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Operator identification and Feichtinger's algebra

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Abstract

The goal in channel operator identification is to obtain complete knowledge of an operator modelling a communication channel by observing the image of a finite number of input signals. It was proved by Kozek and Pfander that identifiability can be related to the spreading support of the operator modelling the channel, as was conjectured by Kailath in 1963. The extended result proved in this paper shows that the collection of identifiable operators can be chosen to be a closed subspace of a Banach space, which includes as examples the identity operator, small perturbations of the identity, and convolution operators with compactly supported kernels as examples of identifiable channels. These examples exceed the scope of the results from Kozek and Pfander. The Feichtinger algebra and its dual arise naturally in this extension further illustrating the enduring usefulness of Feichtinger's work.

Key words and phrases : Operator identification, Channel measurements, Feichtinger algebra, spreading function, bandlimited Kohn–Nirenberg symbols, communications engineering.

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1 Introduction

The goal of this paper is to extend a result of [14] to a more general setting that includes a variety of natural examples not covered by the original result. The new setting is based on the Feichtinger algebra $S_0(\mathbb{R}^d)$ and its rich and beautiful dual $S_0'(\mathbb{R}^d)$.

The problem of interest is referred to as operator identification (or channel identification or channel measurement) and is described mathematically as follows. Given normed linear spaces X and Y and a collection $\mathcal{H}$ of operators $\mathcal{H} \subset \mathcal{L}(X,Y)$, we say that the collection $\mathcal{H}$ is *identifiable with identifier* $g \in X$ if g induces a bounded and stable linear map $\Phi_g : \mathcal{H} \longrightarrow Y,\ H \mapsto Hg$. That is, if there exist $A, B > 0$ with

$$A\,\|H\|_{\mathcal{H}} \leq \|Hg\|_Y \leq B\,\|H\|_{\mathcal{H}} \tag{1}$$

for all $H \in \mathcal{H}$. The norm $\|\cdot\|_{\mathcal{H}}$ can in principle be arbitrary, however in all examples to be considered the norm satisfies $\|H\|_{\mathcal{L}(X,Y)} \leq \|H\|_{\mathcal{H}}$. This guarantees that for any $g \in X$, the induced operator Φ_g is bounded so that B in (1) always exists. Establishing identifiability is therefore equivalent to finding g so that for some positive A we have $A\,\|H\|_{\mathcal{H}} \leq \|Hg\|_Y$ for all $H \in \mathcal{H}$. This observation gives considerable flexibility in determining when a particular class of operators $\mathcal{H}$ is identifiable. In other words, if $\mathcal{H}$ is known, then the spaces X, Y, and the norm $\|\cdot\|_{\mathcal{H}}$ can in principle be chosen in a number of ways so that (1) will hold. We will give some examples in Section 3 illustrating this principle.

The motivation for considering this problem comes from communications theory. Beginning in the early 1950s a research program was in place whose goal was to understand and model communication channels (e.g., [11], [12], [1] and the references cited therein). One model put forward for such channels was a time–varying linear filter of the form

$$Hf(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta_H(t,\nu)\,T_tM_\nu f(x)\,dt\,d\nu \tag{2}$$

where T_t is a *time-shift*, i.e., $T_tf(x) = f(x-t)$, $t \in \mathbb{R}$ and M_ν is a *frequency-shift*, i.e., $\widehat{M_\nu f}(\gamma) = \widehat{f}(\gamma - \nu)$, $\nu \in \widehat{\mathbb{R}} = \mathbb{R}$ where $\widehat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)\,e^{-2\pi i x\gamma}\,dx$ is the Fourier transform. That is, $M_\nu f(x) = e^{2\pi i\nu x}\,f(x)$. The function $\eta_H(t,\nu)$ is called the spreading function of H and completely determines H. In this context identifiability of a channel means that the spreading function η_H, and hence all characteristics of the channel modelled by H, can be determined by examining a single output Hg for an appropriately chosen g.

In [13], Kailath considered the class of channels whose spreading function vanished outside a rectangle Q_{ab} of the form $Q_{ab} = [0,a]\times[-b/2,b/2]$. The parameter a represents the maximum time-spread of a signal by the given channel (such a channel is said to have *finite memory*) and the parameter b represents the maximum Doppler spread of a signal by the channel. Kailath conjectured that the collection of such channels that were identifiable coincided precisely with those such that $ab \leq 1$. This conjecture was given precise mathematical footing and proved for operators with η in Feichtinger's algebra in [14].

More precisely, it was shown in that paper that the class of Hilbert-Schmidt operators

$$\mathcal{H}_{Q_{ab}} = \{H \in \mathcal{L}(L^2(\mathbb{R}), L^2(\mathbb{R})) \colon \eta_H \in S_0(\mathbb{R}\times\widehat{\mathbb{R}}),\ \operatorname{supp}\eta_H \subseteq Q_{ab}\}, \qquad (3)$$

with $\|H\|_{\mathcal{H}_{Q_{ab}}} = \|\eta_H\|_{L^2}$, the Hilbert-Schmidt norm of H, is identifiable if and only if $ab \leq 1$. The identifier in this case is the delta train $g = \sum_{n\in\mathbb{Z}} \delta_{na}$ and we have taken $X = S_0'(\mathbb{R})$ and $Y = L^2(\mathbb{R})$ in the operator identification formalism described above.

Upon closer inspection, the setting of [14] leaves room for improvement in the following sense. (A) The class $\mathcal{H}_{Q_{ab}}$ is not closed with respect to the Hilbert-Schmidt norm. This means that, in principle, an arbitrarily small perturbation of an operator covered by the theory exceeds the scope of the theory. (B) Hilbert-Schmidt operators are compact. Hence the class $\mathcal{H}_{Q_{ab}}$ does not include the identity operator or small perturbations of the identity, i.e., distortion-free and almost distortion-free channels. (C) The class $\mathcal{H}_{Q_{ab}}$ does not include ordinary convolution operators with compactly supported kernels, i.e. linear, time-invariant channels with finite memory. Indeed, operators of the form (2) are canonical generalizations of such channels.

The main results of this paper (Theorems 5.2 and 5.3) show that the result of [14] remains valid in a sufficiently general setting which includes each of the above situations.

In terms of pseudodifferential operators, we discuss the identifiability of classes of operators with distributional, bandlimited symbols. In fact, the compactly supported spreading functions considered in this paper are exactly the symplectic Fourier transforms of the corresponding and therefore bandlimited Kohn–Nirenberg symbols [6, 14, 20].

