A Comparison of Various MCM Schemes

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Abstract— **In this paper we compare different transmultiplexer structures with respect to the ISI/ICI occuring for typical time–invariant channels. In particular we consider wavelet–type, Gabor-type (the class containing OFDM and DMT) and Wilson-type (offset-QAM/OFDM) transmultiplexer. We present both theoretical results (based on a recently developed perturbation theory of coherent Riesz bases) and numerical simulations.**

Keywords— **OFDM, DMT, Wavelets, Gabor Systems, Wilson Bases, Offset-QAM**

I. Introduction

Multicarrier –modulation (MCM) is among the most popular concepts for data transmission over dispersive communication channels. FFT–based versions of MCM [1] are the basis for the wireless standard HIPERLAN/2 [2] and the digital subscriber line standard ADSL [3]. The latter case means baseband modulation called discrete multi–tone (DMT) which is up to a frequency shift mathematically equivalent to the passband realization called orthogonal frequency division multiplex (OFDM).

One of the key ideas underlying these standard MCM realizations is the use of a guard time that contains a cyclic prefix which in essence converts the action of the linear time–invariant channel to a cyclic convolution. Such a cyclic convolution is diagonalized by the DFT, hence the equalization reduces to a simple scalar multiplication (frequency domain equalization). The simple equalization comes at the cost of (i) a loss of modulation efficiency (redundancy of transmission signal) and (ii) poor spectral concentration of the subcarriers.

Alternative approaches to MCM are based on filterbank (wavelet) theory, which opens up quite different avenues to highly structured and thus efficiently realizable transmission signal sets. The most prominent structures correspond either to a linear (“constant-B”) or logarithmic (“constant-Q”) type partitioning of the frequency axis. In [4] the authors suggest the use of what they call discrete wavelet multitone (DWMT). However, the concrete filter bank design of [4] is obviously constant-B in contrast to the constant-Q type wavelet transform defined by the mathematical community [5], [6]. DWMT can be characterized as consisting of paruni-tary DFT filter banks at the transmitter and receiver. This implies in particular nonredundancy of the transmission signal with the undesired consequence of intersymbol/interchannel interference (ISI/ICI).

A different nonredundant transmission signal set has been suggested in [7] consisting of so–called channel adapted wavelet packets which depart from a strictly logarithmic or linear frequency scale. However, the level of adaptivity in [7] is unrealistically high and neglects practical problems (such as the enormous overhead when changing a transmultiplexer during data transmission). Another approach with total adaptivity on a sound information theoretical basis has been proposed in [8] but it leads to general filter banks (full matrix multiplications) which is not practically feasible.

Redundancy in the transmission signal set corresponds to Hilbert space completeness within the band. Incomplete, nonorthogonal systems of transmission signals for a constant-B type (FFT–based) MCM scheme where introduced in [9]. Another interesting alternative to the standard OFDM scheme is offset-QAM/OFDM scheme which is claimed to yield excellent spectral concentration [10] with nonredundant transmission signal.

In this paper we report about recent mathematical results [11] concerning the robustness of prominent function systems w.r.t. linear distortions (perturbation) caused by typical time–invariant channels. We show that the Gabor structure underlying OFDM, DMT, DWMT is matched to time–invariant channels in a deep mathematical sense (consistent with the intuitive motivation that led to MCM schemes). On the other hand we show that one can exclude existence of a “magic wavelet” that outperforms existing MCM schemes w.r.t. implementation efficiency (computational cost of modulation and equalization).

II. The Shift–Invariant MCM Setup

The transmission signal of a shift–invariant MCM scheme as depicted in Figure 1 can be formulated as

$$x(t) = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} c_{k,l} g_l(t-kT)$$

where $c_{k,l}$ are information bearing complex coefficients (“QAM symbols”) and $g_l(t)$ is a finite set of transmission pulses. The received signal $y(t)$ is given by a linearly distorted version of the transmission signal $x(t)$ and additive noise $n(t)$:

$$y(t) = (\mathbf{H}x)\,(t) + n(t)$$

The standard receiver strategy is based on an inner product representation of the received QAM symbols:

$$\hat{c}_{k,l} = \int_{-\infty}^{\infty} y(t)\overline{\gamma_l(t-kT)}dt = \langle y, \mathbf{T}_{kT}\gamma_l\rangle$$

where we have introduced the receiver pulse shapes $\gamma_l(t)$ and a time-shift operator $\mathbf{T}_\tau$ acting as $(\mathbf{T}_\tau t)(t) \stackrel{def}{=} x(t-\tau)$. To achieve perfect reconstruction in the case of an ideal channel one has to require a biorthogonality condition as follows:

$$\langle \mathbf{T}_{kT} g_l, \mathbf{T}_{k'T}\gamma_{l'}\rangle = \delta_{k,k'}\delta_{l,l'}$$

