IDENTIFICATION OF OPERATORS WITH BANDLIMITED SYMBOLS

W. KOZEK [†¶] AND G. E. PFANDER[‡¶]

Abstract. Underspread and overspread operators are Hilbert–Schmidt operators with strictly bandlimited Kohn–Nirenberg symbol. In this paper, we prove a classical conjecture concerning the necessity of the underspread condition for the identifiability of such operator classes, and, in doing so, we exhibit a new uncertainty principle phenomenon in the time–frequency analysis of operators.

Key words. Underspread and overspread operators, Gabor frame operators, bandlimited Kohn-Nirenberg symbol, spreading function.

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1. Introduction.

Identification of incompletely known linear operators based on the observation of a restricted number of input and corresponding output signals is an important goal in many applied sciences. In *communications engineering*, for instance, identifying the transmission channel can help to adjust signal synthesis at the transmitter and signal analysis at the receiver. This is possible in wired communications, since a linear *time-invariant* system is a convolution operator, and — leaving numerical instability of deconvolution aside — is completely determined by its action on a single function.

Underspread and overspread operators on the other hand are *time-varying* Hilbert–Schmidt operators. They act on a space of d-dimensional signals, but the corresponding kernels of time–varying operators are essentially $2d$-dimensional so that a single observation of its action cannot uniquely determine the operator unless one has additional *a priori* knowledge of the operator class at hand in the form of certain constraints.

Hilbert–Schmidt operators can be represented as a weighted superposition of *translation operators* T_t, $t \in \mathbb{R}$, with $T_t f(x) = f(x-t)$, $x \in \mathbb{R}^d$, and *modulation operators* M_ν, $\nu \in \widehat{\mathbb{R}}$, with $M_\nu f(x) = f(x)e^{2\pi i \nu \cdot x}$, $x \in \mathbb{R}^d$, i.e., as an operator valued integral

$$H = \int\int \eta_H(t,\nu)\, T_t M_\nu \, dt\, d\nu \,. \tag{1.1}$$

Underspread and overspread operators are characterized by the property that the support of their *spreading function* η_H in (1.1) is contained in a rectangular parallelepiped. Such an operator is called *underspread* if the volume of the rectangular parallelepiped does not exceed one, and it is *overspread* otherwise, conditions, which are intimately related to uncertainty phenomena in time–frequency analysis. The Kohn–Nirenberg symbol of a Hilbert–Schmidt operator is the symplectic Fourier transformation of the respective spreading function, and, consequently, it is bandlimited in the case of an underspread or overspread operator.

The identification of underspread and overspread operators is important in various areas of electrical engineering and applied mathematics, including radar/sonar measurements and mobile radio communications, which we now briefly describe.

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The principle of *radar/sonar measurements* is to send out a signal modulated onto an electromagnetic/acoustic wave and to deduce information about a (generally) moving target from an echo of the signal [Sko80]. In simple *range-Doppler estimation* the target is modelled as a pure time-frequency shift and distance ("range") and velocity ("Doppler-shift") are estimated. A more precise model of the physical phenomenon is the *doubly-spread target model.* Here, the reflection is described as a continuous superposition of time-frequency shifts which arise since the target causes different reflections whose distance and velocity vary over a certain interval of the real-line. Unambiguous identification of the target was realized to depend on the product of the range and Doppler uncertainty, a fact that led to the terminology of underspread and overspread targets [Gre68]. Qualitatively speaking, overspread targets are those where the inherent uncertainty of the model is larger than the amount of information gathered by observing the reflected signal [VT71].

In *mobile radio communication*, the transmitted signal typically undergoes multiple reflections with different time–delay (corresponding to translation operators) and Doppler–shift (corresponding to modulation operators). The action of such channels on the signal can be modelled by underspread and overspread operators [VT71]. In order to obtain reliable communication, it is necessary to gather knowledge about channels by means of observations of transmitted and received signals to identify the channel operator (channel sounding) [MMH$^+$02, MGO03, LKS03].

Starting in the late 1950s, Thomas Kailath analyzed the identifiability of operators with restricted time and frequency spread [Kai59, Kai62, Kai63]. In engineering terms and without detailing a mathematical setup, Kailath proclaimed that a collection of communication channels which are characterized by having common maximum delay a and common maximum Doppler spread b, would be identifiable by a single input signal if and only if $ab \leq 1$, i.e., if and only if the operator class is underspread. To prove the necessity of the underspread condition, Kailath provided ingenious arguments based on the comparison of the degrees of freedom of operators (which approximate underspread operators), and degrees of freedom of the output signal. To compare finite dimensions, Kailath used the theoretical construct of a bandlimited input signal with finite duration.

Being aware of the mathematical shortcomings of his approach, and understanding the work of Slepian, Landau, and Pollak on "the dimensions of the space of essentially time- and bandlimited functions" [SP61, LP61, LP62], Kailath conjectured that the underspread condition $ab \leq 1$ is necessary in general [Kai62].

We shall prove Kailath's conjecture in Section 3 of this paper using the mathematical framework which is described in Section 2. In Section 4, we shall describe connections between the critical density in Gabor theory and the *critical spread* $ab = 1$ in the theory of operators with bandlimited symbols. We proof an identification result for Gabor frame operators in Section 4.1, and compare this result and Kailath's conjecture to uncertainty principles in time–frequency analysis in Section 4.2.

In Section 5, we shall extend our identifiability result to higher dimensions and include classes of operators which have restricted but not necessarily rectangular spreading support. These results are based on the representation theory of the reduced Weyl–Heisenberg group, a fact which indicates close connections of our results to quantum mechanics.

2. Preliminaries. The goal of operator/system identification is to locate, for given normed linear spaces X and Y and a normed linear space of bounded linear operators $\mathcal{H} \subset \mathcal{L}(X, Y)$, an element $f \in X$ which induces a bounded and stable linear map $\Phi_f : \mathcal{H} \longrightarrow Y,\ H \mapsto Hf$ (see Figure 2.1). Consequently, we call $\mathcal{H}$ *identifiable by $f \in X$, if there exist $A, B > 0$ with $A\,\|H\|_{\mathcal{H}} \leq \|Hf\|_Y \leq B\,\|H\|_{\mathcal{H}}$ for all $H \in \mathcal{H}$.*

In Section 2.1 and Section 2.2, we shall describe the operator spaces $\mathcal{H}$, the domain spaces X, and the target spaces Y that are considered in this paper. In Section 2.3, we shall present some techniques from Gabor analysis which will be used in this paper.

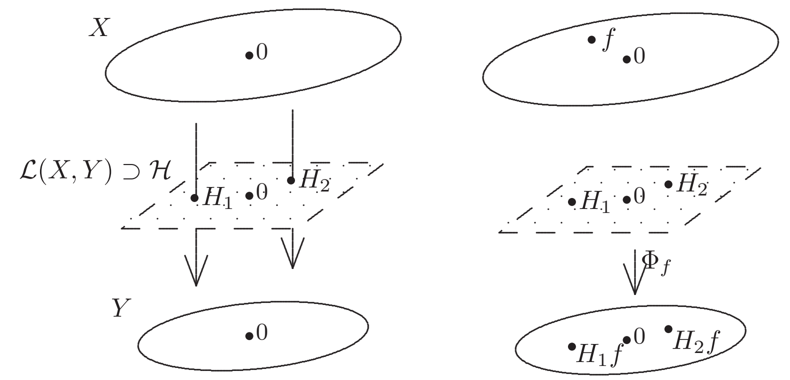

FIG. 2.1. *The goal of operator identification: find $f \in X$ such that $\Phi_f : \mathcal{H} \longrightarrow Y$ is bounded and stable.*

2.1. Hilbert–Schmidt operators with bandlimited symbols. We shall use Hilbert–Schmidt operators which act on the Hilbert space $L^2(\mathbb{R}^d)$ of complex valued and square integrable functions as model of physical time–varying linear systems, as they appear in radar and in mobile communications [FL96, Yoo02, Str05] .

A Hilbert–Schmidt operator $H : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is given by

$$Hf(x) = \int \kappa_H(x,t) f(t)\, dt = \int \kappa_H(x, x-t) f(x-t)\, dt \quad (a.e.),$$

with kernel $\kappa_H \in L^2(\mathbb{R}^{2d})$. The space of Hilbert–Schmidt operators $HS(L^2(\mathbb{R}^d))$ is itself a Hilbert space with inner product $\langle H_1, H_2\rangle_{\mathrm{HS}} = \langle \kappa_{H_1}, \kappa_{H_2}\rangle_{L^2}$ [Die70, Gaa73].

Underspread and overspread operators are Hilbert–Schmidt operators which satisfy two constraints: First, they have restricted delay, i.e., $\kappa_H(x, x-t)$ vanishes for large $|t|$, say for $|t| > \frac{a}{2} > 0$. Consequently, if f satisfies $\operatorname{supp} f \subseteq [0, T]$, then $\operatorname{supp} Hf \subseteq [-\frac{a}{2}, T+\frac{a}{2}]$. Second, underspread and overspread operators have the property that they are almost time–invariant, i.e., that their characteristics change only slowly over time. A comparison to the time–invariant convolution operators K given by $Kf(x) = \int \kappa_K(t) f(x-t)\, dt$ — whose kernel κ_K is independent of the time variable x — leads us to quantify the *slow variance* of an operator H by means of a Paley–Wiener type support condition on its spreading function which is given by

$$\eta_H(t,\nu) = \int \kappa_H(x, x-t) e^{-2\pi i \nu x}\, dx. \quad (a.e.)$$

In fact, underspread and overspread operators have the property that $\eta_H(t,\nu)$ vanishes for large $|\nu|$, say for $|\nu| > \frac{b}{2} > 0$.

Combining the aforementioned time and frequency spread conditions on H leads to the condition

$$\operatorname{supp} \eta_H \subseteq Q_{a,b} = [-\tfrac{a}{2}, \tfrac{a}{2}]^d \times [-\tfrac{b}{2}, \tfrac{b}{2}]^d \tag{2.1}$$

for some $a, b > 0$. An operator which satisfies (2.1) for $a, b > 0$ is called underspread if $ab \leq 1$ and overspread if $ab > 1$.

The spreading function η_H of a Hilbert–Schmidt operator H leads to a representation of H as an operator valued integral by means of (1.1). Here and in the following, operator valued integrals shall be interpreted weakly, i.e., $\int H(z)dzf,\ f \in L^2(\mathbb{R})$, is given by means of

$$\Big\langle \int H(z)dzf,\, g\Big\rangle_{L^2(\mathbb{R}^d)} = \int \langle H(z)f, g\rangle_{L^2(\mathbb{R}^d)}\, dz\,, \quad \text{for all } g \in L^2(\mathbb{R}^d).$$

Equation (1.1) illustrates that support restrictions on η_H reflect limitations on the maximal time and frequency shifts which the input signals undergo, a fact which emphasizes the usefulness of η_H in the time–frequency analysis of operators.

The condition (2.1) on a Hilbert–Schmidt operator H is a band–limitation on its Kohn–Nirenberg symbol σ_H which is given by

$$\sigma_H(x,\xi) = \int \kappa_H(x, x-y)\, e^{-2\pi i y\xi}\, dy = \int\int \eta_H(t,\nu) e^{2\pi i(x\nu - t\xi)}\, dt\, d\nu \ \ (a.e.) \tag{2.2}$$

[KN65, Fol89].

To prove that a class of Hilbert–Schmidt operators whose spreading functions satisfy (2.1) for fixed $a, b > 0$ with $ab \leq 1$ is identifiable necessitates the use of *Shah distributions* (also called *combfunctions* or *delta train*) $ш_a = \sum_{n\in\mathbb{Z}^d} \delta_{an}$, $a > 0$ as identifiers (see Section 2.2). Since not all Hilbert–Schmidt operators in $\mathcal{L}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$ can be extended to a space of distributions containing the Shah distribution, we need to restrict ourselves to operators which satisfy a regularity condition on their kernels. Here, we choose Hilbert–Schmidt operators with kernels in the Feichtinger algebra $S_0(\mathbb{R}^{2d})$, a Banach algebra of test–functions which is discussed in detail in Section 2.2. In fact, if $\kappa_H \in S_0(\mathbb{R}^{2d})$, then the Hilbert–Schmidt operator H extends to $S_0'(\mathbb{R}^d)$ with $ш_a \in S_0'(\mathbb{R}^d)$ [FZ98]. We set

$$\mathcal{H} = \big\{ H \in HS(L^2(\mathbb{R}^d)) :\ \kappa_H \in S_0(\mathbb{R}^{2d}) \big\}\,, \tag{2.3}$$

and, as discussed above, we consider operator classes with restricted spreading, i.e., we consider operator classes of the form

$$\mathcal{H}_M = \{ H \in \mathcal{H} :\ \operatorname{supp} \eta_H \subseteq M \}\ , \quad M \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d. \tag{2.4}$$

Note that $\mathcal{H}$ and $\mathcal{H}_M$, $M \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, are not closed as linear subspaces of the space of Hilbert–Schmidt operators, and that $\mathcal{H}_M \subseteq \mathcal{H}_{M'}$ if $M \subseteq M'$.