The paper is structured as follows. In Section 2 we review some basic results about the Feichtinger algebra $S_0(\mathbb{R}^d)$ and its dual $S_0'(\mathbb{R}^d)$ that are used in this paper. In Section 3 we present three examples of operator identification with the view of showing how the notion of the spreading function and its support relates to the identifiability of operator classes in a variety of settings. It also illustrates the naturalness of considering spreading functions that are actually distributions. This section also contains an outline of the proof of the identifiability theorem from [14] which will be generalized. In Section 4 we define the generalized operator class $\mathcal{H}_M$ of operators with compactly supported spreading functions in $S_0'(\mathbb{R}\times\widehat{\mathbb{R}})$. We gather together some general properties of such operators and in particular show that the identification operator Φ_g is bounded for any $g \in S_0'(\mathbb{R})$. Section 5 contains the proof of the main results of this paper.

2 Preliminaries

In this section we collect some basic results on the Feichtinger algebra, its dual, and other Wiener amalgam spaces that will be used throughout this paper. Here and in the following, $A(\mathbb{R}^d)$ denotes the space of all functions with absolutely integrable Fourier transform with norm $\|f\|_A = \|\widehat{f}\|_{L^1}$, and $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class of smooth, rapidly decreasing functions.

Definition 2.1 *The Feichtinger algebra is*

$$S_0(\mathbb{R}^d)=\Big\{f\in L^2(\mathbb{R}^d) : V_{\varphi_0}f(t,\nu)=\int f(x)\, e^{-2\pi i\nu(x-t)}\overline{\varphi_0}(x-t)\, dx \,\in L^1(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)\Big\}$$

where $V_{\varphi_0}f$ is called short time Fourier transform of f with respect to the gaussian window $\varphi_0(x) = e^{-\pi\|x\|_2^2}$, $x \in \mathbb{R}^d$. We set $\|f\|_{S_0} = \|V_{\varphi_0}f(\cdot,\cdot)\|_{L^1(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}$.

For $1 \leq p \leq \infty$, define the Wiener amalgam space *$W\big(A(\mathbb{R}^d), l^p\big)$ to be the Banach space of functions that are locally in $A(\mathbb{R}^d)$ and whose local $A(\mathbb{R}^d)$ norms behave like an l^p sequence. Specifically, given $\psi \in A(\mathbb{R}^d)$ with compact support satisfying $\sum_{n\in\mathbb{Z}^d} T_n\psi \equiv 1$, define*

$$\|f\|_{W(A,l^p)} = \|\{\|f\cdot T_n\psi\|_A\}_{n\in\mathbb{Z}^d}\|_{l^p}.$$

Different functions ψ produce equivalent norms for $W\big(A(\mathbb{R}^d), l^p\big)$. Similarly, we define the Wiener amalgam spaces $W\big(\mathcal{F}L^\infty(\mathbb{R}^d), l^p\big)$ of tempered distributions for $1 \leq p < \infty$ and $\|f\|_{\mathcal{F}L^\infty} = \|\widehat{f}\|_{L^\infty}$.

In Section 4, we shall use that

$$S_0'(\mathbb{R}^d) = W\big(A(\mathbb{R}^d), l^1\big)' = W\big(\mathcal{F}L^\infty(\mathbb{R}^d), l^\infty\big)$$

and

$$W\big(A(\mathbb{R}^d), l^\infty\big)' = W\big(\mathcal{F}L^\infty(\mathbb{R}^d), l^1\big)$$

For more details and history see [4, 5, 7] and the references therein.

Throughout this paper, the pairing $\langle\cdot,\cdot\rangle$ is sesquilinear, that is, linear in the first variable and conjugate linear in the second variable. It will be used to denote L^2–inner products as well as sesquilinear (S_0, S_0') and (S_0', S_0) duality brackets. Our choice of sesquilinear duality brackets rather than bilinear duality brackets is done for notational simplicity only. It has, at most, cosmetic effects on the mathematical statements made in this paper.

The following proposition describes a number of equivalent norms on $S_0(\mathbb{R}^d)$ that will be used in this paper. Here and in the following $A(f) \asymp B(f)$ means that there exist positive constants c, C which do not depend on f, such that $c\, A(f) \leq B(f) \leq C\, A(f)$.

Proposition 2.2

1. $S_0(\mathbb{R}^d) = W\big(A(\mathbb{R}^d), l^1\big)$. *For any compactly supported $\psi \in A(\mathbb{R}^d)$ with $\sum_{n\in\mathbb{Z}^d} T_n\psi = 1$ we have*

$$\sum_{n\in\mathbb{Z}^d} \|f\cdot T_n\psi\|_A = \|f\|_{W(A,l^1)} \asymp \|f\|_{S_0}, \quad f \in S_0(\mathbb{R}^d).$$

2. *For $\psi \in S_0(\mathbb{R}^d)$ such that (ψ, a, b), $a, b > 0$, is an $L^2(\mathbb{R}^d)$–Gabor frame, we have*

$$\|\{\langle f, T_{bn}M_{am}\psi\rangle\}_{m,n\in\mathbb{Z}^d}\|_{l^1} \asymp \|f\|_{S_0}, \quad f \in S_0(\mathbb{R}^d)$$

and

$$\|\{\langle f, T_{bn}M_{am}\psi\rangle\}_{m,n\in\mathbb{Z}^d}\|_{l^\infty} \asymp \|f\|_{S_0'}, \quad f \in S_0'(\mathbb{R}^d).$$

Proposition 2.2, part *1*, is Theorem 3.2.6, page 130, in[7]. Part *2* follows from [8], Theorem 13.6.1, page 298, for $ab \in \mathbb{Q}$ and from [9] in the general case.

Some of the results in later sections will rely on properties of the *Zak transform*. We collect the important properties used in this paper in Proposition 2.4 (see [8] for proofs and further results).

