Regular Operator Sampling for Parallelograms

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***Abstract*—Operator sampling considers the question of when operators of a given class can be distinguished by their action on a single probing signal. The fundamental result in this theory shows that the answer depends on the area of the support S of the so-called spreading function of the operator (i.e., the symplectic Fourier transform of its Kohn-Nirenberg symbol). $|S| < 1$ then identification is possible and when $|S| > 1$ it is impossible. In the critical case when $|S| = 1$, the picture is less clear. In this paper we characterize when *regular* operator sampling (that is, when the probing signal is a periodically-weighted delta train) is possible when S is a parallelogram of area 1.**

I. Operator Sampling and the Spreading Function

The goal in *operator identification* is to determine an operator completely from its action on a single input function or distribution. If the operator models a linear communication channel, then the problem is one of *channel identification*. It is well-known that a time-invariant communication channel is completely determined by its action on a unit impulse. Identification of time-variant channels has its roots in the work of T. Kailath and P. Bello ([5], [6], [7], [1]), and has been significantly extended in the past decade by the authors and others (see [8], [12], [16]). If the input function g is a weighted delta-train, viz. $g = \sum_{\lambda\in\Lambda} c_\lambda\, \delta_\lambda$ where $\Lambda \subseteq \mathbb{R}$ is discrete, then we refer to operator identification as *operator sampling*, and if g is a periodically-weighted delta train of the form $g = \sum_n c_n\, \delta_{nT}$ for some $T > 0$ and periodic sequence $c = (c_n)$, we refer to operator identification as *regular operator sampling*. Regular operator sampling is a generalization of classical sampling in the case when the operator class in question is the class of multiplication operators (see [13]).

In this paper we restrict our attention to the class of Hilbert-Schmidt operators H on $L^2(\mathbb{R})$. Any such operator can be represented as a pseudodifferential operator as

$$Hf(x) = \int \sigma_H(x,\xi)\widehat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi.$$

$\sigma_H(x,\xi) \in L^2(\mathbb{R}^2)$ is the *Kohn-Nirenberg* (KN) symbol of H. The *spreading function* $\eta_H(t,\nu)$ of the operator H is the *symplectic Fourier transform* of the KN symbol, viz.

$$\eta_H(t,\nu) = \iint \sigma_H(x,\xi)\, e^{-2\pi i(\nu x - \xi t)}\, dx\, d\xi$$

and we have the representation

$$Hf(x) = \iint \eta_H(t,\nu)\, \mathcal{T}_t\, \mathcal{M}_\nu f(x)\, d\nu\, dt$$

where $\mathcal{T}_t f(x) = f(x-t)$ is the *time-shift operator* and $\mathcal{M}_\nu f(x) = e^{2\pi i\nu x}\, f(x)$ is the *frequency-shift operator*. Given a set $S \subseteq \mathbb{R}^2$, we define the *operator Paley-Wiener space* $OPW(S)$ to be the set of all Hilbert-Schmidt operators H on $L^2(\mathbb{R})$ with $\operatorname{supp} \eta_H \subseteq S$.

Definition 1.1: Given $S \subseteq \mathbb{R}^2$, we say that the operator class $OPW(S)$ is *weakly identifiable* if there exists a distribution g such that every $H \in OPW(S)$ is completely determined by Hg, that is, if the operator $\Phi_g \colon OPW(S) \to L^2(\mathbb{R})$ given by $H \mapsto Hg$ is injective. We say that $OPW(S)$ is *strongly identifiable* if there exist constants A, $B > 0$ such that for all $H \in OPW(S)$,

$$A\|H\|_{HS} \le \|Hg\|_2 \le B\,\|H\|_{HS}. \tag{1}$$

Note that strong identifiability implies that the operator H depends in a stable way on the output Hg.

The following theorem was proved in [12], (see also [13]).

Theorem 1.2: Let $S \subseteq \mathbb{R}^2$ be compact. If $|S| < 1$ then $OPW(S)$ is strongly identifiable by regular operator sampling, and if $|S| > 1$ then $OPW(S)$ is not weakly identifiable.