2.2. The Feichtinger Algebra. Introduced in [Fei81], Feichtinger's Banach algebra $S_0(\mathbb{R}^d)$ of test functions gained popularity in the growing field of Gabor analysis which is discussed in Section 2.3. The usefulness of $S_0(\mathbb{R}^d)$ stems from the fact that it is the smallest Banach space allowing a meaningful time–frequency analysis, which, as a consequence, extends to its respectively large dual Banach space $S_0'(\mathbb{R}^d)$. In fact, the L^2–Fourier transform, the modulation operators M_ν, $\nu \in \widehat{\mathbb{R}}^d$, and the translation operators T_t, $t \in \mathbb{R}^d$, which are all unitary on the Hilbert space $L^2(\mathbb{R}^d)$, are isometric isomorphisms on the Feichtinger algebra $S_0(\mathbb{R}^d)$, and, therefore, on its dual $S_0'(\mathbb{R}^d)$. The Feichtinger algebra $S_0(\mathbb{R}^d)$ can be continuously embedded in any Banach space with these properties and which contains at least one, and therefore all, non–trivial Schwartz function [FZ98].

Note that we chose to work with the Banach spaces $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$ as supposed to the Fréchet space of Schwartz functions $\mathcal{S}(\mathbb{R}^d) \subset S_0(\mathbb{R}^d)$ and its dual $\mathcal{S}'(\mathbb{R}^d) \supset S_0'(\mathbb{R}^d)$ of tempered distributions for the convenience of expressing continuity (boundedness) and openess (stability) of linear operators by means of norm inequalities. We would like to point out that the results in this paper are consequences of the structure of the identification problem at hand, and not of topological subtleties.

There exist various ways of defining $S_0(\mathbb{R}^d)$, and equally many different equivalent norms for $S_0(\mathbb{R}^d)$. Here, we shall give a definition based on the space of Lebesgue measurable and integrable functions $L^1(\mathbb{R}^d)$, the space of Fourier transforms of functions in $L^1(\mathbb{R}^d)$, which is denoted by $A(\mathbb{R}^d)$ and which is equipped with the Banach–space structure of $L^1(\mathbb{R}^d)$ by means of $\|\widehat{f}\|_A = \|f\|_{L^1}$[Kat76], and the space of absolutely summable sequences $l^1(\mathbb{Z}^d)$.

The *Feichtinger algebra* $S_0(\mathbb{R}^d)$ coincides with the *Wiener amalgam space* $W(A(\mathbb{R}^d), l^1(\mathbb{Z}^d))$. Consequently, we have $f \in S_0(\mathbb{R}^d)$ if and only if f is locally in $A(\mathbb{R}^d)$ with global decay of l^1–type, i.e., given any compactly supported $\psi \in A(\mathbb{R}^d)$ with $\sum_{n\in\mathbb{Z}^d} T_n\psi = 1$ we have $f \in S_0(\mathbb{R}^d)$ if and only if $\sum_{n\in\mathbb{Z}^d} \|f \cdot T_n\psi\|_A < \infty$, and

$$\|f\|_{S_0} = \sum_{n\in\mathbb{Z}^d} \|f \cdot T_n\psi\|_A$$

is a norm on $S_0(\mathbb{R}^d)$. Moreover, $S_0(\mathbb{R}^d)$ is a Banach algebra under convolution and pointwise multiplication.

The dual space $S_0'(\mathbb{R}^d)$ of the Feichtinger algebra satisfies $S_0'(\mathbb{R}^d) = W(A'(\mathbb{R}^d), l^\infty(\mathbb{Z}^d))$ since the class of compactly supported functions in $A(\mathbb{R}^d)$ is dense in $A(\mathbb{R}^d)$ [FG85]. Hence, $S_0'(\mathbb{R}^d)$ contains *Dirac's delta* $\delta : f \mapsto f(0)$ and *Shah distributions* $ш_a = \sum_{n\in\mathbb{Z}^d} \delta_{an}$, where $\delta_{na} = T_{na}\delta$ and $a > 0$. We set $ш = ш_1$.

2.3. Gabor analysis. Most techniques applied in this paper originate from Gabor analysis.

Gabor introduced the concept of coherent states to electrical engineering independently of quantum theory [Gab46, Grö01]. Hence, we shall simply call the family

$$(g, a, b) = \{M_{kb}T_{la}g\}_{k,l\in\mathbb{Z}^d}$$

of coherent states a *Gabor system*.

One of the basic results of Gabor analysis is the fact that there exists $g \in L^2(\mathbb{R}^d)$ such that (g, a, b) is an orthonormal basis for $L^2(\mathbb{R}^d)$ if and only if $ab = 1$. For example, the Gabor system $(\mathbf{1}_{[0,a)}, a, b)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$, where $\mathbf{1}_A(x) = 1$ for $x \in A$ and $\mathbf{1}_A(x) = 0$ else.

If $ab > 1$ the system (g, a, b) is not complete. However, if $ab > 1$ then there exists $g \in L^2(\mathbb{R})$ such that the (g, a, b)–synthesis map $D_g : l^2(\mathbb{Z}^2) \longrightarrow L^2(\mathbb{R})$, $\{c_{k,l}\} \mapsto \sum c_{k,l}M_{kb}T_{la}g$ is well–defined, bounded, and stable, i.e., (g, a, b) is a Riesz basis for its closed linear span, $\overline{\mathrm{span}(g, a, b)}$, in $L^2(\mathbb{R}^d)$; and, hence, there exist $A, B > 0$ such that

$$A\|\{c_{k,l}\}\|_{l^2} \leq \| \sum_{k,l\in\mathbb{Z}^d} c_{k,l} M_{kb} T_{la} g\|_{L^2} \leq B\|\{c_{k,l}\}\|_{l^2} \quad \text{for all } \{c_{k,l}\} \in l^2(\mathbb{Z}^{2d})\,. \tag{2.5}$$

For $ab < 1$, the system (g,a,b), $g \in L^2(\mathbb{R})$ is overcomplete, i.e., there exists a non–trivial coefficient sequence $\{c_{k,l}\} \in l^2(\mathbb{Z}^{2d}) \setminus \{0\}$ such that $\sum c_{k,l} M_{kb} T_{la} g = 0$ in $L^2(\mathbb{R}^d)$. Nevertheless, for an appropriate choice of g, e.g., g being a Gaussian, the (g,a,b)–analysis operator $C_g = D_g^* : L^2(\mathbb{R}) \longrightarrow l^2(\mathbb{Z}^2)$, $f \mapsto \{\langle f, M_{kb} T_{la} g\rangle\}$ is well–defined, bounded, and stable, i.e., (g,a,b) forms a frame for $L^2(\mathbb{R}^d)$; and, hence, there exists $A, B > 0$ such that

$$A\|f\|_{L^2}^2 \leq \sum |\langle f, M_{kb} T_{la} g\rangle|^2 \leq B\|f\|_{L^2}^2 \quad \text{for all } f \in L^2(\mathbb{R}^d)\,. \tag{2.6}$$

As a consequence of (2.6), every $f \in L^2(\mathbb{R}^d)$ has a stable representation

$$f = \sum_k \sum_l c_{k,l} M_{kb} T_{la} g \ \text{ in } L^2(\mathbb{R}^d)\,,$$

in terms of the frame (g,a,b), where the coefficients $\{c_{k,l}\} \in l^2(\mathbb{Z}^2)$ can be chosen by means of inner products, i.e., $c_{k,l} = \langle f, M_{kb} T_{la} \gamma\rangle$, where (γ, a, b) is a so–called dual frame of (g,a,b).

More details on time–frequency analysis with some relevance to this paper can be found in [Grö01].

Operator–theoretic applications of Gabor theory as presented in this paper have drawn increasing interest in applied harmonic analysis, see, for example, [Dau88, HRT97, FK98, Koz98, RT98, Lab01, FN03, CG03, Hei03, GLM04].

3. Identification of underspread and overspread operators. We shall first prove Kailath's conjecture for operators acting on functions defined on the real line, i.e., we choose $d = 1$. The identification problem is given by the operator space $\mathcal{H}_{Q_{a,b}}$, $a, b > 0$, which is defined in (2.3) and (2.4), where $M = Q_{a,b} = [-\frac{a}{2}, \frac{a}{2}] \times [-\frac{b}{2}, \frac{b}{2}]$. The linear space $\mathcal{H}_{Q_{a,b}}$ is equipped with the Hilbert–Schmidt norm and its elements map $X = S_0'(\mathbb{R})$ to $Y = L^2(\mathbb{R})$ [FK98].

The Lebesgue measure $a{\cdot}b$ of the set $Q_{a,b}$ plays a crucial role in determining the identifiability of $\mathcal{H}_{Q_{a,b}}$. The main result of our paper is

Theorem 3.1. *The set $\mathcal{H}_{Q_{a,b}}$ is identifiable, i.e., there is $f \in S_0'(\mathbb{R})$ such that $\Phi_f : \mathcal{H}_{Q_{a,b}} \longrightarrow L^2(\mathbb{R})$ is bounded and stable, where $\mathcal{H}_{Q_{a,b}}$ is equipped with the Hilbert–Schmidt norm, if and only if $ab \leq 1$.*

First, we shall give a proof of the long understood identifiability of $\mathcal{H}_{Q_{a,b}}$ for $ab \leq 1$.

3.1. Sufficiency of $ab \leq 1$ for the identifiability of $\mathcal{H}_{Q_{a,b}}$. Our proof of the sufficiency of the underspread condition is based on the unitarity of the Zak transformations $Z_c : L^2(\mathbb{R}) \longrightarrow L^2(Q_{c,\frac{1}{c}})$, $c > 0$, which are defined by

$$Z_c f(t,\nu) = c^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} f(t - cn) e^{2\pi i c n \nu}, \quad (a.e.)\ (t,\nu) \in Q_{c,\frac{1}{c}}\,,$$

and the following lemma.

Lemma 3.2. *For $H \in \mathcal{H}$ we have*

$$Z_c \circ H \perp\!\!\!\perp_c(t,\nu) = c^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \eta_H(t - cn, \nu - \tfrac{m}{c}) e^{2\pi i (\nu - \frac{m}{c}) t}, \quad (t,\nu) \in Q_{c,\frac{1}{c}}\,.$$

Proof. For $x \in \mathbb{R}$ we have $H\sqcup\!\sqcup\!\sqcup_c(x) = \langle \sqcup\!\sqcup\!\sqcup_c, \kappa_H(x,\cdot)\rangle = \sum_{k\in\mathbb{Z}} \kappa_H(x, ck)$. Using in succession the Tonelli–Fubini Theorem, the formula

$$\kappa_H(x,y) = \int \eta_H(x-y,\nu)e^{2\pi i\nu x}\,d\nu, \quad (x,y)\in\mathbb{R}^2,$$

two substitutions, and the Poisson summation formula [Grö01], page 250, we obtain for $(t,\nu)\in Q_{c,\frac{1}{c}}$ that

$$\begin{aligned} Z_c \circ H\sqcup\!\sqcup\!\sqcup_c(t,\nu) &= c^{\frac{1}{2}} \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \kappa_H(t-cl, ck)e^{2\pi i cl\nu} \\ &= c^{\frac{1}{2}} \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \int \eta_H(t-cl-ck,\omega)e^{2\pi i(cl\nu+\omega(t-cl))}\,d\omega \\ &\overset{\substack{\xi=\nu+\omega\\ n=k+l}}{=} c^{\frac{1}{2}} \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \int \eta_H(t-cn,\xi+\nu)e^{2\pi i(\xi+\nu)t}e^{-2\pi i cl\xi}\,d\xi \\ &= c^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \eta_H(t-cn,\nu-\tfrac{m}{c})e^{2\pi i(\nu-\frac{m}{c})t}. \end{aligned}$$

∎

A standard periodization argument leads to the sufficiency of $ab \le 1$ for the identifiability of $\mathcal{H}_{Q_{a,b}}$. In fact, the following theorem shows that for $f = \sqcup\!\sqcup\!\sqcup_a \in S_0'(\mathbb{R})$ we have $\Phi_f : \mathcal{H}_{Q_{a,b}} \longrightarrow L^2(\mathbb{R})$, where $\mathcal{H}_{Q_{a,b}}$ is equipped with the Hilbert–Schmidt norm, is bounded and stable whenever $ab \le 1$.

