Frame expansions for Gabor multipliers

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Abstract

Discrete Gabor multipliers are composed of rank one operators. We shall prove, in the case of rank one projection operators, that the generating operators for such multipliers are either Riesz bases (exact frames) or not frames for their closed linear spans. The same dichotomy conclusion is valid for general rank one operators under mild and natural conditions. This is relevant since discrete Gabor multipliers have an emerging role in communications, radar, and waveform design, where redundant frame decompositions are increasingly applicable.

Key words: Gabor multipliers, Gabor frames, symplectic Fourier transforms

1 Introduction

Inspired and initiated by von Neumann in quantum mechanics [1], pp.405 ff., and Gabor in communications and acoustics [2], decompositions of functions $f \in L^2(\mathbb{R}^d)$, such as

$$f = \sum_{(x,\xi)\in\Lambda} \langle f, M_\xi T_x h\rangle \, M_\xi T_x g \,, \tag{1}$$

have become a fundamental tool in time–frequency analysis and applications dealing with time–varying spectra, e.g., [3–8]. In (1), $g, h \in L^2(\mathbb{R}^d)$ are given

[1] John J. Benedetto was supported in part by NSF DMS Grant 0139759.
[2] Götz E. Pfander was supported by the Max–Kade Foundation.

square integrable functions on Euclidean space $\mathbb{R}^d$, $\Lambda \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is a full rank lattice (such as $\mathbb{Z}^{2d}$) where $\widehat{\mathbb{R}}^d$ is $\mathbb{R}^d$ considered as a spectral domain, T_x is the *translation operator* $T_x k(y) = k(y-x)$, M_ξ is the *modulation operator* $M_\xi k(y) = e^{2\pi i y\cdot\xi}\, k(y)$, $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\mathbb{R}^d)$, and convergence is in $L^2(\mathbb{R}^d)$.

The expansion (1) can be written operator theoretically in terms of the resolution of the identity $Id_{L^2} : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ as

$$Id_{L^2} = \sum_{\lambda\in\Lambda} \rho(\lambda) P_{g,h}\,, \tag{2}$$

where the rank one operator $P_{g,h} : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is defined by $f \mapsto \langle f, h\rangle\, g$, and where $\rho(\lambda)P_{g,h}$ is the composition (conjugation)

$$\rho(\lambda)P_{g,h} = \pi(\lambda) \circ P_{g,h} \circ \pi(\lambda)^*\,, \tag{3}$$

for $\pi(\lambda)k(y) = M_\xi T_x k(y)$, $\lambda = (x,\xi) \in \Lambda$, and for the adjoint U^* of the unitary operator U. The equivalence of (1) and (2) follows from the elementary calculation

$$\pi(\lambda) \circ P_{g,h} \circ \pi(\lambda)^* f(y) = \langle f, \pi(\lambda)h\rangle\, \pi(\lambda)g\,(y)\,. \tag{4}$$

Further, the right side of (4) is

$$\int_{\mathbb{R}^d} f(z)\, e^{-2\pi i(z-y)\cdot\xi} g(y-x)\overline{h(z-x)}\, dz\,;$$

and so, from (3) and (4), $\rho(\lambda)P_{g,h}$ is a Hilbert–Schmidt operator with kernel $k_\lambda(y,z) = e^{-2\pi i(z-y)\cdot\xi} g(y-x)\overline{h(z-x)}$, where $\lambda = (x,\xi)$. We denote $P_{g,g}$ by P_g.

In this context, the "dichotomy" theorem we shall prove, under mild necessary conditions on g and h, is that $\{\rho(\lambda)P_{g,h}\}_{\lambda\in\Lambda}$ *is a Riesz basis for its closed linear span in the space* $HS(\mathbb{R}^d)$ *of Hilbert–Schmidt operators, or it is not a frame for this span*, see Theorem 3.1.

The reason for the abstraction to the setting of $HS(\mathbb{R}^d)$ for our theorem is the emerging importance of *Gabor multipliers* G_m, which are formally defined by a weighted version of (2), namely

$$G_m = \sum_{\lambda\in\Lambda} m_\lambda\, \rho(\lambda)P_{g,h}, \quad m_\lambda \in \mathbb{C} \text{ for } \lambda \in \Lambda, \tag{5}$$

e.g., [9–11], and the revitalization of underspread operators in the mathematical community, e.g., [12].

As a concluding application of our dichotomy result and the inherent characterization of Riesz bases of the form $\{\rho(\lambda)P_{g,h}\}_{\lambda\in\Lambda}$, we shall describe the role of the volume of the lattice Λ in (5) in terms of operator identification. In fact, we shall show that with natural hypotheses *the Gabor multiplier class spanned by* $\{\rho(\lambda)P_{g,h}\}_{\lambda\in\Lambda}$ *is identifiable if the volume of* Λ *is greater than one and not identifiable if the volume of* Λ *is less than one*, see Theorem 5.2.

We begin in Section 2 with mathematical preliminaries concerning Gabor analysis, Hilbert–Schmidt operators, and shift invariant spaces. Section 3 contains a precise statement and proof of our dichotomy result, mentioned above, as well as some related results. Section 4 is devoted to relevant examples, and Section 5 to Gabor multipliers and identification.

2 Preliminaries

Throughout this paper we shall use standard notation from harmonic analysis and in particular Gabor analysis as found in [7]. For example, we shall use the unitary Fourier transformation $\mathcal{F}$ on $L^2(\mathbb{R}^d)$ which is normalized to satisfy $\mathcal{F}g(\gamma) = \widehat{g}(\gamma) = \int g(y)e^{-2\pi i y\cdot\gamma}\,dy$, $\gamma \in \widehat{\mathbb{R}}^d$, for $g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

In addition, the notation $P \asymp R$ on X is used to abbreviate the following statement: there exist $A, B > 0$ such that for all $x \in X$ we have $AP(x) \leq R(x) \leq BP(x)$.

2.1 Gabor analysis

The short time Fourier transform of $f \in L^2(\mathbb{R}^d)$ with respect to a window function $g \in L^2(\mathbb{R}^d) \setminus \{0\}$ is given by

$$V_g f(\lambda) = \langle f, \pi(\lambda)g\rangle = \int_{\mathbb{R}^d} f(y)\overline{g(y-x)}\, e^{-2\pi i y\cdot\xi}\,dy, \quad \lambda = (x,\xi) \in \mathbb{R}^d\times\widehat{\mathbb{R}}^d.$$

We have $V_g f \in L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$ and $\|V_g f\|_{L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} = \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}$. Further, we can *synthesize* $f \in L^2(\mathbb{R}^d)$, using translates and modulates of any $h \in L^2(\mathbb{R}^d)$ with $\langle h, g\rangle = 1$, in the sense that the integral $\int_\lambda V_g f(\lambda)\pi(\lambda)h\,d\lambda$ converges weakly to f.

A central goal in Gabor analysis is to find $g, h \in L^2(\mathbb{R}^d)$ and full rank lattices $\Lambda \subset \mathbb{R}^d\times\widehat{\mathbb{R}}^d$ which allow a *discretization* of the reconstruction formula $f \equiv$

$\int_\lambda V_g f(\lambda)\pi(\lambda)h\,d\lambda$ of the form

$$f = \sum_{\lambda\in\Lambda} V_g f(\lambda)\,\pi(\lambda)h, \quad f \in L^2(\mathbb{R}^d), \tag{6}$$

with convergence in $L^2(\mathbb{R}^d)$ and where g and h are independent of f.

A discussion of the validity of (6) entails Bessel sequences, Gabor frames, and Riesz bases, notions we now define.

