IDENTIFICATION AND SAMPLING OF STOCHASTIC OPERATORS

Götz Pfander[1], *Pavel Zheltov*[1]

[1]School of Engineering and Science,
Jacobs University Bremen, 28759 Bremen, Germany
Emails: {g.pfander, p.zheltov}@jacobs-university.de

ABSTRACT

The classical Shannon-Nyquist theorem allows regular sampling of bandlimited signals. Recently, this result was generalized to sampling of channels with delay-Doppler occupancy pattern of area less than one, resolving Bello's conjecture in positive. In this paper, it is shown that stochastic channels possess a similar property, namely, that we can characterize the channel as long as the support of its scattering function has area less than one. This holds for rectangular as well as non-rectangular regions, allowing sampling of channels classically considered overspread.

Keywords— stochastic, sampling, identification, operators, spreading function, scattering function, estimation.

1. INTRODUCTION

Operator identification is a common goal in several applied disciplines. In radar detection, we are interested in the identification of a target. A known waveform is transmitted towards a target, and the echo is used to recognize it. In wireless as well as wired communications and acoustic telemetry, the properties of the channel between the transmitter and the receiver have to be known to facilitate the communication. Similarly, in signal processing it is usually assumed that a signal is passed through a filter, whose parameters have to be determined from the output. Commonly, such systems are modeled with a time-variant linear operator acting on a space of signals. Due to Doppler shifts and multi-path propagation the echo is a sum of time-frequency shifts of the sent signal. More generally, we observe a continuum of scatterers. Then the channel is represented by an operator with a superposition integral

$$(Hf)(x) = \iint \eta(t,\nu)\mathcal{T}_t M_\nu f(x)\,dt\,d\nu,$$

where $\mathcal{T}_t$ is a *time-shift* by t, that is $\mathcal{T}_t f(x) = f(x-t)$, $t \in \mathbb{R}$, M_ν is a *frequency shift* or *modulation* given by $M_\nu f(x) = e^{2\pi i\nu x} f(x)$ (it follows that $\widehat{M_\nu f(x)} = \hat{f}(\omega-\nu)$ for all $\omega \in \hat{\mathbb{R}} = \mathbb{R}$). The function $\eta(t,\nu)$ is called a *(Doppler-delay) spreading function of* H. If we denote the time-impulse response of H related to $\eta(t,\nu)$ via

$$\eta(t,\nu) = \int h(t,x)e^{-2\pi i\nu(x-t)}\,dx$$

then we obtain the linear time-variant (LTV) operator model:

$$(Hf)(x) = \int h(x,t)f(x-t)\,dt.$$

The conditions under which such operator (or, equivalently, its spreading function) is identifiable have long been of interest to the scientific community. In 1963, T.Kailath [7] realizes that for a deterministic time-variant channel to be identifiable, it is necessary and sufficient that the product BL of the maximum time delay L and maximum Doppler spread B is not greater than 1. If $BL < 1$, the channel is said to be *underspread*, and *overspread* if $BL > 1$ [15, 7]. His arguments have been made precise in Kozek and Pfander [9].

Following in Kailath footsteps, P.Bello in his seminal paper [3] lays the groundwork for channel sampling and characterization tools and vocabulary, and in [4] he argues that it is not the product BL that matters for identification of a deterministic time-variant channel, but rather the area of what he calls an *occupancy pattern* $\operatorname{supp}\eta_H$, that is, a not necessarily rectangular support of the spreading function. In [11] this conjecture has been given a mathematical framework and was proven using tools from Gabor analysis. See section 2, Theorem 4 for a full statement of this result. This has profound importance as it significantly extends the class of systems that can be considered "underspread" and identifiable, a fact which is particularly of interest in the field of sonar communication [8]. The speed of sound is much lower than electromagnetic waves, which leads to time delays up to several seconds and Doppler spreads in the tens of Hertz for high-frequency channels [2].

An alternative model, also set up and discussed in [6, 7, 3, 4], is the stochastic time-variant operator. In this setting, $\boldsymbol{h}(t,x)$, and hence, the spreading function $\boldsymbol{\eta}(t,\nu)$, is a sample function of some random process. A popular assumption, stemming from Bello and Proakis [3, 14], is the wide-sense stationarity with uncorrelated scattering (WSSUS) of the channel. Namely, the autocorrelation function of $\boldsymbol{\eta}$ is of the form

$$\begin{aligned} R_{\boldsymbol{\eta}}(t,\nu;t',\nu') &= \mathbb{E}\left\{\boldsymbol{\eta}(t,\nu)\boldsymbol{\eta}^*(t',\nu')\right\} \\ &= \delta(t-t')\delta(\nu-\nu')P_{\boldsymbol{\eta}}(t,\nu), \end{aligned}$$

where $P_{\boldsymbol{\eta}}(t,\nu)$ is known as the *scattering function* of H. It completely characterizes the second-order statistics of $\boldsymbol{\eta}$ and represents the power spectral density of the transfer function of the channel.

