|^2_{L^2(\mathbb{R}^d,dx)} &= a\int |F_y(x)|^2 dx \\ &\leq \sum_{\lambda\in\Lambda_1} |\langle F_y, \pi(\lambda)g_1\rangle|^2 \\ &\leq b\|F_y\|^2_{L^2(\mathbb{R}^d,dx)} \\ &= b\int |F_y(x)|^2 dx \end{aligned}$$

We also want to show that the function $\phi_\lambda(y) = \langle F_y, \pi(\lambda)g_1\rangle \in L^2(\mathbb{R}^d, dy)$. Essentially $\phi_\lambda(y) = V_{g_1}F_y(\lambda)$. We compute in turn

$$\begin{aligned} \|\phi_\lambda\|^2_{L^2(\mathbb{R}^d,dy)} &= \int |\langle F_y, \pi(\lambda)g_1\rangle|^2 dy \\ &\leq \int \sum_{\lambda'\in\Lambda_1} |\langle F_y, \pi(\lambda')g_1\rangle|^2\, dy \leq b\int \|F_y\|^2_{L^2(\mathbb{R}^d,dx)} dy \qquad (34) \\ &= b\int\int |f(x,y)|^2 dxdy \\ &= b\|f\|_2^2 < \infty \qquad (35) \end{aligned}$$

Thus, $\phi_\lambda \in L^2(\mathbb{R}^d, dy)$. Next we rearrange terms in (33) so that we try to estimate by Tonnelli the following summation

$$\begin{aligned} &\sum_{(\lambda,\mu)\in\Lambda_1\times\Lambda_2} \left|\iint f(x,y)\, \overline{\pi(\lambda)g_1(x)\pi(\mu)g_2(y)}\, dxdy\right|^2 \\ =\ &\sum_{(\lambda,\mu)\in\Lambda_1\times\Lambda_2} \left|\int \langle F_y, \pi(\lambda)g_1\rangle \overline{\pi(\mu)g_2(y)}\, dy\right|^2 \qquad (36) \\ =\ &\sum_{\lambda\in\Lambda_1}\sum_{\mu\in\Lambda_2} \left|\int \phi_\lambda(y)\overline{\pi(\mu)g_2(y)}\, dy\right|^2 \qquad (37) \\ =\ &\sum_{\lambda\in\Lambda_1}\sum_{\mu\in\Lambda_2} |\langle\phi_\lambda, \pi(\mu)g_2\rangle|^2 \qquad (38) \end{aligned}$$

Because $\phi_\lambda \in L^2(\mathbb{R}^d, dy)$ and (g_2, Λ_2) is a frame for $L^2(\mathbb{R}^d, dy)$, there exists constants $c, d > 0$ such that

$$c\|\phi_\lambda\|^2_{L^2(\mathbb{R}^d,dy)} \leq \sum_{\mu\in\Lambda_2} |\langle\phi_\lambda, \pi(\mu)g_2\rangle|^2 \leq d\|\phi_\lambda\|^2_{L^2(\mathbb{R}^d,dy)}, \quad \lambda \in \Lambda_1 \qquad (39)$$

Using (39) into the summations of (38), we obtain

$$\begin{aligned} c\sum_{\lambda\in\Lambda_1}\|\phi_\lambda\|^2_{L^2(dy)} &\le \sum_{(\lambda,\mu)\in\Lambda_1\times\Lambda_2}\left|\iint f(x,y)\,\overline{\pi(\lambda)g_1(x)\pi(\mu)g_2(y)}\,dxdy\right|^2 \\ &\le d\sum_{\lambda\in\Lambda_1}\|\phi_\lambda\|^2_{L^2(dy)} \end{aligned} \tag{40}$$

These are a preliminary upper and lower bounds on the quantity we want. Furthermore,

$$\|\phi_\lambda\|^2_{L^2(dy)} = \int |\langle F_y, \pi(\lambda)g_1\rangle|^2 dy$$