Let $g \in L^2(\mathbb{R}^d)$ and let $\Lambda \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be a full rank lattice. Formally, consider the discrete analysis operator C_g defined by

$$C_g : L^2(\mathbb{R}^d) \longrightarrow l^2(\Lambda),\ f \mapsto \{V_g f(\lambda)\}_{\lambda\in\Lambda},$$

and the discrete synthesis operator $T_g = C_g^*$ defined by

$$T_g : l^2(\Lambda) \longrightarrow L^2(\mathbb{R}^d),\ \{c_\lambda\}_{\lambda\in\Lambda} \mapsto \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)g\,.$$

The set $(g,\Lambda) = \{\pi(\lambda)g\}_{\lambda\in\Lambda}$ is called a *Gabor system*; and a Gabor system is a *Bessel sequence* if C_g is a well–defined linear operator in which case both C_g and T_g are bounded. A Bessel sequence (g,Λ) is a *frame* for $L^2(\mathbb{R}^d)$ if C_g is also stable, i.e., if $\|f\|_{L^2(\mathbb{R}^d)} \asymp \|C_g f\|_{l^2(\Lambda)}$ for $f \in L^2(\mathbb{R}^d)$, and it is a *Riesz basis* (*bounded unconditional basis*) for its closed linear span in $L^2(\mathbb{R}^d)$ if T_g is stable in addition to being bounded, i.e., if $\|\{c_\lambda\}\|_{l^2(\Lambda)} \asymp \|T_g\{c_\lambda\}\|_{L^2(\mathbb{R}^d)}$, for $\{c_\lambda\} \in l^2(\Lambda)$.

The right hand side of (6) is well defined if the Gabor systems (g,Λ) and (h,Λ) are Bessel sequences. Further, if (g,Λ) and (h,Λ) are frames, then the operator

$$S_{g,h} : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d),\ f \mapsto \sum_{\lambda\in\Lambda} V_g f(\lambda)\,\pi(\lambda)h$$

is an algebraic and topological isomorphism. If (g,Λ) and (h,Λ) are frames, and $S_{g,h} = I : f \mapsto f$,we say that (h,Λ) is a *dual frame* of (g,Λ) [7].

Fundamental to the analysis of Gabor systems (g,Λ) is the *volume* $|\Lambda|$ of the full rank lattice Λ, which is given by $|\Lambda| = |\det A|$ where A is chosen such that $A\mathbb{Z}^{2d} = \Lambda$. In fact, if (g,Λ) is a Riesz basis for its closed linear span in $L^2(\mathbb{R}^d)$, then $|\Lambda| \geq 1$; and if (g,Λ) is a frame for $L^2(\mathbb{R}^d)$, and therefore complete in $L^2(\mathbb{R}^d)$, then $|\Lambda| \leq 1$.

In the case that Λ has *critical density*, i.e., if $|\Lambda| = 1$, and (g,Λ) is a frame for $L^2(\mathbb{R}^d)$, then (g,Λ) is automatically a Riesz basis for $L^2(\mathbb{R}^d)$, or, equivalently, an *exact frame* for $L^2(\mathbb{R}^d)$, i.e., (g,Λ) ceases to be a frame if any one of its elements is removed. In case $|\Lambda|<1$, any frame (g,Λ) for $L^2(\mathbb{R}^d)$ is *non–exact* (*overcomplete*), and one can remove any finite number of elements from (g,Λ) and the resulting family remains a frame for $L^2(\mathbb{R}^d)$. Further, if $|\Lambda|<1$, then, for any $g\in L^2(\mathbb{R}^d)$, there exists non–trivial $\{c_\lambda\}_{\lambda\in\Lambda}\in l^2(\Lambda)\setminus\{0\}$ for which $0=\sum_{\lambda\in\Lambda}c_\lambda\pi(\lambda)g$ in $L^2(\mathbb{R}^d)$.

The uncertainty principle provides insight into any decomposition such as (6) [13–18]. For example, in the case of Gabor systems one manifestation of the uncertainty principle is the Balian Low theorem [19–21], which asserts that if (g,Λ) is a Riesz basis for $L^2(\mathbb{R}^d)$, in which case we have $|\Lambda|=1$, then g cannot be well localized in time and frequency, in the sense that

$$\Big(\int |y\,g(y)|^2\,dy\Big)\cdot\Big(\int |\eta\,\widehat{g}(\eta)|^2\,d\eta\Big)=\infty \tag{7}$$

must occur.

We shall sometimes use the Feichtinger algebra $S_0(\mathbb{R}^d)$ in place of $L^2(\mathbb{R}^d)$. $S_0(\mathbb{R}^d)$ is the Banach algebra composed of those functions $f\in L^2(\mathbb{R}^d)$ with the property that $V_{g_0}f\in L^1(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$ for the Gaussian $g_0(x)=e^{-\|x\|^2}$, $x\in\mathbb{R}^d$. The norm $\|f\|_{S_0(\mathbb{R}^d)}=\|V_{g_0}f\|_{L^1(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}$ gives $S_0(\mathbb{R}^d)$ a Banach algebra structure under pointwise multiplication and/or convolution. For equivalent definitions of $S_0(\mathbb{R}^d)$, as well as basic theory, see [22].

2.2 Hilbert–Schmidt operators

A Hilbert–Schmidt operator $H\in HS(\mathbb{R}^d)$ is a compact integral operator on $L^2(\mathbb{R}^d)$, i.e., H is defined by

$$Hf(x)=\int \kappa_H(x,t)f(t)\,dt=\int \kappa_H(x,x-t)f(x-t)\,dt\quad a.e.,\quad f\in L^2(\mathbb{R}^d),$$

with kernel $\kappa_H\in L^2(\mathbb{R}^{2d})$. The space of Hilbert–Schmidt operators is a Hilbert space with inner product $\langle H_1,H_2\rangle_{\mathrm{HS}}=\langle\kappa_{H_1},\kappa_{H_2}\rangle_{L^2}$ [23,24]. For any orthonormal basis $\{e_i\}_{i\in I}$ of $L^2(\mathbb{R}^d)$ we have

$$\|H\|_{HS}^2=\langle H,H\rangle_{\mathrm{HS}}=\sum_{i\in I}\|He_i\|^2_{L^2(\mathbb{R}^d)},$$

and therefore $\|H\|_{HS}\geq\|H\|_{\mathcal{L}}$ where $\|\cdot\|_{\mathcal{L}}$ denotes the operator norm of $H\in\mathcal{L}(L^2(\mathbb{R}^d),L^2(\mathbb{R}^d))$.

Our use of families of Hilbert–Schmidt operators is carried out on a symbolic level. For any Hilbert–Schmidt operator H with kernel $\kappa_H\in L^2(\mathbb{R}^{2d})$, the

Kohn–Nirenberg symbol σ_H of H is defined as

$$\sigma_H(\lambda) = \sigma_H(x,\xi) = \int_{\mathbb{R}^d} \kappa_H(x, x-y)\, e^{-2\pi i y\cdot\xi}\, dy \quad a.e.$$

[25]. The operator H can then be expressed using the Kohn–Nirenberg symbol by means of the formula

$$Hf(x) = \int_{\widehat{\mathbb{R}}^d} \sigma_H(x,\xi)\widehat{f}(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi \quad a.e.$$

Critical to our analysis is the fact that the linear operator $K : \kappa_H \mapsto \sigma_H$ is the composition of a partial Fourier transformation and a volume preserving axis transformation. Hence, K is unitary and, consequently,

$$\langle H_1, H_2\rangle_{HS} = \langle \kappa_{H_1}, \kappa_{H_2}\rangle_{L^2(\mathbb{R}^{2d})} = \langle \sigma_{H_1}, \sigma_{H_2}\rangle_{L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}. \tag{8}$$