The problem of identifying the channel can still be posed in this stochastic setting to determine the channels parameters $\boldsymbol{h}(t,x)$ or $\boldsymbol{\eta}(t,\nu)$ based on the echo from the transmission of a known sounding signal. A classical problem that is closely related to identifying the operator is to estimate the scattering function and other functions alike. There, the quantity of interest is not the operator itself, but its average behavior. A multitude of authors, including Kailath and Bello have suggested methods to do that [1, 5, 10], two common types being deconvolution and direct measurement methods. However, it is clear that the complete identification of the stochastic spreading function guarantees identification of the scattering function exactly, or a straightforward estimator of it via ensemble averaging.

The evolution of the condition on the channel that guarantees the identification follows the development of the deterministic case. A brief comment of Kailath suggests sufficiency of $BL \le 1$, and Bello treats this question as a side matter, more interested in developing the estimator for $\boldsymbol{\eta}(t,\nu)$ when the output has been contaminated by additive noise. The reconstruction formula due to Pfander and Walnut [11] allows us to recover a *sample* spreading function from the echo. We generalize this result to the stochastic operator setting. We can prove that it is possible to recover the stochastic spreading function of the channel in the mean-square sense provided that the support of the scattering function $\operatorname{supp} P_{\boldsymbol{\eta}}(t,\nu)$ occupies a region of area less than one. We provide an explicit reconstruction formula for it from the periodic samples of the echo of a specially constructed weighted delta train sounding signal.

2. DETERMINISTIC CHANNELS

The Shannon-Nyquist theorem allows sampling of signals which are bandlimited, that is, whose Fourier transform is supported on a bounded set $[-\Omega/2,\Omega/2]$. Formally, any such function f from the Paley-Wiener space $\mathrm{PW}([-\Omega/2,\Omega/2])$ is completely characterized by its periodic samples taken at a rate at least $T=\Omega^{-1}$, where

$$\mathrm{PW}(S) := \{f \in L^2(\mathbb{R}) : \quad \operatorname{supp} \hat{f} \subseteq S\}.$$

The reconstruction of f is achieved with

$$f(x) = \sum_{n\in\mathbb{Z}} f(nT)\operatorname{sinc} T(x-n),$$

where $\operatorname{sinc}(x) := (\pi x)^{-1}\sin \pi x$ (which is notable for its Fourier transform $\chi_{[-1/2,1/2]}(x)$).

In the context of the sampling theory of operators, the operators that allow an analogous reconstruction formula [13] must be "bandlimited" to a region in the time-frequency plane whose area is less than one. We will need the following definitions.

Definition 1. *The operator Paley-Wiener space is defined as*

$$\mathrm{OPW}(S) := \{H : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \text{ such that } \sigma_H \in L^2(\mathbb{R}^2), \quad \operatorname{supp} \mathcal{F}_s\sigma_H \subseteq S\},$$

where $\mathcal{F}_s : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is the symplectic Fourier transform given by

$$\mathcal{F}_s f(t,\nu) = \iint f(x,\xi) e^{-2\pi i(\nu x - \xi t)}\, dx\, d\xi$$

and $\sigma_H(\nu,\xi)$ is the Kohn-Nirenberg symbol of the operator H which is defined by

$$\sigma_H(\nu,\xi) := \mathcal{F}_s^{-1}\eta(t,\nu).$$

In other words, the space $\mathrm{OPW}(S)$ contains precisely those operators whose occupancy region, that is, the support of the spreading function $\eta(t,\nu)$, is contained in S. It is important to notice that due to the compact support of $\eta(t,\nu)$, it is possible to extend the domain of H to include some tempered distributions, in particular, the delta train $\text{Ш}_T(x) = \sum_{k\in\mathbb{Z}} \delta_{kT}$ (see [12, Prop.4.2]).

In the case where the occupancy region is rectangular, the exemplary result is the following:

Theorem 2. *For $H \in \mathrm{OPW}([0,T]\times[-\Omega/2,\Omega/2])$ such that $T\Omega \le 1$, we can reconstruct the kernel*

$$\boldsymbol{\kappa}_H(t+x,x) = \sum_{n\in\mathbb{Z}} (H \sum_{k\in\mathbb{Z}} \delta_{kT})(t+nT)\operatorname{sinc} T(x-n)$$

with convergence in $L^2(\mathbb{R}^2)$.

