ABSTRACT

Title of Dissertation: PERIODIC WAVELET TRANSFORMS

AND

PERIODICITY DETECTION

Götz Eduard Pfander, Doctor of Philosophy, 1999

Dissertation directed by: Professor John J. Benedetto
Department of Mathematics

The theory and some applications of *piecewise constant wavelets* are developed in this thesis.

Non–normalized continuous wavelet transforms of periodic functions $f \in L^\infty(\mathbf{R})$, which are taken with respect to *piecewise constant wavelets*, are periodic in time and in scale. We shall use this fact, to develop a method to detect periodic components in noisy signals $s = f + N$, where f is a known periodic signal and N noise. Our method is based on the redundancy in continuous wavelet transforms and their discrete counterparts, as well as on waveletgram averaging techniques.

We shall investigate our discretized version of the continuous wavelet transformation, to obtain conditions, which imply that the analyzing elements in our wavelet transformation form a frame for $l^2(\mathbf{Z})$. The same is done to obtain frames $l^2(\mathbf{Z}^d)$.

All wavelets with the property, that the non–normalized wavelet transforms of periodic functions $f \in L^\infty(\mathbf{T})$ are periodic in scale and time, are classified. Our result is generalized to higher dimensions.

PERIODIC WAVELET TRANSFORMS

AND

PERIODICITY DETECTION

by

Götz Eduard Pfander

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Advisory Committee:

Professor John J. Benedetto, Chairman/Advisor
Professor Der-Chen Chang
Professor Raymond Johnson
Professor Peter Wolfe
Professor Steven Tretter, Electrical Engineering

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TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction and Motivation

Piecewise constant wavelets were introduced in [?]. The theory and some applications of these generalized Haar wavelets will be developed in this Thesis.

In [?], Benedetto and Collela addressed a component of the problem of predicting epileptic seizures. A satisfactory solution of this problem would provide maximal lead time in which to predict an epileptic seizure [?, ?]. It was shown that spectrograms of electrical potential time series derived from brain activities of patients during seizure episodes exhibit multiple chirps consistent with the relatively simple almost periodic behavior of the observed time-series[?]. In the process, electrocorticogram (ECoG) data was used instead of the more common electroencephalogram (EEG) data. To obtain ECoG data, electrodes are planted directly on the cortex, eliminating some noise. To analyze the periodic components in these time-series, a redundant (non-dyadic) wavelet analysis was used, which the authors referred to as wavelet integer scale processing (WISP). The waveletgram obtained with respect to the Haar wavelet showed, among other things, that the almost periodic behavior in the signal resulted in almost periodic behavior in time as well as in scale in the waveletgram.

In fact, analyzing a periodic signal through the non–normalized continuous Haar wavelet transformation, we obtain a wavelet transform periodic in time and in scale.

Continuing the work of Benedetto and Collela, we realized the origin of this phenomenon (Proposition **??**). This led us to the definition of *piecewise constant wavelets* (Definition **??**).

Motivated by the epileptic seizure problem, we developed a method aimed at detecting periodic behavior inherent in noisy data. In the case of epileptic seizures, we aim at detecting periodic behavior in EEG data taken prior to the seizure, which is similar to the periodic behavior inherent in ECoG or EEG seizure data. The proposed procedure recognizes interindividual different periodic behavior in the electrical brain activity during an epileptic seizure. The method is composed of three steps:

1. ECoG data of an individual patient are analyzed through spectral and

wavelet methods to extract periodic patterns associated with epileptic seizures of a specific patient;

2. Using this knowledge of seizure periodicity, we construct an *optimal piecewise constant wavelet* designed to detect the epileptic periodic patterns of the patient;

3. A fast discretized version of the continuous wavelet transform and waveletgram averaging techniques are used to detect occurrence and period of the seizure periodicities in the preseizure EEG data of the patient; and the algorithm is formulated to provide real time implementation.

Our procedure is generally applicable to detect locally periodic components in signals s which can be modeled as

$$s(t) = A(t)f(h(t)) + N(t), \tag{1.1}$$

$t \in I$, where f is a periodic signal defined on the time interval I, A is a non–negative slowly varying function, and h is strictly increasing with h' slowly varying. N denotes background activity. For example, in the case of ECoG data, N is essentially $1/f$ noise. In the case of EEG data and for t in I, N includes noise due to cranial geometry and densities [?, ?]. In both cases N also includes standard low frequency rhythms [?].

If F is a trigonometric polynomial, then the signals described in (??) have been analyzed by Kronland-Martinet, Seip, Torresani, et al., to deal with the problem of detecting spectral lines in NMR data [?, ?, ?]. Another technique, that of computing critical frequencies in ECoG seizure data using waveletgram striations, was formulated by Benedetto and Colella [?]. These frequencies are related to the *instantaneous frequency* [?] $h'(t)$ of s at t; and, with our period detection and computation problem in mind, $1/h'(t)$ is the *instantaneous period* of s at t.

We shall approach the general case (??) with a method similar to the three step procedure we proposed to detect epileptic seizures [?]:

1. Non–noisy data are analyzed through spectral and wavelet methods to extract specific periodic patterns of interest, i.e., f;

2. We construct an *optimal piecewise constant wavelet* designed to detect f;

3. Using our discretized version of the continuous wavelet transform and waveletgram averaging techniques, we detect occurrence and period of these periodicities in real time.

The mathematical background for this approach to periodicity detection is supplied in Chapter ??. Chapter ?? also defines and explains the notation used in this thesis.

In Chapter ?? we define *piecewise constant wavelets* and state their key property of periodicity preservation: the non–normalized wavelet transforms of periodic functions taken with respect to a piecewise constant wavelet are periodic in time and in scale. We shall show also, that *odd* and *even piecewise constant wavelets* force additional structure on wavelet transforms of periodic functions.

We shall then present one possible approach to design a piecewise constant wavelet which is particularly apt to detect a specific, known periodic pattern in a signal. This is done in Chapter **??**. The wavelets we obtain, we shall call *optimal piecewise constant wavelets*.

The main advantage of our method over Fourier analysis methods, is the flexibility at hand. This flexibility can be used to request additional properties of the piecewise constant wavelet of choice. How to do this is shown in Chapter **??**.

Our approach to detect periodicities in scale and time in a wavelet transform is based on calculating certain averages in the wavelet transform, as is illustrated in Chapter **??**. Using this method, we obtain knowledge of appearance and period of a specific periodic component in the analyzed signal.

The redundancy of discretized versions of the continuous wavelet transformation offers some robustness to noise, but requires more calculations than needed to calculate a dyadic wavelet transform. In Chapter **??**, we shall present a fast cascade algorithm which reduces the number of calculations needed to compute a wavelet transform significantly if our analyzing wavelet is a piecewise constant wavelet.

In Chapter **??** we shall continue to discuss our discretized version of the continuous wavelet transformation. It is shown that, under some mild conditions, the analyzing functions used in our transformation form a frame for $l^2(\mathbf{Z})$. This result is generalized to $l^2(\mathbf{Z}^d)$.

The two main advantages of using piecewise constant wavelets are the fact that the non–normalized wavelet transforms of periodic functions are periodic in time and in scale, and the fact that the discretized version of the continuous wavelet transformation introduced in Chapter **??** allows a fast computation of the wavelet transform of any function. The question arises whether piecewise constant wavelets are the only wavelets with these two properties.

The algorithm presented in Chapter **??** relies strongly on the fact that the wavelet used is piecewise constant. In Chapter **??**, we shall classify all wavelets with the property that the non–normalized wavelet transforms of periodic functions are periodic in time and periodic in scale. We shall also generalize this result to higher dimensions.

Chapter 2

Background and Notation

The term wavelet does not possess a unique definition. The following definition is appropriate for our needs.

Definition 2.1. *A wavelet is a complex valued function* $\psi \in L^1(\mathbf{R})$ *such that*

$$\int_{\mathbf{R}} \psi(t)\, dt = 0.$$

We shall use an $L^p(\mathbf{R})$–normalized wavelet transformation W^ψ defined as

$$\begin{aligned} W^\psi : L^\infty(\mathbf{R}) &\longrightarrow C(\mathbf{R}\times\mathbf{R}^+) \\ f &\mapsto W_f^\psi : (b,a) \mapsto a^{-1/p}\int_{\mathbf{R}} f(t)\psi(\frac{t-b}{a})\, dt \end{aligned} \tag{2.1}$$

In some cases though, we shall use a non–normalized wavelet transform, which is obtained by dropping the normalization factor $a^{-1/p}$.

The wavelets we shall focus on are elements of $L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$ and they will have compact support. Hence, we can extend the domain of the wavelet transformation to $L^1_{loc}(\mathbf{R})$. $L^1_{loc}(\mathbf{R})$ is the space of locally integrable functions, that is, the restriction of $f \in L^1_{loc}(\mathbf{R})$ to any compact set is an integrable function. Of interest will be be the subspaces $L^p(\mathbf{T}_T), p \geq 1$ of $L^1_{loc}(\mathbf{R})$ of T periodic functions with the property that $\int_{\mathbf{T}_T} |f|^p < \infty$. Similar conventions are used in higher dimensions. Even though the d-dimensional torus $\mathbf{T}^d_T$ will always be thought of as a subspace of $\mathbf{R}^d$, we shall make the proper algebraic identifications, and, hence, $\mathbf{T}^d_T \setminus \{0\}$ will denote the torus without all its corners.

Further, ∂A is the boundary of a set A and A^o is the interior of a set A.

For $k = (k_1, \ldots, k_d) \in \mathbf{Z}^d$ we shall write $0 \leq k < N$ if $0 \leq k_i < N$ for all $i = 1, \ldots, d$. This approach will be carried over to intervals, namely, $(x, y) = (x_1, y_1)\times\ldots\times(x_d, y_d)$, where $x, y \in \mathbf{R}^d$. We set $|x| = (|x_1|, \ldots, |x_d|)$ and $\text{sign}(x) = (\text{sign}(x_1), \ldots, \text{sign}(x_d))$ for $x \in \mathbf{R}^d$.

Certainly, a polynomial p in one complex variable, is said to divide a polynomial q, if there exists a polynomial r such that $p \cdot r = q$.

In Chapter **??** it will be necessary to make a distinction between the Fourier transformation $\widehat{\ }: L^1(\mathbf{R}^d) \longrightarrow C_0(\mathbf{R^d})$ defined on $L^1(\mathbf{R}^d)$ and the Fourier transformation $\mathcal{F} : L^2(\mathbf{R}^d) \longrightarrow L^2(\mathbf{R^d})$ defined on $L^2(\mathbf{R}^d)$.

The Hilbert transform of a function f is formally defined by

$$\mathcal{H}(f)(t) = \lim_{\epsilon \to 0} \int_{|t-u| \le \epsilon} \frac{f(u)}{t-u}\, du.$$

For $f \in L^2(\mathbf{R})$ this limit exists for almost every $t \in \mathbf{R}$. In Chapter **??** we shall use the following standard result [**?**, **?**]:

THEOREM 2.2. *$\mathcal{H} : L^2(\mathbf{R}) \longrightarrow L^2(\mathbf{R})$ is a well-defined isometry. The Hilbert transformation and the Fourier transformation are related by*

$$\mathcal{H}(f) = \mathcal{F}^{-1}\left(-i \operatorname{sign}(\cdot) \cdot (\mathcal{F}(f))\right)$$

for all $f \in L^2(\mathbf{R})$.

In order to generalize some results to higher dimensions, we need some not necessarily standard notation.

For $a = (a_1, \ldots, a_d) \in \mathbf{R}^d$ we define $p(a) = a_1 \cdot \ldots \cdot a_d$. For the vectors $b = (b_1, \ldots, b_d)$, $t = (t_1, \ldots, t_d) \in \mathbf{R}^d$, and $a = (a_1, \ldots, a_d) \in \mathbf{R}^d \setminus S$, where

$$S = \{x \in \mathbf{R}^d : p(x) = 0\},$$

we define the vectors $\frac{t-b}{a} \in \mathbf{R}^d$ and $t \star a \in \mathbf{R}^d$ component wise by

$$\frac{t-b}{a} = \begin{pmatrix} \frac{t_1-b_1}{a_1} \\ \vdots \\ \frac{t_d-b_d}{a_d} \end{pmatrix} \quad \text{and} \quad t \star a = \begin{pmatrix} t_1 \cdot a_1 \\ \vdots \\ t_d \cdot a_d \end{pmatrix}.$$

If $0 \neq a \in \mathbf{R}$ we let

$$\frac{t-b}{a} = \begin{pmatrix} \frac{t_1-b_1}{a} \\ \vdots \\ \frac{t_d-b_d}{a} \end{pmatrix}.$$

In the case $a \in \mathbf{R}$, $\star$–multiplication reduces to scalar multiplication.