THEOREM 3.3. *The operator family $\mathcal{H}_M = \{H \in \mathcal{H} : \ \operatorname{supp}\eta_H \subseteq M\}$ can be identified with the identifier $\sqcup\!\sqcup\!\sqcup_c$ if and only if the interior M° of M satisfies*

$$M^\circ \ \cap \bigcup_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \Big(M^\circ + (cn, \tfrac{m}{c})\Big) = \emptyset, \tag{3.1}$$

i.e., if and only if M° is contained in a fundamental domain of the lattice $c\mathbb{Z}\times\frac{1}{c}\mathbb{Z}$. In particular, $\mathcal{H}_{Q_{a,b}}$, $a,b>0$, is identifiable with $\sqcup\!\sqcup\!\sqcup_c$ if and only if $a \le c$ and $ab \le 1$.

Note that Theorem 3.3 classifies all sets M with the property that $\mathcal{H}_M$ can be identified using the tempered distribution $\sqcup\!\sqcup\!\sqcup_c$, $c>0$. No result regarding the necessity of the underspread condition $ab\le 1$ for the identifiability of $\mathcal{H}_{Q_{a,b}}$ by any other $f\in S_0'(\mathbb{R})$ has been obtained.

Figure 3.1 is a picture proof of Theorem 3.3 for $M = Q_{c,\frac{1}{c}}$, $c>0$. Details in the case $c=1$ are given below.

Proof of Theorem 3.3. For ease of notation, we shall only provide a proof of Theorem 3.3 for $c=1$. The general case follows from Theorem 5.4.

First, we show that if (3.1) holds, then $\sqcup\!\sqcup\!\sqcup$ identifies $\mathcal{H}_M$. Set $Q = Q_{1,1}$ and let $A_{m,n} = M^\circ \cap \big(Q+(m,n)\big)$ and $B_{m,n} = A_{m,n} - (m,n) \subseteq Q$. Then $B_{m,n}\cap B_{m',n'} = \emptyset$ for $(m,n)\ne(m',n')$, since else $M^\circ \cap \big(M^\circ + (m-m', n-n')\big) \ne \emptyset$. Further, the spreading function η_H of each $H\in\mathcal{H}_M$ is continuous, and, therefore, $\eta_H(t,\nu)=0$ for all $(t,\nu)\notin \bigcup_{m,n} A_{m,n}$. We conclude that $\{(t,\nu)\in Q : \ Z\circ H\sqcup\!\sqcup\!\sqcup(t,\nu)\ne 0\} \subseteq$

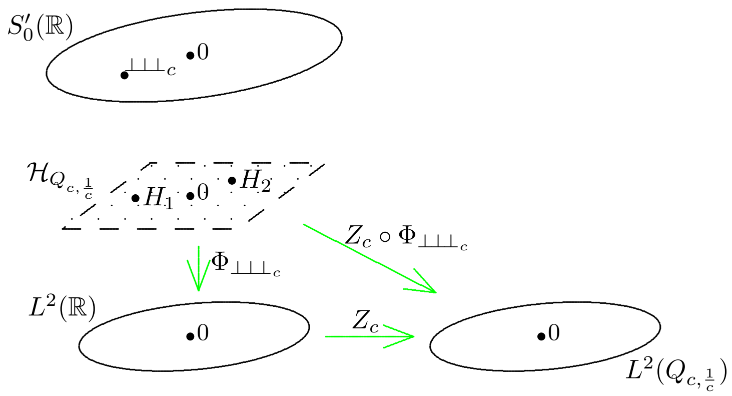

FIG. 3.1. *Sketch of the proof of the identifiability of $\mathcal{H}_{Q_{c,\frac{1}{c}}}$, $c > 0$, using as identifier $\perp\!\!\perp\!\!\perp_c$. The Zak transform Z_c is unitary and, therefore, bounded and stable, and $Z_c \circ \Phi_{\perp\!\!\perp\!\!\perp_c}$ maps $\mathcal{H}_{Q_{c,\frac{1}{c}}}$ into $L^2(Q_{c,\frac{1}{c}})$ and is bounded and stable as well. We conclude that $\Phi_{\perp\!\!\perp\!\!\perp_c}$ is bounded and stable on $\mathcal{H}_{Q_{c,\frac{1}{c}}}$, i.e., $\perp\!\!\perp\!\!\perp_c$ identifies $\mathcal{H}_{Q_{c,\frac{1}{c}}}$.*

$\bigcup_{m,n} B_{m,n} \subseteq Q$. For $H \in \mathcal{H}_M$ we calculate

$$\begin{aligned}\|H\|_{HS} &= \|\eta_H\|_{L^2(\mathbb{R}\times\widehat{\mathbb{R}})} = \sum_{m,n} \|\eta_H\|_{L^2(A_{m,n})} = \sum_{m,n} \|T_{(-m,-n)}\eta_H\|_{L^2(B_{m,n})} \\ &= \sum_{m,n} \|Z \circ H\perp\!\!\perp\!\!\perp\|_{L^2(B_{m,n})} = \|Z \circ H\perp\!\!\perp\!\!\perp\|_{L^2(Q)} = \|H\perp\!\!\perp\!\!\perp\|_{L^2(\mathbb{R})} \\ &= \|\Phi_{\perp\!\!\perp\!\!\perp} H\|_{L^2(\mathbb{R})}\,; \qquad (3.2)\end{aligned}$$

and, by definition, $\mathcal{H}_M$ allows identification with identifier $\perp\!\!\perp\!\!\perp$.

Let us now assume that $M^\circ \cap \bigcup_{(m,n)\neq 0} M^\circ{+}(m,n) \neq \emptyset$, and show that $\mathcal{H}_M$ is not identifiable. In this case, there exists $(t_0,\nu_0) \in M^\circ \cap \bigcup_{(m,n)\neq 0} \big(M^\circ{+}(m,n)\big)$, $\frac{1}{2} > \epsilon > 0$, and $(n_0,m_0) \in \mathbb{Z}^{2d}$ with $B_\epsilon(t_0,\nu_0) \subset M^\circ \cap \big(M^\circ{+}(n_0,m_0)\big)$, where $B_\epsilon(t_0,\nu_0) = \{(t,\nu) : \|(t,\nu)-(t_0,\nu_0)\|_\infty < \epsilon\}$. Hence $B_\epsilon(t_0-n_0,\nu_0-m_0) \subset M^\circ$. Choose $0 \neq \widetilde{\eta} \in A(\mathbb{R}^{2d}) \subset S_0(\mathbb{R}^{2d})$ with $\operatorname{supp}\widetilde{\eta} \subset B_\epsilon(t_0,\nu_0)$, and define $H \in \mathcal{H}_M$ by means of $\eta(t,\nu) = \widetilde{\eta}(t,\nu) - \widetilde{\eta}(t+n_0,\nu+m_0)e^{2\pi i t m_0} \not\equiv 0$, $(t,\nu) \in \mathbb{R}\times\widehat{\mathbb{R}}$. We obtain

$$\begin{aligned}Z\circ H\perp\!\!\perp\!\!\perp(t,\nu) &= \sum_{m,n\in\mathbb{Z}} \eta(t{-}n,\nu{-}m)e^{2\pi i(\nu-m)t} \\ &= \sum_{m,n\in\mathbb{Z}} \Big(\widetilde{\eta}(t{-}n,\nu{-}m) - \widetilde{\eta}(t{-}n{+}n_0,\nu{-}m{+}m_0)e^{2\pi i(t-n)m_0}\Big)\, e^{2\pi i(\nu-m)t} \\ &= \Big(\sum_{m,n\in\mathbb{Z}} \widetilde{\eta}(t{-}n,\nu{-}m)e^{2\pi i(\nu-m)t}\Big) \;\; - \\ &\qquad \Big(\sum_{m,n\in\mathbb{Z}} \widetilde{\eta}(t{+}n_0{-}n,\nu{+}m_0{-}m)e^{2\pi i(t+n_0)m_0}e^{2\pi i(\nu-m)t}\Big) \;=\; 0\,.\end{aligned}$$

The injectivity of the Zak transformation implies $H\perp\!\!\perp\!\!\perp = 0$, contradicting the injectivity of $\Phi_{\perp\!\!\perp\!\!\perp}$ and therefore the identifiability of $\mathcal{H}_M$ by $\perp\!\!\perp\!\!\perp$. ∎

Note that equation (3.2) implies that $\Phi_{\perp\perp\perp}$, which is *a-priori* defined on $\mathcal{H}_M = \{H \in \mathcal{H} :\ \operatorname{supp} \eta_H \subseteq M\} \subset \mathcal{H} \subset HS(L^2(\mathbb{R}))$ where M is a fundamental domain of $\mathbb{Z} \times \mathbb{Z}$, can be isometrically extended to its HS–closure $\overline{\mathcal{H}_M} = \{H \in HS(L^2(\mathbb{R})) :\ \operatorname{supp} \eta_H \subseteq M\}$. Certainly, not all $H \in HS(L^2(\mathbb{R}))$ extend in this fashion to $S_0'(\mathbb{R})$, and, hence, we must continue to focus our attention on operators with kernels in the Feichtinger algebra, i.e., on operator classes contained in $\mathcal{H} = \big\{H \in HS(L^2(\mathbb{R})) :\ \kappa_H \in S_0(\mathbb{R}^2)\big\}$.

3.2. Necessity of $ab \leq 1$ for the identifiability of $\mathcal{H}_{Q_{a,b}}$. We shall show that for $ab > 1$ and *every* $f \in S_0'(\mathbb{R})$, the well–defined operator $\Phi_f : \mathcal{H}_{Q_{a,b}} \longrightarrow L^2(\mathbb{R})$ is *not* stable.

To obtain this result, we shall equip $l_0(\mathbb{Z}^2)$ with the topology induced by the l^2–norm and use the fact that $ab > 1$ to construct a bounded and stable synthesis operator $E :\ l_0(\mathbb{Z}^2) \to \mathcal{H}_M$ in Lemma 3.4, and a bounded and stable (g, a', b')–analysis operator $C_g : L^2(\mathbb{R}) \longrightarrow l^2(\mathbb{Z}^2)$ in the proof of Theorem 3.6, with the property that the compositions

$$C_g \circ \Phi_f \circ E :\ l_0(\mathbb{Z}^2) \longrightarrow l^2(\mathbb{Z}^2), \quad f \in S_0'(\mathbb{R})$$

are not stable. The stability of E and C_g implies that all operators $\Phi_f : \mathcal{H}_{Q_{a,b}} \longrightarrow L^2(\mathbb{R})$, $f \in S_0'(\mathbb{R})$, must not be stable, showing that $\mathcal{H}_{Q_{a,b}}$ is not identifiable for $ab > 1$.

We shall now construct the aforementioned synthesis operator E. For $ab > 1$, we choose $\lambda \in \mathbb{R}$ with $1 < \lambda^4 < a\,b$. Using a product–convolution operator $P : f \mapsto (f * \eta_1)\,\check{\eta_2}$ as prototype operator, we define the embedding operator E by means of

$$E :\ l_0(\mathbb{Z}^2) \to \mathcal{H}_M, \quad \{\sigma_{k,l}\} \mapsto \sum_{k,l} \sigma_{k,l}\ M_{k\lambda\alpha} T_{l\lambda\beta} P T_{-l\lambda\beta} M_{-k\lambda\alpha}\,,$$

where we chose $\alpha = \frac{1}{a}$ and $\beta = \frac{1}{b}$ for simplicity of notation. The choice of λ allows us to construct $P \in \mathcal{H}_{Q_{a,b}}$ in Lemma 3.4 such that $\{M_{k\lambda\alpha} T_{l\lambda\beta} P T_{-l\lambda\beta} M_{-k\lambda\alpha}\}_{k,l\in\mathbb{Z}}$ is a Riesz basis for its closed linear space in the Hilbert space of Hilbert–Schmidt operators, and, as consequence of (2.5), E is stable. In addition to the Riesz property, P is designed in Lemma 3.4 to satisfy a time–frequency localization property which will play a central role in the proof of our main result.

LEMMA 3.4. *Fix $\lambda > 1$ with $1 < \lambda^4 < a\,b$ and choose $\eta_1, \eta_2 \in \mathcal{S}(\mathbb{R})$ with values in $[0, 1]$ and*

$$\eta_1(t) = \begin{cases} 1 & \text{for } |t| \leq \frac{a}{2\lambda} \\ 0 & \text{for } |t| \geq \frac{a}{2} \end{cases} \quad \text{and} \quad \eta_2(\nu) = \begin{cases} 1 & \text{for } |\nu| \leq \frac{b}{2\lambda} \\ 0 & \text{for } |\nu| \geq \frac{b}{2} \end{cases}.$$

The operator $P \in \mathcal{H}_{Q_{a,b}}$ defined by $\eta_P = \eta_1 \otimes \eta_2$ has the properties:
a) The synthesis operator

$$E :\ l_0(\mathbb{Z}^2) \to \mathcal{H}_M, \quad \{\sigma_{k,l}\} \mapsto \sum_{k,l} \sigma_{k,l}\ M_{k\lambda\alpha} T_{l\lambda\beta} P T_{-l\lambda\beta} M_{-k\lambda\alpha} \tag{3.3}$$

is well–defined, bounded and stable.
b) The operator $P \in \mathcal{H}_M$ is a time–frequency localization operator in the following sense: There exist functions $d_1, d_2 : \mathbb{R} \to \mathbb{R}_0^+$, which decay rapidly at infinity, and which have the property that for all $f \in S_0'(\mathbb{R})$ we have $|Pf(x)| \leq \|f\|_{S_0'}\, d_1(x)$, $x \in \mathbb{R}$ and $|\widehat{Pf}(\xi)| \leq \|f\|_{S_0'}\, d_2(\xi)$, $\xi \in \widehat{\mathbb{R}}$.

