Sparse Finite Gabor Frames for Operator Sampling

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***Abstract*—We derive some interesting properties of finite Gabor frames and apply them to the *sampling* or *identification* of operators with bandlimited Kohn-Nirenberg symbols, or equivalently those with compactly supported spreading functions. Specifically we use the fact that finite Gabor matrices are full Spark for an open, dense set of window vectors to show the existence of periodically weighted delta trains that identify simultaneously large operator classes. We also show that sparse delta trains exist that identify operator classes for which the spreading support has small measure.**

I. Introduction

A. Operator Sampling

The goal in *operator identification* is to determine an operator completely from its action on a single input function or distribution. If the operator models a linear (time-varying) communication channel, then the problem is one of *channel identification* and can be thought of as a generalization of the fact that the impulse response of a time-invariant communication channel modeled as a convolution operator can be determined from the response of the channel to a unit impulse. The question of determining which operators can be identified was addressed in foundational and pioneering work of T. Kailath ([3], [4], [5]) and P. Bello ([1]) who determined that the identifiability of a communication channel is characterized by the area of the support of its so-called *spreading function*. This work has been extended and placed on a firm mathematical footing in [6] and [8].

To be specific and to fix ideas for this paper, we restrict our attention to the class of Hilbert-Schmidt operators H on $L^2(\mathbf{R})$. Any such operator can be represented as a pseudodifferential operator as

$$Hf(x) = \int \sigma_H(x,\xi)\widehat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi.$$

$\sigma_H(x,\xi) \in L^2(\mathbf{R}^2)$ is the *Kohn-Nirenberg* (KN) symbol of H. The spreading function $\eta_H(t,\nu)$ of the operator H is the *symplectic Fourier transform* of the KN symbol, viz.

$$\eta_H(t,\nu) = \iint \sigma_H(x,\xi)\, e^{-2\pi i(\nu x-\xi t)}\, dx\, d\xi$$

and we have the representation

$$Hf(x) = \iint \eta_H(t,\nu)\, T_t\, M_\nu f(x)\, d\nu\, dt$$

where $T_t f(x) = f(x-t)$ is the *time-shift operator* and $M_\nu f(x) = e^{2\pi i\nu x} f(x)$ is the *frequency-shift operator*. In this sense, an operator H whose spreading function has compact support can be said to have a *bandlimited symbol*. This motivates the following definition. Given a compact set $S \subseteq \mathbf{R}^2$, we define the *operator Paley-Wiener space* $OPW(S)$ to be the set of all Hilbert-Schmidt operators H on $L^2(\mathbf{R})$ with $\operatorname{supp}\eta_H \subseteq S$. Identifiability of an operator H therefore means informally that there exists a distribution g such that H is completely determined by Hg. To be more precise, suppose that $\mathcal{H}$ is some class of linear operators with common domain. We say that g *identifies* $\mathcal{H}$ if whenever $H_1, H_2 \in \mathcal{H}$ and $H_1 g = H_2 g$ (or equivalently $(H_1 - H_2)g = 0$) then $H_1 = H_2$. If $\mathcal{H}$ is a linear space, then g identifies $\mathcal{H}$ if and only if $H \in \mathcal{H}$ and $Hg = 0$ implies $H = 0$. However, these notions are not equivalent if $\mathcal{H}$ is not a linear space.

The following theorem was proved in [8] following Bello's work.

Theorem 1. If $|S| < 1$ then $OPW(S)$ is identifiable, and if $|S| > 1$ then $OPW(S)$ is not identifiable. In the former case an identifier has the form $g = \sum_n c_n\, \delta_{nT}$ for some $T > 0$ and periodic sequence $c = (c_n)$.

Since in this case, the operator is being "sampled" by a succession of evenly-spaced weighted impulses, and because the theory bears many formal analogies to the classical sampling of bandlimited functions, this procedure is called *operator sampling*. Indeed, it is shown in [9] that classical sampling is in fact a special case of operator sampling.