Chapter 3

Piecewise Constant Wavelets

We shall introduce the notion of piecewise constant wavelets. There are two main reasons to restrict ourselves to these generalized Haar functions. First, they allow a fast computation of a discretized version of the continuous wavelet transformation by means of a cascade algorithm. The second reason is presented in Proposition **??** and Proposition **??**.

DEFINITION 3.1. *A piecewise constant wavelet of degree M is a complex valued function $\psi \in L^1(\mathbf{R})$ such that $\int_{\mathbf{R}} \psi(t)\,dt = 0$, and such that there exist $M \in \mathbf{R}$ and $s_i \in \mathbf{R}$ with $\psi|_{[s_i,s_{i+1})} = c_i \in \mathbf{C}$ for all $i \in \mathbf{Z}$ and $Ms_i \in \mathbf{Z}$ for all $i \in \mathbf{Z}$.*

Note that the assumptions imply that piecewise constant wavelets are bounded, and that the coefficients are summable, i.e., $\{c_i\}_{i\in\mathbf{Z}} \in l^1(\mathbf{Z})$.

The first observation in our approach to period detection and computation is the following fact.

PROPOSITION 3.2. *Let $f \in L^1(\mathbf{T}_T)$, i.e., f is T–periodic and integrable on $[0,T]$, and let ψ be a piecewise constant wavelet of degree M. Then*

$$a^{1/p} W_f^{\psi}(b,a) = \int_{\mathbf{R}} f(t)\psi(\frac{t-b}{a})\,dt$$

is T–periodic in b and MT–periodic in a.

This theorem can be proven via a direct calculation. A similar calculation is carried out in the proof of Proposition **??**. Proposition **??** is also a corollary of Theorem **??**.

I shall provide an example to illustrate Theorem **??**.

EXAMPLE 3.3. We choose $f \in L^1(\mathbf{T}_T)$ to be a simple sine function

$$f(\cdot) = \sin(2\pi(\gamma \cdot + \theta)),$$

where $\gamma, \theta \in \mathbf{R}$, and we choose our analyzing wavelet to be the centered Haar wavelet

$$\psi = \mathbf{1}_{[-\frac{1}{2},0)} + \mathbf{1}_{[0,\frac{1}{2})}.$$

In this case, we obtain the non–normalized wavelet transform

$$a^{\frac{1}{p}} W_f^{\psi}(b,a) = a^{-\frac{1}{p}} \frac{2}{\pi\gamma} \sin^2(\frac{\pi\gamma a}{2}) \cos(2\pi(\gamma b + \theta)),$$

$b \in \mathbf{R}$, $a \in \mathbf{R}^+$. A segment of $a^{\frac{1}{p}} W_f^{\psi}(b,a)$ for $\theta = 0$ and $\gamma = 1$ is shown in Figure **??**.

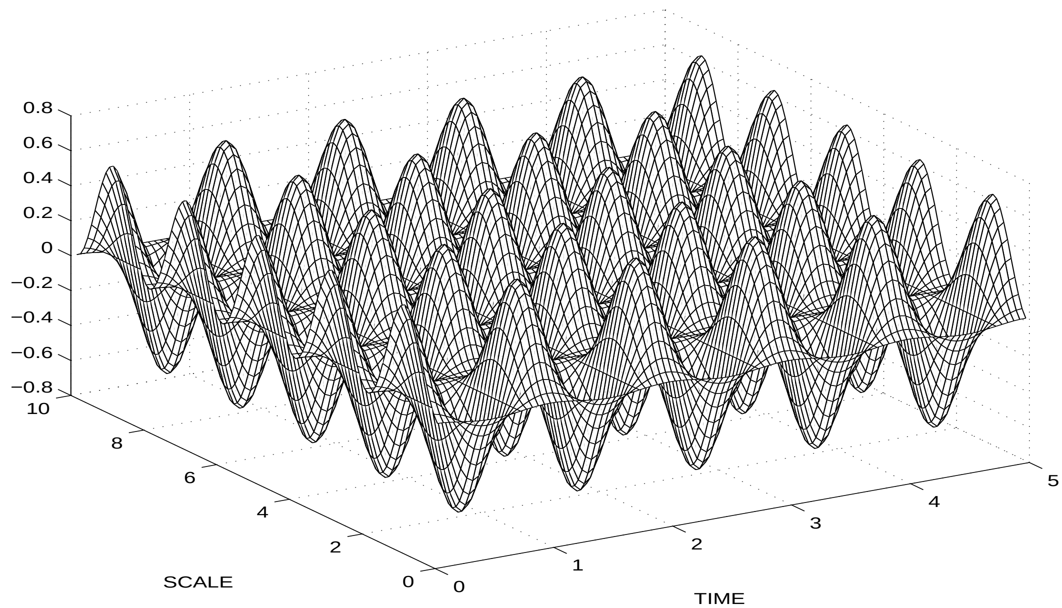

Figure 3.1. Non–normalized Haar wavelet transform of a sine function.

Figure **??** illustrates the cause for periodicity in time, based on our example where f is a simple sine function and ψ is the centered Haar wavelet. For a fixed scale, moving the wavelet by a full period across time, does not change the innerproduct

$$\langle f, \psi(\frac{\cdot - b}{a}) \rangle = a^{\frac{1}{p}} W_f^{\psi}(b,a).$$

Figure **??** illustrates the cancellations leading to periodicity in scale in this example. In general, these are due to the fact that $\sum c_i = 0$.

Proposition **??** implies that if the signal s has the particular form $s(t) = Af(ct)$ for constants A and c, then the relative maxima of $a^{1/p} W_s^{\psi}(b,a)$ form a lattice in time–scale space. The horizontal (time) distance between two neighboring vertices of the lattice is $1/c$, and the vertical (scale) distance between two neighboring vertices is M/c.

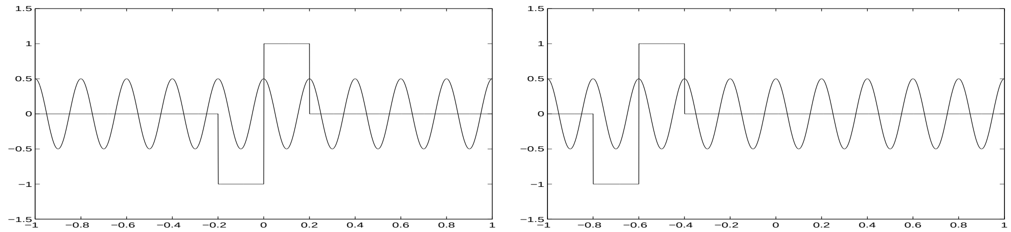

Figure 3.2. Periodicity in time.

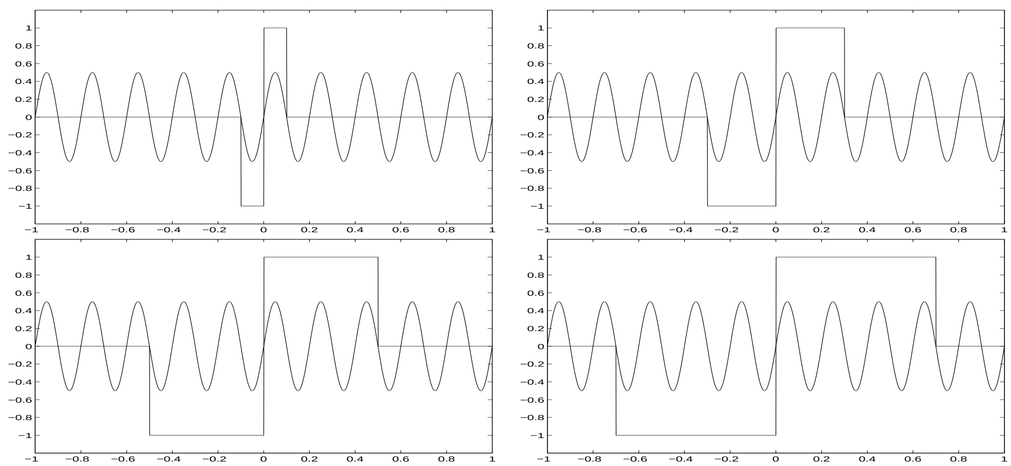

Figure 3.3. Periodicity in scale is caused by cancellations in the continuous wavelet transform.

This regularity displays redundancy in the following way: Each rectangle of size $1/c \times M/c$ in the waveletgram contains all the information in the waveletgram.

Additional structure of ψ can force additional features upon the wavelet transform of periodic functions, as can be seen in the following proposition:

PROPOSITION 3.4. *Let $f \in L^1(\mathbf{T}_T)$, and let ψ be a piecewise constant wavelet of degree M.*

a. Suppose ψ is even, i.e. $\psi(-t) = \psi(t)$ for $t \in \mathbf{R}$. Then

$$a^{\frac{1}{p}} W_f^{\psi}(b,a) = \int_{\mathbf{R}} f(t)\psi(\frac{t-b}{a})\,dt = -(MT-a)^{\frac{1}{p}} W_f^{\psi}(b, MT-a)$$

for $0 < a < MT$.

b. Suppose ψ is odd, i.e. $-\psi(-t) = \psi(t)$ for $t \in \mathbf{R}$. Then

$$a^{\frac{1}{p}} W_f^{\psi}(b,a) = (MT-a)^{\frac{1}{p}} W_f^{\psi}(b, MT-a)$$

for $0 < a < MT$.

Proof. a. Since ψ is a piecewise constant wavelet of degree M and since ψ is even, there exist $s_i \in \mathbf{R}$ such that $\psi|_{[s_{-(i+1)}, s_{-i})} = \psi|_{[s_i, s_{i+1})} = c_i$, $i \in \mathbf{Z}$ and $c_i \in \mathbf{C}$, and $s_{-i} = -s_i$ for $i \in \mathbf{Z}$.

We compute

$$
\begin{aligned}
(MT-a)^{1/p} \quad & W_f^{\psi}(b, MT-a) + a^{1/p} W_f^{\psi}(b,a) \\
&= \int_{\mathbf{R}} f(t)\psi(\frac{t-b}{MT-a})\,dt + \int_{\mathbf{R}} f(t)\psi(\frac{t-b}{a})\,dt \\
&= \sum_{i\geq 0} c_i(\int_{(MT-a)s_i+b}^{(MT-a)s_{i+1}+b} f(t)\,dt + \int_{(MT-a)s_{-(i+1)}+b}^{(MT-a)s_{-i}+b} f(t)\,dt \\
&\quad + \int_{as_i+b}^{as_{i+1}+b} f(t)\,dt + \int_{as_{-(i+1)}+b}^{as_{-i}+b} f(t)\,dt) \\
&= \sum_{i\geq 0} c_i(\int_{MTs_i-as_i+b}^{MTs_{i+1}-as_{i+1}+b} f(t)\,dt + \int_{-MTs_{i+1}+as_{i+1}+b}^{-MTs_i+as_i+b} f(t)\,dt \\
&\quad + \int_{as_i+b}^{as_{i+1}+b} f(t)\,dt + \int_{-as_{i+1}+b}^{-as_i+b} f(t)\,dt) \\
&= \sum_{i\geq 0} c_i(\int_{-as_i+b}^{-as_{i+1}+b+MT(s_{i+1}-s_i)} f(t)\,dt + \int_{as_{i+1}+b}^{as_i+b+MT(s_{i+1}-s_i)} f(t)\,dt \\
&\quad + \int_{as_i+b}^{as_{i+1}+b} f(t)\,dt + \int_{-as_{i+1}+b}^{-as_i+b} f(t)\,dt) \\
&= \sum_{i\geq 0} c_i(\int_{-as_{i+1}+b}^{-as_{i+1}+b+MT(s_{i+1}-s_i)} f(t)\,dt + \int_{as_i+b}^{as_i+b+MT(s_{i+1}-s_i)} f(t)\,dt) \\
&= \sum_{i\geq 0} c_i(\int_0^{MT(s_{i+1}-s_i)} f(t)\,dt + \int_0^{MT(s_{i+1}-s_i)} f(t)\,dt) \\
&= \sum_{i\geq 0} c_i 2M(s_{i+1}-s_i)\int_0^T f(t)\,dt \\
&= M\int_{\mathbf{R}} \psi(t)\,dt \int_0^T f(t)\,dt \\
&= 0.
\end{aligned}
$$

b. Since ψ is odd, there exist $s_i \in \mathbf{R}$ such that $-\psi|_{[s_{-(i+1)},s_{-i})} = \psi|_{[s_i,s_{i+1})} = c_i$, $i \in \mathbf{Z}$ and $c_i \in \mathbf{C}$. and $s_{-i} = -s_i$ for $i \geq 0$.