II. Regular operator sampling in the $|S| < 1$ case

Here we will outline the proof of the sufficiency part of Theorem 1.2 as it appears in [13]. The proof relies on the notion of a *rectification* of the set S.

Definition 2.1: Let $S \subseteq \mathbb{R}^2$, $|S| \le 1$, $T > 0$, and $L \in \mathbb{N}$ be given, and let $\Omega = 1/(LT)$. We say that S admits a (T, L)-*rectification* if

(a) S is contained in a fundamental domain of the lattice $(1/\Omega)\mathbb{Z} \times (1/T)\mathbb{Z}$, and
(b) the set

$$S^\circ = \bigcup_{(k,\ell)\in\mathbb{Z}^2} S + (k/\Omega, \ell/T) \tag{2}$$

meets at most L rectangles of the form $R_{q,m} = [0,T] \times [0,\Omega] + (qT, m\Omega)$, $0 \le q, m < L$.

Note that if S is compact then for all $T > 0$ sufficiently small and $L \in \mathbb{N}$ sufficiently large, Definition 2.1(a) is

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satisfied. Also note that since $|S| < 1$, for all $T > 0$ sufficiently small and $L \in \mathbb{N}$ sufficiently large, S can be covered by rectangles from a $T \times \Omega$ grid whose total area is less than 1. By choosing T and L for which both hold it follows that a (T, L)-rectification for S exists.

Next we have the following lemma that uses the Zak transform, a fundamental tool of time-frequency analysis (see [3]).

Definition 2.2: The non-normalized Zak Transform is defined for $f \in \mathcal{S}(\mathbb{R})$, and $a > 0$ by

$$Z_a f(t,\nu) = \sum_{n\in\mathbb{Z}} f(t - an)\, e^{2\pi i a n \nu}.$$

Now we define a variant of the periodization of a bivariate function which arises naturally in operator sampling, called the *quasiperiodization*. The fundamental property of the quasiperiodization that it shares with the ordinary periodization is that a function supported in a fundamental domain of a lattice can be recovered from its quasiperiodization with respect to that lattice (see [13]).

Definition 2.3: Given a bivariate function $f(t,\nu)$ and parameters $T, \Omega > 0$, define the *$(1/\Omega,\, 1/T)$–quasiperiodization* of f, denoted f^{QP}, by

$$f^{QP}(t,\nu) = \sum_k \sum_\ell f(t + k/\Omega, \nu + \ell/T)\, e^{-2\pi i \nu k/\Omega} \tag{3}$$

whenever the sum is defined.

Lemma 2.4: Let $T, \Omega > 0$ be given such that $T\Omega = 1/L$ for some $L \in \mathbb{N}$, let (c_n) be a period-L sequence. Then with $g = \sum_n c_n\, \delta_{nT}$, $(t,\nu) \in \mathbb{R}^2$, and $p = 0,\, 1,\, \ldots,\, L{-}1$,

$$\begin{aligned} &e^{-2\pi i \nu T p}\, (Z_{1/\Omega} \circ H) g(t + Tp, \nu) \\ &= \Omega \sum_{q,\, m=0}^{L-1} (\mathcal{T}^q\, \mathcal{M}^m c)_p\, e^{-2\pi i \nu T q}\, \eta_H^{QP}(t + Tq, \nu + \Omega m). \end{aligned} \tag{4}$$

Here $\mathcal{T}$ as an operator on $\mathbb{C}^L$ represents a shift of indices modulo L, that is,

$$\mathcal{T}(x_0,\, x_1,\, \ldots,\, x_{L-1}) = (x_{L-1},\, x_0,\, x_1,\, \ldots,\, x_{L-2})$$

and $\mathcal{M}$ as an operator on $\mathbb{C}^L$ represents modulation, that is, with $\omega = e^{2\pi i/L}$

$$\mathcal{M}(x_0,\, x_1,\, \ldots,\, x_{L-1}) = (\omega^0 x_0,\, \omega^1 x_1,\, \ldots,\, \omega^{L-1} x_{L-1}).$$

In what follows, we will frequently abuse notation by identifying a vector $c \in \mathbb{C}^L$ with the period-L sequence $c = (c_n)$ in the obvious way.