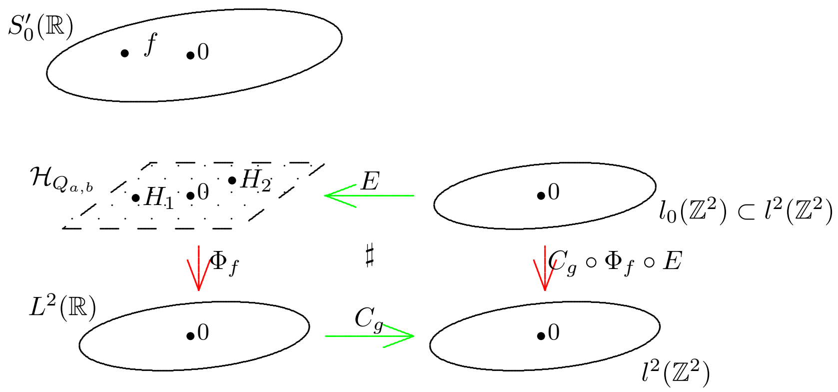

FIG. 3.2. *Sketch of the proof that $\mathcal{H}_{Q_{a,b}}$ is not identifiable if $ab > 1$. We show that for all $f \in S_0'(\mathbb{R})$, the bounded operator $C_g \circ \Phi_f \circ E$ is not stable. The synthesis operator E and the analysis operator C_g are stable, hence, stability of $C_g \circ \Phi_f \circ E$ must fail at Φ_f.*

Proof. a) Observe that for any $(s, \omega) \in \mathbb{R}\times\widehat{\mathbb{R}}$ and $f \in S_0(\mathbb{R})$ we have

$$\begin{aligned} M_\omega T_s P T_{-s} M_{-\omega} f &= \int\int \eta_P(t,\nu) M_\omega T_s\, T_t M_\nu\, T_{-s} M_{-\omega} f \, dt\, d\nu \\ &= \int\int \eta_P(t,\nu) e^{2\pi i(\omega t - s\nu)} T_t M_\nu f \, dt\, d\nu, \end{aligned}$$

Hence, for E defined in (3.3) and any $\{\sigma_{k,l}\} \in l_0(\mathbb{Z}^2)$, we have $E\{\sigma_{k,l}\} \in \mathcal{H}_{Q_{a,b}}$ with

$$\eta_{E\{\sigma_{k,l}\}}(t,\nu) = \eta_P(t,\nu) \sum_{k,l\in\mathbb{Z}} \sigma_{k,l}\, e^{2\pi i(k\lambda\alpha t - l\lambda\beta\nu)}, \quad (t,\nu) \in \mathbb{R}\times\widehat{\mathbb{R}}. \tag{3.4}$$

We consider $l_0(\mathbb{Z}^2)$ as a subspace of $l^2(\mathbb{Z}^2)$ and observe that E is stable, since

$$\|E\{\sigma_{k,l}\}\|_{HS} = \|\eta_{E\{\sigma_{k,l}\}}\|_{L^2} \geq \|\eta_{E\{\sigma_{k,l}\}}\, \mathbf{1}_{[-\frac{a}{2\lambda},\frac{a}{2\lambda}]\times[-\frac{b}{2\lambda},\frac{b}{2\lambda}]}\|_{L^2} = \tfrac{ab}{\lambda^2}\, \|\{\sigma_{k,l}\}\|_{l^2}.$$

The boundedness of E follows from a similar calculation.
b) For $f \in S_0(\mathbb{R})$ and $x \in \mathbb{R}$ we have

$$\begin{aligned} |Pf(x)| &= \left| \int\int \eta_P(t,\nu) e^{2\pi i \nu x} f(x-t)\, dt\, d\nu \right| \\ &= \left| \int \eta_2(\nu) e^{2\pi i\nu x}\, d\nu \right| \, \left| \int \eta_1(t) f(x-t)\, dt \right| \\ &\leq |\widehat{\eta}_2(-x)| \; \|f\|_{S_0'} \|\eta_1\|_{S_0}. \end{aligned} \tag{3.5}$$

The function $d_1(x) = |\widehat{\eta}_2(-x)|\, \|\eta_1\|_{S_0}$ decays rapidly at infinity, i.e., $d_1(x) \to 0$ as $|x| \to \infty$ faster than any power of $\frac{1}{x}$, since $\widehat{\eta}_2 \in \mathcal{S}(\mathbb{R})$. Further, the inequality $|Pf(x)| \leq \|f\|_{S_0'}\, d_1(x)$, $x \in \mathbb{R}$, extends to general $f \in S_0'(\mathbb{R})$, since $S_0(\mathbb{R})$ is w*–dense in $S_0'(\mathbb{R})$.

To establish a rapidly decaying bound on $|\widehat{Pf}|$, $f \in S_0'(\mathbb{R}^d)$, we first assume

$f \in S_0(\mathbb{R})$ and calculate for $\xi \in \widehat{\mathbb{R}}$

$$
\begin{aligned}
|\widehat{Pf}(\xi)| &= \left| \int \widehat{\eta}_2(-x) \int \eta_1(t) f(x-t)\, dt\, e^{-2\pi i \xi x}\, dx \right| \\
&= \left| \int \widehat{\eta}_2(-x) \int \widehat{\eta}_1(\gamma) \widehat{f}(\gamma) e^{-2\pi i x(\xi-\gamma)}\, d\gamma\, dx \right| \\
&= \left| \int \eta_2(\xi-\gamma) \widehat{\eta}_1(\gamma) \widehat{f}(\gamma)\, d\gamma \right| \qquad (3.6) \\
&\leq \|f\|_{S_0'} \|\eta_2(\xi-\ \cdot)\widehat{\eta}_1(\cdot)\|_{S_0}. \qquad (3.7)
\end{aligned}
$$

The application of the theorem of Tonelli and Fubini to obtain (3.6) is valid for $f \in S_0(\mathbb{R})$, and the validity of (3.7) extends once more to general $f \in S_0'(\mathbb{R})$.

We claim that $d_2(\xi) = \|\eta_2(\xi-\ \cdot)\widehat{\eta}_1(\cdot)\|_{S_0}$ is rapidly decaying. Since the Feichtinger algebra $S_0(\mathbb{R})$ equals the Wiener amalgam space $W(A(\mathbb{R}), l^1(\mathbb{Z}))$, we choose $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}) \subset A(\mathbb{R})$ with supp $\widehat{\varphi} \subseteq [-1,+1]$, and $\sum_{n\in\mathbb{Z}} \mathrm{T}_n \widehat{\varphi} \equiv 1$, and observe that $(\|\widehat{\eta}_1 \cdot \mathrm{T}_n \widehat{\varphi}\,\|_A)_{n\in\mathbb{Z}}$, decays rapidly, i.e., for any $k \in \mathbb{N}$ exists $C_k > 0$ such that

$$
\left\|g \cdot \mathrm{T}_n \widehat{\varphi}\right\|_A = \int \left| \int \eta_1(x) e^{-2\pi i x n} \varphi(t-x)\, dx \right| dt \leq C_k (1+n^2)^{-k/2},\ n \in \mathbb{Z}, \qquad (3.8)
$$

[Grö01], page 228. For $k \in \mathbb{N}$ we choose C_k satisfying (3.8) and calculate

$$
\begin{aligned}
d_2(\xi) = \|\eta_2(\xi-\ \cdot)\widehat{\eta}_1(\cdot)\|_{S_0} \ &\leq\ C \sum_{n\in\mathbb{Z}} \|\,\mathrm{T}_n \widehat{\varphi}(\cdot)\ \eta_2(\xi-\ \cdot)\widehat{\eta}_1(\cdot)\|_A \\
&= C \sum_{\xi-1-\frac{b}{2}<n<\xi+1+\frac{b}{2}} \|\,\mathrm{T}_n \widehat{\varphi}(\cdot)\ \eta_2(\xi-\ \cdot)\widehat{\eta}_1(\cdot)\|_A \\
&\leq C\,\|\eta_2\|_A \sum_{\xi-1-\frac{b}{2}<n<\xi+1+\frac{b}{2}} \|\,\mathrm{T}_n \widehat{\varphi}(\cdot)\ \widehat{\eta}_1(\cdot)\|_A \\
&\leq C\,C_k\,\|\eta_2\|_A \,\lceil 2+b \rceil \left(1+ \min\left\{ \lceil \xi - 1 - \tfrac{b}{2} \rceil^2, \lfloor \xi + 1 + \tfrac{b}{2} \rfloor^2 \right\} \right)^{-k/2} \\
&\leq \widetilde{C}(1+\xi^2)^{-k/2}.
\end{aligned}
$$

□

Lemma 3.5 is technical but of upmost importance in the proof of Theorem 3.6. It generalizes the fact that $m \times n$ matrices with $m < n$ have a non–trivial kernel and, therefore, are not stable, to operators acting on $l^2(\mathbb{Z}^2)$. In fact, the bi–infinite matrices $M = (m_{j',j})_{j',j\in\mathbb{Z}^2}$ considered in Lemma 3.5 are not dominated by its diagonal $m_{j,j}$ — which would correspond to square matrices — but by a skew diagonal $m_{j,\lambda j}$, with $\lambda > 1$.

LEMMA 3.5. *Given $M = (m_{j',j}) : l^2(\mathbb{Z}^2) \to l^2(\mathbb{Z}^2)$. If there exists a monotonically decreasing function $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $w(x) = O\left(x^{-2-\delta}\right)$, $\delta > 0$, and constants $\lambda > 1$ and $K_0 > 0$ with $|m_{i,j}| < w(\|\lambda j' - j\|_\infty)$ for $\|\lambda j' - j\|_\infty > K_0$, then M is not stable.*

Proof. First, we show that if $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $w(x) = O\left(x^{-2-\delta}\right)$ is monotonically decreasing, then

$$
\sum_{K\geq 1} K \sum_{k\geq K} k\, w(k)^2 < \infty\,. \qquad (3.9)
$$

Inequality (3.9) is proven using the Riemann integral criterium for sums. To this end, we pick continuous $v \in L^\infty(\mathbb{R}^+)$ with $w(x) \leq v(x)x^{-2-\delta}$ and observe that

$$\sum_{K\geq 1} K \sum_{k\geq K} k\, w(k)^2 \leq \int_0^\infty x \int_x^\infty y\, w(y)^2\, dy\, dx \leq \int_0^\infty x \int_x^\infty v(y)^2 y^{-3-2\delta}\, dy\, dx$$
$$\leq \frac{\|v\|_\infty^2}{2+2\delta} \int_0^\infty x^{-1-2\delta}\, dx < \infty\,.$$

Now, we shall use (3.9) to show that $\inf_{x\in l^2(\mathbb{Z}^2)} \left\{ \frac{\|Mx\|_{l^2}}{\|x\|_{l^2}} \right\} = 0$. To this end, fix $\epsilon > 0$ and pick $K_1 > K_0$ with

$$\sum_{K\geq K_1} K \left(\sum_{k\geq K} k\, w(k)^2 \right) \leq 2^{-6}\epsilon^2\,.$$

Pick $N \in \mathbb{N}$ with $\widetilde{N} := \lceil \frac{N}{\lambda} \rceil + K_1 < N$ and define

$$\widetilde{M} = (m_{j',j})_{\|j'\|_\infty \leq \widetilde{N}, \|j\| \leq N} : \mathbb{C}^{(2N+1)^2} \to \mathbb{C}^{(2\widetilde{N}+1)^2}.$$

The matrix $\widetilde{M}$ has a non-trivial kernel since $(2\widetilde{N}+1)^2 < (2N+1)^2$, so we can choose $\widetilde{x} \in \mathbb{C}^{(2N+1)}$ with $\|\widetilde{x}\|_2 = 1$ and $\widetilde{M}\widetilde{x} = 0$. Define $x \in l^2(\mathbb{Z}^2)$ according to $x_j = \widetilde{x}_j$ if $\|j\|_\infty \leq N$ and $x_j = 0$ else.

By construction we have $\|x\|_{l^2} = 1$, and $(Mx)_{j'} = 0$ for $\|j'\|_\infty \leq \widetilde{N}$.