Since, in addition to (8), the Kohn-Nirenberg symbol of $\rho(\lambda)H = \pi(\lambda)\circ H\circ \pi(\lambda)^*$ for $H \in HS(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}^d\times\widehat{\mathbb{R}}^d$ satisfies $\sigma_{\rho(\lambda)H} = T_\lambda \sigma_H$, we obtain that $\{\rho(\lambda)P_{g,h}\}_{\lambda\in\Lambda}$ *is a frame or Riesz basis for its closed linear span in* $HS(\mathbb{R}^d)$ *if and only if* $\{T_\lambda\sigma_{P_{g,h}}\}_{\lambda\in\Lambda}$ *is a frame or Riesz basis for its closed linear span in* $L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$. The question of asking if $\{T_\lambda\sigma_{P_{g,h}}\}_{\lambda\in\Lambda}$ is a frame or Riesz basis for the closed shift invariant space generated by $\{T_\lambda\sigma_{P_{g,h}}\}_{\lambda\in\Lambda}$ in $L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$ can be answered using zero set criteria for spectral periodizations, e.g., [17,26–28] and Theorem 2.1.

2.3 Shift–invariance of functions defined on phase space

We have reduced the analysis of sequences $\{\rho(\lambda)P_{g,h}\}_{\lambda\in\Lambda}$ of Hilbert–Schmidt operators in $HS(\mathbb{R}^d)$ to the analysis of function sequences $\{T_\lambda\sigma_{P_{g,h}}\}_{\lambda\in\Lambda}$ in $L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$. Since the sequences $\{\rho(\lambda)P_{g,h}\}_{\lambda\in\Lambda}$ are defined on phase space, we shall state a symplectic version of Theorem 1.4.1 in [17] as Theorem 2.1, part b. The Fourier version of Theorem 2.1, part a, is well–known and elementary to prove; and so the proof of Theorem 2.1, part a, is also straightforward.

The *symplectic Fourier transformation*, $\mathcal{F}_s$, of functions defined on the phase space $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ is formally defined as follows:

$$\mathcal{F}_s : L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d) \longrightarrow L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d), \quad f \mapsto \mathcal{F}_s f : \lambda \mapsto \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} f(\lambda')e^{-2\pi i[\lambda',\lambda]}\, d\lambda',$$

where

$$[\lambda',\lambda] = [(x',\xi'),(x,\xi)] = x'\cdot\xi - \xi'\cdot x, \quad \lambda,\lambda' \in \mathbb{R}^d\times\widehat{\mathbb{R}}^d \tag{9}$$

is the standard symplectic form on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

The dual lattice of Λ with respect to the standard symplectic form on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is the so–called *adjoint lattice* $\Lambda^\circ \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ of Λ; and it is defined by the rule: $\lambda' \in \Lambda^\circ$ if and only if $[\lambda', \lambda] \in \mathbb{Z}$ for all $\lambda \in \Lambda$. Therefore, if $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ then we have $\Lambda^\circ = \frac{1}{b}\mathbb{Z} \times \frac{1}{a}\mathbb{Z}$, and, in general, we have $|\Lambda^\circ| = |\Lambda|^{-1}$ [29,30].

To illustrate the important role of the adjoint lattice and, consequently, the symplectic Fourier transformation, in time–frequency analysis, we mention the fact that (h, Λ) *is a Riesz basis for its closed linear span in* $L^2(\mathbb{R}^d)$ *if and only if* (h, Λ°) *is a frame for* $L^2(\mathbb{R}^d)$ [31,30].

In the following, P_Λ denotes periodization by the lattice Λ, i.e., $P_\Lambda F(\lambda) = \sum_{\lambda' \in \Lambda} F(\lambda - \lambda')$, $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d \big/ \Lambda$.

Theorem 2.1 *Given* $F \in L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ *and a full rank lattice* $\Lambda \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

a. *The family* $\{T_\lambda F\}_{\lambda \in \Lambda}$ *is a Riesz basis for its closed linear span in* $L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ *if and only if* $P_{\Lambda^\circ} |\mathcal{F}_s F|^2 \asymp 1$ *a.e. on* $\mathbb{R}^d \times \widehat{\mathbb{R}}^d \big/ \Lambda^\circ$.
b. *The family* $\{T_\lambda F\}_{\lambda \in \Lambda}$ *is a frame for its closed linear span in* $L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ *if and only if* $P_{\Lambda^\circ} |\mathcal{F}_s F|^2 \asymp 1$ *a.e. on* $\left(\mathbb{R}^d \times \widehat{\mathbb{R}}^d \big/ \Lambda^\circ\right) \setminus \{x : P_{\Lambda^\circ} |\mathcal{F}_s F|^2(x) = 0\}$.

Theorem 2.1 and the material of Section 2.2 illustrate that the analysis of $\{T_\lambda \sigma_{P_{g,h}}\}_{\lambda \in \Lambda}$ in $L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ and therefore of $\{\rho(\lambda) P_{g,h}\}_{\lambda \in \Lambda}$ in $HS(\mathbb{R}^d)$ requires only a thorough investigation of $P_{\Lambda^\circ} |\mathcal{F}_s \sigma_{P_{g,h}}|^2$ on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d \big/ \Lambda^\circ$.

To this end, note that for any rank one operator $P_{g,h}$ we have

$$\sigma_{P_{g,h}}(\lambda) = \sigma_{P_{g,h}}(x, \xi) = \int_{\mathbb{R}^d} g(x) \overline{h}(x - y)\, e^{-2\pi i y \cdot \xi}\, dy = e^{-2\pi i x \cdot \xi}\, g(x)\, \widehat{\overline{h}}(\xi) \quad a.e.$$

and therefore

$$\begin{aligned} \mathcal{F}_s \sigma_{P_{g,h}}(\lambda) = \mathcal{F}_s \sigma_{P_{g,h}}(x, \xi) &= \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}}^d} g(x')\, \widehat{\overline{h}}(\xi') e^{-2\pi i (x' \cdot \xi' + x' \cdot \xi - x \cdot \xi')}\, dx'\, d\xi' \\ &= \int_{\mathbb{R}^d} g(x')\, \overline{h(x' - x)} e^{-2\pi i x' \cdot \xi}\, dx' = V_h g(x, \xi) = V_h g(\lambda). \qquad (10) \end{aligned}$$

The results of Section 2.2 and Section 2.3 allow us to prove Theorem 3.1.

3 Results

In [9], Feichtinger proved that *if* (g, Λ) *is a Gabor frame for* $L^2(\mathbb{R}^d)$ *generated by* $g \in S_0(\mathbb{R}^d)$, *then* $\{\rho(\lambda) P_g\}_{\lambda \in \Lambda}$ *is a Riesz basis for its closed linear span* *in* $HS(\mathbb{R}^d)$ *if and only if* $|\Psi| \asymp 1$ *where* $\Psi(\chi) = \sum_{\lambda\in\Lambda} |V_g g(\lambda)|^2 e^{2\pi i\lambda\cdot\chi}$, $\chi \in \mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda$.

Theorem 3.1, part *a*, is essentially Feichtinger's theorem; and is, itself, the motivation for Theorem 3.1, parts *b* and *c*. Theorem 3.1, part *b*, and Theorem 3.2 are precise statements of our main theorem which was stated without hypotheses in Section 1. We emphasize that this is a *dichotomy theorem*, asserting that $\{\rho(\lambda)P_{g,h}\}_{\lambda\in\Lambda}$ is either a Riesz basis or not a frame for its closed linear span in $HS(\mathbb{R}^d)$.