We compute

$$
\begin{aligned}
(MT-a)^{1/p} \qquad & W_f^\psi(b, MT-a) - a^{1/p} W_f^\psi(b,a) \\
= \ & \int_{\mathbf{R}} f(t)\psi(\frac{t-b}{MT-a})\,dt - \int_{\mathbf{R}} f(t)\psi(\frac{t-b}{a})\,dt \\
= \ & \sum_{i\geq 0} c_i(\int_{(MT-a)s_i+b}^{(MT-a)s_{i+1}+b} f(t)\,dt - \int_{(MT-a)s_{-(i+1)}+b}^{(MT-a)s_{-i}+b} f(t)\,dt \\
& - \int_{as_i+b}^{as_{i+1}+b} f(t)\,dt + \int_{as_{-(i+1)}+b}^{as_{-i}+b} f(t)\,dt) \\
= \ & \sum_{i\geq 0} c_i(\int_{MTs_i-as_i+b}^{MTs_{i+1}-as_{i+1}+b} f(t)\,dt - \int_{-MTs_{i+1}+as_{i+1}+b}^{-MTs_i+as_i+b} f(t)\,dt \\
& - \int_{as_i+b}^{as_{i+1}+b} f(t)\,dt + \int_{-as_{i+1}+b}^{-as_i+b} f(t)\,dt) \\
= \ & \sum_{i\geq 0} c_i(\int_{-as_i+b}^{-as_{i+1}+b+MT(s_{i+1}-s_i)} f(t)\,dt - \int_{as_{i+1}+b}^{as_i+b+MT(s_{i+1}-s_i)} f(t)\,dt \\
& - \int_{as_i+b}^{as_{i+1}+b} f(t)\,dt + \int_{-as_{i+1}+b}^{-as_i+b} f(t)\,dt) \\
= \ & \sum_{i\geq 0} c_i(\int_{-as_{i+1}+b}^{-as_{i+1}+b+MT(s_{i+1}-s_i)} f(t)\,dt - \int_{as_i+b}^{as_i+b+MT(s_{i+1}-s_i)} f(t)\,dt) \\
= \ & \sum_{i\geq 0} c_i(\int_0^{MT(s_{i+1}-s_i)} f(t)\,dt - \int_0^{MT(s_{i+1}-s_i)} f(t)\,dt) \\
= \ & 0.
\end{aligned}
$$

Note that in part b we did not use the fact that $\int_{\mathbf{R}} \psi(t)\,dt = 0$ explicitly. Nevertheless, $\int_{\mathbf{R}} \psi(t)\,dt = 0$ since ψ is odd. $\square$

Can we use these observations to detect occurrence and period of periodic components in a signal? For example, can we develop a method which detects the bi–periodic structure in waveletgrams, and might these methods have an advantage over simpler methods applied directly to a periodic signal?

We shall present one approach based on Proposition **??** to detect periodic components in signals in Chapter **??** and Chapter **??**. There, we shall attempt to detect lattice patterns of relative maxima in waveletgrams and to measure the distance between those points to disclose the periodic behavior of the signal. Averaging techniques should then reduce the effect of noise in the case $s(t) = Af(ct) + N(t)$.

We also have some flexibility at hand. In fact, the wavelet transform of a fixed periodic signal can be manipulated by choosing a specific piecewise constant wavelet intelligently. One way of doing so is illustrated in Chapter **??**.

Chapter 4

Constructing Optimal Piecewise Constant Wavelets

We shall construct piecewise constant wavelets to detect a specific periodic function f in a noisy signal $s(t) = Af(ct) + N(t)$, $t\imath I \subset \mathbf{R}$. An optimal piecewise constant wavelet is chosen such that we achieve good readability of the waveletgram W_f^ψ.

To apply averaging methods to detect bi–periodic behavior in W_s^ψ (Chapter **??**), we want W_f^ψ to be well localized. This will result in a lattice pattern of relative maxima in time–scale space. For a given periodic signal f, we shall show the existence of an optimal piecewise constant wavelet, which guarantees these relative maxima to be as large as possible.

4.1 Existence of Optimal Piecewise Constant Wavelets

Before being more precise with respect to the term *optimal* piecewise constant wavelet, we need to introduce certain restrictions.

We begin by letting M=1 and by fixing $N \in \mathbf{N}$. We consider piecewise constant wavelets ψ^c with compact support of the form

$$\psi^c|_{[i,i+1)} = c_i \text{ for } i = 0, \dots, N-1, \quad c = (c_0, c_1, \dots, c_{N-1}) \in \mathbf{C}^N. \tag{4.1}$$

Additionally, we require

$$0 = \int_{\mathbf{R}} \psi^c(t)\, dt = \sum_{i=0}^{N-1} c_i, \tag{4.2}$$

and we normalize ψ^c so that

$$\|\psi^c\|_{L^2(\mathbf{R})} = \|c\|_{l^2(\mathbf{C}^N)} = 1. \tag{4.3}$$

Equation (**??**) allows us to achieve the periodicity properties asserted in Proposition **??**. Note that (**??**) is equivalent to the condition that

$$c \in H = \{x \in \mathbf{C}^N : \sum_{i=0}^{N-1} x_i = \langle x, (1, 1, \dots, 1, 1) \rangle = 0\}.$$

H is an $N-1$ dimensional subspace, i.e., a hyperplane. Equation (**??**) is a standard normalization constraint in constructing wavelets. For ψ^c it can be expressed as

$$c \in S^{2N-1} = \{x \in \mathbf{C}^N : \|x\|_{l^2(\mathbf{C}^N)} = 1\}.$$

We shall design a wavelet which has a clear single peak in the $(0, T] \times (0, MT] = (0, T] \times (0, T]$ cell of the waveletgram. The following theorem shows how to achieve a maximal peak.

THEOREM 4.1. *Let $p > 1$ and $f \in L^\infty(\mathbf{R})$, or $p \geq 1$, $f \in L^1(\mathbf{T}_T)$, and each $x \in \mathbf{R}$ is a Lebesque point of f. Let $N \in \mathbf{N}$.*
a. There exist $(b_0, a_0) \in \mathbf{R} \times \mathbf{R}^+$ such that

$$a_0^{-\frac{1}{p}} \|P_H(k_{b_0,a_0})\|_{l^2(\mathbf{C}^N)} = \max_{(b,a) \in \mathbf{R} \times \mathbf{R}^+} a^{-\frac{1}{p}} \|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}$$

where $k_{b,a} = (k_{b,a,0}, \dots k_{b,a,N-1}) \in \mathbf{C}^N$ is defined by

$$k_{b,a,i} = \int_{ia+b}^{(i+1)a+b} f(t)\, dt$$

and P_H is the orthogonal projection of $\mathbf{C}^N$ onto the hyperplane H.
b. For this (b_0, a_0) we set

$$c_0 = \frac{P_H(k_{b_0,a_0})}{\|P_H(k_{b_0,a_0})\|_{l^2(\mathbf{C}^N)}}.$$

The piecewise constant wavelet ψ^{c_0} satisfies (**??**),(**??**), *and* (**??**), *and*

$$|W_f^{\psi^{c_0}}(b_0, a_0)| \geq |W_f^{\psi^c}(b, a)| \tag{4.4}$$

for all $(b, a) \in \mathbf{R} \times \mathbf{R}^+$ and all ψ^c satisfying (**??**),(**??**),(**??**).

Before proving Theorem **??**, we shall recall some elementary facts concerning orthogonal projections of finite dimensional vector spaces and provide a useful lemma.

Suppose U is a subspace of the finite dimensional vector space V. For every vector $v \in V$ there are unique vectors $u \in U$ and $w \in U^\perp = \{v \in V : \langle u, v \rangle = 0 \text{ for all } u \in U\}$ such that $v = u + w$. The *orthogonal projection* P_U on V is defined by $P_U(v) = u$, where u is chosen as above.

Let $\{e_1, e_2, \ldots, e_l\}$ be an orthonormal basis of U and $\{e_1, \ldots, e_l, e_{l+1}, \ldots, e_n\}$ be an orthonormal basis of V. For $v \in V$ we have $v = \sum_{i=1}^n \langle v, e_i\rangle e_i$ and, by definition and the orthogonality of $\{e_1, \ldots, e_n\}$,

$$P_U(v) = \sum_{i=1}^{l} \langle v, e_i\rangle e_i = v - \sum_{i=l+1}^{n} \langle v, e_i\rangle e_i.$$

LEMMA 4.2. *Let U be an l-dimensional subspace of $\mathbf{C}^N$. Let $v \in \mathbf{C}^N$ and let P_U be the orthogonal projection of $\mathbf{C}^N$ onto U. Then*

$$|\langle u, v\rangle| \leq \langle \frac{P_U(v)}{\|P_U(v)\|_{l^2(\mathbf{C}^N)}}, v\rangle$$

for all $u \in U \cap S^{2N-1}$.

Proof. Using the Cauchy–Schwartz inequality, we obtain for all $u \in U \cap S^{2N-1}$

$$\begin{aligned} |\langle u, v\rangle| &= |\langle u, P_U(v)\rangle| \\ &\leq \|u\|_{l^2(\mathbf{C}^N)} \|P_U(v)\|_{l^2(\mathbf{C}^N)} \\ &= \|P_U(v)\|_{l^2(\mathbf{C}^N)} \\ &= \langle \frac{P_U(v)}{\|P_U(v)\|_{l^2(\mathbf{C}^N)}}, P_U(v)\rangle \\ &= \langle \frac{P_U(v)}{\|P_U(v)\|_{l^2(\mathbf{C}^N)}}, v\rangle. \end{aligned}$$

□

Proof of Theorem **??**.
STEP 1. Let us first fix $(b, a) \in (0, T] \times (0, T]$. We want to construct $c_{b,a} \in \mathbf{C}^N$ such that

$$|W_f^{\psi^{c_{b,a}}}(b, a)| \geq |W_f^{\psi^c}(b, a)| \tag{4.5}$$

for all ψ^c satisfying conditions (**??**),(**??**),(**??**). After finding $c_{b,a}$ we shall choose the "optimal" (b_0, a_0) and let $c = c_{b_0,a_0}$.