In the general case of the non-rectangular regions we need the following definition to measure the spread factor.

Definition 3. *Let $S \subset \mathbb{R}\times\hat{\mathbb{R}}$ be bounded and let μ be the Lebesgue measure on $\mathbb{R}\times\hat{\mathbb{R}}$. We define Jordan inner and outer measures of S via*

$$\mu^-(S) = \inf\{\mu(U) : U \supseteq S, U \text{ is a finite union of rectangles}\}$$

$$\mu^+(S) = \sup\{\mu(U) : U \subseteq S, U \text{ is a finite union of rectangles}\}$$

If $\mu^+(S) = \mu^-(S)$, we say that S is *Jordan measurable*. Clearly, every Jordan measurable set is Lebesgue measurable, and $\mu^+(S) = \mu^-(S) = \mu(S)$.

The method of proof in the non-rectangular case requires S to be bounded: $S \subset [0,T]\times[-\Omega/2,\Omega/2]$ with $T\Omega$ not necessarily smaller than 1. It would be interesting to extend this result to the unbounded domains.

By [11, Proposition 2.2], there exists a prime number $L > T\Omega$ such that S can be covered with L integer translations of the base rectangle $R = [0,\frac{T}{L})\times[-\Omega/2,-\Omega/2+\frac{1}{T})$.

$$S \subseteq \bigcup_{j=1}^{L} R + (\frac{q_jT}{L},\frac{k_j}{T}), \quad \text{such that } (q_j,k_j)\in\mathbb{Z}^2.$$

The following key theorem proven by Pfander and Walnut [11, 13] extends the Kailath's rectangular underspread condition $T\Omega < 1$ to the essentially arbitrary regions.

Theorem 4. *Let $S \subset \mathbb{R} \times \hat{\mathbb{R}}$ as above. If $\mu^-(S) > 1$, then $OPW(S)$ is not identifiable. If $\mu^+(S) < 1$, then $\mathrm{OPW}(S)$ is identifiable via operator sampling, and the identifier is of the form*

$$g = \sum_n c_n \delta_{n/L},$$

where $L > T\Omega$ a prime number, and $\{c_n\}$ is an appropriately chosen $L-$periodic sequence. Moreover, there exist coefficients $a_{j,k}$ determined by the choice of the sequence $\{c_n\}$ such that

$$h_H(t,x) = \sum_{j=0}^{L-1} \sum_{k \in \mathbb{Z}} a_{j,k} (Hg)(t - \frac{q_i - k}{L}) \times\ r_j(t) \varphi_j(x - t - \frac{q_j + k}{L}) \quad (1)$$

unconditionally in $L^2(\mathbb{R}^2)$.

Here, the $r_j(t)$ and $\varphi_j(x)$ are "mollified rectangular" functions such that

$$r_j(t) \hat{\varphi}_j(\nu) = 1$$

on R_{q_j,m_j} and vanishes outside a small neighborhood of it.

3. STOCHASTIC CHANNELS

It is well known that the bandlimited stationary stochastic processes can be sampled in the same way as the deterministic functions.

Theorem 5. *If $\boldsymbol{x}(t)$ is bandlimited, that is, its power density spectrum is integrable and $S(\omega) = 0$ for all $|\omega| > \Omega/2$, then we can recover $\boldsymbol{x}(t)$ from the samples taken at a rate $T = \Omega^{-1}$*

$$\boldsymbol{x}(t) \overset{m.s.}{=} \sum_{n \in \mathbb{Z}} \boldsymbol{x}(nT) \operatorname{sinc} T(x - n).$$

It turns out that the sampling of operators can be similarly generalized to a class of stochastic operators. Here we report on our results.

Definition 6. *We will say $\boldsymbol{H}$ belongs to a stochastic operator Paley-Wiener space, or $H \in StOPW(S)$ if its scattering function $P_{\boldsymbol{\eta}}(t,\nu)$ is supported on S:*

$$\mathrm{StOPW}(S) := \{\boldsymbol{H} : \boldsymbol{h}(x,t) \textit{ is WSSUS random process, and } \operatorname{supp} P_{\boldsymbol{\eta}}(t,\nu) \subseteq S, \quad P_{\boldsymbol{\eta}}(t,\nu) \in L^1(\mathbb{R}^2)\}.$$

From the definition of the scattering function it follows that $|\boldsymbol{\eta}(t,\nu)|^2 \overset{m.s.}{=} 0$ for all $(t,\nu) \notin S$. In addition, a sample spreading function $\boldsymbol{\eta}(t,\nu) \in L^2(\mathbb{R}^2)$ almost surely, and the operator $\boldsymbol{H}$ is well-defined. In parallel with the development of the sampling theory of deterministic operators, we first focus on the case when the support of the scattering function is rectangular. A direct analogue of Theorem 2 is the following.