Theorem 3.1 *Let* $g, h \in L^2(\mathbb{R}^d)$ *and let* $\Lambda \subset \mathbb{R}^d\times\widehat{\mathbb{R}}^d$ *be a full rank lattice.*

a. The family $\{\rho(\lambda)P_{g,h}\}_{\lambda\in\Lambda}$ *is a Riesz basis for its closed linear span in* $HS(\mathbb{R}^d)$ *if and only if* $P_{\Lambda^\circ}|V_h g|^2 \asymp 1$ *on* $\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda^\circ$.

b. If (g,Λ) *and* (h,Λ) *are Bessel sequences in* $L^2(\mathbb{R}^d)$*, then* $\{\rho(\lambda)P_{g,h}\}_{\lambda\in\Lambda}$ *is either a Riesz basis or not a frame for its closed linear span in* $HS(\mathbb{R}^d)$.

c. If $g, h \in S_0(\mathbb{R}^d)\setminus\{0\}$*, then there exists* $r > 0$ *such that, for all* $\alpha > r > 0$, $\{\rho(\alpha\lambda)P_{g,h}\}_{\lambda\in\Lambda}$ *is a Riesz basis for its closed linear span in* $HS(\mathbb{R}^d)$.

In the case of $g = h$, we can drop the Bessel sequence condition in Theorem 3.1, part *b.*, and we obtain the following result.

Theorem 3.2 *Let* $g \in L^2(\mathbb{R}^d)$ *and let* $\Lambda \subset \mathbb{R}^{2d}$ *be a full rank lattice.* $\{\rho(\lambda)P_g\}_{\lambda\in\Lambda}$ *is either a Riesz basis or not a frame for its closed linear span in* $HS(\mathbb{R}^d)$.

For the calculations in the proofs of Theorem 3.1 and Theorem 3.2, we need the following simple facts.

Lemma 3.3 *For* $g, h \in L^2(\mathbb{R}^d)$ *we have* $\mathcal{F}_s\,|V_h g|^2 = V_h h\,\overline{V_g g}$. *For* $g = h$, *this is* $\mathcal{F}_s\,|V_g g|^2 = |V_g g|^2$.

Lemma 3.3 is proven in [18], page 17.

Lemma 3.4 *Let* $F_n : \mathbb{R}^d \longrightarrow \mathbb{R}^+$, $n \in \mathbb{N}$, *be continuous functions with* $\sum_{n\in\mathbb{N}} F_n(x) \le B$ *a.e. Then* $\sum_{n\in\mathbb{N}} F_n(x) \le B$ *for all* $x \in \mathbb{R}^d$.

Proof. If there is x_0 for which $\infty \ge A = \sum_{n=1}^{\infty} F_n(x_0) > B$, then there exists $N \in \mathbb{N}$ such that

$$G_N(x_0) = \sum_{n=1}^{N} F_n(x_0) \ge \tfrac{1}{2}B + \tfrac{1}{2}\min\{A, B+1\}.$$

Since G_N is continuous, there exists an open set $V \subset \mathbb{R}^d$ such that

$$\sum_{n=1}^{\infty} F_n \ge G_N > \tfrac{3}{4}B + \tfrac{1}{4}\min\{A, B+1\} > B$$

on V, and this is a contradiction. $\square$

The crucial lemma to prove Theorem 3.1, part b, and Theorem 3.2 is the following result.

Lemma 3.5 *Let Λ be a full rank lattice in $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$, and let $g,h \in L^2(\mathbb{R}^d)$ with $P_\Lambda|V_h g|^2 \in L^\infty(\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda)$. If $h = g$, or if (g,Λ°) and (h,Λ°) are Bessel sequences, then $P_\Lambda|V_h g|^2 = \Phi$ a.e. for some function Φ which is continuous on $\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda$.*

Proof. We shall twice apply the Poisson Summation Formula for the symplectic Fourier transform. To this end, we define the symplectic Fourier transformation on $L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda)$ as follows:

$$\mathcal{F}_s : L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda) \longrightarrow l^2(\Lambda^\circ),\ \mathcal{F}_s F(\lambda) = \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda} F(\lambda')e^{-2\pi i[\lambda',\lambda]}\, d\lambda'\,.$$

For $F \in L^1(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$ with $P_\Lambda F \in L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda)$ and $\lambda \in \Lambda^\circ$, we have

$$\begin{aligned}\mathcal{F}_s\, P_\Lambda F\,(\lambda) &= \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda} \Big(\sum_{\lambda''\in\Lambda} F(\lambda'-\lambda'')\Big)\, e^{-2\pi i[\lambda',\lambda]}\, d\lambda' \\ &= \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} F(\lambda')\, e^{-2\pi i[\lambda',\lambda]}\, d\lambda' \ = \mathcal{F}_s F\,(\lambda).\end{aligned}$$

Therefore, the Poisson Summation Formula,

$$P_\Lambda F = |\Lambda^\circ| \sum_{\lambda\in\Lambda^\circ} \mathcal{F}_s F(\lambda)\, e^{2\pi i[\,\cdot\,,\lambda]}\,, \tag{11}$$

with convergence of the right hand side in $L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda)$, is valid. We apply (11) and Lemma 3.3 to $|V_h g|^2 \in L^1(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$ with $P_\Lambda|V_h g|^2 \in L^\infty(\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda) \subset L^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda)$ to obtain

$$P_\Lambda|V_h g|^2(\lambda') = |\Lambda^\circ| \sum_{\lambda\in\Lambda^\circ} V_h h(\lambda)\,\overline{V_g g(\lambda)}\; e^{2\pi i[\lambda',\lambda]} \quad a.e.\ \lambda' \in \mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda\,. \tag{12}$$

If (g,Λ°) and (h,Λ°) are Bessel sequences, then $\{V_h h(\lambda)\}_{\lambda\in\Lambda^\circ}, \{V_g g(\lambda)\}_{\lambda\in\Lambda^\circ} \in l^2(\Lambda^\circ)$ and, consequently, $\{V_h h(\lambda)\overline{V_g g(\lambda)}\}_{\lambda\in\Lambda^\circ} \in l^1(\Lambda^\circ)$. Hence, the right hand side of (12) is absolutely convergent and so it is continuous on $\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda$.

Let us now turn to the case $h = g$ and $P_\Lambda|V_g g|^2 \in L^\infty(\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda)$. $P_\Lambda|V_g g|^2$ bounded and $|V_g g|^2$ continuous and positive imply $P_\Lambda|V_g g|^2(\lambda) \le \|P_\Lambda|V_g g|^2\|_{L^\infty}$ for all $\lambda \in \mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda$ by Lemma 3.4. In particular, $P_\Lambda|V_g g|^2(0,0) \in \mathbb{R}$, i.e.,

$\{|V_g g|^2(\lambda)\}_{\lambda\in\Lambda} \in l^1(\Lambda)$. Since $\mathcal{F}_s|V_g g|^2 = |V_g g|^2$, the adjoint version of equation (11) implies that $\{|V_g g|^2(\lambda)\}_{\lambda\in\Lambda}$ is the symplectic Fourier transform of $P_{\Lambda^\circ}|V_g g|^2$ and, therefore,

$$P_{\Lambda^\circ}|V_g g|^2(\lambda') = |\Lambda| \sum_{\lambda\in\Lambda} |V_g g|^2(\lambda)\, e^{2\pi i[\lambda',\lambda]} \quad a.e.\ \lambda' \in \mathbb{R}^d\times\widehat{\mathbb{R}}^d\big/\Lambda^\circ\,. \tag{13}$$

The right hand side of (13) is continuous and therefore bounded. Hence, $P_{\Lambda^\circ}|V_g g|^2 \in L^\infty(\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda)$ and, applying Lemma 3.4 again, we conclude that $P_{\Lambda^\circ}|V_g g|^2$ is bounded everywhere. In particular, we have $\{|V_g g|^2(\lambda)\}_{\lambda\in\Lambda^\circ} \in l^1(\Lambda^\circ)$.