Note that for $c \in \mathbf{C}^N$

$$W_f^{\psi^c}(b, a) = a^{-\frac{1}{p}} \sum_{i=0}^{N-1} c_i \int_{ia+b}^{(i+1)a+b} f(t)\, dt.$$

Setting

$$k_{b,a,i} = \int_{ia+b}^{(i+1)a+b} f(t)\, dt$$

and $k_{b,a} = (k_{b,a,0}, \dots k_{b,a,N-1})$, we have

$$W_f^{\psi^c}(b,a) = a^{-\frac{1}{p}} \sum_{i=0}^{N-1} c_i k_{b,a,i} = a^{-\frac{1}{p}} \langle c, k_{b,a} \rangle. \tag{4.6}$$

Note that conditions (**??**) and (**??**) on ψ^c are equivalent to the following restriction on c:

$$c \in \{x \in \mathbf{C}^N : \sum x_i = 0, \|x\|_{l^2(\mathbf{C}^N)} = 1, \} = H \cap S^{2N-1}.$$

Given the vector $k_{b,a}$ we can optimize (**??**) by projecting $k_{b,a}$ onto the hyperplane H and normalizing the result (Lemma **??**), i.e., letting $P_H : \mathbf{C}^N \longrightarrow \mathbf{C}^N$ be the orthogonal projection of $\mathbf{C}^N$ onto H, we obtain as best choice of $c_{b,a}$,

$$c_{b,a} = \frac{P_H(k_{b,a})}{\|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}},$$

and $\psi^{c_{b,a}}$ fulfills (**??**). ■

Explicitly, we have

$$\begin{aligned}
P_H(k_{b,a}) &= k_{b,a} - \langle k_{b,a}, N^{-\frac{1}{2}} \quad (1,1,\dots,1)\rangle N^{-\frac{1}{2}} \quad (1,1,\dots,1) \\
&= k_{b,a} - \frac{1}{N} \sum_{i=0}^{N-1} k_{b,a,i} \quad (1,1,\dots,1) \\
&= k_{b,a} - \frac{1}{N} \sum_{i=0}^{N-1} \int_{ia+b}^{(i+1)a+b} f(t)\,dt \quad (1,1,\dots,1) \\
&= k_{b,a} - \frac{1}{N} \int_{b}^{Na+b} f(t)\,dt \quad (1,1,\dots,1),
\end{aligned}$$

and therefore

$$c_{b,a,i} = \frac{k_{b,a,i} - \frac{1}{N}\int_b^{Na+b} f(t)\,dt}{\|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}},$$

$i = 0, \dots, N-1$ are the optimal choices of values for the piecewise constant wavelet in the case that b and a are fixed.
STEP 2. It remains to show the existence of (b_0, a_0) such that

$$|W_f^{\psi^{c_{b_0,a_0}}}(b_0,a_0)| \geq |W_f^{\psi^{c_{b,a}}}(b,a)| \tag{4.7}$$

for all $(b,a) \in \mathbf{R} \times \mathbf{R}^+$ where c_{b_0,a_0} and $c_{b,a}$ are chosen as above. This, together with (**??**), will conclude the proof, see (**??**).

Since $c_{b,a} \in H$, we have

$$
\begin{aligned}
|W_f^{\psi^{c_{b,a}}}(b,a)| &= |a^{-\frac{1}{p}} \langle c_{b,a}, k_{b,a} \rangle| \\
&= a^{-\frac{1}{p}} |\langle c_{b,a}, P_H(k_{b,a}) \rangle \\
&= a^{-\frac{1}{p}} |\langle \frac{P_H(k_{b,a})}{\|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}}, P_H(k_{b,a}) \rangle| \\
&= a^{-\frac{1}{p}} \, \|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)} \,,
\end{aligned}
$$

and hence, we need to show the existence of (b_0, a_0) such that

$$
a_0^{-\frac{1}{p}} \, \|P_H(k_{b_0,a_0})\|_{l^2(\mathbf{C}^N)} \geq a^{-\frac{1}{p}} \, \|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}
$$

for all $(b,a) \in \mathbf{R} \times \mathbf{R}^+$.

To see this, first note that $\|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}$ is T periodic in b. This is the case since $k_{b,a}$ is T periodic in b, i.e., for $i = 1, \ldots, n$ we have

$$
\begin{aligned}
k_{b+T,a,i} &= \int_{ia+b+T}^{(i+1)a+b+T} f(t)\, dt = \int_{ia+b}^{(i+1)a+b} f(u-T) du = \int_{ia+b}^{(i+1)a+b} f(u) du \\
&= k_{b,a,i}.
\end{aligned}
$$

$\|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}$ is also T periodic in a, in fact, we compute

$$\begin{aligned}
\|P_H(k_{b,a+T})\|^2_{l^2(\mathbf{C}^N)} &= \sum_{i=0}^{N-1}\Big(\int_{i(a+T)+b}^{(i+1)(a+T)+b} f(t)\,dt - \frac{1}{N}\int_b^{N(a+T)+b} f(t)\,dt\Big)^2 \\
&= \sum_{i=0}^{N-1}\Big(\int_{ia+iT+b}^{(i+1)a+iT+T+b} f(t)\,dt - \frac{1}{N}\int_b^{Na+b} f(t)\,dt \\
&\quad -\frac{1}{N}\int_{Na+b}^{Na+b+NT} f(t)\,dt\Big)^2 \\
&= \sum_{i=0}^{N-1}\Big(\int_{ia+b}^{(i+1)a+T+b} f(t)\,dt - \frac{1}{N}\int_b^{Na+b} f(t)\,dt \\
&\quad -\frac{1}{N}N\int_0^T f(t)\,dt\Big)^2 \\
&= \sum_{i=0}^{N-1}\Big(\int_{ia+b}^{(i+1)a+b} f(t)\,dt + \int_0^T f(t)\,dt \\
&\quad -\frac{1}{N}\int_b^{Na+b} f(t)\,dt - \int_0^T f(t)\,dt\Big)^2 \\
&= \sum_{i=0}^{N-1}\Big(\int_{ia+b}^{(i+1)a+b} f(t)\,dt - \frac{1}{N}\int_b^{Na+b} f(t)\,dt\Big)^2 \\
&= \|P_H(k_{b,a})\|^2_{l^2(\mathbf{C}^N)}.
\end{aligned}$$

Since $a^{-\frac{1}{p}}$ is monotonely decreasing for $a \to \infty$ and by the periodicity of $\|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}$ in time and scale, it suffices to show the existence of $(b_0, a_0) \in [0,T] \times (0,T]$ such that

$$a_0^{-\frac{1}{p}} \|P_H(k_{b_0,a_0})\|_{l^2(\mathbf{C}^N)} \geq a^{-\frac{1}{p}} \|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}$$

for all $(b,a) \in [0,T] \times (0,T]$.

It is easy to see that $a^{-\frac{1}{p}} \|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}$ is continuous on $[0,T] \times (0,T]$ and we shall show that, if $p > 1$ and $f \in L^\infty(\mathbf{R})$, or $p \geq 1$, $f \in L^1(\mathbf{T}_T)$, and each $x \in \mathbf{R}$ is a Lebesque point of f, $a^{-\frac{1}{p}} \|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}$ has a continuous extension to $[0,T] \times [0,T]$ and therefore it obtains a maximum on $[0,T] \times [0,T]$. We shall further show that this maximum is obtained in $[0,T] \times (0,T]$. In fact, we shall verify that for all $b \in \mathbf{R}$

$$\lim_{a \to 0^+} a^{-\frac{1}{p}} \|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)} = 0. \tag{4.8}$$

To see this, we need to consider two cases. Recall that the ith entry in the vector $P_H(k_{b,a})$ is given by $a^{-\frac{1}{p}}(k_{b,a,i} - \frac{1}{N}\int_b^{Na+b} f(t)\,dt)$.

CASE 1. Let $p > 1$ and $f \in L^\infty(\mathbf{R})$. We compute for $b \in \mathbf{R}$

$$\begin{aligned}
0 &\leq \lim_{a\to 0^+} |a^{-\frac{1}{p}}(k_{b,a,i} - \frac{1}{N}\int_b^{Na+b} f(t)\,dt)| \\
&= \lim_{a\to 0^+} a^{1-\frac{1}{p}}|\frac{1}{a}\int_{ia+b}^{(i+1)a+b} f(t)\,dt - \frac{1}{aN}\int_b^{Na+b} f(t)\,dt| \\
&\leq \lim_{a\to 0^+} a^{1-\frac{1}{p}}(1+N)\,\|f\|_{L^\infty(\mathbf{R})} \\
&= 0.
\end{aligned}$$

CASE 2. Let $p \geq 1$ and let $f \in L^1(\mathbf{T}_T)$, such that each $x \in \mathbf{R}$ is a Lebesque point of f.

For $p = 1$ and for all $b \in \mathbf{R}$ we note that

$$\begin{aligned}
\lim_{a\to 0^+} a^{-1}(k_{b,a,i} &- \frac{1}{N}\int_b^{Na+b} f(t)\,dt) \\
&= \lim_{a\to 0^+} a^{-1}(\int_{ia+b}^{(i+1)a+b} f(t)\,dt - \frac{1}{N}\int_b^{Na+b} f(t)\,dt) \\
&= \lim_{a\to 0^+} a^{-1}(\int_b^{(i+1)a+b} f(t)\,dt \\
&\quad -\int_b^{ia+b} f(t)\,dt) - \frac{1}{N}\int_b^{Na+b} f(t)\,dt) \\
&= (i+1)\lim_{a\to 0^+}\frac{1}{(i+1)a}\int_b^{(i+1)a+b} f(t)\,dt \\
&\quad -i\lim_{a\to 0^+}\frac{1}{ia}\int_b^{ia+b} f(t)\,dt - \lim_{a\to 0^+}\frac{1}{N}\int_b^{Na+b} f(t)\,dt \\
&= (i+1)\lim_{h\to 0^+}\frac{1}{h}\int_b^{b+h} f(t)\,dt \\
&\quad -i\lim_{h\to 0^+}\frac{1}{h}\int_b^{b+h} f(t)\,dt - \lim_{h\to 0^+}\frac{1}{h}\int_b^{b+h} f(t)\,dt \\
&= 0.
\end{aligned}$$

Using the addition property of limits in the third step of the previous calculation is a priori valid for almost every b. This is the case since $f \in L^1(\mathbf{T}_T)$ and therefore the limit $\lim_{h\to 0^+}\frac{1}{h}\int_b^{b+h} f(t)\,dt$ exist for all Lebesque points $b \in \mathbf{R}$. Therefore, by hypothesis, the limit exists everywhere.

For $p > 1$ we have $a^{-\frac{1}{p}} = a^{-1}a^{1-\frac{1}{p}}$. Since $a^{1-\frac{1}{p}} \to 0$ as $a \to 0^+$ we have

$$\lim_{a\to 0^+} a^{1-\frac{1}{p}}|a^{-1}(k_{b,a,i} - \frac{1}{N}\int_b^{Na+b} f(t)\,dt)| = 0.$$

In both cases, the componentwise convergence of $\lim_{a\to 0^+} a^{-1}P_H(k_{b,a})$, together with the continuity of norms and the fact that$\|av\| = |a|\,\|v\|$, gives (**??**).

Let $(b,a) \in \mathbf{R}\times\mathbf{R}^+$ and let ψ^c satisfy (**??**),(**??**),(**??**), using (**??**) and (**??**) we obtain

$$\begin{aligned}
|W_f^{\psi^{c_{b_0,a_0}}}(b_0,a_0)| &= a_0^{-\frac{1}{p}}\,\|P_H(k_{b_0,a_0})\|_{l^2(\mathbf{C}^N)} \\
&\geq a^{-\frac{1}{p}}\,\|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)} \\
&= |W_f^{\psi^{c_{b,a}}}(b,a)| \\
&\geq |W_f^{\psi^c}(b,a)|. \qquad (4.9)
\end{aligned}$$

■

□

Our result leads to the following construction algorithm for optimal piecewise constant wavelets. First find b and a such that

$$\begin{aligned}
a^{-\frac{1}{p}}\,\|P_H(k_{b,a})\|^2_{l^2(\mathbf{C}^N)} &= a^{-\frac{1}{p}}\sum_{i=0}^{N-1}(k_{b,a,i} - \frac{1}{N}\int_b^{Na+b} f(t)\,dt)^2 \\
&= a^{-\frac{1}{p}}\sum_{i=0}^{N-1}(\int_{ia+b}^{(i+1)a+b} f(t)\,dt - \frac{1}{N}\int_b^{Na+b} f(t)\,dt)^2
\end{aligned}$$

is maximal. Then let

$$\begin{aligned}
c_i &= c_{b,a,i} = \frac{k_{b,a,i} - \frac{1}{N}\sum_{i=0}^{N-1} k_{b,a,i}}{\|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}} \\
&= \frac{\int_{ia+b}^{(i+1)a+b} f(t)\,dt - \frac{1}{N}\int_b^{Na+b} f(t)\,dt}{\|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}}.
\end{aligned}$$

If $a_0 = T/N$, $\psi_{b,T/N}^{c_{b,T/N}}$ fills out exactly one period of f, i.e., supp $\psi_{b,a} =$ "supp f". In this special case we have

$$\frac{1}{N}\int_b^{NT/N+b} f(t)\,dt = \frac{1}{N}\int_b^{T+b} f(t)\,dt = \frac{1}{N}\int_0^{T} f(t)\,dt$$

which is independent of b.