B. Gabor Matrices

Definition 2. Given $L \in \mathbf{N}$, let $\omega = e^{2\pi i/L}$ and define the *translation operator* T on $(x_0, \ldots, x_{L-1}) \in \mathbf{C}^L$ by

$$Tx = (x_{L-1}, x_0, x_1, \ldots, x_{L-2}),$$

and the *modulation operator* M on $\mathbf{C}^L$ by

$$Mx = (\omega^0 x_0, \omega^1 x_1, \ldots, \omega^{L-1} x_{L-1}).$$

Define the *full Gabor system matrix* $G(c)$ to be the $L \times L^2$ matrix given by

$$G(c) = [\ D_0\, W_L \mid D_1\, W_L \mid \cdots \mid D_{L-1}\, W_L\ ]$$

where D_k is the diagonal matrix with diagonal $T^k c$, and where W_L is the $L \times L$ Fourier matrix $W_L = (e^{2\pi i nm/L})_{n,m=0}^{L-1}$.

The *finite Gabor system with window* c is the collection $\{M^p T^q c\}_{q,p=0}^{L-1}$.

For generic vectors $c \in \mathbf{C}^L$, the finite Gabor system with window c is a tight frame for $\mathbf{C}^L$, and in fact is a so-called *full Spark frame*. In particular the following holds.

Theorem 3. [7] Suppose that L is prime. Then there is an open, dense set of $c \in \mathbf{C}^L$ with the property that every square submatrix of $G(c)$ has nonzero determinant. In particular, this implies that every collection of columns in $G(c)$ has full rank.

Outline of Proof: Given any $N \times N$ submatrix, M, of $G(c)$, $\det(M)$ is a homogeneous polynomial of degree L in the variables $c_0, c_1, \ldots, c_{L-1}$, and it is sufficient to show that this polynomial does not vanish identically, and for that it suffices to show that there is a monomial in $\det(M)$ with a nonzero coefficient. In the proof such a monomial, p_M, is constructed recursively as follows.

If $N = 1$ then M is simply a multiple of a single variable c_j and we define $p_M = c_j$. If $N > 1$, let c_j be the variable of lowest index appearing in M. Choose any entry of M in which c_j appears, eliminate from M the row and column containing that entry, and call the remaining matrix M'. Define $p_M = c_j\, p_{M'}$. The coefficient of p_M in $\det M$ is a product of minors of W_L. Since L is prime, by Chebotarëv's Theorem, such minors never vanish.

C. Operator Sampling and Gabor Matrices

To illustrate the connection between operator sampling and Gabor matrices, we outline the proof of the sufficiency direction of Theorem 1 ([8], [9]).

Suppose that $|S| < 1$ is compact, and assume without loss of generality that S is contained in the first quadrant. Choose L prime so large that $S \subseteq [0, \sqrt{L}]^2$ and S meets no more than L rectangles of the form

$$R_{q,m} = [0, 1/\sqrt{L}]^2 + (q/\sqrt{L}, m/\sqrt{L}).$$

In the sequel, let $T\Omega = 1/\sqrt{L}$. For any sequence $c = (c_n)$ with period L, a straightforward calculation ([9], [8]) yields

$$\overline{Z}(t, \nu) = G(c)\, \overline{\eta}(t, \nu) \tag{1}$$

where

$$\overline{Z}(t, \nu) = [e^{-2\pi i pq/L}\, e^{-2\pi i \nu T p}\, (Z_{1/\Omega} \circ H) g(t + Tp, \nu)]_{p=0}^{L-1},$$

$Z_{1/\Omega} f(t, \nu) = \sum_{n \in \mathbf{Z}} f(t - n/\Omega)\, e^{2\pi i n \nu/\Omega}$ is the *Zak transform*, and

$$\overline{\eta}(t, \nu) = [e^{-2\pi i qm/L}\, e^{-2\pi i \nu T q}\, \eta_H(t + Tq, \nu + \Omega m)]_{q,m=0}^{L-1}.$$

Note that (1) is a linear system of L equations in L^2 unknowns, the coefficients of which are a discrete Gabor system. Because S meets no more than L rectangles $R_{q,m}$, (1) reduces to a system of L equations in L unknowns, with the reduced matrix $G_0(c)$ an $L \times L$ submatrix of $G(c)$. We now invoke Theorem 3 to assert that there is a choice of $c \in \mathbf{C}^L$ making $G_0(c)$ invertible.

II. Operators with Unknown Spreading Support

Theorem 3 says that the set of vectors c for which every square submatrix of $G(c)$ is invertible is dense and open. Since there are only finitely many such submatrices, there exists a dense, open set of $c \in \mathbf{C}^L$ such that all square submatrices of $G(c)$ are invertible. Hence c can be chosen independently of S, depending only on L.