Replacing Λ by Λ° in (13) and repeating the argument above, we conclude that $\{|V_g g|^2(\lambda)\}_{\lambda\in(\Lambda^\circ)^\circ} \in l^1\left((\Lambda^\circ)^\circ\right)$.

The argument is completed by observing that $(\Lambda^\circ)^\circ = \Lambda$, [30], page 257, and therefore

$$P_{\Lambda}|V_g g|^2(\lambda') = |\Lambda^\circ| \sum_{\lambda\in\Lambda^\circ} |V_g g|^2(\lambda)\, e^{2\pi i[\lambda',\lambda]} \quad a.e.\ \lambda' \in \mathbb{R}^d\times\widehat{\mathbb{R}}^d\big/\Lambda\,,$$

where the right hand side is continuous on $\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda$. □

Proof of Theorem 3.1. *a.* The equivalence in Theorem 3.1, part *a*, follows directly from (8), (10), and Theorem 2.1, part *a*. Alternatively it can be derived simply by using Feichtinger's result in [9] which we mentioned at the beginning of Section 3, since Lemma 3.3 and (11) imply $P_{\Lambda^\circ}|V_g g|^2 = \mathcal{F}_s^{-1}\{|V_g g(\lambda)|^2\}_{\lambda\in\Lambda}$. For $g \neq h$, Feichtinger's criterion requires an analysis of the lower bound of $|\mathcal{F}_s^{-1}\{V_h h(\lambda)\overline{V_g g(\lambda)}\}_{\lambda\in\Lambda}|$, which is made significantly easier by means of our observation that

$$P_{\Lambda^\circ}|V_h g|^2 = \mathcal{F}_s^{-1}\{V_h h(\lambda)\overline{V_g g(\lambda)}\}_{\lambda\in\Lambda}.$$

b. In order for $\{\rho(\lambda)P_{g,h}\}_{\lambda\in\Lambda}$ to form a non–exact frame, $G = P_{\Lambda^\circ}|V_h g|^2$ would have to be bounded, vanish on a set of positive measure and be bounded away from zero off this set. Obviously, this criteria cannot be fulfilled for continuous G, and Lemma 3.5 therefore implies Theorem 3.1, part *b*, for the case that (g,Λ) and (h,Λ) are Bessel sequences.

c. To prove Theorem 3.1, part *c*, let us observe that $g,h \in S_0$ implies that $P_{\Lambda^\circ}|V_g g|^2$ converges absolutely and uniformly on $\mathbb{R}^d\times\widehat{\mathbb{R}}^d/\Lambda$ [7], page 255. Since $V_g g$ is uniformly continuous for any $g \in L^2(\mathbb{R}^d)$, we conclude that $P_{\Lambda^\circ}|V_g g|^2$ is continuous and bounded.

Since $g,h \neq 0$ there exists $\lambda_0 \in \mathbb{R}^d\times\widehat{\mathbb{R}}^d$ such that $V_h g(\lambda_0) \neq 0$. Using the continuity of $V_h g$, we can find an open, convex, and bounded set V such that $|V_h g|^2 \geq \frac{1}{2}|V_h g|^2 > 0$ on V. For any full rank lattice Λ we can now choose $r > 0$ so that V contains a fundamental domain of $r^{-1}\Lambda^\circ$, guaranteeing a lower bound of $P_{\alpha^{-1}\Lambda^\circ}|V_g g|^2$ for any $\alpha \geq r$. The fact that $(\alpha\Lambda)^\circ = \alpha^{-1}\Lambda^\circ$ completes the proof. $\square$

Proof of Theorem 3.2. See the proof of Theorem 3.1, part *b*. $\square$

Remark 3.6 The hypotheses in Theorem 3.1, part *b*, and Theorem 3.1, part *c*, can be weakened considerably. For example, we could replace the Bessel sequence hypothesis on (g, Λ) and (h, Λ) in Theorem 3.1, part *b*, with the hypothesis that $\mathcal{F}_s^{-1}\{V_h h(\lambda)\overline{V_g g}\}_{\lambda\in\Lambda^\circ}$ be continuous on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d / \Lambda$; and the hypothesis that $g, h \in S_0(\mathbb{R}^d)$ in Theorem 3.1, part *c*, can be replaced with the hypothesis $|V_h g|^2 \in S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

4 Examples

Let us now provide examples illustrating our results for the case $d = 1$. The first example in the case of a Gaussian was pointed out to us by Hans Feichtinger.

Example 4.1 For $g \in S_0(\mathbb{R})$ with $V_g g(\lambda) \neq 0$ for all $\lambda \in \mathbb{R}\times\widehat{\mathbb{R}}$, e.g., let g be a Gaussian, say $g(x) = e^{-x^2}$, we have that $\{\rho(\lambda)P_{g,g}\}_{\lambda\in\Lambda}$ is a Riesz basis in $HS(\mathbb{R})$ for any full rank lattice Λ.

Example 4.2 There exist non–exact frames in $HS(\mathbb{R})$ of the form $\{\rho(\lambda)H\}_{\lambda\in\Lambda}$ where H is not rank one. For example, we may define H by means of its Kohn–Nirenberg symbol by choosing $\mathcal{F}_s\sigma_H = \mathbf{1}_{[0,1]^2}$ and $\Lambda = \frac{1}{2}(\mathbb{Z}\times\mathbb{Z})$. Since $\{M_\lambda \mathcal{F}_s\sigma_H\}_{\lambda\in\Lambda}$ forms a non–exact frame for its closed linear span in $L^2(\mathbb{R}\times\widehat{\mathbb{R}})$, so does $\{T_\lambda\sigma_H\}$; and, therefore, $\{\rho(\lambda)H\}_{\lambda\in\Lambda}$ forms a non–exact frame for its closed linear span in $HS(\mathbb{R})$.

Note that any such example implies $\sigma_H \notin S_0(\mathbb{R}\times\widehat{\mathbb{R}})$, since otherwise $P_{\Lambda^\circ}|\sigma_H|^2$ is continuous.

Example 4.3 There also exist non–exact frames in $HS(\mathbb{R})$ of rank one operators with smooth kernels, e.g., let g_0 be a Gaussian and set $\Lambda = \{(n, \frac{m}{2})\}_{m,n\in\mathbb{Z}}$. Then $(g_0\otimes g_0, \Lambda\otimes\Lambda)$ is a frame for $L^2(\mathbb{R}\times\widehat{\mathbb{R}})$ and we obtain that $\{\pi(\lambda)P_g\pi(\lambda')\}_{\lambda,\lambda'\in\Lambda}$ is a non–exact frame (for its closed linear span in $HS(\mathbb{R})$) composed of rank one operators.

Example 4.4 There exist $g \in S_0(\mathbb{R})$ and $\Lambda \subset \mathbb{R}\times\widehat{\mathbb{R}}$ such that $\{\rho(\lambda)P_g\}_{\lambda\in\Lambda}$ is not a frame for its closed linear span in $HS(\mathbb{R})$. Consider any $g \in S_0(\mathbb{R})$

such that $\operatorname{supp} g \subseteq [0,1]$; then $V_g g(x,\xi) = 0$ for $|x| > 1$. If $\Lambda = \beta\mathbb{Z}\times\frac{1}{4}\mathbb{Z}$, $\beta > 0$, we have $\Lambda^\circ = 4\mathbb{Z}\times\frac{1}{\beta}\mathbb{Z}$. Since in this case $P_{\Lambda^\circ}|V_g g|^2(\lambda) = 0$ whenever $\lambda \in [2,4]\times[0,\frac{1}{\beta}] \subset \mathbb{R}\times\widehat{\mathbb{R}}/4\mathbb{Z}\times\frac{1}{\beta}\mathbb{Z}$, we have that $\{\rho(\lambda)P_g\}_{\lambda\in\beta\mathbb{Z}\times\frac{1}{4}\mathbb{Z}}$, $\beta > 0$ is not a frame for its closed linear span in $HS(\mathbb{R})$.