Letting

$$\mathbf{Z}_{Hg}(t,\nu)_p = (Z_{1/\Omega} \circ H) g(t + pT, \nu)\, e^{-2\pi i \nu p T} \tag{5}$$

and

$$\boldsymbol{\eta}_H(t,\nu)_{(q,m)} = \Omega\, \eta_H^{QP}(t + qT, \nu + m\Omega)\, e^{-2\pi i \nu q T}\, e^{-2\pi i q m/L}, \tag{6}$$

we have that

$$\mathbf{Z}_{Hg}(t,\nu)_p = \sum_{q,m=0}^{L-1} G(c)_{p,(q,m)}\, \boldsymbol{\eta}_H(t,\nu)_{(q,m)} \tag{7}$$

where $G(c)$ is the $L \times L^2$ *Gabor system matrix* given by $[G(c)]_{p,(q,m)} = (T^q\, M^m c)_p$.

Identifiability of $OPW(S)$ thus reduces to the question of whether the underdetermined linear system (7) can be solved. By restricting (t,ν) to the basic rectangle $[0,T] \times [0,\Omega]$, we observe that at each point, at most L entries of $\boldsymbol{\eta}_H(t,\nu)_{(q,m)}$ do not vanish, so that solving (7) reduces to solving an $L \times L$ linear system. One piece of the puzzle remains, namely ensuring that this system is always solvable. This is the content of the following lemma.

Lemma 2.5: ([9], [10]) For every $L \in \mathbb{N}$ there exists a dense, open subset of $c \in \mathbb{C}^L$ such that every $L \times L$ submatrix of $G(c)$ has full rank.

III. Regular operator sampling and lattice tilings.

The following characterization of operator identification by regular operator sampling appears in [13].

Theorem 3.1: Let $g = \sum_{n\in\mathbb{Z}} c_n \delta_{nT}$ with period L sequence $c = (c_n)$ chosen so that every $L \times L$ submatrix of $G(c)$ has full rank and let $\Omega = 1/(LT)$. For $S \subseteq \mathbb{R}^2$ the following are equivalent.

(i) The operator class $OPW(S)$ is weakly identifiable by regular operator sampling with identifier g.

(ii) The operator class $OPW(S)$ is strongly identifiable by regular operator sampling with identifier g.

(iii) S is a subset of a fundamental domain of the lattice $(1/\Omega)\mathbb{Z} \times (1/T)\mathbb{Z}$, that is,

$$\sum_{k,\ell} \chi_{S+(k/\Omega,\ell/T)} \le 1 \quad \textit{a.e.} \tag{8}$$

and S periodized by the lattice $T\mathbb{Z} \times \Omega\mathbb{Z}$ is at most an L-cover, that is

$$\sum_{k,\ell} \chi_{S+(kT,\ell\Omega)} \le L \quad \textit{a.e.} \tag{9}$$

It is clear that (8) and (9) are satisfied if S admits a (T, L)-rectification, but the converse is not true (see [13]). The key observation here is that if S admits a (T, L)-rectification then the linear system (7) reduces to the same $L \times L$ submatrix of $G(c)$ for each (t,ν). However (8) and (9) allow for the linear system (7) to change depending on the point (t,ν).

Note that (9) implies that $|S| \le 1$. This by itself however is not sufficient for S to be identifiable by regular operator sampling. Indeed there are examples of sets S with arbitrarily small area such that (8) is not satisfied for any choice of $T > 0$ or $L \in \mathbb{N}$ (see [13]). However, Theorem 3.1 implies that under the assumption that S is compact, $|S| < 1$ suffices for strong identifiability by regular operator sampling. This leaves open the question of identifiability when $|S| = 1$.