Theorem 7. *For $\boldsymbol{H} \in \mathrm{StOPW}([0,T) \times [-\Omega/2, \Omega/2])$ such that $T\Omega \leq 1$, we can reconstruct the kernel for all $x, t \in \mathbb{R}$:*

$$\boldsymbol{\kappa}(x + t, x) \overset{m.s.}{=} \chi_{[0,T)}(t) \sum_{n \in \mathbb{Z}} (\boldsymbol{H} \sum_{k \in \mathbb{Z}} \delta_{kT}) \operatorname{sinc} \Omega(x - nT).$$

A case when the support of the scattering function S is not confined to a rectangle of area one is rather more interesting. If $S \subset [T_0, T_0 + T) \times [\Omega_0, \Omega_0 + \Omega)$, with $T\Omega$ not necessarily less than 1, as in Theorem 4, there exists a prime number $L > T\Omega$ and a finite union of the integer translations of the base rectangle $R = [T_0, T_0 + \frac{T}{L}) \times [\Omega_0, \Omega_0 + \frac{1}{T})$ that cover S completely:

$$S \subseteq \bigcup_{j=1}^{L} R + (\frac{q_j T}{L}, \frac{k_j}{T}), \quad \text{such that } (q_j, k_j) \in \mathbb{Z}^2.$$

Theorem 8. *Given a stochastic operator $\boldsymbol{H} \in \mathrm{StOPW}(S)$ with a spreading function representation*

$$\boldsymbol{H} f(x) = \iint \boldsymbol{\eta}(t,\omega) \mathcal{T}_t M_\omega f(x)\, dt\, d\omega$$

and S as above, $\mu^+(S) < 1$, there exists a vector of complex weights $c \in \mathbb{C}^L$ such that we can reconstruct $\boldsymbol{H}$ from its response to a $L-$periodic $c-$weighted delta train

$$f(x) = \sum_{q=0}^{L-1} c_q \sum_{m \in \mathbb{Z}} \delta_{m + \frac{q}{L} + \frac{T_0}{T}}$$

via

$$\eta(x,\nu) \overset{m.s.}{=} \sum_{j=0}^{L-1} \sum_{p=0}^{L-1} a_{pj}(c) \times (Z \circ \boldsymbol{H}(f))(\frac{x - T_0}{T} + \frac{p - q_j}{L}, (\nu - \Omega_0)T - m_j) \times e^{2\pi i((\nu - \Omega_0)T - m_j)(p - q_j)/L} \chi_R(x - \frac{q_j T}{L}, \nu - \frac{k_j}{T}), \quad (2)$$

where $Zf(x,\nu)$ is the Zak transform

$$Zf(x,\nu) = \sum_{n \in \mathbb{Z}} f(x - n) e^{2\pi i n \nu},$$

χ_R is the characteristic function

$$\chi_R(x,\nu) = \begin{cases} 1, & (x,\nu) \in R, \\ 0, & (x,\nu) \notin R, \end{cases}$$

and the coefficients $a_{j,k}$ are determined by the choice of the sequence $\{c_n\}$.

Similarly to the Theorem 4, a characteristic function can be replaced with an infinitely smooth function

$$r_j(t) \hat{\varphi}_j(\nu) = \begin{cases} 1, & (x,\nu) \in E, \\ 0, & \text{outside of a small neighborhood of } E. \end{cases}$$

The coefficients $a_{j,k}$ are given with an explicit construction via straightforward inversion of the matrix populated by those time-frequency shifts of the vector c that correspond to the positions of the covering rectangles $R+(\frac{q_j T}{L}, \frac{k_j}{T})$. The further details can be found in [13].

We note that $\mu^+(S) < 1$ is a sufficient condition for the identification of the operator by the sampling formula (2). Necessity of $\mu^-(S) > 1$ for the identification of the class $\mathrm{StOPW}(S)$ follows from Theorem 4 by Pfander and Walnut. In the degenerate case when the stochastic operator is actually deterministic, the supports of $\eta(t,\nu)$ and $P_{\boldsymbol{\eta}}(t,\nu)$ coincide:

$$|\eta(t,\nu)|^2 = \mathbb{E}\left\{\boldsymbol{\eta}(t,\nu)\boldsymbol{\eta}^*(t,\nu)\right\} = \delta(0)\delta(0)P_{\boldsymbol{\eta}}(t,\nu),$$

and thus $\mathrm{OPW}(S) \subset \mathrm{StOPW}(S)$.

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