Therefore, equation (40) can be rewritten as

$$\begin{aligned} c\sum_{\lambda\in\Lambda_1}\int |\langle F_y, \pi(\lambda)g_1\rangle|^2 dy &\le \sum_{\lambda,\mu}\left|\iint f(x,y)\,\overline{\pi(\lambda)g_1(x)\pi(\mu)g_2(y)}\,dxdy\right|^2 \\ &\le d\sum_{\lambda\in\Lambda_1}\int |\langle F_y, \pi(\lambda)g_1\rangle|^2 dy \end{aligned} \tag{41}$$

and from (35), we know already that

$$\int \sum_{\lambda'\in\Lambda_1} |\langle F_y, \pi(\lambda')g_1\rangle|^2\, dy < \infty$$

so first Tonelli's and then Fubini's theorem allow us to interchange the order of summation and integration in (41), so we obtain

$$\begin{aligned} & c\int \sum_{\lambda\in\Lambda_1} |\langle F_y, \pi(\lambda)g_1\rangle|^2 dy \\ &\le \sum_{(\lambda,\mu)\in\Lambda_1\times\Lambda_2}\left|\iint f(x,y)\,\overline{\pi(\lambda)g_1(x)\pi(\mu)g_2(y)}\,dxdy\right|^2 \\ &\le d\int \sum_{\lambda\in\Lambda_1} |\langle F_y, \pi(\lambda)g_1\rangle|^2 dy \end{aligned} \tag{42}$$

Since the quantities on the left- and right-hand side of the inequality signs can be simplified using the frame inequalities for (g_1, Λ_1) in $L^2(\mathbb{R}^d, dx)$, we obtain consequently

$$\begin{aligned} a\int |f(x,y)|^2 dx &\le \sum_{\lambda\in\Lambda_1} |\langle F_y, \pi(\lambda)g_1\rangle|^2 \le b\int |f(x,y)|^2 dx \\ a\iint |f(x,y)|^2 dxdy &\le \int\sum_{\lambda\in\Lambda_1} |\langle F_y, \pi(\lambda)g_1\rangle|^2 dy \le b\iint |f(x,y)|^2 dxdy \end{aligned}$$

A substitution in (41) gives us finally,

$$ac \iint |f(x,y)|^2 dxdy \leq \sum_{(\lambda,\mu)\in\Lambda_1\times\Lambda_2} \left| \iint f(x,y)\, \overline{\pi(\lambda)g_1(x)\pi(\mu)g_2(y)}\, dxdy \right|^2 \quad (43)$$

$$\leq bd \iint |f(x,y)|^2 dxdy \quad (44)$$

Therefore, we have proved that in (37) Tonnelli's theorem is applicable, which brings us to the desired result. □

5.2 Symplectic capacity

We introduce a concept from symplectic geometry which might turn useful for making classifications such as that in Theorem 5.1. The ball centered at the origin and of radius r in $\mathbb{R}^{2d}$ is denoted by B_r^{2d}.

Examples given in the following section as well as general experience with time-frequency analysis, imply that Theorem 5.1 might actually generalize to higher dimensions in a canonical way, if we change the criterion on the density of the lattice Λ by a concept involving the ideas of a symplectic capacity. This is defined for bodies in $\mathbb{R}^d$, see [Hof90], and this definition would yield that each lattice (a discrete set) has symplectic capacity 0 in the classical sense. The following represent some attempts at re-defining this quantity

Definition 5.4 *'Capacity of a lattice' can be defined in two possible ways:*

1. *The symplectic capacity of a lattice Λ is given by*

$$c(\Lambda) = c(M\mathbb{Z}^{2d}) = \sup_{\mathcal{P} \text{ symplectic plane}} \left\{ \text{ area} \left(M\mathbb{T}^d \cup \mathcal{P} \right) \right\},$$

that is, the area of largest intersection of a fundamental domain with a symplectic plane.