The structure and amount of ISI/ICI is governed by the effective channel matrix:

$$Q_{g,\gamma,\mathbf{H}}(k-k',l,l') \stackrel{def}{=} \langle \mathbf{H}\mathbf{T}_{kT} g_l, \mathbf{T}_{k'T}\gamma_{l'}\rangle\,, \tag{II.1}$$

which (as suggested by our notation) can be shown to be block–Toeplitz. In the ideal case the matrix is diagonal

$$Q_{g,\gamma,\mathbf{H}}(k-k',l,l') = \lambda_l \delta_{l,l'}\delta_{k,k'}$$

and, for zero noise, the symbols can be recovered by a scalar multiplication $c_{k,l} = \frac{1}{\lambda_l}\hat{c}_{k,l}$.

Fig. 1. The considered MCM scheme

Practically important shift-invariant biorthogonal systems are defined by the action of unitary operators on one specific prototype pulse (mother wavelet). In particular we consider one of the following structures:

- *Gabor systems* correspond to a rectangular tiling of the time–frequency plane (constant–B), the g_l are modulated versions of a prototype function g_0:

$$g_l(t) = g_0(t)e^{i2\pi\frac{\rho}{T}lt}. \tag{II.2}$$

Note that in order to have existence of orthonormal bases one has necessarily $\rho \geq 1$.

- The real-valued *Wilson* bases [10] have a structure related to but different from the WH systems ($m \in [0, M-1]$, $N = 4M+1$): $\quad g_0(t) = g(t)\,,$

$$\begin{aligned} g_{m,1}(t) &= g(t)\,\sqrt{2}\cos(2\pi\tfrac{2m}{T}x), \\ g_{m,2}(t) &= g(t-\tfrac{T}{2})\,\sqrt{2}\cos(2\pi\tfrac{2m-1}{T}x)\,, \\ g_{m,3}(t) &= g(t)\,\sqrt{2}\sin(2\pi\tfrac{2m-1}{T}x), \\ g_{m,4}(t) &= g(t-\tfrac{T}{2})\,\sqrt{2}\sin(2\pi\tfrac{2m}{T}t)\,, \end{aligned}$$

Wilson bases correspond to an offset–QAM/OFDM which allows FFT–based realization. A recently developed theory allows the design of pulses $g(t)$ with improved frequency localization [10].

- Dyadic *Wavelet* bases [6] are defined as ($m \in [0, M]$, $n \in [0, 2^m - 1]$, $N = 2^{M+1} - 1$):

$$g_m^{(n)}(t) = 2^{m/2} g_0\big(2^m(t - n\frac{T}{2^m})\big).$$

Wavelet bases are known to combine relatively good frequency localization with a fast computation algorithm (in principle faster than FFT).

We assume throughout this paper that the channel distortion corresponds to a translation invariant system, i.e., (all integrals are over $\mathbb{R}$ and Fourier transforms of signals are denoted by capital letters)

$$(\mathbf{H}\,x)(t) = (h{*}x)(t) = \int_{t'} h(t-t')\,x(t')\,dt'$$

for some $h \in \boldsymbol{L}^2(\mathbb{R})$. Since h and thus $\mathbf{H}$ is not fixed, but varies from case to case, we consider the following ensemble of possible impulse responses:

$$\mathcal{H} = \Big\{h \in \boldsymbol{L}^2(\mathbb{R}) : \operatorname{supp} h \subseteq [-\tfrac{\tau_0}{2}, +\tfrac{\tau_0}{2}], \\ \int_t |h(t)|^2\, x\, dx = 0\,, \sup|H(f)| = 1\Big\}\,.$$

The three conditions imposed on h seem realistic for the following reasons:

- The receiver does not know when the transmission starts, so he has to fix the time $T = 0$ in some way. Since this is equivalent to choosing some translate of h, we may as well fix h to have vanishing first moment.
- Although h does not have compact support, we may cut off at some point and treat the influence of the remaining part of h as noise.
- The condition $\sup|H(f)| = 1$ corresponds to perfect gain control.