Definition 2.3 *Given $f \in S_0(\mathbb{R}^d)$ we define the* Zak transform *of f, denoted Zf, on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, by*

$$Zf(x,\omega) = \sum_{k\in\mathbb{Z}^d} f(x-k)\, e^{2\pi i k\cdot\omega}.$$

Proposition 2.4 *Let $f \in S_0(\mathbb{R}^d)$ be given.*

1. *The series defining Zf in Definition 2.3 converges in the $L^2([0,1]^{2d})$ norm.*

2. *For $n \in \mathbb{Z}^d$, $Zf(x, \omega + n) = Zf(x,\omega)$ and $Zf(x+n,\omega) = e^{2\pi i n\cdot\omega}\, Zf(x,\omega)$. These identities are referred to as the* quasiperiodicity conditions*.*

3. *Z extends to a unitary operator from $L^2(\mathbb{R}^d)$ onto $L^2([0,1]^{2d})$.*

4. *$f(x) = \int_{[0,1]^d} Zf(x,\omega)\, d\omega$.*

5. *If $f \in \mathcal{S}(\mathbb{R}^d)$ then $Zf \in C^\infty(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ and if $F \in C^\infty(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ is quasiperiodic then there is a unique $f \in \mathcal{S}(\mathbb{R}^d)$ such that $Zf = F$.*

6. *$V_{\perp\!\perp\!\perp} f(x,\omega) = Zf(x,\omega)$ where $\perp\!\perp\!\perp$ denotes the* Shah distribution *or* delta train*, $\perp\!\perp\!\perp = \sum_{n\in\mathbb{Z}^d} \delta_n$.*

Proof. Assertions 1-4 are taken directly from [8], Section 8.2. Assertion 5 follows by direct calculation where for $f \in S_0(\mathbb{R})$ and $g \in S_0'(\mathbb{R})$, $V_g f(x,\omega) = \langle f, T_x M_\omega g\rangle$ is interpreted as a sesquilinear (S_0, S_0') duality bracket. □

3 Examples of operator identification via the spreading function

In this section, we will include some examples of identifiable classes of operators in various settings where the criterion for identification is related to conditions on the support of the spreading function, appropriately interpreted, of the operator. In other words we write the operator as a superposition of time and frequency shifts and define the coefficients of such a superposition as the spreading function for the operator. This illustrates that such representations of operators are very natural for considering questions of identifiability.

3.1 Finite dimensions, function spaces on $\mathbb{Z}_n$

Here, we choose $X = Y = \mathbb{C}^n = (\mathbb{Z}_n)^{\mathbb{C}}$ with $\mathbb{Z}_n = \{0, \ldots, n-1\}$ and $n \in \mathbb{N}$. We consider the operator spaces as matrix spaces, i.e., $\mathcal{M} \subseteq \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n) \equiv Mat(n{\times}n)$.

Since we are working in finite dimensions, we have $\mathcal{M}$ is identifiable if there exists a vector $x \in \mathbb{C}^n$ such that for all $M \in \mathcal{M}$, $Mx = 0$ implies $M = 0$.

Definition 3.1 *Set $\omega = e^{2\pi i/n}$. The translation operator T is the unitary operator on $\mathbb{C}^n$ given by $Tx = T(x_0, \ldots, x_{n-1}) = (x_{n-1}, x_0, x_1, \ldots, x_{n-2})$, and the modulation operator M is the unitary operator defined by $Mx{=}M(x_0, \ldots, x_{n-1}){=}(\omega^0 x_0, \omega^1 x_1, \ldots, \omega^{n-1} x_{n-1})$. We set $\pi(\lambda) = T^k M^l$ for $\lambda = (k, l)$. The* spreading function *of a matrix $H \in Mat(n{\times}n)$ given by $H = (h_{i,j})_{i,j=0}^{n-1}$, denoted $\eta_H \in \mathbb{C}^{n^2}$, is defined by*

$$\eta_H(k, l) = \frac{1}{n} \sum_{m=0}^{n-1} h_{m,m-k}\, \omega^{-ml}$$

for l, $k = 0, \ldots, n-1$, where here and in the following, indices are taken modulo n.

A straightforward calculation establishes the following lemma.

Lemma 3.2 *The family of operators $\{T^k M^l\}_{(l,k)\in\mathbb{Z}_n\times\mathbb{Z}_n} \subseteq Mat(n{\times}n)$ is a basis for $Mat(n \times n)$. In particular,*

$$H = \sum_{l=0}^{n-1}\sum_{k=0}^{n-1} \eta_H(l, k) T^k M^l = \sum_{\lambda\in\mathbb{Z}_n\times\mathbb{Z}_n} \eta_H(\lambda)\pi(\lambda),$$

where $\eta_H \in \mathbb{C}^{n^2}$ is the spreading function of H.

Definition 3.3 *For $\Lambda \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ define*

$$\mathcal{H}_\Lambda = \operatorname{span}\{\pi(\lambda) : \ \lambda \in \Lambda\} = \{H \in Mat(n \times n) : \operatorname{supp} \eta_H \subseteq \Lambda\}.$$

The following theorem was proved in [15] and relies on a general result about the robustness of finite Gabor frames.

Theorem 3.4 *For n prime, $\mathcal{H}_\Lambda$ is identifiable if and only if $|\Lambda| \leq n$.*

Numerical experiments suggest that Theorem 3.4 holds for any n, but the general result for n not prime is still open.

3.2 Gabor frame operators

The relationship between Gabor frame operators and overspread and underspread operators of the form (2) was investigated for the first time in [14]. For the purposes of this paper, Gabor frame operators provide a good example of the connection between spreading function supports and identifiability. They also illustrate how spreading functions in S_0' arise naturally in time-frequency analysis.

For $\varphi,\gamma\in S_0(\mathbb{R})$ and $a,b>0$, define the *Gabor frame operator* $S^{a,b}_{\varphi,\gamma}$ on $L^2(\mathbb{R})$ by

$$S^{a,b}_{\varphi,\gamma}f=\sum_{k,l\in\mathbb{Z}}\langle f,T_{la}M_{kb}\varphi\rangle\,T_{la}M_{kb}\gamma.$$

Janssen's representation (see [8], Section 7.2) allows us to write the Gabor frame operator as a superposition of time and frequency translates as

$$S^{a,b}_{\varphi,\gamma}f=(ab)^{-1}\sum_{n,m\in\mathbb{Z}}\langle\gamma,M_{\frac{m}{a}}T_{\frac{n}{b}}\varphi\rangle\,M_{\frac{m}{a}}T_{\frac{n}{b}}f.$$

In this sense the frame operator $S^{a,b}_{\varphi,\gamma}$ has the form (2) with

$$\eta_{S^{a,b}_{\varphi,\gamma}}=(ab)^{-1}\sum_{n,m\in\mathbb{Z}}\langle\gamma,M_{\frac{m}{a}}T_{\frac{n}{b}}\varphi\rangle\;\delta_{\frac{n}{b}}\!\otimes\delta_{\frac{m}{a}}$$

which is an element of $S_0'(\mathbb{R}\times\widehat{\mathbb{R}})$.