Definition 4. Given $\Sigma > 0$ and $0 \le \Delta \le 1$, define the operator class $\mathcal{H}_\Sigma(\Delta)$ to be the collection of operators H in $OPW([-\Sigma, \Sigma]^2)$ such that $\operatorname{supp} \eta_H$ is contained in no more than Σ Jordan regions (that is, Jordan curves together with their interiors) with total area no more than $\Delta - 1/\Sigma$, and whose boundaries have total length no more than Σ.

Note that $\mathcal{H}_\Sigma(\Delta)$ is not a linear space, but has the property that the spreading supports admit uniformly good coverings by squares. A more general version of the following theorem appears in [9] (see [2] for a characterization in the case of fixed L.).

Theorem 5. Let $\Sigma > 0$ be given. Then for every sufficiently large prime L, there is a $c \in \mathbf{C}^L$ such that with $g = \sum_n c_n\, \delta_{n/\sqrt{L}}$, if $H \in \mathcal{H}_\Sigma(1)$ and $Hg = 0$, then $H = 0$. It follows immediately that if $\Delta \le 1/2$, then whenever $H_1, H_2 \in \mathcal{H}_\Sigma(\Delta)$, and $H_1 g = H_2 g$ then $H_1 = H_2$.

Proof: An argument in [9] shows that if $H \in \mathcal{H}_\Sigma(1)$, then the conditions in Definition 6 on $\operatorname{supp} \eta_H$ imply that as long as $1/\sqrt{L} + 1/L < 1/(4\Sigma^2)$, then $\operatorname{supp} \eta_H$ is guaranteed to meet at most L rectangles $R_{q,m}$. Since L is now fixed, we can choose $c \in \mathbf{C}^L$ with the property that all square submatrices of $G(c)$ are invertible. This combined with (1) implies the result.

The conclusion of Theorem 5 is not sufficient by itself to allow the recovery from Hg of the spreading function of H. However, it is shown in [9] and in [2] that if $\Delta \le 1/2$, and $H \in \mathcal{H}_\Sigma(\Delta)$, the support set of H can be determined and H can be stably recovered from Hg. Heckel and Boelcskei go further in [2] and show that for almost every operator $H \in \mathcal{H}_\Sigma(\Delta)$ with $\Delta \le 1$, the support set of H can be determined and H can be stably recovered from Hg. Once the support set is known, explicit formulas for reconstructing the spreading function and impulse response of H from Hg are given in [9].

III. Efficient Operator Sampling

It is easy to see that any c satisfying the conclusion of Theorem 3 must have full support, that is, $\|c\|_0 = L$ where $\|c\|_0$ is the number of nonzero entries in c. However, for a given operator class, there is an advantage to choosing a c that has minimal support. First, having some or most of the c_k vanish would mean that the matrix $G(c)$ in (1) would be sparse, and hence the reduced matrix $G_0(c)$ that must be inverted to recover the spreading function would be sparse as well. In fact, the quantity $\|c\|_0/L$ is the fraction of nonzero entries in $G(c)$. Second, a vector c with small support would mean that the identifier g would require fewer deltas to be transmitted per unit time. This is analogous to reducing the "sampling rate" in operator sampling. Third, note that

$$\operatorname{supp} Hg(x) \subseteq \bigcup_{y \in supp(g)} \operatorname{supp} \kappa(x, y)$$

and hence that if $\|c\|_0$ is small, in particular if in each period c vanishes but for a few contiguous indices, then in each time interval of length TL, Hg would have small support thereby reducing the amount of time spent measuring the channel.

A. Invertibility of Gabor Submatrices.

Definition 6. Let $G(c)$ be an $L \times L^2$ Gabor system matrix, and let $G_0(c)$ be an $L \times N$ submatrix of $G(c)$ corresponding to a collection of $N \leq L$ columns of $G(c)$. Define

$$\mu = \min\{\|c\|_0 \colon G_0(c) \text{ has full rank}\}.$$

We associate to $G_0(c)$ the L-tuple $\tau = (\tau_0, \tau_1, \ldots, \tau_{L-1})$, where τ_k is the number of columns of $G_0(c)$ chosen from the submatrix $D_k W_L$. The total number of columns chosen is given by $\|\tau\|_1$, the number of submatrices $D_k W_L$ from which any columns are chosen by $\|\tau\|_0$, and the largest number of columns chosen from any submatrix $D_k W_L$ by $\|\tau\|_\infty$.