Example 4.5 We now illustrate that Theorem 3.1, part c, does not hold for arbitrary $g, h \in L^2(\mathbb{R})$.

Let $g = \sum_{k=1}^\infty T_k\, D_{k^{\frac{3}{2}}}\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]} \in L^2(\mathbb{R})$ where $D_a f(x) = f(ax)$. For example, we have $T_4\, D_{4^{\frac{3}{2}}}\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]} = \mathbf{1}_{[-\frac{1}{8}+5,\frac{1}{8}+5]}$ and $\|T_k\, D_{k^{\frac{3}{2}}}\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}\|_1 = \|T_k\, D_{k^{\frac{3}{2}}}\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}\|_2^2 = k^{-\frac{3}{2}}$.

We shall prove that $\{\rho(\lambda)P_g\}_{\lambda\in K(\mathbb{Z}\times\mathbb{Z})}$ is not a frame for its closed linear span in $HS(\mathbb{R})$ for any $K \in \mathbb{N}$ by showing that, for any $K \in \mathbb{N}$,

$$P_{\frac{1}{K}(\mathbb{Z}\times\mathbb{Z})}|V_g g|^2 \notin L^\infty\Big(\mathbb{R}\times\widehat{\mathbb{R}}/\tfrac{1}{K}(\mathbb{Z}\times\mathbb{Z})\Big)\,. \tag{14}$$

To this end observe that

$$\begin{aligned} V_g g(n,0) &= \int g(x)g(x-n)\,dx = \int \sum_{k=n+1}^\infty T_k D_{k^{\frac{3}{2}}}\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(x)dx = \sum_{k=n+1}^\infty k^{-\frac{3}{2}} \\ &> \int_{n+2}^\infty x^{-\frac{3}{2}}\,dx = 2(n+2)^{-\frac{1}{2}}. \end{aligned}$$

Therefore

$$\begin{aligned} P_{\frac{1}{K}(\mathbb{Z}\times\mathbb{Z})}|V_g g|^2(0,0) &= \sum_{n,m\in\mathbb{Z}} |V_g g(0-\tfrac{n}{K}, 0-\tfrac{m}{K})|^2 \geq \sum_{n\in\mathbb{Z}} |V_g g(n,0)|^2 \\ &\geq 2\sum_{n=1}^\infty (n+2)^{-1} = \infty. \end{aligned}$$

An application of Lemma 3.4 gives (14).

Remark 4.6 In [32] it is shown that for $g_1, g_2 \in S_0(\mathbb{R})$, and $f_1, f_2 \in L^2(\mathbb{R})$ we have $V_{g_1}f_1 \cdot \overline{V_{g_2}f_2} \in S_0(\mathbb{R}\times\widehat{\mathbb{R}})$.

Example 4.5 on the other hand shows that there exists $g \in L^2(\mathbb{R})$ such that $|V_g g|^2 \notin S_0(\mathbb{R}\times\widehat{\mathbb{R}})$ since for g constructed in Example 4.5 we have

$$\infty = P_{\mathbb{Z}\times\mathbb{Z}}|V_g g|^2(0) = \langle\, |V_g g|^2, \sum_{n\in\mathbb{Z}^2}\delta_n\rangle$$

with $\sum_{n\in\mathbb{Z}}\delta_n \in S_0'(\mathbb{R}\times\widehat{\mathbb{R}})$.

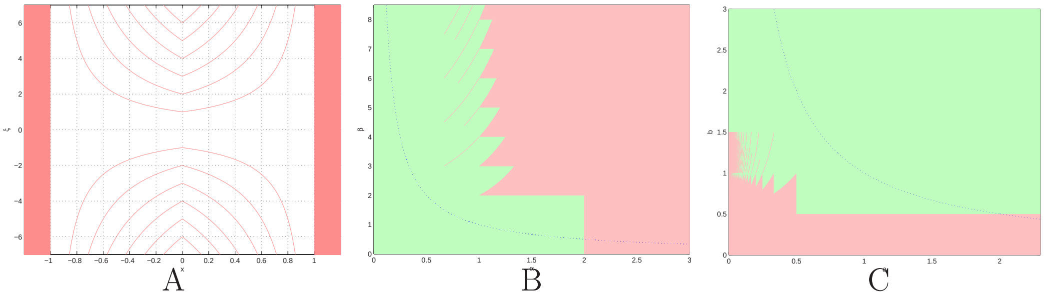

Fig. 1. A: Zero set of $V_{\mathbf{1}_{[0,1]}}\mathbf{1}_{[0,1]}$ (dark). B: Set containing all pairs (α,β), $\alpha,\beta>0$ such that $P_{\alpha\mathbb{Z}\times\beta\mathbb{Z}}|V_g g|^2$ has a root (dark). We include as reference the curve $\alpha\beta=1$ (dashed). C: Set containing all pairs (a,b), $a,b>0$ such that $\{\rho(an,bm)P_g\}_{n,m\in\mathbb{Z}}$ is not a Riesz basis in $HS(\mathbb{R})$ (dark). The curve $ab=1$ is included (dashed).

Example 4.7 We shall now consider a classical example, namely $g_c=\mathbf{1}_{[0,c]}$ and $\Lambda=a\mathbb{Z}\times b\mathbb{Z}$, $a,b>0$. The question for which a,b,c the Gabor system $(g_c,a\mathbb{Z}\times b\mathbb{Z})$ is a frame has been analyzed extensively by Janssen [33].

Note that $\{\rho(an,bm)P_{g_c}\}_{n,m\in\mathbb{Z}}$ is a frame or Riesz basis for its closed linear span for $g_c=\mathbf{1}_{[0,c]}$ if and only if $\{\rho(\frac{a}{c}n,bcm)P_{g_1}\}_{n,m\in\mathbb{Z}}$ is the same. Hence, we shall analyze the function $g=g_1=\mathbf{1}_{[0,1]}$, see Figure 1 and 2. In this case,

$$V_g g(x,\xi)=\begin{cases}\int_0^{1+x}e^{-2\pi i t\xi}\,dt, & \text{for } \ -1\le\ x\ \le 0\\ \int_x^1 e^{-2\pi i t\xi}\,dt, & \text{for } \quad 0\le\ x\ \le 1\\ 0, & \text{for } \qquad\quad |x|\ge 1\,,\end{cases}$$

and therefore

$$|V_g g|^2(x,\xi)=\begin{cases}\frac{\sin^2\pi(1-|x|)\xi}{\pi^2\xi^2} & \text{for } |x|\le 1\\ 0, & \text{for } |x|\ge 1\,.\end{cases}$$

Thus, $P_{\Lambda^\circ}|V_g g|^2=P_{\frac{1}{b}\mathbb{Z}\times\frac{1}{a}\mathbb{Z}}|V_g g|^2$ is continuous and bounded and we can rule out the existence of non–exact frames for any a,b.