Note that the optimization process depends on the choice of p.

4.2 Examples

Example **??** and Example **??** illustrate how to apply Theorem **??**.

Example 4.3. Figure **??**.A shows the 1–periodic signal $\sin(2\pi x) + \sin(4\pi x) + \sin(6\pi x) + \sin(8\pi x) + \sin(10\pi x) + \sin(12\pi x)$, sampled at 20 samples per unit. Fixing $N = 8$, we calculate $k(b, a) = \|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}$ for this signal. The result is displayed in Figure **??**.B.

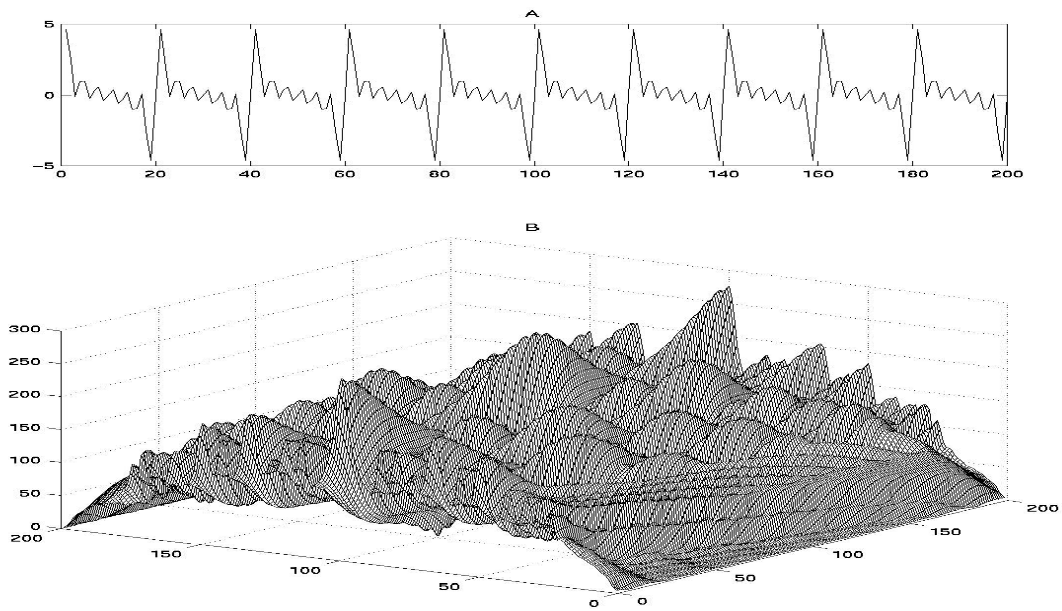

Figure 4.1. A: $\sin(2\pi x)+\sin(4\pi x)+\sin(6\pi x)+\sin(8\pi x)+\sin(10\pi x)+\sin(12\pi x)$, sampled at 20 samples per unit. B: $k(b, a) = \|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}$ for $N = 8$.

Figure **??** illustrates the dependence of the piecewise constant wavelet on the choice of the normalization constant p. For different p, but the same signal f, we obtain different maxima in $a^{\frac{1}{p}} \|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}$, whose location indicate the optimal piecewise constant wavelet for the L^p – normalized wavelet transformations. Figure **??**.A and Figure **??**.B display the optimal piecewise constant wavelets for $p = 1$ to $p = 2.4$. For $p > 2.4$ we continue to obtain the same wavelet as for $p = 2.4$. The optimal piecewise constant wavelets for $p = 1$, $p = 1.75$, $p = 2$, and $p = 2.2$ are shown separately below Figure **??**.A and **??**.B.

Example 4.4. Theorem **??** is applied to the epileptic seizure problem in Figure **??**. After simulating an expected period, in our case the seizure period of an individual patient, we define the periodic function F associated with the seizure period. F is sampled at 130 samples per period for subsequent calculations with the projection P_H. We choose $N = 5$ and calculate $k(b, a) = \|P_H(k_{b,a})\|_{l^2(\mathbf{C}^N)}$. For the normalization constants $p = 1$, $p = 1.35$, and $p = 2$, we obtain distinct optimal piecewise constant wavelets.

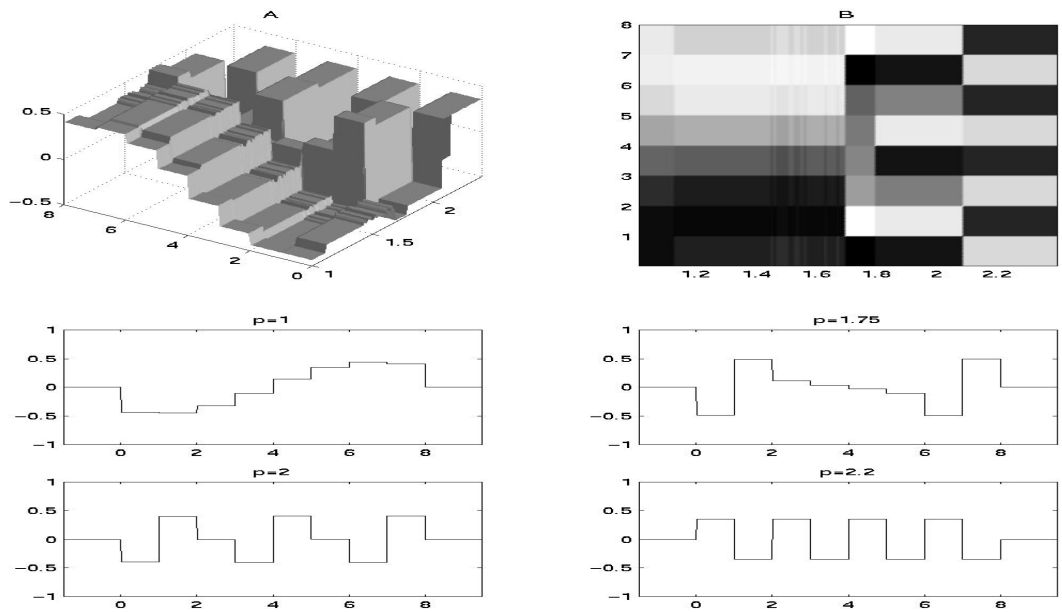

Figure 4.2. Optimal piecewise constant wavelets for $\sin(2\pi x) + \sin(4\pi x) + \sin(6\pi x) + \sin(8\pi x) + \sin(10\pi x) + \sin(12\pi x)$, $p = 1$ to $p = 2.4$, $N = 8$.

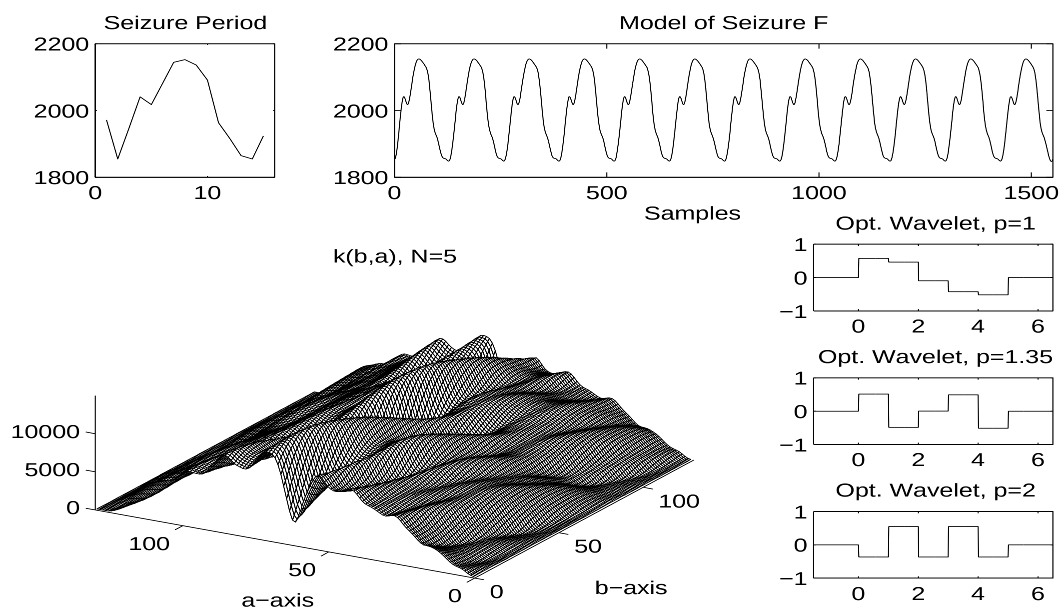

Figure 4.3. Construction of the optimal piecewise constant wavelet in the epileptic seizure case.

Chapter 5

Optimal Piecewise Constant Wavelets with Additional Properties

In some applications, the signal might have features we want to bear in mind when constructing "optimal" piecewise constant wavelets.

PROBLEM 5.1. Suppose, the periodic signal is symmetric or almost "symmetric" with respect to a reference point $t_0 \in [0, T]$ ("odd signal") or symmetric with respect to a reference axis $t = t_0$ ("even signal"). We would like the constructed wavelet to have the corresponding symmetric form, i.e., we would like to construct an optimal even piecewise constant wavelet or an optimal odd piecewise constant wavelet in order to capitalize on Proposition **??**.

PROBLEM 5.2. We would like the waveletgram obtained through the constructed piecewise constant wavelet to be resistant to some specific background behavior in the signal.

PROBLEM 5.3. Our signal might carry two periodic components which we want to analyze separately. Here, the goal is to construct a pair of piecewise constant wavelets which are sensitive in detecting one of the components and overlooking the other.

PROBLEM 5.4. One period of the signal might have parts where it is slowly varying and other parts with high variance. The associated wavelet should focus toward the fast varying part and allow many different values there, while in other parts a few values might be sufficient.

The question of whether we can construct piecewise constant wavelets which take into account a specific feature of the signal has to be answered individually. Nevertheless, a small contribution to the general case can be made.

Theorem **??** generalizes Theorem **??** and presents a useful tool in solving problems as stated above. Proposition **??** and Proposition **??** use this theorem to give useful solution to problems of the kind described in Problem **??** and Problem **??**, respectively. They further illustrate how solutions to some problems can be found. The method is based on Lemma **??** and the fact that the optimization process in Theorem **??** can be applied if we replace H by any subspace U of $\mathbf{C}^N$ with $U \subset H$.

THEOREM 5.5. *Let $p > 1$ and $f \in L^\infty(\mathbf{R})$, or $p \geq 1$, $f \in L^1(\mathbf{T}_T)$, and each $x \in \mathbf{R}$ is a Lebesque point of f. Let $N \in \mathbf{N}$ and let $k_{b,a}$ and H be defined as in Theorem ??. If U is a subspace of $\mathbf{C}^N$, then there exists $(b_0, a_0) \in \mathbf{R} \times \mathbf{R}^+$ such that*

$$a_0^{-\frac{1}{p}} \left\| P_{U\cap H}(k_{b_0,a_0}) \right\|_{l^2(\mathbf{C}^N)} = \max_{(b,a)\in\mathbf{R}\times\mathbf{R}^+} a^{-\frac{1}{p}} \left\| P_{U\cap H}(k_{b,a}) \right\|_{l^2(\mathbf{C}^N)},$$

where $P_{U\cap H}$ is the orthogonal projection of $\mathbf{C}^N$ onto the subspace $U \cap H$. By setting

$$c_0 = \frac{P_{U\cap H}(k_{b_0,a_0})}{\left\| P_{U\cap H}(k_{b_0,a_0}) \right\|_{l^2(\mathbf{C}^N)}},$$

we obtain

$$|W_f^{\psi^{c_0}}(b_0, a_0)| \geq |W_f^{\psi^c}(b, a)|$$

for all $(b, a) \in \mathbf{R} \times \mathbf{R}^+$, $c \in U$, and ψ^c satisfying (??),(??),(??).

Proof. The first steps of the proof of Theorem ?? can easily be generalized to the setting of Theorem ?? by replacing H by $U \cap H$.