In [14], the authors investigate the identifiability of the operator class

$$\mathcal{S}^{a,b}=\Big\{S^{a,b}_{\varphi,\gamma}:\ \varphi\in L^2(\mathbb{R}),\ \gamma\in W(\mathbb{R})\Big\}\ \text{ with }\ \|S^{a,b}_{\varphi,\gamma}\|_{\mathcal{S}^{a,b}}=\Big\|\Big\{\Big\langle\gamma,M_{\frac{m}{a}}T_{\frac{n}{b}}\varphi\Big\rangle\Big\}\Big\|_{l^2}$$

and make the choice

$$X=W(\mathbb{R})=W\big(L^\infty(\mathbb{R}),l^1\big)=\big\{f\in L^2(\mathbb{R}):\|f\|_W=\sum_{k\in\mathbb{Z}}\|f\cdot 1_{[k,k+1)}\|_\infty<\infty\big\}\,,$$

and $Y=L^2(\mathbb{R})$.

Theorem 3.5 *$\mathcal{S}^{a,b}$ is identifiable if and only if $ab\le 1$. Moreover, for any a,b with $ab>1$ and any $g\in W(\mathbb{R})$ exist $\varphi\in L^2(\mathbb{R})$ and $\gamma\in W(\mathbb{R})$ such that $S^{a,b}_{\varphi,\gamma}g=0$.*

The proof of this theorem relies on fundamental and deep results on the relation between the existence of Gabor frames and the lattice density ab. It also suggests a connection between results in the theory of underspread and overspread operators and Gabor theory. For more details and discussion, see [14].

3.3 Hilbert-Schmidt operators.

In [14] the following theorem was proved.

Theorem 3.6 *The operator class $\mathcal{H}_{Q_{ab}}$ given by (3) is identifiable if and only if $ab \leq 1$. In the case of identifiability an identifier is the distribution $g = \sum_{n \in Z} \delta_{na}$.*

We will outline the proof given in [14] here in order to more effectively contrast it with the proof of the more general version in Section 5. Assuming first that $ab \leq 1$, and that $Q_{ab} \subseteq [0,1]^2$, let $g = \sum_{n \in Z} \delta_n = ⊥⊥⊥$. Let $f \in S_0(\mathbb{R}) \subseteq L^2(\mathbb{R})$ with $\|f\|_{L^2} = 1$ be arbitrary. Then for any $H \in \mathcal{H}_{Q_{ab}}$,

$$\langle \Phi_{⊥⊥⊥} H, f \rangle = \langle H ⊥⊥⊥, f \rangle = \langle \eta_H, V_{⊥⊥⊥} f \rangle = \langle \eta_H, Zf \rangle$$

where Zf is the Zak transform of f. By the unitarity of the Zak transform as an operator from $L^2(\mathbb{R})$ onto $L^2(\,[0,1]^2)$, Proposition 2.4, part *3*, it follows that $\|Zf\|_{L^2[0,1]^2} = 1$. Moreover, since $S_0(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ it follows that as f runs through $S_0(\mathbb{R})$, Zf runs through a dense subset of $L^2(\,[0,1]^2)$. Therefore $\|\Phi_{⊥⊥⊥} H\|_{L^2} = \|\eta_H\|_{L^2} = \|H\|_{\mathcal{H}_{Q_{ab}}}$ for all $H \in \mathcal{H}_{Q_{ab}}$ and this is (1).

To show the converse, assume that $ab > 1$. The goal is to show that for any $g \in S_0'(\mathbb{R})$ the operator $\Phi_g \colon \mathcal{H}_{Q_{ab}} \longrightarrow L^2(\mathbb{R})$ is not stable, that is it does not possess a lower bound in the inequality (1). The proof consists of the following steps.

1. Define a synthesis operator $E\colon l_0(\mathbb{Z}^2) \longrightarrow \mathcal{H}_{Q_{ab}}$, where $l_0(\mathbb{Z}^2)$ is the space of finite length sequences equipped with the l^2 norm, by

$$E\,\sigma = E(\{\sigma_{k,l}\}) = \sum_{k,l} \sigma_{k,l} T_{l\lambda/b} M_{k\lambda/a}\, P\, M_{-k\lambda/a} T_{-l\lambda/b}.$$

Here, $\lambda > 1$ is chosen so that $1 < \lambda^4 < ab$ and $P \in \mathcal{H}_{Q_{ab}}$ is a product convolution operator with spreading function

$$\eta_P(t,\nu) = \eta_1(t)\,\eta_2(\nu)$$

where η_1, $\eta_2 \in \mathcal{S}(\mathbb{R})$ take values in $[0,1]$ and satisfy

$$\eta_1(t) = \begin{cases} 1 & \text{for } |t - a/2| \leq a/2\lambda \\ 0 & \text{for } |t - a/2| \geq a/2 \end{cases} \quad \textit{and} \quad \eta_2(\nu) = \begin{cases} 1 & \text{for } |\nu - b/2| \leq b/2\lambda \\ 0 & \text{for } |\nu - b/2| \geq b/2. \end{cases}$$

Then the following can be shown.

(a) The operator E is well-defined and satisfies $\|E\,\sigma\|_{\mathcal{H}_{Q_{ab}}} \asymp \|\sigma\|_{l^2}$.

(b) There exist nonnegative functions d_1 and d_2 on $\mathbb{R}$, decaying rapidly at infinity such that for all $g \in S_0'(\mathbb{R})$, $|Pg(x)| \leq \|g\|_{S_0'}\, d_1(x)$ and $|\widehat{Pg}(\xi)| \leq \|g\|_{S_0'}\, d_2(\xi)$.

2. Define the analysis operator $C_{\varphi_0}\colon L^2(\mathbb{R}) \longrightarrow l^2(\mathbb{Z}^2)$ by

$$C_{\varphi_0} f = \{\langle f\,,\, M_{k\lambda^2/a} T_{l\lambda^2/b}\varphi_0\rangle\}_{k,l\in\mathbb{Z}}$$

where $\varphi_0(x) = e^{-\pi x^2}$ is the Gaussian. It is well-known ([16, 19, 18]) that $\{M_{k\alpha}T_{l\beta}\,\varphi_0\}_{k,l\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$ for every $\alpha\beta < 1$. Consequently C_{φ_0} satisfies $\|C_{\varphi_0} f\|_{l^2} \asymp \|f\|_{L^2(\mathbb{R})}$ since $\lambda^2/a \cdot \lambda^2/b = \lambda^4/ab < 1$.