III. Orthogonal Perturbations

Due to lack of space we cannot perform a detailed statistical analysis of the general MCM-modell (Fig. 1). With regard to noise sensivity we just note that orthonormal systems with arbitrary structure are optimum; biorthogonal Riesz bases used for MCM need to have an excellent condition number which implies $\|g_l - \gamma_l\| \ll 1$ [11]. Hence in what follows we put the focus on orthonormal systems.

For orthonormal systems the total ISI/ICI can be defined as an off–diagonal norm of the effective channel matrix

$$O_{\mathbf{H},g,g} \stackrel{def}{=} \sum_{k=-K}^{K}\sum_{l=0}^{N-1}\sum_{l'=0}^{N-1} |Q_{\mathbf{H},g,g}(k,l,l')|^2\,(1-\delta_k\delta_{l,l'})\,.$$

In order to derive useful estimates for the off–diagonal decay of $Q_{\mathbf{H},g,g}(k,l,l')$ we introduce the orthogonal perturbation of each individual basis member as follows (compare Figure 2).

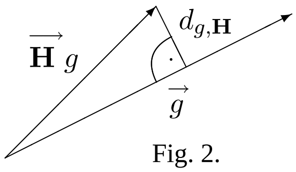

Fig. 2.

$$d^2_{g,\mathbf{H}} \stackrel{def}{=} \|\mathbf{H}g\|^2 - |\langle \mathbf{H}g, g\rangle|^2, \tag{III.1}$$

Here, and in what follows we assume $\langle g, g\rangle = \|g\|^2 = 1$.

With the above definitions, it is straightforward to show

$$O_{\mathbf{H},g,g} \le \sum_{k=-K}^{K} \sum_{l=0}^{N-1} d^2_{g_{k,l},\mathbf{H}},$$

which means that the sum of all orthogonal perturbations bounds the total ISI/ICI.

Since the convolution $\mathbf{H}g = h * g$ corresponds to multiplication in the Fourier domain, $d_{g,\mathbf{H}}$ can be related to the frequency localization of g, as the following theorem shows.

Theorem 1: [11] Let $g, h \in \boldsymbol{L}^2(\mathbb{R})$ with $\|g\|_{\boldsymbol{L}^2} = 1$. Then the orthogonal distortion is given by the following variance

$$d^2_{g,\mathbf{H}} = \mathrm{V}\left\{H(\Xi)\right\},$$

where Ξ is a random variable with probability density $|G|^2$,

$$\mathrm{V}\left\{H(\Xi)\right\} = \int_\xi \left|H(\xi) - \mathrm{E}\{H(\Xi)\}\right|^2 |G(\xi)|^2 \, d\xi$$

with expected value $\mathrm{E}\{H(\Xi)\} = \int_\xi H(\xi)\,|G(\xi)|^2\,d\xi$.

Using the expression given in theorem 1, we can find an upper bound for the orthogonal perturbation $d_{g,\mathbf{H}}$ with $h \in \mathcal{H}$.

Theorem 2: [11] (*Upper Bound*) Assume that h is an impulse response in $\mathcal{H}$. Then we have for $g \in \boldsymbol{L}^2(\mathbb{R})$ with $\|g\|_{\boldsymbol{L}^2} = 1$

$$d^2_{g,\mathbf{H}} \le (\pi\tau_0)^2 \sigma^2_{|G|^2},$$

where $\sigma^2_{|G|^2}$ is the variance of $|G|^2$, i.e.,

$$\sigma^2_{|G|^2} = \int_f (f-\mu)^2\,|G(f)|^2\,df \qquad \text{with}$$

$$\mu = \mu_{|G|^2} = \int_f f\,|G(f)|^2\,df .$$

On the other hand, one must expect that signals which are not well frequency localized potentially undergo a relatively strong orthogonal perturbation.

Fig. 3. Bad localized eigenfunction of $\mathbf{H}$ (eigenvalue H_0)

For a given convolution operator there might be arbitrarily bad localized functions g which are exact eigenfunctions of this specific operator, so $d_{g,\mathbf{H}} = 0$ for this particular h. Such a situation is depicted in Fig. 3. But recall that for practical purposes, we require a family of basis functions that are stable under the action of all $h \in \mathcal{H}$. Therefore, to be able to show that certain bases are inadequate, we want to determine a lower bound of

$$d_g = \sup_{h\in\mathcal{H}} d_{g,\mathbf{H}}.$$

The following theorem is based on the uncertainty principle in so far as it exploits knowledge of a minimum frequency spread given a maximum temporal support length.