Part (1) of the following theorem is proved in [9].

Theorem 7 Suppose that the L-vector τ describes a collection of columns chosen from a full Gabor matrix.

(1) If L is prime then $\mu \leq (\|\tau\|_1 - \|\tau\|_0) + 1$.
(2) For any $L \in \mathbf{N}$, $\mu \geq \|\tau\|_\infty$.

Proof: (1) Let L be prime, and assume that columns are chosen from $G(c)$ according to the vector τ. By definition, there will be at least one column chosen from $\|\tau\|_0$ distinct submatrices $D_k W_L$ of $G(c)$. This means that there are exactly $\|\tau\|_0$ distinct rows in which the variable c_0 formally appears. Choose those rows and the remaining $\|\tau\|_1 - \|\tau\|_0$ rows arbitrarily, and let M be the resulting $\|\tau\|_1 \times \|\tau\|_1$ submatrix. Proceeding now with the construction of the monomial p_M defined above, it follows that p_M will contain exactly $\|\tau\|_0$ factors of c_0 plus at most $\|\tau\|_1 - \|\tau\|_0$ other distinct factors. Hence p_M will be a monomial with at most $\|\tau\|_1 - \|\tau\|_0 + 1$ distinct variables appearing. Hence the variables not chosen can be set to zero and the polynomial $\det M$ will still not vanish identically. Hence there is a choice of c with $\|c\|_0 \leq \|\tau\|_1 - \|\tau\|_0 + 1$ for which $\det M \neq 0$, and the result follows.

(2) Let $L \in \mathbf{N}$ be given and suppose that columns are chosen from $G(c)$ according to the vector τ. Let $\|\tau\|_1$ rows be chosen from the submatrix $G_0(c)$, and call the resulting $\|\tau\|_1 \times \|\tau\|_1$ matrix M. Any diagonal of M must have τ_k entries chosen from τ_k distinct rows of each submatrix $D_k W_L$. Hence every term in the expansion of $\det(M)$ is a multiple of a monomial with at least τ_k distinct variables appearing in it. Therefore, if more than $\|\tau\|_\infty$ of the c_k are zero, then the polynomial $\det(M)$ will vanish identically. Hence $\mu \geq \|\tau\|_\infty$.

Remark (a) The bounds on μ in Theorem 7 cannot be improved. For example, if one column is chosen from distinct submatrices $D_k W_L$, then the vector τ will have $\|\tau\|_1$ non-zero entries each of which is 1 and . Hence $\|\tau\|_1 = \|\tau\|_0$, and $\|\tau\|_\infty = 1$. Letting $c_0 = 1$, $c_1 = c_2 = \cdots = c_{L-1} = 0$, and letting the rows of M be those of $G_0(c)$ in which c_0 appears gives

$$\mu = \|\tau\|_\infty = (\|\tau\|_1 - \|\tau\|_0) + 1.$$

If all $\|\tau\|_1$ columns are chosen from one submatrix $D_k W_L$, then $\|\tau\|_0 = 1$ and $\|\tau\|_1 = \|\tau\|_\infty$. If fewer than $\|\tau\|_1$ of the c_k are nonzero, then any choice of $\|\tau\|_1$ rows of $G_0(c)$ will contain at least one identically zero row. This means that

$$\mu \geq (\|\tau\|_1 - \|\tau\|_0) + 1 = \|\tau\|_1 = \|\tau\|_\infty.$$

Moreover, if L is prime we once again have equality ([7]).

(b) The following example will show that for arbitrarily large L there are vectors τ that avoid both extremes, that is, for any choice of submatrix $G_0(c)$, $\|\tau\|_\infty < \mu < \|\tau\|_1 - \|\tau\|_0 + 1$. More specifically, the following theorem holds.

Theorem 8 For every $L \in \mathbf{N}$ large enough, there is an L-vector τ describing a choice of columns of a full Gabor matrix $G(c)$ such that $\|\tau\|_\infty < \mu$. Moreover, if L is prime, then also $\mu < \|\tau\|_1 - \|\tau\|_0 + 1$.