2. *The linear symplectic capacity $c(\Lambda)$ of a lattice $\Lambda = MZ^{2d}$ is given by the linear symplectic capacity $c(\mathcal{W}_M)$ of the Wigner ellipsoid $\mathcal{W}_M = \{z : \ M^{-1}z \cdot z \leq 1\}$, that is,*

$$c(\Lambda) = \sup\{\pi r^2 : \ A(B_r^{2d}) \subseteq \mathcal{W}_M, \ A \in \mathrm{Sp}(d)\}.$$

At this time, we still do not know how to define the symplectic capacity of a lattice. The above definitions probably have to be corrected. In fact, it is not clear whether our approach is well defined, for example, in the second case whenever $M\mathbb{Z}^4 = M'\mathbb{Z}^4$ for $M \neq M'$, must imply that $c(\mathcal{W}_M) = c(\mathcal{W}_{M'})$. In fact, if we consider $M = I$ and $M' = \begin{pmatrix} \begin{smallmatrix} 1 & 100 \\ 0 & 1 \end{smallmatrix} & 0 \\ 0 & I \end{pmatrix}$, then $c(\mathcal{W}_M) = 1$, while the set $W_{M'}$ is not an ellipsoid but a hyperboloid, so its capacity should be infinite!

5.3 Examples

The following propositions illustrate the peculiar behavior of the Gaussian function in higher dimensions. For a lattice Λ in $\mathbb{R}^{2d}$ parametrized by a diagonal matrix is not difficult to characterize the lattice parameters so that (γ_n, Λ) is a frame for $L^2(\mathbb{R}^d)$.

Proposition 5.5 *Let $\Lambda = \mathbb{Z}^2 \times \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mathbb{Z}^2$. If $a < 1$ and $b < 1$, then (γ_2, Λ) is a frame for $L^2(\mathbb{R}^2)$. If $a = b = 1$, (γ_2, Λ) is complete in $L^2(\mathbb{R}^2)$, but not a frame. If $a > 1$ or $b > 1$, then (γ_2, Λ) is incomplete.*

Proof. If $a < 1, b < 1$, then the result from [Lyu92], [SW92] tells us that $(\gamma_1, \mathbb{Z} \times a\mathbb{Z})$ and $(\gamma_1, \mathbb{Z} \times b\mathbb{Z})$ are frames for $L^2(\mathbb{R})$. Lemma 5.3 implies that (γ_2, Λ) is a frame for $L^2(\mathbb{R}^2)$.

To show completeness of the Gabor system, for $a = b = 1$ we observe that for $(x, \omega) = (x_1, x_2, \omega_1, \omega_2)$

$$Z\gamma_2(x, \omega) = Z\gamma_1(x_1, \omega_1) \cdot Z\gamma_1(x_2, \omega_2) \tag{45}$$

Because $(\gamma_1, \mathbb{Z} \times \mathbb{Z})$ is complete in $L^2(\mathbb{R})$, but not a frame, according to Proposition 9.4.3 in [Chr03], $Z\gamma_1$ vanishes on a set of measure zero in $[0, 1)^2$. Hence, the Zak transform $Z\gamma_2$ vanishes only on a set of zero measure in $[0, 1)^4$. According to Proposition 9.4.3 in [Chr03], $(\gamma_2, \mathbb{Z}^2 \times \mathbb{Z}^2)$ is complete. Furthermore, since $\gamma_2 \in \mathcal{S}(\mathbb{R}^2)$, its Zak transform is continuous. Hence, it is not bounded away from 0 almost everywhere. Proposition 8.3.2 in [Grö01] implies that the Gabor system $(\gamma_2, \mathbb{Z}^2 \times \mathbb{Z}^2)$ is not a frame for $L^2(\mathbb{R}^2)$.