3. Now given $g \in S_0'(\mathbb{R})$, consider the composition operator

$$C_{\varphi_0}\circ \Phi_g \circ E\colon l_0(\mathbb{Z}^2) \longrightarrow l^2(\mathbb{Z}^2).$$

The crux of the proof lies in showing that this composition operator is not stable. Since C_{φ_0} and E are both bounded and stable, it follows that Φ_g cannot be stable. Since $g \in S_0'(\mathbb{R})$ was arbitrary, this completes the proof.

To complete this final step we examine the canonical bi-infinite matrix representation of the above defined composition operator, that is, the matrix $M = (m_{k',l',k,l})$ that satisfies

$$(C_{\varphi_0}\circ \Phi_g \circ E\;\sigma)_{k',l'} = \sum_{k,l} m_{k',l',k,l}\,\sigma_{k,l}.$$

It can be shown that M has the property that for some rapidly decreasing function $w(x)$,

$$|m_{k',l',k,l}| \leq w(\max\{|\lambda k' - k|, |\lambda l' - l|\}). \tag{4}$$

4. The final step in the proof is the following lemma. Its proof can be found in [14] and generalizations can be found in [17].

Lemma 3.7 *Given $M = (m_{j',j})_{j',j\in\mathbb{Z}^2}$. If there exists a monotonically decreasing function $w\colon R_0^+ \longrightarrow R_0^+$ with $w = O(x^{-2-\delta})$, $\delta > 0$, and constants $\lambda > 1$ and $K_0 > 0$ with $|m_{j',j}| < w(\|\lambda j' - j\|_\infty)$ for $\|\lambda j' - j\|_\infty > K_0$, then M is not stable.*

4 The operator class $\mathcal{H}_M$

The goal of this section is to extend results given in [14] to allow larger operator classes. Specifically we want to justify the selection

i. $X(\mathbb{R}) = S_0'(\mathbb{R})$,

ii. $\mathcal{H}_M = \{H : \eta_H \in S_0'(\mathbb{R}\times\widehat{\mathbb{R}})$ and $\operatorname{supp}\eta_H \subset M\}$ with $\|H\|_{\mathcal{H}} = \|\eta_H\|_{S_0'}$ and M compact, and

iii. $Y(\mathbb{R}) = S_0'(\mathbb{R})$

in the operator identification formalism described in Section 1. This choice has the following advantages:

1. $\mathcal{H}_M$ is closed subspace of $S_0'(\mathbb{R}\times\widehat{\mathbb{R}})$ with respect to the norm topology on $S_0'(\mathbb{R}\times\widehat{\mathbb{R}})$.

2. If M contains the origin then the class $\mathcal{H}_M$ contains the identity. This is clear since the spreading function for the identity operator is $\eta_{Id} = \delta_0 \otimes \delta_0$ which is clearly in $S_0'(\mathbb{R}\times\widehat{\mathbb{R}})$.

3. If $h \in S_0(\mathbb{R})$ is given with $\operatorname{supp} h \subseteq [0, a]$ for some $a > 0$ then the convolution operator H_h on $S_0'(\mathbb{R})$ given by
$$H_h f(x) = \int_0^a h(t)\, f(x-t)\, dt$$
is in the class $\mathcal{H}_M$ for any $M \in \mathbb{R}\times\widehat{\mathbb{R}}$ containing $[0, a] \times \{0\}$. Clearly the spreading function for H_h is $\eta_h = h \otimes \delta_0$. Hence the classes under consideration here contain all causal, finite-memory, translation invariant, linear channels.

Our goal is to find $g \in S_0'(\mathbb{R})$ such that $\|Hg\|_{S_0'(\mathbb{R})} \asymp \|\eta_H\|_{S_0'(\mathbb{R}\times\widehat{\mathbb{R}})}$. It turns out that $g = \perp\!\!\!\perp\!\!\!\perp$ also works in this setting. We begin with a classical kernel theorem for operators with $\eta_H \in S_0'(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$ (see [8], page 314).

Theorem 4.1

1. *Every $\eta \in S_0'(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$ defines a bounded operator $H_\eta : S_0(\mathbb{R}^d) \longrightarrow S_0'(\mathbb{R}^d)$ by setting*
$$\langle H_\eta g, f\rangle = \langle \eta, V_g f\rangle, \quad f, g \in S_0(\mathbb{R}^d).$$

2. *Conversely, for any bounded operator $H : S_0(\mathbb{R}^d) \longrightarrow S_0'(\mathbb{R}^d)$ there exists $\eta_H \in S_0'(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$ such that*
$$\langle Hg, f\rangle = \langle \eta_H, V_g f\rangle, \quad f, g \in S_0(\mathbb{R}^d).$$

Proposition 4.2

1. *For $f \in S_0(\mathbb{R}^d)$ and $g \in S_0'(\mathbb{R}^d)$, we have $V_g f \in W\big(A(\mathbb{R}^d\times\widehat{\mathbb{R}}^d), l^\infty\big) \subseteq A_{loc}(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$.*

2. *Any bounded linear operator $H : S_0(\mathbb{R}^d) \longrightarrow S_0'(\mathbb{R}^d)$ with $\operatorname{supp} \eta_H$ compact extends to a bounded linear operator $H : S_0'(\mathbb{R}^d) \longrightarrow S_0'(\mathbb{R}^d)$.*

Proof. Part *1* follows from

$$\|V_g f\|_{W(A,l^\infty)} \leq \|f\|_{S_0(\mathbb{R}^d)} \, \|g\|_{S_0'(\mathbb{R}^d)}$$

for $f \in S_0(\mathbb{R}^d)$, $g \in S_0'(\mathbb{R}^d)$ which is Lemma 4.1 for $p = q = \infty$ and $m = \nu \equiv 1$ in [2] .

To see part *2*, let $H \in \mathcal{H}_M$ with $\eta_H \in S_0'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d) = W\big(\mathcal{F}L^\infty(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), l^\infty\big)$, $\operatorname{supp} \eta_H \subseteq M$, M compact. Choose any $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\sum_n T_n\psi \equiv 1$, let

$$F = \{(n,m) \in \mathbb{Z}^{2d} : \ \operatorname{supp} T_n\psi \otimes T_m\psi \cap M \neq \emptyset\}$$

and note that F is a finite set since M is compact and ψ is compactly supported. The fact that

$$\|(T_n\psi \otimes T_m\psi)\, \eta_H\|_{\mathcal{F}L^\infty} \neq 0$$

for only finitely many (n,m) implies $\eta_H \in W\big(\mathcal{F}L^\infty(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), l^1\big)$, which, together with $V_g f \in W\big(A(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), l^\infty\big)$ gives $\langle Hg, f\rangle = \langle \eta_H, V_g f\rangle$ is well-defined.