In this case, it is easy to show that equality must hold in (9). This in turn is equivalent to the statement that the collection of sets $\{S + (kT, \ell\Omega) \colon k,\, \ell \in \mathbb{Z}\}$ forms an exact L-cover of the plane. The case of interest to us in this paper is when S is a parallelogram, that is, when there exists an invertible 2×2 matrix A such that $S = A[0,1]^2$. If in addition $|S| = 1$, then equality in (9) implies that

$$\begin{aligned} L &= \sum_{k,\ell} \chi_{A[0,1]^2 + (kT,\ell\Omega)}(Ax) \\ &= \sum_{k,\ell} \chi_{[0,1]^2 + A^{-1}(kT,\ell\Omega)}(x). \end{aligned}$$

Therefore, in the parallelogram case, (9) is equivalent to the statement that

$$\left\{ [0,1]^2 + A^{-1} \begin{pmatrix} T & 0 \\ 0 & \Omega \end{pmatrix} \mathbb{Z}^2 \right\} \qquad (10)$$

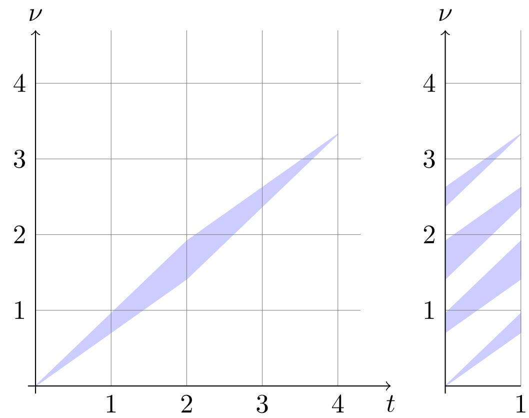
forms an exact L-cover of the plane. Since the cover involves only shifts by the lattice $A_0\mathbb{Z}^2$ where $A_0 = A^{-1} \left(\begin{smallmatrix} T & 0 \\ 0 & \Omega \end{smallmatrix} \right)$, it forms what is known as an *L-fold lattice tiling of the plane*. A considerable literature exists on this subject, going back to a conjecture of Minkowski [11], equivalent to the statement that any lattice tiling of $\mathbb{R}^n$ by unit cubes contains two cubes that share an $(n-1)$-dimensional face. This result was proved by Hajós in [4] and in addition he proved that the Minkowski conjecture also holds for all L-fold lattice tilings in dimension $n \leq 3$. This latter result was in fact proved a few years earlier by Furtwängler [2]. For the purposes of this paper, we need only that for an L-fold lattice tiling of $\mathbb{R}^2$, two squares must share an edge. For more information on this topic, see for example [15], [14], [17].

IV. Regular operator sampling for parallelograms

In [13] some attention is given to the case of operator sampling when the spreading support S is a parallelogram or can be rectified by parallelograms. In that paper an example is given in which S is a parallelogram with $|S| = 1$ such that $OPW(S)$ can be identified by operator sampling but not by regular operator sampling (See Figure 1). Specifically, if $A = \left(\begin{smallmatrix} 2 & 2 \\ \sqrt{2} & \sqrt{2}+1/2 \end{smallmatrix} \right)$, then $OPW(A[0,1]^2)$ can be identified by a non-periodically-weighted delta train, but not by a periodically-weighted delta train for any value of T and L. The main result of this paper characterizes when regular operator sampling of $OPW(S)$ is possible when S is a parallelogram of unit area.

Theorem 4.1: Suppose that $S = A[0,1]^2$ where $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix} \right)$ with $\det(A) = 1$. Then there exist $T > 0$ and $L \in \mathbb{N}$ such that (8) and (9) hold if and only if $a_{11}a_{21}$ or $a_{21}a_{22}$ is rational.

Proof: $(\Longrightarrow)$. Suppose that (9) holds for some $T > 0$ and $L \in \mathbb{N}$ and let Ω satisfy $T\Omega = 1/L$. As observed above, (9) is equivalent to the statement that (10) forms an L-fold lattice tiling of the plane by unit squares, and hence two such squares must share an edge.



(a) (b)

Fig. 1. (a) The the operator class $OPW^2(S)$ with $S = (2,\ 2\ ;\ \sqrt{2},\ \sqrt{2} + 1/2)[0,1]^2$ whose area equals 1 is identifiable by a (non-periodically) weighted delta train. It is not identifiable using regular operator sampling. (b) $T = 1$ periodization of S. For periodic operator sampling to succeed with S having area 1, we require that the T, Ω periodization of S leads to an exact L cover of the time-frequency plane. Close examination of the periodization of S shows that this is not possible.