It remains to show that the maximum exists. For this, note that we proved that $k_{b,a}$ is T periodic in b. This implies that $P_{U\cap H}(k_{b,a})$ is T periodic in b. Essentially, we also showed that $P_H(k_{b,a})$ is T periodic in a. By the definition of orthogonal projections we have

$$P_{U\cap H}(k_{b,a}) = P_U(P_H(k_{b,a}))$$

and therefore $P_{U\cap H}(k_{b,a})$ is T periodic in a.

We can conclude the existence of the maximum by continuing to follow the proof of Theorem ?? and by using the fact that

$$\left\| P_{U\cap H}(k_{b,a}) \right\|_{l^2(\mathbf{C}^N)} \leq \left\| P_H(k_{b,a}) \right\|_{l^2(\mathbf{C}^N)}.$$

$\square$

Solving a given problem can be approached by defining the subspace U such that $c \in U$ if and only if ψ^c has the desired properties. Of course, such a subspace might not exist.

The problem described in Problem ?? can be quantified and resolved in the following way.

PROPOSITION 5.6.

a. For $k = 1, \ldots, N$, define $v_k \in \mathbf{C}^{2N}$ by $v_k^i = \delta_{i,k} - \delta_{2N-i+1,k}$ for $i = 1, \ldots, 2N$. Let $U^e = \operatorname{span}\{v_1, \ldots, v_N\}^\perp$. Then $c \in U$ if and only if ψ^c is even.

b. For $k = 1, \ldots, N$, define $v_k \in \mathbf{C}^{2N}$ by $v_k^i = \delta_{i,k} + \delta_{2N-i+1,k}$ for $i = 1, \ldots, 2N$. Let $U^o = \operatorname{span}\{v_1, \ldots, v_N\}^\perp$. Then $c \in U$ if and only if ψ^c is odd.

Proof. We shall prove part a. The proof of part b is similar.

Clearly, $c \in U^e$ if and only if $c \perp v_k$ for $k = 1, \ldots, N$, i.e.,

$$0 = \langle c, v_k \rangle = c_k - c_{2N-k+1} \quad \text{for} \quad k = 1, \ldots, N$$

This holds if and only if $c_k = c_{2N-k+1}$, that is, if and only if ψ^c is even. $\square$

Let us discuss further possible features of piecewise constant wavelets. The property $c \in H$ implies that if f is a constant function, then $W_f^{\psi^c}(b,a) = 0$ for all $(b,a) \in \mathbf{R} \times \mathbf{R}^+$. The tophat wavelet ψ^{top} is defined by $top = \frac{1}{\sqrt{6}}(1, -2, 1)$ and has the property that if f is a linear function, $W_f^{\psi^{top}}(b,a) = 0$ for all $(b,a) \in \mathbf{R} \times \mathbf{R}^+$. The Haar wavelet does not posses this property. We are led to the question of whether it is possible to construct a subspace $U \subset \mathbf{C}^N$ such that for any $c \in U$ we have the property that $W_f^{\psi^c}(b,a) = 0$ for all $(b,a) \in \mathbf{R} \times \mathbf{R}^+$ and for any polynomial f of degree less or equal some given n. The answer to this question is affirmative as the following proposition shows.

Proposition 5.7. *For $n \le N-2$, define $v_k = (1^k, 2^k, 3^k, \ldots, N^k) \in \mathbf{C}^N$ for $k = 0, \ldots, n$ and $U^n = \operatorname{span}\{v_0, \ldots, v_n\}^\perp$. Then $c \in U^n$ if and only if ψ^c has the property that for any polynomial f of degree less or equal n $W_f^{\psi^c}(b,a) = 0$ for $(b,a) \in \mathbf{R} \times \mathbf{R}^+$.*

Proof. For $n \le N-2$, let

$$M^n = \{c \in \mathbf{C}^N : W_f^{\psi^c} \equiv 0 \text{ for any polynomial } f \text{ with } \deg(f) \le n\}.$$

We need to show that $M^n = U^n$ for $n \le N-2$. Note that M^n is a vector space for $n \le N-2$. In fact, if c, $d \in M^n$, and $\lambda \in \mathbf{C}$, then

$$\begin{aligned}
W_f^{\psi^{\lambda c+d}}(b,a) &= a^{-\frac{1}{p}} \int \psi^{\lambda c+d}(\frac{t-b}{a}) f(t)\, dt \\
&= a^{-\frac{1}{p}} \sum_{i=0}^{N-1} (\lambda c_i + d_i) \int_{ia+b}^{(i+1)a+b} f(t)\, dt \\
&= \lambda\, a^{-\frac{1}{p}} \sum_{i=0}^{N-1} c_i \int_{ia+b}^{(i+1)a+b} f(t)\, dt + a^{-\frac{1}{p}} \sum_{i=0}^{N-1}\ d_i \int_{ia+b}^{(i+1)a+b} f(t)\, dt \\
&= \lambda W_f^{\psi^c}(b,a) + W_f^{\psi^d}(b,a) \\
&= 0 + 0 = 0.
\end{aligned}$$

Clearly, by definition, $M^n \subset M^{n-1}$ for $1 \le n \le N-2$, and $U^n \subset U^{n-1}$ for $1 \le n \le N-2$; in fact, $\operatorname{span}\{v_0, \ldots, v_{n-1}\} \subset \operatorname{span}\{v_0, \ldots, v_{n-1}, v_n\}$, and therefore

$$U^n = \operatorname{span}\{v_0, \ldots, v_{n-1}, v_n\}^\perp \subset \operatorname{span}\{v_0, \ldots, v_{n-1}\}^\perp = U^{n-1}.$$

Also, $\dim U^n + 1 = \dim U^{n-1}$ for $1 \leq n \leq N-2$, since $\{v_0, \ldots, v_k\}$ is a set of $k+1$ linear independent vectors, and therefore

$$\begin{aligned}
\dim U^n &= N - \dim(\operatorname{span}\{v_0, \ldots, v_{n-1}, v_n\}) = N - (n+1) \\
&= N - n - 1 = N - (n - 1 + 1) - 1 \\
&= N - \dim(\operatorname{span}\{v_0, \ldots, v_{n-1}\}) - 1 \\
&= \dim U^{n-1} - 1.
\end{aligned}$$

We shall prove that $M^n = U^n$ for $1 \leq n \leq N-2$ by induction.

For $n = 0$, we have $U^0 = H = M^0$ and the result holds. Let us assume the result is true for $n-1$, i.e., $M^{n-1} = U^{n-1}$. By the induction hypothesis and the definition of M^n, we have $M^n \subset M^{n-1} = U^{n-1}$.

To show $M^n = U^n$, it suffices to prove

$$U^n \subset M^n \tag{5.1}$$

and

$$U^{n-1} \setminus M^n \neq \emptyset. \tag{5.2}$$

In fact, assuming (**??**) and (**??**), we have $U^n \subset M^n \subsetneq U^{n-1}$, and, since $\dim U^n + 1 = \dim U^{n-1}$ and M^n is a vector space, we obtain $U^n = M^n$.

To show (**??**), i.e., $U^n \subset M^n$, let us first calculate $W_f^{\psi^c}$ for $f(t) = t^n$ and $c \in \mathbf{C}^N$:

$$
\begin{aligned}
W_f^{\psi^c}(b,a) &= a^{-\frac{1}{p}} \sum_{i=0}^{N-1} c_i \int_{ia+b}^{(i+1)a+b} t^n \, dt \\
&= a^{-\frac{1}{p}} \frac{1}{n+1} \sum_{i=0}^{N-1} c_i ((ia+b+a)^{n+1} - (ia+b)^{n+1}) \\
&= a^{-\frac{1}{p}} \frac{1}{n+1} \sum_{i=0}^{N-1} c_i (\sum_{k=0}^{n+1} \binom{n+1}{k} (ia+b)^{n+1-k} a^k - (ia+b)^{n+1}) \\
&= a^{-\frac{1}{p}} \frac{1}{n+1} \sum_{i=0}^{N-1} c_i \sum_{k=0}^{n} \binom{n+1}{k} (ia+b)^k a^{n+1-k} \\
&= a^{-\frac{1}{p}} \frac{1}{n+1} \sum_{i=0}^{N-1} c_i \sum_{k=0}^{n} \binom{n+1}{k} \sum_{l=0}^{k} \binom{k}{l} (ia)^l b^{k-l} a^{n+1-k} \\
&= a^{-\frac{1}{p}} \frac{1}{n+1} \sum_{i=0}^{N-1} \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n+1}{k} \binom{k}{l} c_i (ia)^l b^{k-l} a^{n+1-k} \\
&= a^{-\frac{1}{p}} \frac{1}{n+1} \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n+1}{k} \binom{k}{l} b^{k-l} a^{n+1-k} a^l \sum_{i=0}^{N-1} c_i i^l \\
&= a^{-\frac{1}{p}} \frac{1}{n+1} \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n+1}{k} \binom{k}{l} b^{k-l} a^{n+1-k+l} \langle c, v_l \rangle \\
&= a^{-\frac{1}{p}} \frac{1}{n+1} \sum_{l=0}^{n} \sum_{k=l}^{n} \binom{n+1}{k} \binom{k}{l} b^{k-l} a^{n+1-k+l} \langle c, v_l \rangle \\
&= a^{-\frac{1}{p}} \frac{1}{n+1} \sum_{l=0}^{n} S_l(b,a) \langle c, v_l \rangle,
\end{aligned}
$$

where

$$
S_l(b,a) = \sum_{k=l}^{n} \binom{n+1}{k} \binom{k}{l} b^{k-l} a^{n+1-k+l}
$$

for $l = 0, \ldots, n$.

Now, let $c \in U^n$ and let f be a polynomial with $\deg f \leq n$. Then $f(t) = g(t) + \lambda t^n$, where g is a polynomial with $\deg g \leq n-1$ and $\lambda \in \mathbf{C}$.

Since $U^n \subset U^{n-1}$ implies $c \in U^{n-1}$, we compute

$$\begin{aligned}
W_f^{\psi^c}(b,a) &= a^{-\frac{1}{p}} \int \psi^c(\frac{t-b}{a}) f(t)\, dt \\
&= a^{-\frac{1}{p}} \int \psi^c(\frac{t-b}{a}) (g(t) + \lambda t^n)\, dt \\
&= a^{-\frac{1}{p}} \int \psi^c(\frac{t-b}{a}) g(t)\, dt + \int \psi^c(\frac{t-b}{a}) \lambda t^n)\, dt \\
&= W_g^{\psi^c}(b,a) + 0 = 0,
\end{aligned}$$

and therefore $U^n \subset M^n$, i.e., (**??**) is obtained.

It remains to prove (**??**), i.e., to show that

$$U^{n-1} \setminus M^n \neq \emptyset.$$

In fact, let c in $U^{n-1} - U^n \neq \emptyset$. For $f(t) = t^n$ we have

$$\begin{aligned}
S_l(0,1) &= \sum_{k=l}^{n} \binom{n+1}{k} \binom{k}{l} 0^{k-l} 1^{n+1-k+l} \\
&= \binom{n+1}{n} \binom{n}{l} 1 \\
&= (n+1) \binom{n}{l}
\end{aligned}$$

for $l = 0, \ldots, n$, and, therefore,

$$\begin{aligned}
W_f^{\psi^c}(0,1) &= 1^{-\frac{1}{p}} \frac{1}{n+1} \sum_{l=0}^{n} S_l(0,1) \langle c, v_l \rangle \\
&= \frac{1}{n+1} \sum_{l=0}^{n} (n+1) \binom{n}{l} \langle c, v_l \rangle \\
&= \binom{n}{n} \langle c, v_n \rangle \\
&= \langle c, v_n \rangle \\
&\neq 0.
\end{aligned}$$

This holds since $c \in U^{n-1}$, and therefore $\langle c, v_l \rangle = 0$ for $l = 0, \ldots, n-1$, and since $c \notin U^n$, and therefore $\langle c, v_n \rangle \neq 0$. This proves that $c \notin M^n$, and therefore $c \in U^{n-1} - M^n$. □

EXAMPLE 5.8. For $N = 3$, $u_\pm = \pm\frac{1}{\sqrt{6}}(1, -2, 1)$ define the only normalized real valued piecewise constant wavelet with 2 vanishing moments. Earlier, we referred to u_+ as tophat wavelet.