Further, using Proposition 2.2, part *1*, we conclude that

$$\begin{aligned}
|\langle Hg, f\rangle| &= |\langle \eta_H, V_g f\rangle| \\
&= \Big|\Big\langle \eta_H, \sum_{(n,m)\in F} (T_n\psi \otimes T_m\psi) V_g f \Big\rangle\Big| \\
&\leq \sum_{(n,m)\in F} |\langle \eta_H, (T_n\psi \otimes T_m\psi) V_g f\rangle| \\
&\leq \|\eta_H\|_{S_0'} \sum_{(n,m)\in F} \|(T_n\psi \otimes T_m\psi) V_g f\|_{S_0} \\
&\leq C\, \|\eta_H\|_{S_0'} \sum_{(n,m)\in F} \ \sum_{(k,l)\in F} \|(T_k\psi \otimes T_l\psi)(T_n\psi \otimes T_m\psi) V_g f\|_A \\
&\leq C\, \|\eta_H\|_{S_0'} \Big(\sum_{(k,l)\in F} \|T_k\psi \otimes T_l\psi\|_A \Big) \Big(\sum_{(n,m)\in F} \|(T_n\psi \otimes T_m\psi) V_g f\|_A \Big) \\
&\leq C\, \|\eta_H\|_{S_0'}\, |F|\, \|\psi\|_A^2\, |F| \sup_{n,m} \|(T_n\psi \otimes T_m\psi) V_g f\|_A \\
&\leq C\, |F|^2\, \|\psi\|_A^2\, \|\eta_H\|_{S_0'}\, \|V_g f\|_{W(A,l^\infty)} \\
&\leq C\, |F|^2\, \|\psi\|_A^2\, \|\eta_H\|_{S_0'}\, \|f\|_{S_0}\, \|g\|_{S_0'}.
\end{aligned}$$

Taking the supremum over all $f \in S_0(\mathbb{R})$ on both sides gives

$$\|Hg\|_{S_0'} \leq C\, |F|\, \|\eta_H\|_{S_0'}\, \|g\|_{S_0'}. \tag{5}$$

□

Proposition 4.3 *Let M be compact. Any $g \in S_0'(\mathbb{R}^d)$ induces a bounded linear operator*

$$\Phi_g : \mathcal{H}_M \longrightarrow S_0'(\mathbb{R}^d), \quad H \mapsto Hg.$$

Proof. According to (5), we have

$$\|\Phi_g\|_{\mathcal{L}(\mathcal{H}_M,S_0')} \le C\|g\|_{S_0'}\ \|\eta\|_{S_0'}$$

where C depends on the volume of M. □

5 Identifiability of the class $\mathcal{H}_M$

Our initial goal is to show that the class $\mathcal{H}_M$ with M compact is identifiable if $M = [a,a'] \times [b,b']$ with $|M| < 1$. We will assume without loss of generality that $M \equiv M_\epsilon = [\epsilon, 1-\epsilon]^2$ for some $0 < \epsilon < 1/2$ (see [14]) and we will use the identifier

$$g = \sum_{n\in\mathbb{Z}} \delta_n \equiv \perp\!\!\perp\!\!\perp.$$

In particular we will show that the operator

$$\Phi_{\perp\!\perp\!\perp} : \mathcal{H}_M \longrightarrow S_0'(\mathbb{R}), \quad H \mapsto H\perp\!\!\perp\!\!\perp$$

satisfies

$$\|H\perp\!\!\perp\!\!\perp\|_{S_0'(\mathbb{R})} \asymp \|H\|_{\mathcal{H}_M} = \|\eta_H\|_{S_0'(\mathbb{R}\times\widehat{\mathbb{R}})}. \tag{6}$$

The proof will be an adjustment of the proofs in [14] and will rely on well-known properties of the Zak transform which was described in Proposition 2.4 and on the equality $\langle H\perp\!\!\perp\!\!\perp, f\rangle = \langle \eta_H, Zf\rangle$. In fact, in Section 3.3 we obtained $\|H\perp\!\!\perp\!\!\perp\|_{L^2} = \|\eta_H\|_{L^2} = \|H\|_{HS}$ simply from $\langle H\perp\!\!\perp\!\!\perp, f\rangle = \langle \eta_H, Zf\rangle$ and the unitarity of $Z : L^2(\mathbb{R}) \longrightarrow L^2(\,[0,1]^2)$. Now, an argument is needed to relate $\|H\perp\!\!\perp\!\!\perp\|_{S_0'}$ and $\|\eta_H\|_{S_0'}$. This is given in Lemma 5.1. Its usefulness stems from the fact that the action of $\eta_H \in S_0'(\mathbb{R}\times\widehat{\mathbb{R}})$ with $\operatorname{supp}\eta_H \subseteq M_\epsilon$ is completely determined by $\langle \eta_H, F\rangle$, $F \in M_{\epsilon/2} \cap C_c^\infty(\mathbb{R}\times\widehat{\mathbb{R}})$.

Note that the results in [14] are valid in higher dimensions and the proofs below can be generalized to this setting as well.

Lemma 5.1 *Let $F \in C_c^\infty(\mathbb{R}\times\widehat{\mathbb{R}})$ with $\operatorname{supp} F \subseteq M_{\epsilon/2}$. Then there is an $f \in S_0(\mathbb{R})$ such that $Zf = F$ on $[0,1]^2$ and a $C > 0$ independent of f such that,*

$$\|f\|_{S_0} \le C\,\|F\|_{S_0}. \tag{7}$$

Proof. With F as above, the quasiperiodic extension of F to all of $\mathbb{R}\times\widehat{\mathbb{R}}$ is in $C^\infty(\mathbb{R}\times\widehat{\mathbb{R}})$. Hence by Proposition 2.4, part *5*, there is an $f \in \mathcal{S}(\mathbb{R}) \subseteq S_0(\mathbb{R})$ such that $Zf = F$ on $[0,1]^2$.