If a vertical edge is shared, this implies that for some $\left(\begin{smallmatrix} p \\ q \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} r \\ s \end{smallmatrix} \right) \in \mathbb{Z}^2$, and $\alpha \in \mathbb{R}$,

$$A_0 \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \text{ and } A_0 \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 1+\alpha \\ \beta \end{pmatrix}.$$

Subtracting implies that for some $\left(\begin{smallmatrix} n \\ m \end{smallmatrix} \right) \in \mathbb{Z}^2$,

$$A_0 \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} nT \\ m\Omega \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

and $nTm\Omega = \frac{nm}{L} = a_{11}a_{21}$.

If a horizontal edge is shared, then by the same argument we have for some $n,\, m \in \mathbb{Z}$,

$$A_0 \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

from which it follows that

$$\begin{pmatrix} nT \\ m\Omega \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

and hence that $nTm\Omega = \frac{nm}{L} = a_{12}a_{22}$.

$(\Longleftarrow)$. In order to show that (8) and (9) hold, we will show that there exists $T > 0$ and $L \in \mathbb{N}$, with $T\Omega = 1/L$, and a matrix B with integer entries and $\det(B) = L$ such that the collection

$$\left\{ [0,1]^2 + A^{-1} \begin{pmatrix} T & 0 \\ 0 & \Omega \end{pmatrix} B\,\mathbb{Z}^2 \right\} \qquad (11)$$

tiles the plane. Assuming that we can do this, we have that

$$\sum_{k,\ell} \chi_{[0,1]^2 + A^{-1} \left(\begin{smallmatrix} T & 0 \\ 0 & \Omega \end{smallmatrix} \right) B \left(\begin{smallmatrix} k \\ \ell \end{smallmatrix} \right)}(x) = 1 \text{ a.e.}$$

which implies that

$$\sum_{k,\ell} \chi_{A[0,1]^2+\left(\begin{smallmatrix} T & 0 \\ 0 & \Omega \end{smallmatrix}\right)B\left(\begin{smallmatrix} k \\ \ell \end{smallmatrix}\right)}(x) = 1 \text{ a.e.}$$

By restricting the sum to only those $(k,\ell) \in \mathbb{Z}^2$ such that $B\left(\begin{smallmatrix} k \\ \ell \end{smallmatrix}\right) = \left(\begin{smallmatrix} Ln \\ Lm \end{smallmatrix}\right)$, in which case

$$\begin{pmatrix} T & 0 \\ 0 & \Omega \end{pmatrix} B \begin{pmatrix} k \\ \ell \end{pmatrix} = \begin{pmatrix} nLT \\ m/T \end{pmatrix},$$

it follows that

$$\sum_{n,m} \chi_{A[0,1]^2+(nLT,m/T)} \leq 1$$

which is (8).

To see that (9) holds, note that since the shifts of $[0,1]^2$ by the vectors $A^{-1}\left(\begin{smallmatrix} T & 0 \\ 0 & \Omega \end{smallmatrix}\right)B\mathbb{Z}^2$, tile the plane, and since $\det(B) = L$, the subgroup $\mathbb{Z}^2/B\mathbb{Z}^2$ consists of exactly L cosets. Therefore the full collection of shifts of $[0,1]^2$ by $A^{-1}\left(\begin{smallmatrix} T & 0 \\ 0 & \Omega \end{smallmatrix}\right)\mathbb{Z}^2$ tiles the plane exactly L times. Hence (9) holds.

It remains only to determine the matrix B in each case. Suppose first that $a_{11}a_{21} = 0$. If $a_{11} = 0$ then $a_{21} \neq 0$. Let $T = 1/a_{21}$ and $\Omega = a_{21}/L$ for any $L \in \mathbb{N}$. In this case, let $B = \left(\begin{smallmatrix} 0 & -1 \\ L & 0 \end{smallmatrix}\right)$ so that

$$A^{-1}\begin{pmatrix} T & 0 \\ 0 & \Omega \end{pmatrix} B = \begin{pmatrix} 1 & -a_{22}/a_{21} \\ 0 & 1 \end{pmatrix}$$

and it follows that the collection (11) tiles the plane.