To estimate $(Mx)_{j'}$ for $\|j'\|_\infty > \widetilde{N}$, we fix $K > K_1$ and $j' \in \mathbb{Z}^d$ with $\|j'\|_\infty = \lceil \frac{N}{\lambda} \rceil + K$. We have $\|\lambda j'\|_\infty \geq N + K\lambda$ and $\|\lambda j' - j\|_\infty \geq K\lambda \geq K$ for all $j \in \mathbb{Z}^d$ with $\|j\|_\infty \leq N$, and, therefore,

$$\begin{array}{rcccl} |(Mx)'_j|^2 & = & \left|\sum_{\|j\|_\infty \leq N} m_{j',j} x_j\right|^2 & \leq & \|x\|_2^2 \sum_{\|j\|_\infty \leq N} |m_{j',j}|^2 \\ & \leq & \sum_{\|j\|_\infty \leq N} w(\|\lambda j' - j\|_\infty)^2 & \leq & \sum_{\|j\|_\infty \geq K} w(\|j\|_\infty)^2 \\ & = & 2^2 \sum_{k\geq K} 2k\, w(k)^2 & = & 2^3 \sum_{k\geq K} k\, w(k)^2. \end{array}$$

Finally, we can compute

$$\begin{aligned} \|Mx\|_{l^2}^2 = \sum_{j'\in\mathbb{Z}^d} |(Mx)'_j|^2 \quad &= \sum_{\|j'\|_\infty \geq \lceil \frac{N}{\lambda} \rceil + K_1} |(Mx)_{j'}|^2 \\ = \; 2^3 \sum_{\|j'\|_\infty \geq \lceil \frac{N}{\lambda} \rceil + K_1} \; &\sum_{k\geq \|j'\|_\infty} k\, w(k)^2 \leq 2^6 \sum_{K\geq \lceil \frac{N}{\lambda} \rceil + K_1} K \sum_{k\geq K} k\, w(k)^2 \leq \epsilon^2 \end{aligned}$$

and obtain $\|Mx\|_{l^2} \leq \epsilon$. Since ϵ was chosen arbitrarily and $\|x\|_{l^2} = 1$, we have $\inf_{x\in l^2(\mathbb{Z}^2)} \left\{ \frac{\|Mx\|_{l^2}}{\|x\|_{l^2}} \right\} = 0$ and M is not stable. $\square$

Now all pieces are in place to state and prove the main contribution of this paper.

THEOREM 3.6. *For $a, b > 0$ with $ab > 1$, $\mathcal{H}_{Q_{a,b}}$ is not identifiable.*

Proof. Fix $a, b > 0$ with $ab > 1$ and choose λ, η_1, η_2, P, and E as in Lemma 3.4.

To construct the aforementioned stable (g, a', b')–analysis operator C_g, we choose as Gabor atom the Gaussian $g_0 : \mathbb{R} \to \mathbb{R}^+, \quad x \mapsto e^{-\pi x^2}$. Lyubarski [Lyu92] and Seip and Wallsten [SW92, Sei92] have shown that $(g_0, a', b') = \{M_{ka'}T_{lb'}g_0\}$ is a frame for any $a', b' > 0$ with $a'b' < 1$, and, hence, we conclude that the analysis map given by

$$C_{g_0} : L^2(\mathbb{R}) \to l^2(\mathbb{Z}^2), \quad f \mapsto \big\{\langle f, M_{k\lambda^2\alpha}T_{l\lambda^2\beta}g_0\rangle\big\}_{k,l}$$

is bounded and stable since $\lambda^2\beta \cdot \lambda^2\alpha = \frac{\lambda^4}{ab} < 1$.

Let us now fix $f \in S_0'(\mathbb{R})$ and consider the composition

$$\begin{array}{ccccccc} l_0(\mathbb{Z}^2) & \overset{E}{\to} & \mathcal{H}_M & \overset{\Phi_f}{\to} & L^2 & \overset{C_{g_0}}{\to} & l^2(\mathbb{Z}^2) \\ \{\sigma_{k,l}\} & \mapsto & E\{\sigma_{k,l}\} & \mapsto & E\{\sigma_{k,l}\}\, f & \mapsto & \big\{\,\langle\, E\{\sigma_{k,l}\}\, f,\ M_{k'\lambda^2\alpha}T_{l'\lambda^2\beta}g_0\,\rangle\,\big\}_{k',l'}\ . \end{array}$$

The bi–infinite matrix

$$M = \Big(m_{k',l',k,l}\Big) = \Big(\langle\, M_{k\lambda\alpha}T_{l\lambda\beta}PT_{-l\lambda\beta}M_{-k\lambda\alpha}f,\ M_{k'\lambda^2\alpha}T_{l'\lambda^2\beta}g_0\,\rangle\Big)\,,$$

represents the operator $C_{g_0} \circ \Phi_f \circ E$ with respect to the canonical basis of $l^2(\mathbb{Z}^2)$, since

$$\begin{aligned} \Big(C_{g_0} \circ \Phi_f \circ E\ \{\sigma_{k,l}\}\Big)_{k',l'} &= \langle \sum_{k,l} \sigma_{k,l} M_{k\lambda\alpha}T_{l\lambda\beta}PT_{-l\lambda\beta}M_{-k\lambda\alpha}f,\ M_{k'\lambda^2\alpha}T_{l'\lambda^2\beta}g_0\,\rangle \\ &= \sum_{k,l} \langle\, M_{k\lambda\alpha}T_{l\lambda\beta}PT_{-l\lambda\beta}M_{-k\lambda\alpha}f,\ M_{k'\lambda^2\alpha}T_{l'\lambda^2\beta}g_0\,\rangle\, \sigma_{k,l} \\ &= \sum_{k,l} m_{k',l',k,l}\ \sigma_{k,l}\ . \end{aligned}$$

In order to use Lemma 3.5 to show that M, and, therefore, $C_{g_0} \circ \Phi_f \circ E$ is not stable, we have to obtain bounds on the matrix entries of M. Lemma 3.4, part b, will provide us with these bounds. In fact, for $k, l, k', l' \in \mathbb{Z}$, we have

$$\begin{aligned} |m_{k',l',k,l}| &= \big|\langle\, M_{k\lambda\alpha}T_{l\lambda\beta}PT_{-l\lambda\beta}M_{-k\lambda\alpha}f,\ M_{k'\lambda^2\alpha}T_{l'\lambda^2\beta}\, g_0\,\rangle\big| \\ &\le \langle\, T_{l\lambda\beta}\,|PT_{-l\lambda\beta}M_{-k\lambda\alpha}f|\,,\ T_{l'\lambda^2\beta}\,|g_0|\,\rangle \\ &\le \|f\|_{S_0'}\ d_1 * g_0\,(\lambda\beta(\lambda l' - l))\,, \end{aligned}$$

and

$$\begin{aligned} |m_{k',l',k,l}| &= \big|\langle\, T_{k\lambda\alpha}M_{-l\lambda\beta}\big(PT_{-l\lambda\beta}M_{-k\lambda\alpha}f\big)\widehat{\ },\ T_{k'\lambda^2\alpha}M_{-l'\lambda^2\beta}\,\widehat{g_0}\ \rangle\big| \\ &\le \langle\, T_{k\lambda\alpha}\,\big|\big(PT_{-l\lambda\beta}M_{-k\lambda\alpha}f\big)\widehat{\ }\big|\ ,\ T_{k'\lambda^2\alpha}\,|g_0|\,\rangle \\ &\le \|f\|_{S_0'}\ d_2 * g_0(\lambda\alpha(\lambda k' - k)). \end{aligned}$$

In these calculations, we used that $g_0 \ge 0$, $\widehat{g_0} = g_0$, and $g_0(-x) = g_0(x)$, and the Parseval–Plancherel identity. Since d_1, d_2, and g_0 decay rapidly, so do $d_1 * g_0$ and $d_2 * g_0$. We set

$$w(x) = \|f\|_{S_0'}\ \max\big\{d_1 * g_0(\lambda\beta x),\ d_1 * g_0(-\lambda\beta x),\ d_2 * g_0(\lambda\alpha x),\ d_2 * g_0(-\lambda\alpha x)\big\}.$$

and obtain $|m_{k',l',k,l}| \le w\big(\max\{|\lambda k' - k|,\ |\lambda l' - l|\}\big)$ with $w(x) = O\left(x^{-n}\right)$ for $n \in \mathbb{N}$. Lemma 3.5 implies that M is not stable, and, by construction, we can conclude that $C_{g_0} \circ \Phi_f \circ E$ and thus Φ_f is not stable. □

Note that Lemma 3.5 is crucial for the understanding of Theorem 3.6: For any $f \in S_0'$, the operator $C_g \circ \Phi_f \circ E : l_0(\mathbb{Z}^2) \longrightarrow l^2(\mathbb{Z}^2)$, and, therefore, the operator $\Phi_f : \mathcal{H}_{Q_{a,b}} \longrightarrow L^2(\mathbb{R})$, is not stable as a result of the non–quadratic structure of the canonical matrix representation of $C_g \circ \Phi_f \circ E$. The validity of Lemma 3.5 does not depend on the choice of (reasonable) topologies on domain and range, in fact, a more general version of Lemma 3.5 can be found in [Pfa05].

4. Gabor frame operators, underspread operators, and uncertainty. The proof of Kailath's conjecture in Section 3 relies strongly on the existence of a Schwartz function $g \in \mathcal{S}(\mathbb{R})$ such that (g, a, b) is a Gabor frame for given $a, b > 0$ with $ab < 1$. In Section 4.1 we shall discuss the role of the critical density $ab = 1$ in the identification of Gabor frame operators and analogies of underspread and Gabor frame operators. Interpretations of the results in Section 3 and Section 4.1 as consequences of uncertainty in time–frequency analysis are given in Section 4.2.

As in Section 3, we choose to work in Section 4 in the one dimensional setting.

4.1. Identification of Gabor frame operators. For appropriate $g, h \in L^2(\mathbb{R})$, e.g., for $g, h \in S_0(\mathbb{R})$, and $a, b > 0$, the *Gabor frame operator* $S^{a,b}_{g,h} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is given by

$$S^{a,b}_{g,h} f = D_h \circ C_g \; f = \sum_{k,l \in \mathbb{Z}} \langle f, M_{kb} T_{la} g \rangle \, M_{kb} T_{la} h \,, \quad f \in L^2(\mathbb{R}) \,.$$

Let us compare the spreading function representation of Hilbert–Schmidt operators given in (1.1) with *Janssen's representation* of the Gabor frame operator, which is

$$S^{a,b}_{g,h} f = (ab)^{-1} \sum_{m,n \in \mathbb{Z}} \left\langle h, M_{\frac{m}{a}} T_{\frac{n}{b}} g \right\rangle M_{\frac{m}{a}} T_{\frac{n}{b}} f \,, \quad f \in L^2(\mathbb{R})$$

[Jan95, Grö01]. Both types of operators are superpositions of time–frequency shifts, and, hence, we shall refer to the tempered distribution

$$(ab)^{-1} \sum_{m,n \in \mathbb{Z}} \left\langle h, M_{\frac{m}{a}} T_{\frac{n}{b}} g \right\rangle \; \delta_{\frac{n}{b}} \otimes \delta_{\frac{m}{a}} \qquad \in S_0'(\mathbb{R} \times \widehat{\mathbb{R}})$$

as spreading function of the Gabor frame operator $S^{a,b}_{g,h}$.

On a formal level, the relationship between Gabor frame operators and underspread and overspread operators is striking: the spreading functions of Gabor frame operators are supported (as distributions) on a full rank lattice $\frac{1}{b}\mathbb{Z} \times \frac{1}{a}\mathbb{Z}$ in the time–frequency plane $\mathbb{R} \times \widehat{\mathbb{R}}$, whereas the spreading functions of underspread and overspread operators are supported on a fundamental domain of such a lattice (see Figure 4.1). The duality of *compact* and *discrete* locally compact abelian groups suggests that results in the theory of underspread and overspread operators might lead to analogous results in Gabor analysis and vice versa.

The correspondence of underspread and overspread operators to Gabor frame operators has not yet been fully explored. To initiate research in this direction, we shall show in Theorem 4.1 that identifiability of a canonically defined class of Gabor frame operators with fixed lattice $a\mathbb{Z} \times b\mathbb{Z}$ is equivalent to the existence of $f \in L^2(\mathbb{R})$ such that (f, a, b) is a Gabor frame for $L^2(\mathbb{R})$. As in Section 3, we need to define a domain X and classes of Gabor frame operators $\mathcal{S}^{a,b}$ with some care in order to have X sufficiently large to allow identification for $ab \leq 1$, and X small enough to allow for an easy proof of the non–identifiability in case of $ab > 1$.