If $a > 1$ or $b > 1$, say $b > 1$, then $(\gamma_1, \mathbb{Z} \times b\mathbb{Z})$ is incomplete in $L^2(\mathbb{R})$. Hence we can choose $f_1 \in L^2(\mathbb{R}^d)$, $f_1 \neq 0$ such that $V_{\gamma_1} f_1(m_1, bm_1) = 0$ for all $(m_1, n_1) \in \mathbb{Z}^2$. Then for any $f_2 \in L^2(\mathbb{R})$, $f_2 \neq 0$, the STFT

$$V_{\gamma_2}(f_1 \otimes f_2)(m_1, m_2, an_1, bn_2) = V_{\gamma_1}(m_1, an_1) V_{\gamma_1}(m_2, bn_2) = 0. \tag{46}$$

But $f_1 \otimes f_2 \neq 0$, so $(\gamma_2, \mathbb{Z} \times \mathbb{Z} \times a\mathbb{Z} \times b\mathbb{Z})$ is incomplete. □

A more general statement in this spirit is

Proposition 5.6 *The Gabor system $(\gamma_n, \odot_{j=1}^{2d} \lambda_j \mathbb{Z})$ is incomplete in $L^2(\mathbb{R}^d)$ if for some $j, j' : |j - j'| = d, |\lambda_j \lambda_{j'}| > 1$, complete but not a frame if for some $j, j' : |j - j'| = d, |\lambda_j \lambda_{j'}| = 1$, and a frame for $L^2(\mathbb{R}^d)$ if for all $j, j' : |j - j'| = d, |\lambda_j \lambda_{j'}| < 1$.*

Proof. The proof combines Lemma 5.3 with the ideas from the proof of Proposition 5.5. □

Remark: In this case the criterion 'symplectic capacity of a lattice' as defined by Definition 5.4, 2. coincides with the condition of Proposition 5.6. We illusrate this in more detail. By definition, the linear symplectic capacity is invariant under symplectic maps, that is, $c(U) = c(SU)$ for any set U and $S \in Sp(d)$. Hence, rather than considering the Wigner ellipsoid $\mathcal{W}_M$, we can consider $\mathcal{W}_{SM}$, where we choose

$$S = \operatorname{diag}\left(\sqrt{\frac{\lambda_{d+1}}{\lambda_1}}, \dots, \sqrt{\frac{\lambda_d}{\lambda_{2d}}}, \sqrt{\frac{\lambda_1}{\lambda_{d+1}}}, \dots, \sqrt{\frac{\lambda_d}{\lambda_{2d}}}\right) \in \mathrm{Sp}(d)$$

Hence, for $M = \operatorname{diag}(\lambda_1, \dots, \lambda_{2d})$, and $M' = SM$, we have

$$\begin{aligned}
(M')^{-1}z \cdot z &= \begin{pmatrix} (\lambda_1\lambda_{d+1})^{-\frac{1}{2}}\, x_1 \\ \vdots \\ (\lambda_d\lambda_{2d})^{-\frac{1}{2}}\, x_d \\ (\lambda_1\lambda_{d+1})^{-\frac{1}{2}} \xi_1 \\ \vdots \\ (\lambda_d\lambda_{2d})^{-\frac{1}{2}}\, \xi_d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \\ \xi_1 \\ \xi_2 \\ \vdots \\ \xi_d \end{pmatrix} \\
&= (\lambda_1\lambda_{d+1})^{-\frac{1}{2}}\, (x_1^2 + \xi_1^2) + \ldots + (\lambda_d\lambda_{2d})^{-\frac{1}{2}}\, (x_d^2 + \xi_d^2),
\end{aligned}$$

and we can conclude that $c(M\mathbb{Z}^{2d}) = c(M'\mathbb{Z}^{2d}) = \pi \left(\max\{(\lambda_j\lambda_{j+d})^{-\frac{1}{2}}\}\right)^{-2} = \pi \min\{|\lambda_j\lambda_{j+d}|\}$.