Proof: In order to construct this vector τ, first choose P, $R \in \mathbf{N}$ such that $P \leq R$ and

$$\frac{R + P - 1}{RP} < \frac{1}{2}.$$

Note that these imply that at least $R \geq P \geq 3$. Given $L \in \mathbf{N}$ with $L \geq 9$, we can write $L = PR + j$ uniquely for some $0 \leq j \leq R - 1$. Define the L-vector τ as follows. Let $\tau_k = 2$ for $0 \leq k \leq R-1$, and for $k = mR - 1$, $2 \leq m \leq P$, and let $\tau_k = 0$ otherwise. Then $\|\tau\|_0 = R + P - 1$, $\|\tau\|_\infty = 2$, and $\|\tau\|_1 = 2(R + P - 1)$. We will show that $\|\tau\|_\infty = 2 < 3 \leq \mu$ and that in case L is also prime, $\mu \leq R < R + P = \|\tau\|_1 - \|\tau\|_0 + 1$. We describe the matrix $G_0(c)$ pictorially in the figure below. Each column in the figure that starts with $N - k$ represents two columns chosen from the submatrix $D_k W_L$. A generalized diagonal of the matrix $G_0(c)$ corresponds to the choice of two indices from each column and one from each row.

$$\begin{array}{ccccccc}
0 & N-1 & \ldots & N-R+1 & N-2R+1 & \ldots & N-PR+1 \\
1 & 0 & \ldots & N-R+2 & N-2R+2 & \ldots & N-PR+2 \\
2 & 1 & \ldots & N-R+3 & N-2R+3 & \ldots & N-PR+3 \\
3 & 2 & \ldots & N-R+4 & N-2R+4 & \ldots & N-PR+4 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
R-2 & R-3 & \ldots & N-1 & N-R-1 & \ldots & N-(P-1)R-1 \\
R-1 & R-2 & \ldots & 0 & N-1 & \ldots & N-(P-2)R-1 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
2R-1 & 2R-2 & \ldots & R & 0 & \ldots & N-(P-3)R-1 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
3R-1 & 3R-2 & \ldots & 2R & R & \ldots & N-(P-4)R-1 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
PR-1 & PR-2 & \ldots & (P-1)R & (P-2)R & \ldots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
N-1 & N-2 & \ldots & N-R & N-2R & \ldots & N-PR
\end{array},$$

In order to see the first inequality, let $G_0(c)$ be an $L \times 2(R + P - 1)$ matrix described by τ. Specifically, we choose 2 columns from each submatrix $D_k W_L$ of $G(c)$ for all those

k for which $\tau_k = 2$. Now suppose that $\|c\|_0 = 2$ and assume without loss of generality that c_0 and c_{k_0} are the only non-zero entries of c. We will show that any choice of $2(R+P-1)$ rows of $G_0(c)$ will contain a zero row, which will imply that $\mu \geq 3$.

Note that each of the variables c_0 and c_{k_0} appears in at most $R+P-1$ rows of $G_0(c)$. Therefore, if a choice of $2(R+P-1)$ rows of $G_0(c)$ were not to contain a zero row, then we must be able to choose $R+P-1$ rows containing c_0 and an additional $R+P-1$ rows containing c_{k_0}. We will show that this is not possible by showing that there must be at least one row of $G_0(c)$ in which both c_0 and c_{k_0} appear. Specifically, we will show that all of the variables $c_1, c_2, \ldots, c_{L-1}$ appear at least once in the first R rows of $G_0(c)$. Clearly, c_0 also appears in each of these rows.

In the pair of columns of $G_0(c)$ chosen from the matrix D_0W_L, the variables $c_1, \ldots, c_{R-1}$ appear in the first R rows. Given $1 \leq m \leq P$, consider the pair of columns of $G_0(c)$ chosen from the matrix $D_{mR-1}W_L$. It is not hard to see that in the first R rows of these columns, the variables $c_{(P-m)R+j+1}, \ldots, c_{P-(m-1))R+j}$ appear. Consequently, as m runs from 1 through P, all of the variables $c_{j+1}, \ldots, c_{PR+j}$ will appear in the first R rows of $G_0(c)$. This completes the first part of the proof.

Now suppose that L is prime. We will show that $\mu \leq R$ by showing that we can choose $2(R+P-1)$ rows of $G_0(c)$ in such a way that the monomial p_M of the resulting square matrix M, as described in the proof of Theorem 3, will have no more than R distinct variables c_j appearing in it.