In order to show (7), we expand F in an appropriately chosen $S_0(\mathbb{R}\times\widehat{\mathbb{R}})$–Gabor frame. This expansion will lead to an $S_0(\mathbb{R})$–Gabor frame expansion of f and a comparison of the corresponding l^1–coefficient sequences will give (7).

To this end, let $\phi_1 \in C_c^\infty(\mathbb{R})$ be a nonnegative function such that $\operatorname{supp}\phi_1 \subseteq [-\epsilon/2, 1+\epsilon/2]$, $\phi_1 \equiv 1$ on $[\epsilon/2, 1-\epsilon/2]$ and $\sum_n |T_n\phi_1|^2 \equiv 1$, and let $\phi_2 \in C_c^\infty(\mathbb{R})$ be a nonnegative function such that $\operatorname{supp}\phi_2 \subseteq [0,1]$, $\phi_2 \equiv 1$ on $[\epsilon/2, 1-\epsilon/2]$ and $\sum_n |T_{\alpha n}\phi_2|^2 \equiv 1$ where $\alpha = 1-\epsilon/2$. Note that $\{T_n\phi_1\}$ will be used to partition $\mathbb{R}$ while $\{T_{\alpha n}\phi_2\}$ will be used to partition $\widehat{\mathbb{R}}$.

It follows for example from Theorem 4.1.2 of [10] (see also [3]) that the collections

$$\{T_n M_{\beta k}\phi_1\}_{n,k\in\mathbb{Z}}$$

with $\beta = 1/(1+\epsilon)$ and

$$\{T_{\alpha n} M_k \phi_2\}_{n,k\in\mathbb{Z}}$$

are tight frames for $L^2(\mathbb{R})$. It is easy to see that this implies that the collection

$$\{T_{(n,\alpha m)} M_{(\beta k,l)}(\phi_1 \otimes \phi_2)\}_{(n,m),(k,l)\in\mathbb{Z}^2}$$

is a tight frame for $L^2(\mathbb{R}\times\widehat{\mathbb{R}})$. Since $\operatorname{supp} F \subseteq [\epsilon/2, 1-\epsilon/2]^2$, the canonical expansion of F with respect to this frame has the form

$$\begin{aligned} F &= \sum_{k,l} \big\langle F, T_{(n,\alpha m)}\, M_{(\beta k,l)}(\phi_1 \otimes \phi_2)\big\rangle\, T_{(n,\alpha m)}\, M_{(\beta k,l)}(\phi_1 \otimes \phi_2) \\ &= \sum_{k,l} c_{k,l}\, M_{(\beta k,l)}(\phi_1 \otimes \phi_2) \end{aligned}$$

and by Proposition 2.2, part *2*,

$$\|F\|_{S_0} \asymp \sum_{k,l} |c_{k,l}|.$$

For $n \in \mathbb{Z}$, set $f_n(x) = \int_0^1 F(x-n,\xi) e^{2\pi i n\xi}\, d\xi$. We have

$$\operatorname{supp} f_n \subseteq [\epsilon/2, 1-\epsilon/2] + \mathbb{Z}, \tag{8}$$

since for $x \in [-\epsilon/2, \epsilon/2] + \mathbb{Z}$ we have $F(x,\xi) = 0$ for all $\xi \in [0,1]$, implying that $f_n(x) = 0$. Further,

$$\begin{aligned} f_n(x) &= \int_0^1 F(x-n,\xi) e^{2\pi i n\xi}\, d\xi \\ &= \sum_{k,l} c_{k,l} \int_0^1 M_{(\beta k,l)}\phi_1(x-n)\phi_2(\xi)\, e^{2\pi i n\xi}\, d\xi \\ &= \sum_{k,l} c_{k,l} \int_0^1 M_{(\beta k,l+n)}\phi_1(x-n)\phi_2(\xi)\, d\xi = \sum_{k,l} c_{k,l} M_{\beta k} T_n \phi_1(x)\overline{\widehat{\phi_2}(l+n)} \end{aligned}$$

implies that $\operatorname{supp} f_n \subseteq [n-\epsilon/2\,,\, n+1+\epsilon/2]$, and, using (8), we conclude that actually

$$\operatorname{supp} f_n \subseteq [n+\epsilon/2\,,\, n+1-\epsilon/2].$$

For $x \in [n, n+1)$ we have $Zf(x,\xi) = F(x-n,\xi)e^{2\pi i n\xi}$ by definition, and Proposition 2.4, part *3*, implies then $f_n(x) = f(x)$ for $x \in [n, n+1]$. The disjointness of the supports of the f_n gives

$$f = \sum_n f_n = \sum_{k,n} \left(\sum_l c_{k,l} \overline{\widehat{\phi_2}(l+n)} \right) M_{\beta k} T_n \phi_1.$$

Since $\{M_{\beta k} T_n \phi_1\}_{n,k\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ and since $\phi_1 \in C_c^\infty(\mathbb{R})$ it follows from Proposition 2.2, part *2*, that

$$\|f\|_{S_0} \leq C \left\| \sum_l c_{k,l} \overline{\widehat{\phi_2}(l+n)} \right\|_{l^1} \leq C \, \|\{\widehat{\phi_2}(n)\}\|_{l^1} \, \|\{c_{k,l}\}\|_{l^1} \asymp \|F\|_{S_0}.$$

□

Theorem 5.2 *$\mathcal{H}_M$ is identifiable if $M = [a, a'] \times [b, b']$ and $|M| < 1$*

Proof. Without loss of generality, take $M = M_\epsilon = [\epsilon, 1-\epsilon]^2$ with $0 < \epsilon < \frac{1}{2}$. Since $\perp\!\perp\!\perp \in S_0'(\mathbb{R})$, Proposition 4.3 implies that $\Phi_{\perp\perp\perp} : \mathcal{H}_M \longrightarrow S_0'(\mathbb{R})$ is bounded.

We will show directly the existence of the lower bound in (6). Note that given $f \in S_0$,

$$\langle \Phi_{\perp\perp\perp} H, f \rangle = \langle \eta_H, V_{\perp\perp\perp} f \rangle = \langle \eta_H, Zf \rangle,$$

where Zf is the Zak transform of f.