For lattices parametrized by non-diagonal matrices criteria are different. The next example is non-trivial.

Proposition 5.7 *Let* $\Lambda = \mathbb{Z}^2 \times \begin{pmatrix} a & a \\ -b & b \end{pmatrix} \mathbb{Z}^2$. *Then the Gabor system* (γ_2, Λ) *is a frame for* $L^2(\mathbb{R}^2)$ *if* $a, b < \frac{1}{2}$. *If* $a = b = \frac{1}{2}$, (γ_2, Λ) *is complete, but not a frame for* $L^2(\mathbb{R}^2)$. *If* $a, b > \frac{1}{2}$, (γ_2, Λ) *is incomplete.*

Proof. The lattice

$$\Lambda = \mathbb{Z}^2 \times \begin{pmatrix} a & a \\ -b & b \end{pmatrix} \mathbb{Z}^2$$

is separable. A simple calculation shows that for $F = f_1 \otimes f_2 \in L^2(\mathbb{R}^2)$,

$$\begin{aligned}
&V_{\gamma_2}F(m_1, m_2, a(n_1+n_2), b(n_2-n_1)) \\
&\quad = \langle f_1 \otimes f_2, T_{m_1,m_2} M_{a(n_1+n_2),b(n_2-n_1)} \gamma_2 \rangle \\
&\quad = \langle f_1, T_{m_1} M_{a(n_1+n_2)} \gamma_1 \rangle \langle f_2, T_{m_2} M_{b(n_2-n_1)} \gamma_1 \rangle \\
&\quad = V_{\gamma_1} f_1(m_1, a(n_1+n_2)) \cdot V_{\gamma_1} f_2(m_2, b(n_2-n_1))
\end{aligned}$$

If n_1, n_2 are of the same parity, then $n_1 \pm n_2$ is always even, otherwise $n_1 \pm n_2$ is odd. Hence, if

$$a, b > \frac{1}{2}$$

after [Lyu92], [SW92] we can choose a nonzero $f_1 \in L^2(\mathbb{R})$ such that

$$V_{\gamma_1} f_1(m_1, a(n_1+n_2)) = 0, \quad \forall m_1, \forall (n_1, n_2) : 2 \mid n_1 - n_2,$$

and a nonzero $f_2 \in L^2(\mathbb{R})$ such that

$$V_{\gamma_1} f_2(m_2, b(n_2 - n_1)) = 0, \quad \forall m_2, \forall (n_1, n_2) : 2 \nmid n_1 - n_2.$$

Then $F = f_1 \otimes f_2 \neq 0$ but

$$V_\gamma F(m_1, m_2, a(n_1 + n_2), b(n_2 - n_1)) = 0, \quad \forall m_1, m_2, n_1, n_2.$$

This is due to density of the respective Gabor systems being greater than 1. Therefore, the system (γ_2, Λ) is incomplete for all $a, b > \frac{1}{2}$.

We note further that

$$\begin{aligned}\Lambda = &\{(m_1, m_2, 2ak_1, 2bk_2)^T : m_1, m_2, k_1, k_2 \in \mathbb{Z}\} \\ &\cup \{(m_1, m_2, 2ak_1 + a, 2bk_2 + b)^T : m_1, m_2, k_1, k_2 \in \mathbb{Z}\}.\end{aligned}$$

If $a, b = \frac{1}{2}$, the system (γ_2, Λ) is complete in $L^2(\mathbb{R}^2)$, because it is the union of two complete systems in $L^2(\mathbb{R}^2)$. However, it is not a frame for $L^2(\mathbb{R}^2)$. After [Lyu92], [SW92] we can choose $\epsilon > 0$ and $f_1, f_2 \in L^2(\mathbb{R})$ with unit norm such that

$$\sum_{k,l \in \mathbb{Z}} |V_{\gamma_1} f_1(k, l)|^2 < \epsilon, \quad \sum_{k,l \in \mathbb{Z}} |V_{\gamma_1} f_2(k, l + \tfrac{1}{2})|^2 < \epsilon.$$