We choose as domain the Wiener space $W(\mathbb{R})$, i.e.,

$$X = W(\mathbb{R}) = W(L^\infty(\mathbb{R}), l^1(\mathbb{Z})) = \{ f \in L^2(\mathbb{R}) : \|f\|_W = \sum_{k \in \mathbb{Z}} \|f \cdot 1_{[k,k+1)}\|_\infty < \infty \} \,,$$

as range, once more, $Y = L^2(\mathbb{R})$, and, for $a, b > 0$, we consider the operator class

$$\mathcal{S}^{a,b} = \left\{ S^{a,b}_{g,h} : \; g \in L^2(\mathbb{R}), \; h \in W(\mathbb{R}) \right\} \quad \text{with} \quad \|S^{a,b}_{g,h}\|_{\mathcal{S}^{a,b}} = \left\| \left\{ \left\langle h, M_{\frac{m}{a}} T_{\frac{n}{b}} g \right\rangle \right\} \right\|_{l^2} .$$

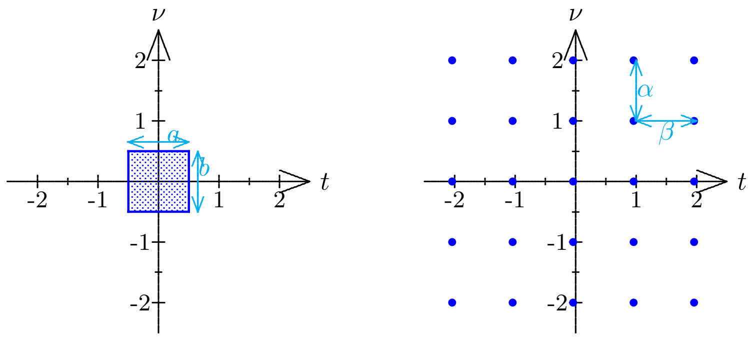

FIG. 4.1. *Support of the spreading symbol of an underspread or overspread operator and distributional support of the spreading symbol of a Gabor frame operator.*

We have

$$\|S^{a,b}_{g,h}f\|_{L^2} \le \sqrt{(a+1)(b+1)}\ \|f\|_W\ \|S^{a,b}_{g,h}\|_{\mathcal{S}^{a,b}},$$

[Grö01], page 107, and, therefore, $\mathcal{S}^{a,b} \subset \mathcal{L}\left(W(\mathbb{R}), L^2(\mathbb{R})\right)$ and $\{\Phi_f :\ f \in W(\mathbb{R})\} \subset \mathcal{L}\left(\mathcal{S}^{a,b}, L^2(\mathbb{R}^d)\right)$, where $\Phi_f : S^{a,b}_{g,h} \mapsto S^{a,b}_{g,h}f$.

THEOREM 4.1. *$\mathcal{S}^{a,b}$ is identifiable if and only if $ab \le 1$. Moreover, for any a, b with $ab > 1$ and any $f \in W(\mathbb{R})$ exist $g \in L^2(\mathbb{R})$ and $h \in W(\mathbb{R})$ such that $S^{a,b}_{g,h}f = 0$.*

Note that identification of $\mathcal{S}^{a,b}$ does not require to uncover g and h in $S^{a,b}_{g,h}$, but only to obtain the coefficients $\left\{\left\langle h, M_{\frac{n}{a}}T_{\frac{m}{b}}g\right\rangle\right\}$ in Janssen's representation of the Gabor frame operator $S^{a,b}_{g,h}$.

Proof of Theorem 4.1. To show the identifiability of $\mathcal{S}^{a,b}$ for $ab \le 1$, we use the fact that for any $ab \le 1$ exists $f \in W(\mathbb{R})$ such that $(f, a, b) = \{M_{kb}T_{la}f\}$ is a frame for $L^2(\mathbb{R})$. For example, if $ab < 1$ we may choose the Gaussian $f = g_0 : \mathbb{R} \to \mathbb{R}^+,\ x \mapsto e^{-\pi x^2}$, with $g_0 \in \mathcal{S}(\mathbb{R}) \subset W(\mathbb{R})$ and for $ab = 1$ we could choose $f = \mathbf{1}_{[0,a)} \in W(\mathbb{R})$. The Ron–Shen duality principle implies that (f, a, b) is a frame for $L^2(\mathbb{R})$ if and only if $(f, \frac{1}{b}, \frac{1}{a})$ is a Riesz basis for its closed linear span in $L^2(\mathbb{R})$, i.e., if and only if there exists $A, B > 0$ such that for all $\{d_{m,n}\} \in l^2(\mathbb{Z}^2)$ we have

$$A\|\{d_{m,n}\}\|_{l^2} \le \|\sum_{m,n\in\mathbb{Z}} d_{m,n} M_{\frac{m}{a}} T_{\frac{n}{b}}\, f\|_{L^2} \le B\|\{d_{m,n}\}\|_{l^2}\,, \tag{4.1}$$

[RS97, Grö01]. Replacing $\{d_{m,n}\}$ by $\left\{\left\langle h, M_{\frac{m}{a}}T_{\frac{n}{b}}g\right\rangle\right\} \in l^2(\mathbb{Z}^2)$ in (4.1) shows that any f with (f, a, b) is a frame for $L^2(\mathbb{R})$ identifies $\mathcal{S}^{a,b}$.

We shall now show that for any $a, b > 0$ with $ab > 1$ and any $f \in W(\mathbb{R})$ exists $g \in L^2(\mathbb{R})$ and $h \in W(\mathbb{R})$ such that $S^{a,b}_{g,h} \in \mathcal{S}^{a,b} \setminus \{0\}$ and $S^{a,b}_{g,h}f = 0$, contradicting that f identifies $\mathcal{S}^{a,b}$. Fix $a, b > 0$ with $ab > 1$ and $f \in W(\mathbb{R})$ and pick $g \in L^2(\mathbb{R})$ such that $g \perp \overline{\text{span}}(f, a, b)$, and, therefore, $f \perp \overline{\text{span}}(g, a, b)$. Let $h = g_0 \in W(\mathbb{R})$ be the Gaussian defined above and observe that $(g_0, \frac{1}{b}, \frac{1}{a})$ is a frame for $L^2(\mathbb{R})$ since $\frac{1}{ab} < 1$. Hence, $\left\{\left\langle h, M_{\frac{m}{a}}T_{\frac{n}{b}}g\right\rangle\right\} = \left\{e^{2\pi i\frac{mn}{ab}}\overline{\left\langle g, M_{-\frac{m}{a}}T_{-\frac{n}{b}}h\right\rangle}\right\} \in l^2(\mathbb{Z}^2) \setminus \{0\}$, i.e., $S^{a,b}_{g,h} \in \mathcal{S}^{a,b}$, $\|S^{a,b}_{g,h}\| \ne 0$, but $S^{a,b}_{g,h}f = \sum \langle f, T_{am}M_{bn}g\rangle\, T_{am}M_{bn}h = 0$. ∎

4.2. Uncertainty. Theorem 4.1 illustrates a strong relationship of critical density in Gabor analysis to the identification of canonically defined classes of Gabor frame operators. The critical density phenomenon in Gabor analysis is well known to be rooted in uncertainty in time–frequency analysis:

- functions cannot be arbitrarily well localized simultaneously in time and frequency, i.e., in phase space, and we can therefore exclude the possibility that there exist Gabor systems (g, a, b) which are Riesz bases for $L^2(\mathbb{R})$ if $ab < 1$, and
- functions cannot represent an area in phase space of volume larger one, in the sense that one cannot construct complete Gabor frames (g, a, b) for $L^2(\mathbb{R})$ if $ab > 1$.

Due to the first of the two limitations described above, Folland refers to a rectangle of volume one in phase space as a "minimal rectangle in phase space" [FS97].

Theorem 3.1 describes a new interpretation of minimal rectangles which plays a role in the time–frequency analysis of operators: an operator, whose spreading symbol is known to be supported in a rectangle in the time–frequency plane can be identified if the rectangle has volume less or equal one, and cannot be identified if the rectangle has volume greater than one. Note that this phenomenon is not a direct consequence of the fact that we cannot construct functions which are arbitrarily well localized in phase space, since, in fact, there exist no support restrictions for the construction of operator symbols or spreading functions in phase space.

Theorem 3.1 and Theorem 4.1 can also be viewed as pull–backs of the critical density phenomenon of "phase space expansions" as described in [Lan93] to operator theory. Any operator output signal can only carry a restricted amount of time–frequency structured information, and therefore, any output signal can only be used to resolve a limited amount of information from an operator. Theorem 3.1 illustrates that this amount of information corresponds to a *minimal rectangle in the spreading domain*. Theorem 4.1 shows that the resolvable amount of information of operators, whose spreading functions have discrete distributional support contained in a lattice $\frac{1}{b}\mathbb{Z}\times\frac{1}{a}\mathbb{Z}$, is connected to the sparsity of the lattice. In fact, all information inherent in such operator can be resolved using a single test–signal if and only if $ab \leq 1$. Note that in the latter case, the Kohn–Nirenberg symbol, which is the symplectic Fourier transformation of the spreading function, is $a\times b$ periodic, i.e., is the periodization of a function supported on a *minimal rectangle of size* $ab \leq 1$ *in phase space.*

We would like to add, that the physical interpretation of the uncertainty principle as a limit to the achievable precision when measuring position and momentum of quantum mechanical objects parallels the identifiability result for underspread and overspread operators nicely, since the latter tells us that we will not be able to identify an overspread operator no matter how smartly a signal is chosen to test the system.

The uncertainty principle phenomena discussed above, among others, can be found in [Fef83, Dau92, Lan93, BHW98, Grö01, Grö03].

5. Generalized spreading constraints. We shall now extend the results stated in Section 3 to higher dimensions and to non–rectangular spreading support sets.

Similarly to the one dimensional case, we have

$$\mathcal{H} = \mathcal{H}(\mathbb{R}^d) = \left\{H \in HS(L^2(\mathbb{R}^d)) : \ \kappa_H \in S_0(\mathbb{R}^{2d})\right\} \subset \mathcal{L}\left(S_0'(\mathbb{R}^d), L^2(\mathbb{R}^d)\right).$$

Once more, we shall use a Zak transformation, namely $Z : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$, $Zf(t,\nu) = \sum_{n\in\mathbb{Z}^d} f(t-n)e^{2\pi i n\cdot\nu}$ for a.e. $(t,\nu) \in Q = Q_{1,1}$, and the Shah distribution

$\perp\!\!\!\perp\!\!\!\perp = \perp\!\!\!\perp\!\!\!\perp_1$. Adjusting Lemma 3.2 accordingly, we obtain

$$Z \circ H\perp\!\!\!\perp\!\!\!\perp(t,\nu) = \sum_{n,m\in\mathbb{Z}^d} \eta_H(t-n,\nu-m)e^{2\pi i(\nu-m)\cdot t} \quad \text{for all } (t,\nu)\in[-\tfrac{1}{2},\tfrac{1}{2}]^{2d}, \qquad (5.1)$$

an identity which leads immediately to

THEOREM 5.1. *$\mathcal{H}_M = \{H \in \mathcal{H} :\ \text{supp}\,\eta_H \subseteq M\}$ is identifiable with identifier $\perp\!\!\!\perp\!\!\!\perp$ if and only if $M^\circ \cap \bigcup_{(m,n)\in\mathbb{Z}^{2d}\setminus(0,0)} \big(M^\circ + (m,n)\big) = \emptyset$.*

The proof of Theorem 5.1 is similar to the proof of Theorem 3.3 and is therefore omitted.

A straightforward generalization of either Theorem 3.3 or Theorem 5.1 leads to the identifiability of $\mathcal{H}_{\mathcal{D}Q}$, $Q = Q_{1,1} = [-\frac{1}{2},\frac{1}{2}]^{2d}$, in the case that $\mathcal{D}$ is a diagonal matrix with diagonal $(a_1,\ldots,a_d,\frac{1}{a_1},\ldots,\frac{1}{a_d}) \in (\mathbb{R}^+)^{2d}$. This observation leads us to the question for which general diagonal or non–diagonal, volume preserving matrices $\mathcal{A} \in SL(2d,\mathbb{R})$, the operator space $\mathcal{H}_{\mathcal{A}Q}$ is identifiable.