Now, in light of the fact that $\operatorname{supp} \eta_H \subseteq M_\epsilon$,

$$\|\eta_H\|_{S_0'} = \sup_F |\langle \eta_H, F \rangle|$$

where the supremum is taken over all $F \in C_c^\infty(\mathbb{R}\times\widehat{\mathbb{R}}) \subseteq S_0(\mathbb{R}\times\widehat{\mathbb{R}})$ with $\|F\|_{S_0} = 1$ and $\operatorname{supp} F \subseteq M_{\epsilon/2}$. By Lemma 5.1, for all such F there is an $f \in S_0$ with $V_{\perp\perp\perp} f = F$ and $\|f\|_{S_0} \leq C \, \|F\|_{S_0(\mathbb{R}\times\widehat{\mathbb{R}})}$. Therefore, for each such F,

$$\begin{aligned} |\langle \eta_H, F \rangle| &= |\langle \eta_H, V_{\perp\perp\perp} f \rangle| = |\langle H \perp\!\perp\!\perp, f \rangle| = |\langle \Phi_{\perp\perp\perp} H, f \rangle| \\ &\leq \|\Phi_{\perp\perp\perp} H\|_{S_0'} \, \|f\|_{S_0} \leq C \, \|\Phi_{\perp\perp\perp} H\|_{S_0'} \, \|F\|_{S_0}. \end{aligned}$$

Taking the supremum of both sides over all such F gives the inequality

$$\|\eta_H\|_{S_0'} = \|H\|_{\mathcal{H}_M} \leq C \, \|\Phi_{\perp\perp\perp} H\|_{S_0'}$$

which is the lower bound of (1). □

Theorem 5.3 *$\mathcal{H}_M$ is not identifiable if $|M| > 1$.*

Proof. The proof proceeds in exactly the same way as the corresponding proof in [14] which was outlined in Section 3. The operator $P \in \mathcal{H}_M$ is defined exactly as before as is the embedding operator $E : l_0 \mapsto \mathcal{H}_M$. E is clearly well-defined. To see that it is also bounded and stable note that the spreading function of the operator $E\sigma$ is given by

$$\eta_{E\sigma}(t,\nu) = \eta_P(t,\nu) \sum_{k,l} \sigma_{k,l}\, e^{2\pi i (k\lambda t/a - l\lambda\nu/b)}.$$

We will consider the collection of functions

$$\{M_{(k\lambda/a, l\lambda/b)}\, \eta_P\}_{k,l}.$$

By a result in [21] (see Theorem 6.5.1 of [8]), for all $c > 0$ sufficiently large the collection

$$\{T_{(k\frac{a}{\lambda}, l\frac{b}{\lambda})} M_{(m/c, n/c)}\, \eta_P\}_{k,l,m,n} \tag{9}$$

is a frame for $L^2(\mathbb{R}^2)$ and since η_P is in S_0, so is its canonical dual $\widetilde{\eta}_P$ (see [9]). Applying now Theorem 3.6.4 of [7] (page 162) it follows that the system (9) forms an (S_0', l^∞)-frame and that the collection

$$\{T_{(cm, cn)} M_{(k\lambda/a, l\lambda/b)}\, \eta_P\}_{k,l,m,n}$$

is an l^∞-*Riesz projection basis in* $S_0'(\mathbb{R})$ (see [7], page 147 for the definition). This means in particular, by taking $(m,n) = (0,0)$ that

$$\left\| \sum_{k,l} \sigma_{k,l}\, M_{(k\lambda/a, -l\lambda/b)}\, \eta_P \right\|_{S_0'(\mathbb{R})} \asymp \|\sigma\|_{l^\infty}$$

which was to be proved.

The synthesis operator C_{φ_0} is also defined as before and it follows from Proposition 2.2, part *2*, that C_{φ_0} is bounded and stable as an operator from $S_0'(\mathbb{R})$ into l^∞.

The matrix M representing the composition $C_{\varphi_0} \circ \Phi_g \circ E$ is also the same and its entries satisfy the growth estimates (4). It therefore suffices to show that, as an operator from $l_0(\mathbb{Z}^2)$ to $l^\infty(\mathbb{Z}^2)$, M is unstable. This amounts to proving a lemma analogous to Lemma 3.7 (see Lemma 5.4 below). This completes the proof. □

Lemma 5.4 *Given* $M = (m_{j',j}) : l^\infty(\mathbb{Z}^d) \to l^\infty(\mathbb{Z}^d)$. *If there exists a monotonically decreasing function* $w \in S_0(\mathbb{R})$ *and constants* $\lambda > 1$ *and* $K_0 > 0$ *with* $|m_{i,j}| < w(\|\lambda j' - j\|_\infty)$ *for* $\|\lambda j' - j\|_\infty > K_0$, *then* M *is not stable.*

Proof. Fix $K \in \mathbb{N}$ such that

$$\sum_{\|j\|_\infty \geq K} w(\|j\|_\infty) < \epsilon,$$

and $N, \widetilde{N} \in \mathbb{N}$ with $N = \lfloor \lambda A \rfloor > K + A = \widetilde{N}$ for some $A \in \mathbb{N}$. We define

$$\widetilde{M} = (m_{j',j})_{\|j'\|_\infty \leq \widetilde{N}, \|j\|_\infty \leq N} : \mathbb{C}^{(2N+1)^d} \to \mathbb{C}^{(2\widetilde{N}+1)^d}.$$

The matrix $\widetilde{M}$ has a non-trivial kernel since $(2\widetilde{N}+1)^d < (2N+1)^d$, so we can choose $\widetilde{x} \in \mathbb{C}^{(2N+1)}$ with $\|\widetilde{x}\|_{l^\infty} = 1$ and $\widetilde{M}\widetilde{x} = 0$. Define $x \in l^\infty(\mathbb{Z}^d)$ according to $x_j = \widetilde{x}_j$ if $\|j\|_\infty \leq N$ and $x_j = 0$ else.

By construction we have $\|x\|_{l^\infty} = 1$, and $(Mx)_{j'} = 0$ for $\|j'\|_\infty \leq \widetilde{N}$. Hence $\|Mx\|_{l^\infty} = \sup\{(Mx)_{j'},\ \|j'\|_\infty > \widetilde{N}\} < \epsilon$, since for $\|j'\|_\infty > \widetilde{N}$, we have

$$\begin{aligned}
|(Mx)_j'| &= \left| \sum_{\|j\|_\infty \leq N} m_{j',j} x_j \right| \\
&\leq \|x\|_{l^\infty} \sum_{\|j\|_\infty \leq N} |m_{j',j}| \\
&\leq \sum_{\|j\|_\infty \leq N} w(\|\lambda j' - j\|_\infty) \\
&\leq \sum_{\|j\|_\infty \geq K} w(\|j\|_\infty) \\
&< \epsilon.
\end{aligned}$$

□

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