Then letting $F = f_1 \otimes f_2$, it is not difficult to see that

$$\begin{aligned}\sum_{m_1, m_2, n_1, n_2} &|V_{\gamma_2} F(m_1, m_2, \tfrac{1}{2}(n_1 + n_2), \tfrac{1}{2}(n_2 - n_1)|^2 \\ &= \sum_{m_1, m_2, \underbrace{n_1, n_2}_{2 \mid n_1 - n_2}} |V_{\gamma_2} F(m_1, m_2, \tfrac{1}{2}(n_1 + n_2), \tfrac{1}{2}(n_2 - n_1)|^2 \\ &+ \sum_{m_1, m_2, \underbrace{n_1, n_2}_{2 \nmid n_1 - n_2}} |V_{\gamma_2} F(m_1, m_2, \tfrac{1}{2}(n_1 + n_2), \tfrac{1}{2}(n_2 - n_1)|^2 \\ &\leq \sum_{k,l \in \mathbb{Z}} |V_{\gamma_1} f_1(k, l)|^2 \cdot \sum_{k,l \in \mathbb{Z}} |V_{\gamma_1} f_2(k, l)|^2 \\ &+ \sum_{k,l \in \mathbb{Z}} |V_{\gamma_1} f_1(k, l + \tfrac{1}{2})|^2 \cdot \sum_{k,l \in \mathbb{Z}} |V_{\gamma_1} f_2(k, l + \tfrac{1}{2})|^2 < 2C\epsilon,\end{aligned}$$

where C is the ℓ^2-norm of the Gabor analysis operator $D_{\gamma_1, \mathbb{Z}^2}$. $D_{\gamma_1, \mathbb{Z}^2}$ is bounded by Proposition 12.2.5 [Grö01] because $\gamma_1 \in M^1(\mathbb{R})$.

Therefore, while the $\|F\|_2 = 1$,

$$\sum_{m_1, m_2, n_1, n_2} |V_{\gamma_2} F(m_1, m_2, \tfrac{1}{2}(n_1 + n_2), \tfrac{1}{2}(n_2 - n_1)|^2 \leq 2C\epsilon,$$

implying that (γ_2, Λ) has no lower frame bound. Thus the system (γ_2, Λ) is not a frame for $L^2(\mathbb{R}^2)$.

If $a, b < \frac{1}{2}$, then (γ_2, Λ) is a frame for $L^2(\mathbb{R}^2)$, because it is the union of two frames for $L^2(\mathbb{R}^2)$. $\square$

Remark: The symplectic capacity in this case is quite difficult to compute. Furthermore, we have not been able to characterize the cases $a > \frac{1}{2}, b < \frac{1}{2}$ or $a < \frac{1}{2}, b > \frac{1}{2}$.

Proposition 5.8 *Let* $\Lambda = \mathbb{Z}^2 \times \begin{pmatrix} ak & a \\ 0 & b \end{pmatrix} \mathbb{Z}^2, k \in \mathbb{N}$. *Then the Gabor system* (γ_2, Λ) *is incomplete if there exists* $l \in \mathbb{N}$ *such that* $a > \frac{l}{k}, b > \frac{k-l}{k}$.

Proof. We split the lattice points of Λ into k disjoint sets: $\Lambda = \{(akm_1 + am_2, bm_2, n_1, n_2) : m_1, m_2, n_1, n_2 \in \mathbb{Z}\}$ according to the remainder of m_2 by division by k, that is

$$\Lambda = \bigcup_{d=0}^{k-1} \{(akm_1 + akm_2' + ad, bkm_2' + bd, n_1, n_2) : m_1, m_2', n_1, n_2 \in \mathbb{Z}\},$$