Theorem 3: [11](*Lower Bound*) For $g \in \boldsymbol{L}^2(\mathbb{R})$, $\|g\|_{\boldsymbol{L}^2} = 1$, with $\operatorname{supp} g \subseteq [\alpha, \alpha+T]$ for some $\alpha \in \mathbb{R}$ and $T > 0$, we have

$$d_g^2 \ge r^2\Big(1 - \frac{4}{3}\,s\,\frac{T}{\tau_0}\Big) \qquad \text{for } \frac{T}{\tau_0} \le \frac{1}{2s},$$

$$\text{and} \qquad d_g^2 \ge \frac{1}{12}\Big(\frac{r\,\tau_0}{s\,T}\Big)^2 \qquad \text{for } \frac{T}{\tau_0} > \frac{1}{2s},$$

with $s \in]0,1[$ and $r \in [\frac{1}{2}, 1[$.

Based on the above mathematical results we now evaluate the orthogonal distortion on a logarithmic scale

$$d'^2_g = 10\log_{10} d_g^2.$$

We consider the structures discussed in Section II and assume validity of a typical maximum support as imposed by the latency constraints for voice communication.

- Gabor bases allow direct application of Theorem 2 because the frequency localization is invariant w.r.t. modulation such that in turn

$$d_g = d_{g_l}. \tag{III.2}$$

For standard pulse functions such as e.g. the Bartlett window satisfying the support constraint we get by a straightforward computation:

$$d'^2_g \approx -20dB$$

- Properly designed Wilson(Offset–QAM/OFDM)–type bases are known for their excellent localization in a *real–valued* sense (i.e., their analytic function shows excellent frequency localization in the sense of Theorem 2). However, in practical OFDM systems we use these basis functions with complex coefficients. In the complex sense the Wilson–type bases do not satisfy a modulation invariance of the orthogonal perturbation (III.2) rather the frequency localizaton decreases with increasing modulation index. Based on Theorem 3 one can show that in a Wilson system with at least 200 carriers there is at least one g_l with $d'^2_{g_l} \ge -8dB$

- In a dyadic wavelet basis, one encounters the problem that, since scaling on the time side results in reverse scaling on the frequency side, the frequency localization gets worse with growing scale index. If we assume to have at least 128 basis functions per symbol period T we need $M \ge 7$ scale indices which based on Theorem 3 leads to

$$d'^2_{g_8^{(n)}} \approx -3dB$$

IV. Simulation Results

To illustrate the theoretical results we consider a numerical experiment involving a noise–free DSL–channel with loop length 2km (the impulse response is plotted in Fig. 4).

Fig. 4. The considered impulse response

Figure 5 shows linearly scaled contour plots of the effective channel matrix for three different orthonormal systems: nonorthogonal Gabor (OFDM with pulse shaping) without cyclic prefix, an orthonormal Wilson basis (the prototype designed according to [10]) and an orthonormal Wavelet basis ("symmlets" defined in [6, p.250]) (symmlets were best performing in the sense of this work among a number of prominent wavelet bases). The poor off–diagonal decay of the wavelet basis is clearly visible; to show the difference between Wilson and Gabor, we have furthermore plotted a 1D slice of $Q_{g,g,\mathbf{H}}(0,l,l')$ in Fig. 6. As expected from the theoretical results, the Wilson basis (broken line) shows relatively poor off–diagonal decay.

Fig. 5. Magnitude of 2D cut $Q_{g,g,\mathbf{H}}(0,l,l')$ for the following bases (a) Gabor , (b) Wilson, (c) Wavelet

Fig. 6. $Q_{g,g,\mathbf{H}}(0,l,20)$ for Gabor (solid), Wilson (broken)

V. Conclusions

We have studied various different transmultiplexer structures with numerically efficient implementation. We have shown that the total ISI/ICI can be bounded by the orthogonal perturbation caused by channel. Among the considered structures, the Gabor structure (constant B-type transmission signal set) turns out to yield optimum perturbation stability. The Gabor structure includes standard OFDM implementations and biorthogonal OFDM with pulse shaping. The optimization of the WH-type transmultiplexers w.r.t. bandwidth efficiency, peak-to-average ratio, robustness and simplicity of equalization for typical channel scenarios is the natural open question for future research.

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