EXAMPLE 5.9. Let us fix $N = 5$ and let us construct a piecewise constant wavelet ψ^c such that $W_f^{\psi^c} \equiv 0$ for any f being a polynomial of degree less or equal 3.

To solve this, we need to find an orthonormal basis of

$$U^3 = \operatorname{span}\{(1,1,1,1,1),(1,2,3,4,5),(1,4,9,16,25),(1,8,27,64,125)\}^{\perp}.$$

Clearly $\dim U^3 = 1$, therefore we are looking for a single vector u, $\|u\|_{l^2(\mathbf{C}^N)} = 1$ such that $u \perp w$ for all $w \in U^{3\perp}$.

For this, using Gram–Schmidt orthogonalization, we construct an orthonormal basis B of W. We obtain

$$B = \{\frac{1}{\sqrt{5}}(1,1,1,1,1), \frac{1}{\sqrt{10}}(-2,-1,0,1,2), \frac{1}{\sqrt{14}}(2,-1,-2,-1,2), \frac{1}{\sqrt{10}}(-1,2,0,-2,1)\}.$$

Again, using Gram-Schmidt orthogonalization to complete B to a orthonormal basis B' of $\mathbf{R}^5$ we get

$$B' = \{\frac{1}{\sqrt{5}}(1,1,1,1,1), \frac{1}{\sqrt{10}}(-2,-1,0,1,2), \frac{1}{\sqrt{14}}(2,-1,-2,-1,2), \frac{1}{\sqrt{10}}(-1,2,0,-2,1), \frac{1}{\sqrt{70}}(1,-4,6,-4,1)\}.$$

This implies $u = \frac{1}{\sqrt{70}}(1,-4,6,-4,1)$.

Chapter 6

Periodicity Detection

In this chapter, we shall develop an approach to detect lattice patterns of relative maxima in periodic waveletgrams. If this pattern is the result of periodic components in the analyzed signal, it can reveal occurrence and period of these components. Our approach is based on averaging methods.

We shall consider both, non–normalized as well as $L^p(\mathbf{R})$-normalized, $1 \leq p < \infty$, versions of the continuous wavelet transformation, i.e.,

$$W_f^{\psi}(b,a) = a^{-\frac{1}{p}} \int_{\mathbf{R}} \psi^c(\frac{t-b}{a}) f(t)\, dt,$$

$(b,a) \in \mathbf{R} \times \mathbf{R}^+$, in the normalized case.

In Section **??**, we shall discuss methods arising for general piecewise constant wavelets (Proposition **??**). In Section **??**, we shall show how to use Proposition **??** if the wavelet we are using is even or odd. Section **??** is devoted examples.

6.1 Using a Piecewise Constant Wavelet

Let f be a T_0–periodic function, and let ψ^c be an even piecewise constant wavelet of degree one, i.e., $\psi^c|_{[i,i+1)} = c_i$ for $i = 0, \ldots, N-1$, $c = (c_0, c_1, \ldots, c_{N-1}) \in \mathbf{C}^N$, N fixed.

Proposition **??** implies that the waveletgram of the non–normalized wavelet transform W_f is identical on each cell

$$[b + iT_0, b + (i+1)T_0] \times [jMT_0, (j+1)MT_0]$$

for $i \in \mathbf{Z}$ and $j \in \mathbf{N_0}$. Figure **??** shows a non–normalized wavelet transform in topographical form.

6.1.1 Non–normalized Wavelet Transform

If $R, Q \in \mathbf{N}$, then the periodicities of the non–normalized wavelet transform imply that

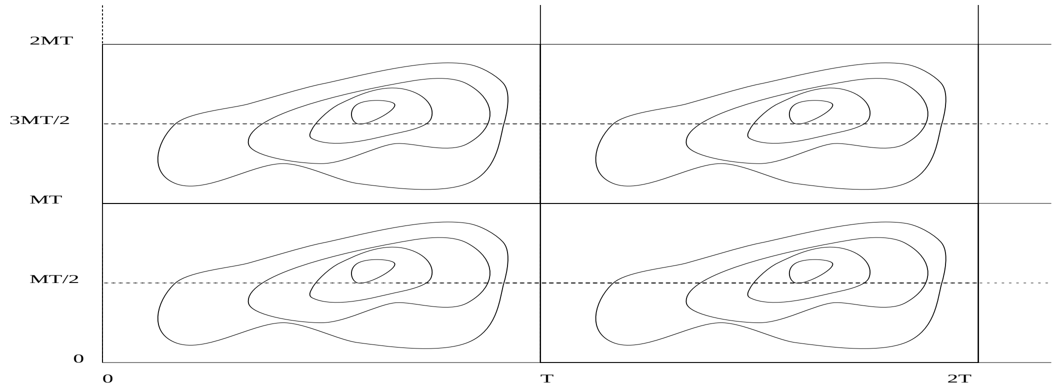

Figure 6.1. Time–scale periodicity in topographical form. .

$$\begin{aligned} W_f^{\psi^c}(b,a) &= \frac{1}{(Q+1)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} W_f^{\psi^c}(b,a) \\ &= \frac{1}{(Q+1)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} W_f^{\psi^c}(b+rT_0, a+qT_0). \end{aligned} \tag{6.1}$$

Suppose we are given a noisy signal s of the form $s(t) = f(t) + N(t)$ where f is T_0–periodic and N is noise. In order to gain knowledge of the period T_0 of f, we define the average

$$U_s^{R,Q}(b,a,T) = \frac{1}{(Q+1)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} W_s^{\psi^c}(b+rT, a+qT)$$

where $T \in \mathbf{R}^+, a \in (0,T)$, and $b \in [0,T)$. Clearly, by Proposition **??** and Equation (**??**), we have

$$U_f^{R,Q}(b,a,T_0) = W_f^{\psi^c}(b,a)$$

for the periodic signal f. Define

$$Z_s^{R,Q}(T) = \sup_{a\in[0,T),b\in[0,T)} |U_s^{R,Q}(b,a,T)|.$$

Therefore,

$$Z_f^{R,Q}(T_0) = \sup_{a\in[0,T),b\in[0,T)} |W_f^{\psi^c}(b,a)|,$$

which we maximized in Chapter **??**. Further, we expect that $Z_f^{R,Q}(T)$ is ”small” for $T \neq k \cdot T_0, k \in \mathbf{N}$ and Q and R large.

Note that for the noisy signal $s = f + N$, we further expect that

$$Z_s^{R,Q}(T_0) \approx \sup_{a,b \in (0,T)} |W_f^{\psi^c}(b,a)|$$

and $Z_s^{R,Q}(T)$ is small if $T \neq T_0$.

6.1.2 Normalized Wavelet Transform

In order to analyze an $L^p(\mathbf{R}),\ 1 \leq p < \infty$ normalized wavelet transform,we define

$$v^Q(a,T) = a^{\frac{1}{p}} \sum_{q=0}^{Q} (a+qT)^{-\frac{1}{p}},$$

where $a, T \in \mathbf{R}^+$. We compute

$$\begin{aligned}
W_f^{\psi^c}(b,a) &= W_f^{\psi^c}(b,a) \frac{1}{v^Q(a,T_0)} a^{\frac{1}{p}} \sum_{q=0}^{Q} (a+qT_0)^{-\frac{1}{p}} \\
&= \frac{1}{v^Q(a,T_0)} \sum_{q=0}^{Q} (a+qT_0)^{-\frac{1}{p}} a^{\frac{1}{p}} W_f^{\psi^c}(b,a) \\
&= \frac{1}{v^Q(a,T_0)} \sum_{q=0}^{Q} (a+qT_0)^{-\frac{1}{p}} (a+qT_0)^{\frac{1}{p}} W_f^{\psi^c}(b,a+qT_0) \\
&= \frac{1}{v^Q(a,T_0)} \sum_{q=0}^{Q} W_f^{\psi^c}(b,a+qT_0) \\
&= \frac{1}{v^Q(a,T_0)} \sum_{q=0}^{Q} \frac{1}{2R+1} \sum_{r=-R}^{R} W_f^{\psi^c}(b+rT_0,a+qT_0) \\
&= \frac{1}{v^Q(a,T_0)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} W_f^{\psi^c}(b+rT_0,a+qT_0).
\end{aligned}$$

As in the non–normalized case, we use this to define

$$V_s^{R,Q}(b,a,T) = \frac{1}{v^Q(a,T)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} W_s^{\psi^c}(b+rT,a+qT)$$

for any signal s, where $T \in \mathbf{R}^+, a \in (0,T)$, and $b \in [0,T)$.

For the T_0–periodic signal f we have

$$V_f^{R,Q}(b,a,T_0) = W_f^{\psi^c}(b,a).$$

Thus, defining

$$Z_s^{R,Q}(T) = \sup_{a\in[0,T),b\in[0,T)} |V_s^{R,Q}(b,a,T)|,$$

for any signal s, we have

$$Z_f^{R,Q}(T_0) = \sup_{a\in[0,T),b\in[0,T)} |W_f^{\psi^c}(b,a)|.$$

Note that, in this case, the assertion that $Z_f^{R,Q}(T)$ is "small" for $T \neq T_0$ and Q, R large, is supported by the fact that if $a, T \in \mathbf{R}^+$, then

$$\begin{aligned} \lim_{Q\to\infty} v^Q(a,T) &= \lim_{Q\to\infty} a^{\frac{1}{p}} \sum_{q=0}^{Q} (a+qT)^{-\frac{1}{p}} \\ &= a^{\frac{1}{p}} \lim_{Q\to\infty} \sum_{q=0}^{Q} \left(\frac{1}{a+qT}\right)^{\frac{1}{p}} \\ &= \infty. \end{aligned}$$

6.2 Using an Even or Odd Piecewise Constant Wavelet

Let f be a T_0 periodic function, and let ψ^c be either an even piecewise constant wavelet of degree 1, i.e., $\psi^c|_{[i,i+1)} = \psi^c|_{[-i-1,-i)} = c_i$ for $i = 0,\ldots,N-1$, $c = (c_0, c_1, \ldots, c_{N-1}) \in \mathbf{C}^N$, or an odd piecewise constant wavelet of degree 1, i.e., $\psi^c|_{[i,i+1)} = -\psi^c|_{[-i-1,-i)} = c_i$ for $i = 0,\ldots,N-1$, $c = (c_0, c_1, \ldots, c_{N-1}) \in \mathbf{C}^N$, where N is fixed.

Due to Proposition **??**, the non–normalized wavelet transform $W_f^{\psi^c}$ is in both cases essentially the same, i.e., the same up to a flip and a sign, on the cells

$$[b + iT_0, b + (i+1)T_0] \times [jMT_0/2, (j+1)MT_0/2]$$

for $i \in \mathbf{Z}$ and $j \in \mathbf{N_0}$. Figure **??** shows the resulting waveletgram.

6.2.1 Non–normalized Wavelet Transform

Now we define the following alternative for averaging:

$$\begin{aligned} U_f^{R,Q}(b,a,T) &= \frac{1}{2(Q+1)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} (W_f^{\psi^c}(b+rT, a+qT) \\ &\quad \pm W_f^{\psi^c}(b+rT, T-a+qT)), \end{aligned}$$

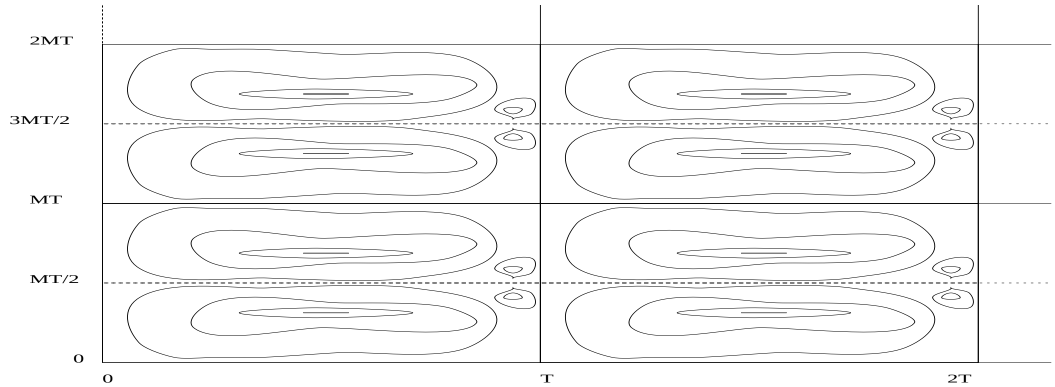

Figure 6.2. Time–scale periodicity for odd or even wavelets in topographical form.

where $T \in \mathbf{R}^+, a \in (0, T/2)$ and $b \in [0, T)$. Here, and in the following, $\pm$ denotes $-$ if ψ^c is even and $+$ if ψ^c is odd.