Elementary calculations show that, for $\alpha=\frac{1}{b}$ and $\beta=\frac{1}{a}$, $P_{\alpha\mathbb{Z}\times\beta\mathbb{Z}}|V_g g|^2$ is *not* bounded below if and only if $\alpha=\frac{2}{3}$ and $\beta=\frac{3}{2}n$ for $n\in\mathbb{N}\setminus\{1\}$, or $\frac{2}{3}<\alpha<1$ and $\beta=\frac{2m}{2-\alpha}$ for $m\in\mathbb{N}\setminus\{1\}$, or $\frac{2k+1}{3k+1}<\alpha<1$ and $\beta=\frac{2k+1}{2-\alpha}$ for $k\in\mathbb{N}\setminus\{1\}$, or $1\le\alpha<2$ and $\beta\in\bigcup_{l\ge 2}^{\infty}[l,\frac{l}{2-\alpha}]$, for $l\in\mathbb{N}\setminus\{1\}$, or $2\le\alpha$.

Hence, $\{\rho(an,bm)P_g\}_{n,m\in\mathbb{Z}}$ is *not* a Riesz basis for its closed linear span in $HS(\mathbb{R})$, and, therefore, by Theorem 3.2, it is not a frame for its closed linear span in $HS(\mathbb{R})$ if and only if $b\le\frac{1}{2}$, or $\frac{1}{2}<b\le 1$ and $a\in\bigcup_{l\ge 2}^{\infty}\left[\frac{2b-1}{bl},\frac{1}{l}\right]$, or $1<b<\frac{3k+1}{2k+1}$ and $a=\frac{2b-1}{(2k+1)b}$ for $k\in\mathbb{N}\setminus\{1\}$, or $1<b<\frac{3}{2}$ and $a=\frac{2b-1}{2mb}$ for $m\in\mathbb{N}\setminus\{1\}$, or $b=\frac{3}{2}$ and $\beta=\frac{2}{3n}$ for $n\in\mathbb{N}\setminus\{1\}$.

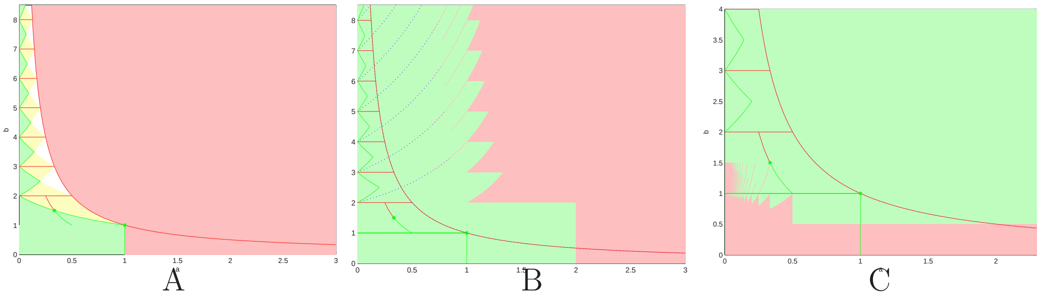

Fig. 2. A: Janssen tie, i.e., set containing pairs (a, b), $a, b > 0$, such that $(1_{[0,1]}, a, b)$ is not a frame (dark), set containing pairs (a, b), $a, b > 0$, such that $(1_{[0,1]}, a, b)$ is a frame (white). In the light area, it is known that $(1_{[0,1]}, a, b)$ is a frame if ab is irrational. B: Superposition of Janssen tie and Figure 1.B. C: Superposition of Janssen tie and Figure 1.C.

5 Gabor multipliers

Multipliers play a central role in functional and harmonic analysis. The theory of multipliers is based on simple pointwise multiplication operators $\mathcal{M}_s : L^2(X) \longrightarrow L^2(X),\ f \mapsto s \cdot f$, where X is a measure space and s is a bounded function defined on X [34].

In applied harmonic analysis, frequency domain multipliers, i.e., convolution operators, $\widehat{\mathcal{M}}_{\widehat{s}} : f \mapsto s * f$ where $\widehat{s * f} = \widehat{s} \cdot \widehat{f}$, are widely used, e.g., to model time–invariant channels in signal processing. Here, we shall consider operators which are composed of an analysis operator $C : L^2(\mathbb{R}^d) \longrightarrow L^2(X)$, whose range consists of real or complex valued functions or sequences, a pointwise multiplication by a fixed function (sequence) s on X, i.e., by the symbol s of the operator, and a synthesis operator $T : L^2(X) \longrightarrow L^2(\mathbb{R}^d)$. For example, we have $\widehat{\mathcal{M}}_{\widehat{s}} = \mathcal{F}^{-1} \circ \mathcal{M}_{\widehat{s}} \circ \mathcal{F}$.

Continuous Gabor multipliers are given by

$$V_h^* \circ \mathcal{M}_F \circ V_g : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d),\ f \mapsto \int_\lambda F(\lambda) \cdot V_g f(\lambda)\ \pi(\lambda) h \, d\lambda,$$

for $g, h \in L^2(\mathbb{R}^d)$, and they are widely discussed in the literature, e.g., in [35–37]. In the following, we shall discuss *discrete Gabor multipliers* which, as noted in the Introduction, are formally given by

$$G_m f = T_h \circ \mathcal{M}_m \circ C_g\, f = \sum_{\lambda \in \Lambda} m_\lambda \, \langle f, \pi(\lambda) h \rangle \, \pi(\lambda) g = \sum_{\lambda \in \Lambda} m_\lambda \, (\rho(\lambda) P_{g,h}) f, \quad (15)$$

for $f \in L^2(\mathbb{R}^d)$, where Λ is a full rank lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, $g, h \in L^2(\mathbb{R}^d)$, and the so–called *upper symbol* $\{m_\lambda\}_{\lambda \in \Lambda}$ is a complex valued sequence [9,10]. The operator G_m in (15) is well defined and bounded on $L^2(\mathbb{R}^d)$, if, for example, the Gabor systems (g,Λ) and (h,Λ) are Bessel sequences and if $\{m_\lambda\} \in l^\infty(\Lambda)$. Thus, (g,Λ) and (h,Λ) are dual Gabor frames for $L^2(\mathbb{R}^d)$ if and only if $G_{\mathbf{1}} = Id_{L^2}$ where $\mathbf{1}_\lambda = 1$ for all $\lambda \in \Lambda$.

Discrete Gabor multipliers on $L^2(\mathbb{R})$ can be used to model time–varying filters in communications engineering. While a convolution operator represents a time–invariant filter which allows the removal of global frequency components in a signal, a Gabor multiplier allows for the decimation of a frequency band $[\Omega_1, \Omega_2]$ during a time interval $[T_1, T_2]$ by setting $m_\lambda = 0$ for $\lambda = (x,\xi) \in [T_1,T_2]\times[\Omega_1,\Omega_2] \cap \Lambda \subset \mathbb{R}\times\widehat{\mathbb{R}}$.

If (g,Λ) is an orthonormal basis of $L^2(\mathbb{R}^d)$, and, therefore $\Lambda = 1$, and if $h = g$, then, discrete Gabor multipliers associated to (g,Λ) are exactly those operators mapping $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ which are represented by bi–infinite diagonal matrices with respect to the orthonormal basis (g,Λ). In this case, the operator G_m in (15) is bounded if and only if $\{m_\lambda\}_{\lambda\in\Lambda}$ is bounded, and G_m is stable if and only if $\{|m_\lambda|^{-1}\}_{\lambda\in\Lambda}$ is well defined and bounded. Nevertheless, families of Gabor multipliers associated to Gabor frames (g,Λ) and (h,Λ) are not simultaneously diagonalizable in general if $|\Lambda| < 1$.