where $m_2 = km_2' + d, 0 \le d \le k-1$. Let $m_1' = m_1 + m_2'$. As there exists $l \in \mathbb{N}$ such that $a > \frac{l}{k}, b > \frac{k-l}{k}$ we can rewrite this as

$$\begin{aligned} \Lambda = &\bigcup_{d=0}^{l-1} \{(akm_1' + ad, bkm_2' + bd, n_1, n_2) : m_1', m_2', n_1, n_2 \in \mathbb{Z}\} \\ &\cup \bigcup_{d=l}^{k} \{(akm_1' + ad, bkm_2' + bd, n_1, n_2) : m_1', m_2', n_1, n_2 \in \mathbb{Z}\}. \end{aligned} \tag{47}$$

Let $F = f \otimes g$. Then the STFT of F with respect to γ_2 factorizes into a product

$$V_{\gamma_2} F = V_{\gamma_1} f(akm_1' + ad, n_1) \cdot V_{\gamma_1} g(bkm_2' + bd, n_2).$$

We shall now suitably choose f, g so that the above product becomes identically zero. The density of the set

$$\bigcup_{d=0}^{l-1} \{(akm_1' + ad, n_1) : m_1', n_1 \in \mathbb{Z}\}$$

equals $\frac{l}{ak}$, while that of

$$\bigcup_{d=l}^{k-1} \{(bkm_2' + bd, n_2) : m_2', n_2 \in \mathbb{Z}\}$$

equals $\frac{k-l}{bk}$ (because these sets are finite unions of translates of the same set). According to our assumptions on a, b both densities are less than 1. Hence after [Lyu92], [SW92] there exists $f \neq 0$ such that

$$V_{\gamma_1} f(akm_1' + ad, n_1) = 0, \quad \forall m_1', n_1 \in \mathbb{Z}, 0 \le d \le l-1;$$

and $g \neq 0$ such that

$$V_{\gamma_1} g(bkm_2' + bd, n_2) = 0, \quad \forall m_2', n_2 \in \mathbb{Z}, l \leq d \leq k-1.$$

This choice of f, g annihilates the STFT $V_{\gamma_2} F$ on the first and the second component of partition (47) of Λ. Therefore, for $F \neq 0, V_{\gamma_2} F$ vanishes on all of Λ implying that the Gabor system (γ_2, Λ) is incomplete. □

Remark: The range of parameters k, l, where the condition from Proposition 5.8 is stronger than the density condition is quite small if $k > 4$. If $k \geq 5$, the only values of l for which

$$\frac{1}{k} > ab > \frac{l}{k} \cdot \frac{k-l}{k}$$

are $l = 1, k-1$ because always $2(k-2) > k$.

Generalizing the ideas underlying Proposition 5.7 leads to a result for lattices Λ with a particular subgroup structure:

Theorem 5.9 *Let $\odot_{i=1}^d A_i \mathbb{Z}^2$ be a subgroup of $\Lambda \subset \mathbb{R}^{2d}$ of index n. If there exist natural numbers $l_i, 1 \leq i \leq d$, such that $\sum_{i=1}^d l_i = n$ and $l_i < \det A_i$, then the system (γ_d, Λ) is incomplete in $L^2(\mathbb{R}^d)$.*

Proof. We split the n cosets of $\odot_{i=1}^d A_i \mathbb{Z}^2$ into d groupings $\Delta_1, \ldots, \Delta_d$ such that $|\Delta_i| = l_i$. Δ_i contains coset representatives denoted by $[\tau]$. We have

$$\Lambda = \bigcup_{i=1}^{d} \bigcup_{[\tau] \in \Delta_i} \{A_1 \mathbb{Z}^2 \times \ldots \times A_d \mathbb{Z}^2\} + [\tau],$$