First, choose the $R+P-1$ rows of $G_0(c)$ in which c_0 appears. For all $1 \leq m \leq P-1$, note that c_1 appears in row $mR+1$, c_2 appears in row $mR+2$ and in general, c_k appears in row $mR+k$ for $k = 1, 2, \ldots, R-1$. Note also that c_0 does not appear in these rows. Therefore, choose those $(P-1)(R-1)$ rows of $G_0(c)$. Note that $(R+P-1)+(P-1)(R-1) = RP > 2(R+P-1)$ by our assumption at the beginning of the proof. This means that by choosing rows in this way, and eliminating some if necessary, we arrive at a square sub-matrix M of $G_0(c)$. The corresponding monomial p_M will have $R+P-1$ factors of c_0 and at most $P-1$ factors of $c_1, c_2, \ldots, c_{R-1}$, resulting in no more than R distinct variables appearing in p_M. Hence $\mu \leq R < R+P = \|\tau\|_1 - \|\tau\|_0 + 1$.

Theorem 9. Let L prime be fixed, and let $N \leq L$. There exists a $c \in \mathbf{C}^L$ with $\|c\|_0 \leq N$ such that for any vector τ with $\|\tau\|_1 = N$ and every $L \times N$ matrix $G_0(c)$ with associated distribution vector τ has full rank. In fact, the collection of all such c considered as vectors in $\mathbf{C}^N$ constitutes a dense, open subset of $\mathbf{C}^N$.

Proof: By Theorem 7, for every vector τ with $\|\tau\|_1 = N$, there is a $c \in \mathbf{C}^L$ with the property that $\|c\|_0 \leq N$ and that $G_0(c)$ has full rank. We will first show that such a c can always be chosen such that $c_N = c_{N+1} = \cdots = c_{L-1} = 0$. To see this, consider a matrix $G_0(c)$, and set c_N through c_{L-1} to zero. In this case, every column of $G_0(c)$ will have N nonvanishing entries. We can therefore follow the algorithm outlined in the proof of Theorem 3 and observe that at each step in the algorithm, there will always be a row of the remaining matrix in which a variable c_j with $0 \leq j \leq N-1$ appears, for if not, this would imply that one of the columns of $G_0(c)$ had fewer than N nonvanishing entries. Choosing now these N rows, and letting M denote the resulting $N \times N$ submatrix of $G_0(c)$, it follows that in the monomial p_M, only variables c_j with $0 \leq j \leq N-1$ will appear and hence the polynomial $\det M$ will be a homogeneous polynomial of degree N in the variables $c_0, c_1, \ldots, c_{N-1}$.

Therefore, any choice of $c_0, c_1, \ldots, c_{N-1}$ that avoids the zero set of the polynomial $\det M$ will ensure that $G_0(c)$ has full rank. The set of such choices constitutes a dense open set in $\mathbf{C}^N$. Since there are only finitely many vectors τ with $\|\tau\|_1 = N$ and only finitely many associated $L \times N$ matrices $G_0(c)$, the collection of such c is the intersection of finitely many dense open subsets of $\mathbf{C}^N$. Since this is also a dense open set, the result follows.

IV. Implications for Operator Sampling

Theorem 10. Let $\Sigma > 0$, $0 \leq \Delta < 1$ be given. Then for every sufficiently large prime L, there is a $c \in \mathbf{C}^L$ with $\|c\|_0/L \leq \Delta$ such that the operator class $\mathcal{H}_\Sigma(\Delta)$ is identifiable by $g = \sum_n c_n \, \delta_{n/\sqrt{L}}$.

Proof. As in the proof of Theorem 5, we can choose a prime L sufficiently large that for any $H \in \mathcal{H}_\Sigma(\Delta)$, $\operatorname{supp} \eta_H$ meets at most ΔL rectangles $R_{q,m}$. For this L, we have seen that it is possible to choose $c \in \mathbf{C}^L$ such that $\|c\|_0 \leq \Delta L$ and such that any collection of no more than ΔL columns of the Gabor matrix $G(c)$ is linearly independent. Hence the operator H is completely determined by Hg where $g = \sum_n c_n \delta_{n/\sqrt{L}}$ and $\|c\|_0/L \leq \Delta$.

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