A contribution to the study of Gabor multipliers in the case $|\Lambda| \neq 1$ is given in terms of operator identification in Theorem 5.2. This result further illuminates the role of the critical density $|\Lambda| = 1$ in the theory of Gabor multipliers. Recall that Figure 1.A and Figure 1.B show that $\{\rho(\lambda)P_{g,h}\}_{\lambda\in\Lambda}$ may or may not be a Riesz basis for its closed linear span in the space $HS(\mathbb{R}^d)$, regardless if $|\Lambda| < 1$, $|\Lambda| = 1$, or $|\Lambda| > 1$.

Definition 5.1 *Let X and Y be normed linear spaces over $\mathbb{C}$; and let $\mathcal{L}(X,Y)$ be the space of bounded linear operators mapping X to Y. A normed space of linear operators $\mathcal{Z} \subset \mathcal{L}(X,Y)$ is* identifiable *if there exists $f \in X$ such that $\|Zf\|_Y \asymp \|Z\|_{\mathcal{Z}}$ for all $Z \in \mathcal{Z}$.*

The operator spaces $\mathcal{Z}$ which are considered here are defined by fixing a full rank lattice Λ in $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ and $g,h \in S_0(\mathbb{R}^d)$ with $\{\rho(\lambda)P_{g,h}\}_{\lambda\in\Lambda}$ a Bessel sequence in $HS(\mathbb{R}^d)$. We set

$$\mathcal{Z} = \mathcal{G}(g,h,\Lambda) = \Big\{G_m = \sum_{\lambda\in\Lambda} m_\lambda\, \rho(\lambda)P_{g,h} :\ \{m_\lambda\} \in l^2(\Lambda)\Big\} \subset HS(\mathbb{R}^d), \tag{16}$$

and choose as norm on $\mathcal{Z}$ the Hilbert–Schmidt norm, i.e., $\|\cdot\|_{\mathcal{Z}} = \|\cdot\|_{HS}$. The operators in $\mathcal{G}(g,h,\Lambda) \subset \mathcal{L}(\, L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)\,)$ extend to $S_0'(\mathbb{R}^d)$ since $g,h \in S_0(\mathbb{R}^d)$, i.e., we have $\mathcal{G}(g,h,\Lambda) \subset \mathcal{L}(\, S_0'(\mathbb{R}^d), L^2(\mathbb{R}^d)\,)$ with domain $X = S_0'(\mathbb{R}^d)$ and range $Y = L^2(\mathbb{R}^d)$.

Theorem 5.2 *Given a full rank lattice Λ in $\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ and $g,h \in S_0(\mathbb{R}^d)$ such* *that $\{\rho(\lambda)P_{g,h}\}_{\lambda\in\Lambda}$ is a Riesz basis for its closed linear span in $HS(\mathbb{R}^d)$.*

a. If $|\Lambda| > 1$ and (g,Λ) and (h,Λ) are Riesz bases for their closed linear span in $L^2(\mathbb{R}^d)$, then $\mathcal{G}(g,h,\Lambda)$ is identifiable.
b. If $|\Lambda| < 1$, then $\mathcal{G}(g,h,\Lambda)$ is not identifiable.

Proof. a. Let $|\Lambda| > 1$ and $g,h \in S_0(\mathbb{R}^d)$ with (g,Λ) and (h,Λ) are Riesz bases for their closed linear span in $L^2(\mathbb{R}^d)$. In order to construct $f \in S_0'(\mathbb{R}^d)$ which identifies $\mathcal{G}(g,h,\Lambda)$, we pick $\widetilde{g} \in S_0(\mathbb{R}^d)$ such that $(\widetilde{g},\Lambda^\circ)$ is a dual frame of (g,Λ°) for $L^2(\mathbb{R}^d)$ [38]. Consequently we have $V_g\widetilde{g}(0) = 1$ and $V_g\widetilde{g}(\lambda) = 0$ if $\lambda \in \Lambda\setminus\{0\}$ [7], page 133, [39]. We have $f = \sum \pi(\lambda)\widetilde{g} \in S_0'(\mathbb{R}^d)$ with weak*–convergence [22], page 141, and, therefore,

$$\begin{aligned}\|G_m f\|_{L^2} &= \Big\| \sum_{\lambda\in\Lambda} m_\lambda \langle f, \pi(\lambda)g\rangle \pi(\lambda)h \Big\|_{L^2} \\ &= \Big\| \sum_{\lambda\in\Lambda} m_\lambda \Big\langle \sum_{\lambda'\in\Lambda} \pi(\lambda')\widetilde{g}, \pi(\lambda)g \Big\rangle \pi(\lambda)h \Big\|_{L^2} \\ &= \Big\| \sum_{\lambda\in\Lambda} m_\lambda \pi(\lambda)h \Big\|_{L^2} \asymp \Big\| \{m_\lambda\} \Big\|_{l^2(\Lambda)} \asymp \|G_m\|_{HS},\end{aligned}$$

since (h,Λ) is a Riesz basis for its closed linear span in $L^2(\mathbb{R}^d)$ and $\{\rho(\lambda)P_{g,h}\}_{\lambda\in\Lambda}$ is a Riesz basis for its closed linear span in $HS(\mathbb{R}^d)$. Hence, f identifies $\mathcal{G}(g,h,\Lambda)$.

b. Let $|\Lambda| < 1$ and $g,h \in S_0(\mathbb{R}^d)$, and suppose that $f \in S_0'(\mathbb{R}^d)$ identifies $\mathcal{G}(g,h,\Lambda)$. Since $\|\{m_\lambda\}\|_{l^2} \asymp \|G_m\|_{HS}$ by hypothesis, identification of $\mathcal{G}(g,h,\Lambda)$ by f is equivalent to the fact that the operator $\Phi_f : l^2(\Lambda) \longrightarrow L^2(\mathbb{R}^d),\ \{m_\lambda\} \mapsto G_m f$ is bounded and stable.

Let $\mathcal{M}$ be the multiplication operator given by $\mathcal{M} : l^2(\Lambda) \longrightarrow l^2(\Lambda),\ \{m_\lambda\} \mapsto \{m_\lambda \cdot \langle f, \pi(\lambda)g\rangle\}$ and observe that we have $\Phi_f = T_h \circ \mathcal{M}$. The multiplication operator $\mathcal{M}$ is bounded since $|\langle f, \pi(\lambda)g\rangle| \le \|f\|_{S_0'}\|g\|_{S_0}$ for all $\lambda \in \Lambda$ and, therefore, $\|\mathcal{M}\{m_\lambda\}\|_{l^2} \le \|f\|_{S_0'}\|g\|_{S_0}\|\{m_\lambda\}\|_{l^2}$. By assumption, we have Φ_f is stable and T_h is bounded, and, hence, $\mathcal{M}$ is stable, i.e., $\{|\langle f, \pi(\lambda)g\rangle|^{-1}\}$ is bounded. This implies that $\mathcal{M}$ is onto as well, and therefore $\mathcal{M}$ is an homeomorphism.

Since $|\Lambda| < 1$, T_h is not stable, and, since M is bounded and onto, this contradicts the assumption that the operator Φ_f is stable. □

Identifiability results such as Theorem 5.2 can be found in [12]. There, it is shown that classes of Hilbert–Schmidt operators which are characterized by a rectangular band limitation of their Kohn–Nirenberg symbols are identifiable if and only if the characterizing rectangle has area less than or equal to one. Similarly, it is shown that classes of Gabor frame operators are identifiable if and only if the generating lattice Λ of time–frequency shifts satisfies $|\Lambda| \leq 1$.

Additional applications of the time frequency analysis of such operators are found in [30,40–49].

Acknowledgement We gratefully acknowledge valuable conversations about this material with Hans Feichtinger.

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