The short-time Fourier transform of the tensor product $\otimes_{i=1}^d f_i$ factorizes, namely

$$V_{\gamma_d}(\otimes_{i=1}^d f_i)(x, \omega) = \prod_{i=1}^{d} V_{\gamma_1} f_i(x_i, \omega_i),$$

where $(x_i, \omega_i) \in A_i \mathbb{Z}^2 + [\tau_i]$, $[\tau_i]$ being the coset representative of $A_i \mathbb{Z}^2$ in the restriction of $\odot_{i=1}^d A_i \mathbb{Z}^2$ to $A_i \mathbb{Z}^2$. As the density of the set

$$U_i = \bigcup_{[\tau] \in \Delta_i} A_i \mathbb{Z}^d + [\tau_i], \quad 1 \leq i \leq d$$

is $l_i D(A_i) < 1$, the results of [Lyu92], [SW92] apply and non-zero functions $f_i \in L^2(\mathbb{R})$ can be chosen so that $V_{\gamma_1} f_i(x_i, \omega_i) = 0$, for all $(x_i, \omega_i) \in U_i$. Then as in Proposition 5.7 we conclude that $V_{\gamma_d}(\otimes_{i=1}^d f_i)$ vanishes on all of Λ, but $\otimes_{i=1}^d f_i \neq 0$. Hence, this Gabor system is incomplete in $L^2(\mathbb{R}^d)$. □

Remark: If Λ satisfies the hypothesis of Theorem 5.9, then the density theorem implies incompleteness if $D(\Lambda) = n \prod_{i=1}^d \frac{1}{\det A_i} < 1$, that is, if $\prod_{i=1}^d \det A_i >$

n. Hence, for Theorem 11 to be effective, we need to combine the condition $\prod_{i=1}^{d} \det A_i \leq n$ with the condition $\det A_i > l_i$ and $\sum_{i=1}^{d} l_i = n$. This leads to

$$\prod_{i=1}^{d} l_i < \sum_{i=1}^{d} l_i. \tag{48}$$

Assuming without loss of generality the order $l_1 \geq l_2 \geq \ldots \geq l_d > 0$, we divide (48) by l_1 and observe that then $\prod_{i=2}^{d} l_i < d$. As all l_i are positive integers, we conclude that $l_2 = l_3 = l_4 = \ldots = l_d = 1$ and $l_1 = n - d + 1$.

Note that theorem 5.9 implies the incompleteness in Proposition 5.7 when both $a, b > \frac{1}{2}$ because $(\mathbb{Z} \times 2a\mathbb{Z}) \odot (\mathbb{Z} \times 2b\mathbb{Z})$ is a subgroup of Λ of index 2 and $l_1 = l_2 = 1 < 2a, 2b$. Next we construct an example for 3-dimensional Gaussian.

Proposition 5.10 *Let* $\Lambda = \mathbb{Z}^3 \times \begin{pmatrix} a & a & 0 \\ -b & b & 0 \\ 0 & 0 & c \end{pmatrix} \mathbb{Z}^3$. *Then the Gabor system* (γ_3, Λ) *is a frame for* $L^2(\mathbb{R}^3)$ *if* $a, b < \frac{1}{2}, c < 1$. *If* $a = b \leq \frac{1}{2}, c = 1$, (γ_3, Λ) *is complete, but not a frame for* $L^2(\mathbb{R}^3)$. *If* $a, b > \frac{1}{2}$ *or* $c > 1$, (γ_3, Λ) *is incomplete.*

Proof. We choose $F = f_1 \otimes f_2 \otimes f_3 \in L^2(\mathbb{R}^3)$ in order to apply a tensor argument as (46). When $a, b > \frac{1}{2}$, the claim follows immediately from Proposition 5.7. When $c > 1$, it suffices to choose f_3 which is in the orthogonal complement of $\{T_m M_{cn} \gamma_1 : m, n \in \mathbb{Z}\}$ and repeat the same line of reasoning.

Whenever $a, b < \frac{1}{2}, c < 1$, then (γ_3, Λ) is a frame, because it is the product of two frames (see Lemma 5.3 and Proposition 5.7). □

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