If $a_{21} = 0$ and $a_{11} \neq 0$, then let $T = a_{11}/L$ and $\Omega = 1/a_{11}$, and let $B = \left(\begin{smallmatrix} L & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then

$$A^{-1}\begin{pmatrix} T & 0 \\ 0 & \Omega \end{pmatrix} B = \begin{pmatrix} 1 & -a_{12}/a_{11} \\ 0 & 1 \end{pmatrix}$$

and the result follows as above.

If neither a_{11} nor $a_{21} = 0$ then $a_{11}a_{21} = M/L$ where $M = M_1M_2$ and M_1 and M_2 are relatively prime. Let $T = a_{11}/M_1$ and $\Omega = a_{21}/M_2$, and choose integers N_1 and N_2 such that $M_1N_2 - M_2N_1 = L$. Letting $B = \begin{pmatrix} M_1 & N_1 \\ M_2 & N_2 \end{pmatrix}$,

$$A^{-1}\begin{pmatrix} T & 0 \\ 0 & \Omega \end{pmatrix} B = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

for some $\alpha \in \mathbb{R}$ and the result follows as before.

A similar argument can be applied when $a_{12}a_{22}$ is rational.

■

V. Conclusion

In this paper we have given a necessary and sufficient condition under which a class of operators whose K-N symbol is bandlimited to a planar parallelogram of unit area can be identified by a periodically-weighted delta train. This sheds some light on the problem of operator sampling for classes of operators whose spreading supports have unit area.

References

[1] P.A. Bello. Measurement of random time-variant linear channels. *IEEE Trans. Comm.*, 15:469–475, 1969.

[2] Ph. Furtwängler. Über Gitter konstanter Dichte. *Monatshe. Math. Phys.*, 43:281–288, 1936.

[3] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston, MA, 2001.

[4] G. Hajós. Über einfache und mehrfache Bedeckung des n-dimensional Raumes mit einem Wurfelgitter. *Math. Z.*, 47:427–467 , 1942.

[5] T. Kailath. Sampling models for linear time-variant filters. Technical Report 352, Massachusetts Institute of Technology, Research Laboratory of Electronics, 1959.

[6] T. Kailath. Measurements on time–variant communication channels. *IEEE Trans. Inform. Theory*, 8(5):229–236, Sept. 1962.

[7] T. Kailath. Time–variant communication channels. *IEEE Trans. Inform. Theory: Inform. Theory. Progress Report 1960–1963*, pages 233–237, Oct. 1963.

[8] W. Kozek and G. E. Pfander. Identification of operators with bandlimited symbols. *SIAM J. Math. Anal.*, 37(3):867–888, 2006.

[9] J. Lawrence, G. E. Pfander, and D. F. Walnut. Linear independence of Gabor systems in finite dimensional vector spaces. *J. Fourier Anal. Appl.*, 11(6):715–726, 2005.

[10] R.-D. Malikiosis. A note on Gabor frames in finite dimensions. http://arxiv.org/abs/1304.7709, preprint, 2013.

[11] H. Minkowski. Geometrie der Zahlen. Teubner, Leipzig, 1896.

[12] G. E. Pfander and D. Walnut. Measurement of time–variant channels. *IEEE Trans. Info. Theory*, 52(11):4808–4820, 2006.

[13] G. E. Pfander and D. Walnut. Sampling and reconstruction of operators. *IEEE Trans. Info. Theory*, submitted, 2014. http://arxiv.org/abs/1503.00628.

[14] R. M. Robinson. Multiple tilings of n-dimensional space by unit cubes. *Math. Z.*, 166:225–264 , 1979.

[15] S. K. Stein. Algebraic tiling. *Amer. Math. Monthly*, 81:445-462, 1974.

[16] D. Walnut, G. Pfander, and T. Kailath. Cornerstones of Sampling of Operator Theory in *Excursions in Harmonic Analysis, the February Fourier Talks at the Norbert Wiener Center, Volume 4*, 2015.

[17] C. Zong. What is known about unit cubes. *Bull. AMS*, 42(2):181-211, 2005.