The underlying idea of obtaining identifiability results on $\mathcal{H}_{\mathcal{A}Q}$ for non–diagonal matrices $\mathcal{A} \in SL(2d,\mathbb{R})$, is to use the canonical correspondence of elements in $\mathcal{H}_{\mathcal{A}Q}$ with elements in $\mathcal{H}_Q$ given by a coordinate transformation in the spreading domain. Theorem 5.3 states that for symplectic $\mathcal{A}$, there exist unitary operators $O_\mathcal{A}$ on $L^2(\mathbb{R}^d)$, such that the following formal calculation of operator valued integrals holds for all $H \in \mathcal{H}_{\mathcal{A}Q}$. Note that here, we set $\mu(t,\nu) = M_\nu T_t$ to obtain

$$\begin{array}{rlcl} H & = \int\int \eta_H(t,\nu)\, M_\nu T_t\ dt\, d\nu & = & \int\int \eta_H(t,\nu)\ \mu(t,\nu)\, dt\, d\nu \\ & = \int\int \eta_H(\mathcal{A}(t,\nu))\ \mu(\mathcal{A}(t,\nu))\ dt\, d\nu & = & \int\int \eta_H(\mathcal{A}(t,\nu))\ O_\mathcal{A}\mu(t,\nu)O_\mathcal{A}^*\ dt\, d\nu \\ & = O_\mathcal{A}\int\int \eta_{H_\mathcal{A}}(t,\nu)\ \mu(t,\nu)\ dt\, d\nu\ O_\mathcal{A}^* & = & O_\mathcal{A}\ H_\mathcal{A}\ O_\mathcal{A}^*\,, \end{array} \qquad (5.2)$$

where $\eta_{H_\mathcal{A}} = \eta_H\circ\mathcal{A}$ and $H_\mathcal{A} \in \mathcal{H}_Q$. We shall see that the *intertwining* operators $O_\mathcal{A} \in U\big(L^2(\mathbb{R}^d)\big)$ in (5.2) extend to $S_0'(\mathbb{R}^d)$ and act isomorphically on $S_0(\mathbb{R}^d)$. The identifiability of $\mathcal{H}_Q$ leads therefore to the identifiability of $\mathcal{H}_{\mathcal{A}Q}$ using as identifier the tempered distribution $O_\mathcal{A}\perp\!\!\!\perp\!\!\!\perp \in S_0'(\mathbb{R}^d)$. See Figure 5.1 for an illustration of this approach.

To gather all $\mathcal{A} \in SL(2d,\mathbb{R})$ which allow for calculations similar to those in (5.2), we turn to the representation theory of the reduced Weyl–Heisenberg group $\mathbb{H}_d^{red}$ which is identical to $\mathbb{R}^d\times\widehat{\mathbb{R}}^d\times\mathbb{T}$ in topology and Haar measure. The group operation on the reduced Weyl–Heisenberg group is

$$\big(t,\nu,e^{2\pi i s}\big)\cdot\big(t',\nu',e^{2\pi i s'}\big) = \big(t+t',\nu+\nu',e^{2\pi i(s+s'+\frac{1}{2}(t'\cdot\nu - t\cdot\nu'))}\big),$$

and its Schrödinger representation on the space of unitary operators on $L^2(\mathbb{R}^d)$ is given by

$$\begin{array}{llllllll} \rho: & \mathbb{H}_d^{red} & \to & U\big(L^2(\mathbb{R}^d)\big) & & & & \\ & (t,\nu,s) & \mapsto & \rho(t,\nu,s): & L^2(\mathbb{R}) & \to & L^2(\mathbb{R}) & \\ & & & & f & \mapsto & \rho(t,\nu,e^{2\pi i s})f: & \mathbb{R}^d \to \mathbb{R}^d \\ & & & & & & & x \mapsto e^{2\pi i(\nu\cdot x)+s}f(x+t)\ . \end{array}$$

Representing H once more as operator valued integral, we obtain

$$\begin{aligned} H = \int\int \eta_H(t,\nu)M_\nu T_t\, dt\, d\nu\ &=\ \int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}}^d}\int_0^1 \eta_H(t,\nu)\rho(-t,\nu,0)\, dt\, d\nu\, ds \\ = \int_{\mathbb{H}_d^{red}} e^{-2\pi i s}\eta_H(-t,\nu)\rho(t,\nu,e^{2\pi i s})\, d\mu(t,\nu,s)\ &=\ \rho(\eta_H^\circ), \end{aligned}$$

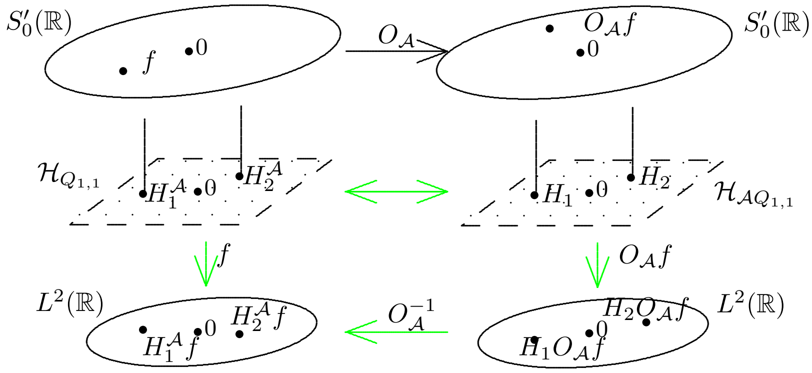

FIG. 5.1. *Identifiability of $\mathcal{H}_{\mathcal{A}Q}$, $\mathcal{A} \in Sp(d,\mathbb{R})$ based on the existence of an intertwining operator $O_{\mathcal{A}}$.*

where $\eta_H^\circ(t,\nu,e^{2\pi i s}) = e^{-2\pi i s}\eta_H(-t,\nu)$. In other words, a Hilbert–Schmidt operator H with $\eta_H \in L^1(\mathbb{R}^{2d})$ is the integrated Schrödinger representations of η_H° with respect to the reduced Weyl–Heisenberg group $\mathbb{H}_d^{red}$ [Fol89, Grö01].

Before listing the relevant results from representation theory in Theorem 5.3, it is now time to define the symplectic group.

DEFINITION 5.2. *The symplectic group $Sp(d,\mathbb{R})$ consists of those matrices $\mathcal{A} \in SL(2d,\mathbb{R})$ that satisfy $\mathcal{A}^\star \left(\begin{smallmatrix} 0 & -I_d \\ I_d & 0 \end{smallmatrix}\right) \mathcal{A} = \left(\begin{smallmatrix} 0 & -I_d \\ I_d & 0 \end{smallmatrix}\right)$, where I_d is the $d \times d$–identity matrix.*

Theorem 5.3, part a) outlines the scope of our approach [Fol89]. Part c) delivers intertwining operators for equivalent representations $\rho \circ \mathcal{A}$ and ρ. Parts d), e), f), and g) describe these operators as products of some elementary operators. This characterization shows us that the a–priori Hilbert space theory applies to the Feichtinger algebra setup used in this paper (see part h)). Part i) covers shifts of the spreading support which allow to extend Theorem 5.4 to affine linear coordinate transformations.

For ease of notation we shall not distinguish between the matrix $\mathcal{A}$ and the corresponding linear map, i.e., we have $\mathcal{A}(t,\nu) = \big((t,\nu)\cdot \mathcal{A}^t\big)^t$.

THEOREM 5.3.

a) Let S operate on $\mathbb{H}_d^{red}$. The induced map $\rho_S = \rho \circ S : \mathbb{H}_d^{red} \longrightarrow U\left(L^2(\mathbb{R}^d)\right)$ is an unitary representation of $\mathbb{H}_d^{red}$ which is unitarily equivalent to the irreducible Schrödinger representation ρ, i.e., there exists an unitary intertwining operator O such that $O\rho(g)O^* = \rho_S(g)$ for all $g \in HW_d^{red}$, if and only if there exists $\mathcal{A} \in Sp(d,\mathbb{R})$ with $S = \widetilde{\mathcal{A}}$ where $\widetilde{\mathcal{A}}$ is given by $\widetilde{\mathcal{A}} : \mathbb{H}_d^{red} \longrightarrow \mathbb{H}_d^{red}, \quad (t,\nu,e^{2\pi i s}) \mapsto (\mathcal{A}(t,\nu), e^{2\pi i s})$.

b) Let $\mathcal{A} \in Sp(d,\mathbb{R})$ and let $\rho_{\mathcal{A}} = \rho \circ \widetilde{\mathcal{A}}$. Then $\rho_{\mathcal{A}}(f) = \rho(f \circ \mathcal{A}^{-1})$ for $f \in L^1(\mathbb{H}_d^{red})$.

c) For $\mathcal{A} \in Sp(d,\mathbb{R})$ exists an unitary operator $O_{\mathcal{A}}$ on $L^2(\mathbb{R}^d)$, with $O_{\mathcal{A}} H O_{\mathcal{A}}^* = \rho(\eta(H)^\circ \circ \widetilde{\mathcal{A}}^{-1})$ for all $H \in HS(L^2(\mathbb{R}^d))$ with $\eta(H) \in L^1(\mathbb{R}^{2d})$.

d) The matrix $\mathcal{I} = \left(\begin{smallmatrix} 0 & I_d \\ -I_d & 0 \end{smallmatrix}\right)$ *together with the subgroups*

$$N = \left\{\left(\begin{smallmatrix} I_d & 0 \\ A & I_d \end{smallmatrix}\right), \quad A = A^*\right\} \text{ and } D = \left\{\left(\begin{smallmatrix} A & 0 \\ 0 & A^{*-1} \end{smallmatrix}\right), \quad A \in GL(n, \mathbb{R})\right\}$$

of $Sp(d, \mathbb{R})$ *generate* $Sp(d, \mathbb{R})$.

e) For $\mathcal{A} = \left(\begin{smallmatrix} 0 & I_d \\ -I_d & 0 \end{smallmatrix}\right)$ *we have* $\mu \circ \mathcal{A}(t, \nu) = \mu(\mathcal{A}(t, \nu)) = \mathcal{F}^{-1}\mu(t, \nu)\mathcal{F}$.

f) For $\mathcal{A} = \left(\begin{smallmatrix} I_d & 0 \\ A & I_d \end{smallmatrix}\right)$ *with* $A = A^*$ *define* C_A *through* $C_A f(x) = e^{-\pi i x^T A x} f(x)$. *Then we have* $\mu \circ \mathcal{A}(t, \nu) = \mu(\mathcal{A}(t, \nu)) = C_A \circ \mu(t, \nu) \circ C_A^*$.

g) For $\mathcal{A} = \left(\begin{smallmatrix} A & 0 \\ 0 & A^{*-1} \end{smallmatrix}\right)$ *let* U_A *be defined by setting* $U_A f(x) = |\det A|^{-\frac{1}{2}} f(A^{-1}x)$. *Then* $\mu \circ \mathcal{A}(t, \nu) = \mu(\mathcal{A}(t, \nu)) = U_A \circ \mu(t, \nu) \circ U_A^*$.

h) The unitary operators $\mathcal{F}$, C_A, *and* U_A *restrict and extend to* $S_0(\mathbb{R})$ *and* $S_0'(\mathbb{R})$ *respectively.*

i) Set $L_{(a,b)} : \mathbb{R}^d \times \widehat{\mathbb{R}}^d \longrightarrow \mathbb{R}^d \times \widehat{\mathbb{R}}^d, \quad (t, \nu) \mapsto (a + t, b + \nu)$. *Then*

$$\mu \circ L_{(a,b)}(t, \nu) = e^{2\pi i \nu a} \mu(a, b)\mu(t, \nu) = e^{2\pi i b t} \mu(t, \nu)\mu(a, b).$$

For details on representation theoretic background, see [Fol89, FK98, Grö01]. Using Theorem 5.3, we obtain

Theorem 5.4. *Let* $S = L_{(a,b)} \circ \mathcal{A}$, $\mathcal{A} \in Sp(d, \mathbb{R})$. *Then* $\mathcal{H}_M$ *is identifiable if and only if* $\mathcal{H}_{SM}$ *is identifiable.*

Proof. Assume that $\mathcal{H}_M$ is identifiable with $f_M \in S_0'(\mathbb{R}^d)$. Theorem 5.3 provides us with an unitary operator $O_\mathcal{A}$ on $L^2(\mathbb{R}^d)$ which extends to $S_0'(\mathbb{R}^d)$. We claim that $O_\mathcal{A} f_M \in S_0'(\mathbb{R}^d)$ identifies $\mathcal{H}_{SM}$. To see this, observe that for all $H \in \mathcal{H}_{SM}$ we have

$$\begin{aligned}
H &= \int\int \eta_H(t, \nu)\ \mu(t, \nu)\, dt\, d\nu \\
&= \int\int \eta_H\big(\mathcal{A}(t, \nu) + (a, b)\big)\ \mu\big(\mathcal{A}(t, \nu) + (a, b)\big)\, dt\, d\nu \\
&= \int\int \eta_H\big(\mathcal{A}(t, \nu) + (a, b)\big)\ e^{2\pi i a(Ct + D\nu)} \mu(a, b)\mu(\mathcal{A}(t, \nu))\ dt\, d\nu \\
&= \int\int \eta_H\big(\mathcal{A}(t, \nu) + (a, b)\big) e^{2\pi i a(Ct + D\nu)}\ \mu(a, b) O_\mathcal{A} \mu(t, \nu) O_\mathcal{A}{}^*\ dt\, d\nu \\
&= \mu(a, b) O_\mathcal{A} \int\int \eta_{H_{\mathcal{A},(a,b)}}(t, \nu)\ \mu(t, \nu)\ dt\, d\nu\ O_\mathcal{A}{}^* \\
&= \mu(a, b) O_\mathcal{A}\ H_{\mathcal{A},(a,b)}\ O_\mathcal{A}{}^*,
\end{aligned}$$

and

$$\begin{aligned}
\|H\, O_\mathcal{A} f_M\|_{L^2(\mathbb{R}^d)} &= \|O_\mathcal{A}{}^* \mu(a, b)^* H O_\mathcal{A} f_M\|_{L^2(\mathbb{R}^d)} \\
&= \|H_{\mathcal{A},(a,b)} f_M\|_{L^2(\mathbb{R}^d)} = \|\eta_{H_{\mathcal{A},(a,b)}}\|_{L^2(\mathbb{R}^{2d})} = \|\eta_H\|_{L^2(\mathbb{R}^{2d})} \equiv \|H\|_{HS}.
\end{aligned}$$

□

For $M = Q$, we can identify $\mathcal{H}_{SQ}$ using the identity in

Corollary 5.5. *Let* $S = L_{(a,b)} \circ \mathcal{A}$ *for some* $(a, b) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ *and* $\mathcal{A} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(d, \mathbb{R})$. *Then for* $H \in \mathcal{H}_{SQ}$ *and for* $(t, \nu) \in \operatorname{supp}(\eta_H)$

$$e^{2\pi i a \cdot (Ct + Dv) + \nu t} Z \circ O_S \circ H \circ O_S^* \perp\!\!\perp\!\!\perp(t, \nu) = \eta_H\big(\mathcal{A}^-1(t - a, \nu - b)\big).$$

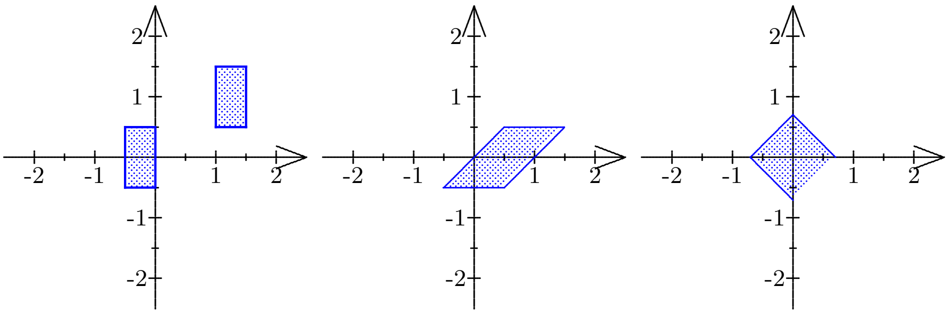

FIG. 5.2. *Examples of sets M such that H_M is identifiable.*

We have shown that identifiability is robust with respect to symplectic coordinate transformations in the spreading domain. This result is rooted in the representation theory of the Weyl–Heisenberg group. Theorem 5.3.i shows that this approach can not be extended to obtain insights on non–symplectic coordinate transformations.

Nevertheless, we should note that for $\mathcal{A} \in SL(2d, \mathbb{R})$ the condition $\mathcal{A} \in Sp(d, \mathbb{R})$ is not necessary for $\mathcal{H}_{\mathcal{A}Q}$ to be identifiable. In fact, the diagonal matrix $\mathcal{D}$ with diagonal $(2, \frac{1}{2}, 1, 1)$ has the property $\mathcal{D} \in SL(4, \mathbb{R}) \setminus Sp(2, \mathbb{R})$, but $\mathcal{H}_{\mathcal{D}Q}$ is identifiable since $\mathcal{D}Q$ is a fundamental domain of the symplectic lattice $\left(\begin{smallmatrix} A & 0 \\ 0 & A^{*-1} \end{smallmatrix}\right) \mathbb{Z}^4$ with $A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$, and therefore an application of Theorem 5.1 and Theorem 5.4 is permissable.

For similar results on non–symplectic lattices in Gabor theory see [Bek04, HW01, HW04].

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REFERENCES

[Bek04] B. Bekka. Square integrable representations, von Neumann algebras and an application to Gabor analysis. *J. Fourier Anal. Appl.*, 10(4):325–349, 2004.

[BHW98] J.J. Benedetto, C. Heil, and D.F. Walnut. Gabor systems and the Balian-Low theorem. In H.G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, pages 85–122. Birkhäuser, Boston, MA, 1998.

[CG03] E. Cordero and K. Gröchenig. Time-frequency analysis of localization operators. *J. Funct. Anal.*, 205(1):107–131, 2003.

[Dau88] I. Daubechies. Time-frequency localization operators: a geometric phase space approach. *IEEE Trans. Inform. Theory*, 34/4, July 1988.

[Dau92] I. Daubechies. *Ten Lectures on Wavelets*. CBMS-NSF Reg. Conf. Series in Applied Math. SIAM, 1992.

[Die70] J. Dieudonné. *Treatise on analysis. Vol. II.* Translated from the French by I. G. MacDonald. Pure and Applied Mathematics, Vol. 10-II. Academic Press, New York, 1970.

[Fef83] C. L. Fefferman. The uncertainty principle. *Bull. Amer. Math. Soc. (N.S.)*, 9(2):129–206, 1983.

[Fei81] H.G. Feichtinger. On a new segal algebra. *Monatsh. f. Math.*, 92:269–289, 1981.

[FG85] H.G. Feichtinger and P. Gröbner. Banach spaces of distributions defined by decomposition methods. I. *Math. Nachr.*, 123:97–120, 1985.

[FK98] H.G. Feichtinger and W. Kozek. Quantization of TF–lattice invariant operators on elementary LCA groups. In H.G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, pages 233–266. Birkhäuser, Boston, MA, 1998.

[FL96] B.H. Fleury and P.E. Leuthold. Radiowave propagation in mobile communications: an overview of european research. *IEEE Commun. Mag.*, 34(2):70–81, 1996.

[FN03] H.G. Feichtinger and K. Nowak. A first survey of Gabor multipliers. In *Advances in Gabor analysis*, Appl. Numer. Harmon. Anal., pages 99–128. Birkhäuser Boston, Boston, MA, 2003.

[Fol89] G.B. Folland. *Harmonic Analysis in Phase Space.* Annals of Math. Studies. Princeton Univ. Press, Princeton, NJ, 1989.

[FS97] G.B. Folland and A. Sitaram. The uncertainty principle: A mathematical survey. *J. Four. Anal. Appl.*, 3(3):207–238, 1997.

[FZ98] H.G. Feichtinger and G. Zimmermann. A space of test functions for Gabor analysis. In H.G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, pages 123–170. Birkhäuser, Boston, MA, 1998.

[Gaa73] S.A. Gaal. *Linear Analysis and Representation Theory.* Springer–Verlag, New York, 1973.

[Gab46] D. Gabor. Theory of communication. *J. IEE, London,*, 93(3):429–457, November 1946.

[GLM04] P.C. Gibson, M.P. Lamoureux, and G.F. Margrave. Representation of linear operators by Gabor multipliers. Preprint, 2004.

[Gre68] P.E. Green. Radar measurements of target scattering properties. In *Radar Astronomy*, pages 1–78. McGraw–Hill, New York, NY, 1968.

[Grö01] K. Gröchenig. *Foundations of Time-Frequency Analysis.* Applied and Numerical Harmonic Analysis. Birkhäuser, Boston, MA, 2001.

[Grö03] K. Gröchenig. Uncertainty principles for time–frequency representations. In H.G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*, pages 11–30. Birkhäuser, Boston, MA, 2003.

[Hei03] C. Heil. Integral operators, pseudodifferential operators, and Gabor frames. In *Advances in Gabor analysis*, Appl. Numer. Harmon. Anal., pages 153–169. Birkhäuser Boston, Boston, MA, 2003.

[HRT97] C. Heil, J. Ramanathan, and P. Topiwala. Singular values of compact pseudodifferential operators. *J. Funct. Anal.*, 150(2):426–452, 1997.

[HW01] D. Han and Y. Wang. Lattice tiling and the Weyl-Heisenberg frames. *Geom. Funct. Anal.*, 11(4):742–758, 2001.

[HW04] D. Han and Y. Wang. The existence of Gabor bases and frames. In *Wavelets, frames and operator theory*, volume 345 of *Contemp. Math.*, pages 183–192. Amer. Math. Soc., Providence, RI, 2004.

[Jan95] A.J.E.M. Janssen. Duality and biorthogonality for Weyl-Heisenberg frames. *J. Four. Anal. Appl.*, 1(4):403–436, 1995.

[Kai59] T. Kailath. Sampling models for linear time-variant filters. Technical Report 352, Massachusetts Institute of Technology. Research Laboratory of Electronics, 1959.

[Kai62] T. Kailath. Measurements on time–variant communication channels. *IEEE Trans. Info. Theory*, 8(5):229–236, Sept. 1962.

[Kai63] T. Kailath. Time–variant communication channels. *IEEE Trans. Info. Theory: Info.Theory. Progress Report 1960–1963*, pages 233–237, Oct. 1963.

[Kat76] Y. Katznelson. *An Introduction to Harmonic Analysis*. Dover, New York, 1976.

[KN65] J.J. Kohn and L. Nirenberg. An algebra of pseudo–differential operators. *Comm. Pure Appl. Math.*, 18:269–305, 1965.

[Koz98] W. Kozek. Adaptation of Weyl–Heisenberg frames to underspread environments. In H.G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, pages 323–352. Birkhäuser, Boston, MA, 1998.

[Lab01] D. Labate. Time-frequency analysis of pseudodifferential operators. *Monatsh. Math.*, 133(2):143–156, 2001.

[Lan93] H.J. Landau. On the density of phase-space expansions. *IEEE Trans. on Information Theory*, 39(4):1152–1156, 1993.

[LKS03] K. Liu, T.A. Kadous, and A.M. Sayeed. Orthogonal time-frequency signaling for doubly dispersive channels. *submitted to the IEEE Transactions on Information Theory*, 2001, revised 2003.

[LP61] H.J. Landau and H.O. Pollak. Prolate spheroidal wave functions, Fourier analysis and uncertainty. II. *Bell System Tech. J.*, 40:65–84, 1961.

[LP62] H.J. Landau and H.O. Pollak. Prolate spheroidal wave functions, Fourier analysis and uncertainty. III. The dimension of the space of essentially time- and band-limited signals. *Bell System Tech. J.*, 41:1295–1336, 1962.

[Lyu92] Y.I. Lyubarskii. Frames in the Bargmann space of entire functions. *Adv.Soviet Math.*, 429:107–113, 1992.

[MGO03] X. Ma, G.B. Giannakis, and S. Ohno. Optimal training for block transmissions over doubly selective wireless fading channels. *IEEE Trans. Signal Proc.*, 51(5):1351–1366, 2003.

[MMH+02] G. Matz, A. F. Molisch, F. Hlawatsch, M. Steinbauer, and I. Gaspard. On the systematic measurement errors of correlative mobile radio channel sounders. *IEEE Trans. Comm.*, 50(5):808–821, May 2002.

[Pfa05] G.E. Pfander. Two results on the invertibility of bi–infinite matrices with applications to Gabor analysis. Preprint, 2005.

[RS97] A. Ron and Z. Shen. Weyl-Heisenberg frames and Riesz bases in $L_2(\mathbf{R}^d)$. *Duke Math. J.*, 89(2):237–282, 1997.

[RT98] R. Rochberg and K. Tachizawa. Pseudodifferential operators, Gabor frames, and local trigonometric bases. In H.G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, pages 171–192. Birkhäuser, Boston, MA, 1998.

[Sei92] K. Seip. Density theorems for sampling and interpolation in the Bargmann-Fock space. I. *J. Reine Angew. Math.*, 429:91–106, 1992.

[Sko80] M.I. Skolnik. *Introduction to Radar Systems*. McGraw-Hill Book Company, New York, 1980.

[SP61] D. Slepian and H. O. Pollak. Prolate spheroidal wave functions, Fourier analysis and uncertainty. I. *Bell System Tech. J.*, 40:43–63, 1961.

[Str05] T. Strohmer. Pseudodifferential operators and Banach algebras in mobile communications. Preprint, 2005.

[SW92] K. Seip and R. Wallstén. Density theorems for sampling and interpolation in the Bargmann-Fock space. II. *J. Reine Angew. Math.*, 429:107–113, 1992.

[VT71] H.L. Van Trees. *Detection, Estimation and Modulation Theory*. Wiley, New York, 1971.

[Yoo02] D.-S. Yoo. *Channel characterization and system design in wireless communications*. PhD thesis, Univ. of Michigan, Ann Arbor, 2002.