By Proposition **??**, we have

$$\begin{aligned} U_f^{R,Q}(b,a,T_0) &= \frac{1}{2(Q+1)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} (W_f^{\psi^c}(b + rT_0, a + qT_0) \\ &\quad \pm W_f^{\psi^c}(b + rT_0, T_0 - a + qT_0)) \\ &= \frac{1}{2(Q+1)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} (W_f^{\psi^c}(b,a) \pm W_f^{\psi^c}(b, T_0 - a)) \\ &= \frac{1}{2(Q+1)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} (W_f^{\psi^c}(b,a) + W_f^{\psi^c}(b,a)) \\ &= W_f^{\psi^c}(b,a) \end{aligned}$$

for any T_0 periodic function f.

We precede as before by defining the test statistic

$$Z_s^{R,Q}(T) = \sup_{a \in [0,T/2), b \in [0,T)} |U_s^{R,Q}(b,a,T)|$$

for any signal s.

6.2.2 Normalized Wavelet Transform

If we are using an $L^p(\mathbf{R}),\ 1 \le p < \infty$ normalized wavelet transform, let us define

$$v^Q(a,T) = a^{\frac{1}{p}} \sum_{q=0}^{Q} ((a + qT)^{-\frac{1}{p}} + (T - a + qT)^{-\frac{1}{p}}), a, T \in \mathbf{R}^+.$$

and as before

$$\begin{aligned} V_s^{R,Q}(b,a,T) &= \frac{1}{v^Q(a,T)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} (W_s^{\psi^c}(b+rT, a+qT) \\ &\quad \pm W_f^{\psi^c}(b+rT, T-a+qT)) \end{aligned}$$

with $T \in \mathbf{R}^+,\ a \in (0,T)$ and $b \in [0,T)$.

For $T = T_0$, we get

$$\begin{aligned} V_s^{R,Q}(b,a,T_0) &= \frac{1}{v^Q(a,T_0)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} (W_s^{\psi^c}(b+rT_0, a+qT_0) \\ &\quad \pm W_f^{\psi^c}(b+rT_0, T_0-a+qT_0)) \\ &= \frac{1}{v^Q(a,T_0)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} (W_s^{\psi^c}(b, a+qT_0) \\ &\quad \pm W_f^{\psi^c}(b, T_0-a+qT_0)) \\ &= \frac{1}{v^Q(a,T_0)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} \\ &\quad ((a+qT_0)^{-\frac{1}{p}}(a+qT_0)^{\frac{1}{p}} W_s^{\psi^c}(b, a+qT_0) \\ &\quad \pm (T_0-a+qT_0)^{-\frac{1}{p}}(T_0-a+qT_0)^{\frac{1}{p}} W_f^{\psi^c}(b, T_0-a+qT_0)) \\ &= \frac{1}{v^Q(a,T_0)(2R+1)} \sum_{r=-R}^{R} \sum_{q=0}^{Q} ((a+qT_0)^{-\frac{1}{p}} a^{\frac{1}{p}} W_s^{\psi^c}(b,a)) \\ &\quad (\pm)(\pm)(T_0-a+qT_0)^{-\frac{1}{p}} a^{\frac{1}{p}} W_f^{\psi^c}(b,a)) \\ &= W_f^{\psi^c}(b,a) \frac{1}{v^Q(a,T_0)(2R+1)} \sum_{r=-R}^{R} a^{\frac{1}{p}} \sum_{q=0}^{Q} ((a+qT_0)^{-\frac{1}{p}} \\ &\quad + (T_0-a+qT_0)^{-\frac{1}{p}}) \\ &= W_f^{\psi^c}(b,a) \frac{1}{v^Q(a,T_0)(2R+1)} \sum_{r=-R}^{R} v^Q(a,T_0) \\ &= W_f^{\psi^c}(b,a). \end{aligned}$$

We can conclude as in Section **??**.

6.3 Examples

In Example **??**, Example **??**, and Example **??** we apply this method to the signals introduced in Example **??** and Example **??**.

Example 6.1. Figure ??.A shows the original signal $\sin(2\pi x) + \sin(4\pi x) + \sin(6\pi x) + \sin(8\pi x) + \sin(10\pi x) + \sin(12\pi x)$, and Figure ??.B represents the absolute value of its Fourier transform. Figure ??.C displays the normalized ($p = 1.75$) wavelet transform of this signal, obtained using the optimal piecewise constant wavelet displayed in Figure ?? ($N = 8$). $Z_F^{R,Q}(T)$ is then calculated for $T = 1, \ldots, 25$ and shown in Figure ??.D. The location of the maximum of Z implies the occurence of the periodic signal with period length of 20 samples.

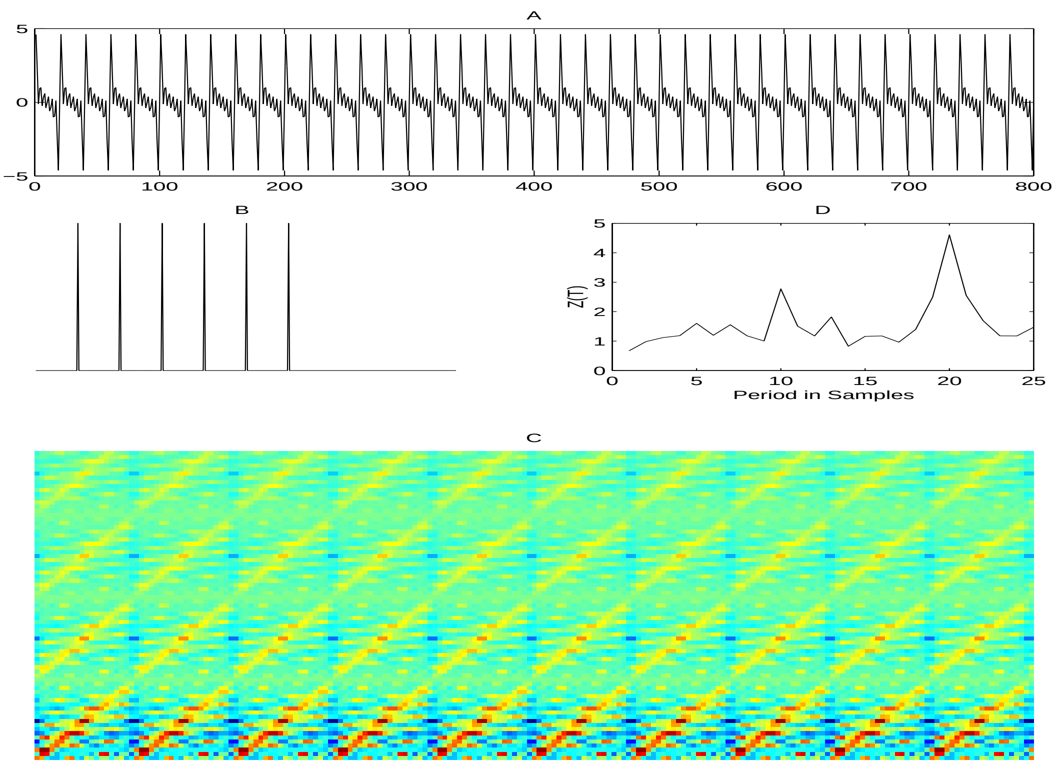

Figure 6.3. A: $\sin(2\pi x)+\sin(4\pi x)+\sin(6\pi x)+\sin(8\pi x)+\sin(10\pi x)+\sin(12\pi x)$. B: Absolute value of its discrete Fourier transform. C: Wavelet transform using the optimal piecewise constant wavelet obtained for $p = 1.75$ and $N = 8$. D: $Z_F^{R,Q}(T)$.

This technique can also be applied successfully to synthesized noisy data as is illustrated in Example ??.

Example 6.2. In this case white noise is added to the signal in Example ?? which is displayed in Figure ??. The same wavelet as in Example ?? is applied. The location of the maximum of Z is clearly visible in Figure ??.D.

Example 6.3. The seizure signal F constructed in Example ?? and shown in Figure ?? has a periodicity characterized by its construction using 13 samples per period. We compute the $p = 1.35$ normalized wavelet transform of F. $Z_F^{R,Q}(T)$

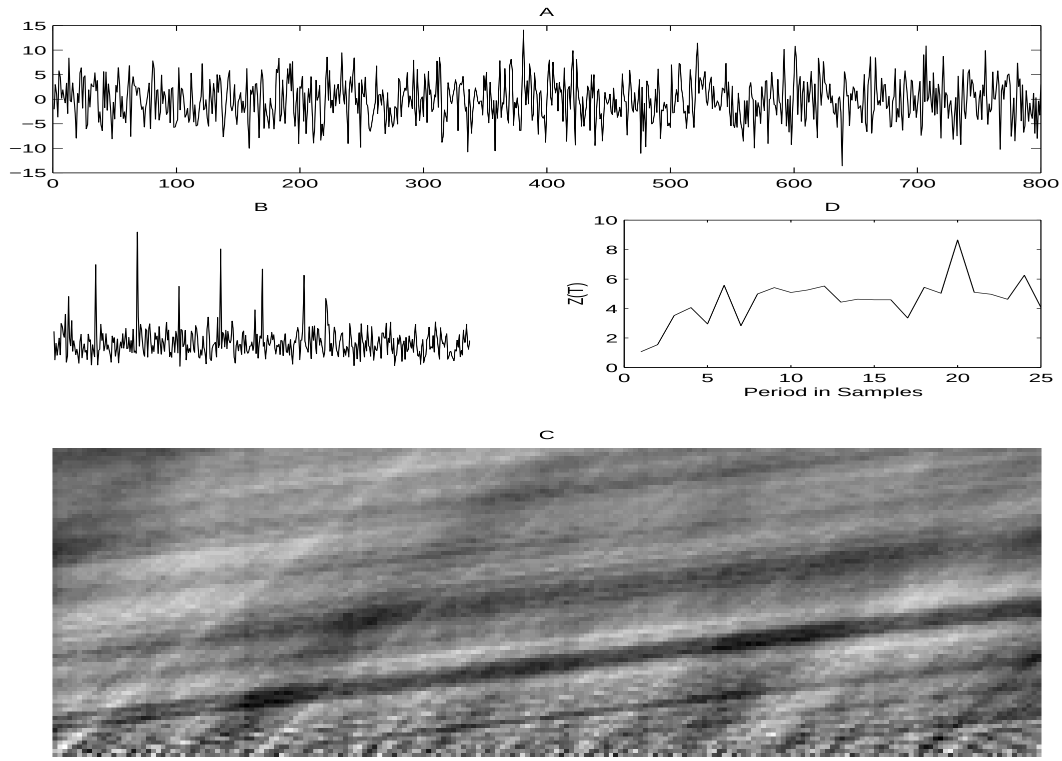

Figure 6.4. A: $\sin(2\pi x)+\sin(4\pi x)+\sin(6\pi x)+\sin(8\pi x)+\sin(10\pi x)+\sin(12\pi x)+$ white noise. B: Absolute value of its discrete Fourier transform. C: Wavelet transform using the optimal piecewise constant wavelet obtained for $p = 1.75$ and $N = 8$ in Figure **??**. D: $Z_F^{R,Q}(T)$.

is then calculated for $T = 1, \ldots, 20$. The maximum of Z in Figure **??** implies the occurrence of the periodic signal with period length of 13 samples.

Remark 6.4. In [**?**], we shall make quantitative estimates for evaluating signal to noise ratios in such experiments.

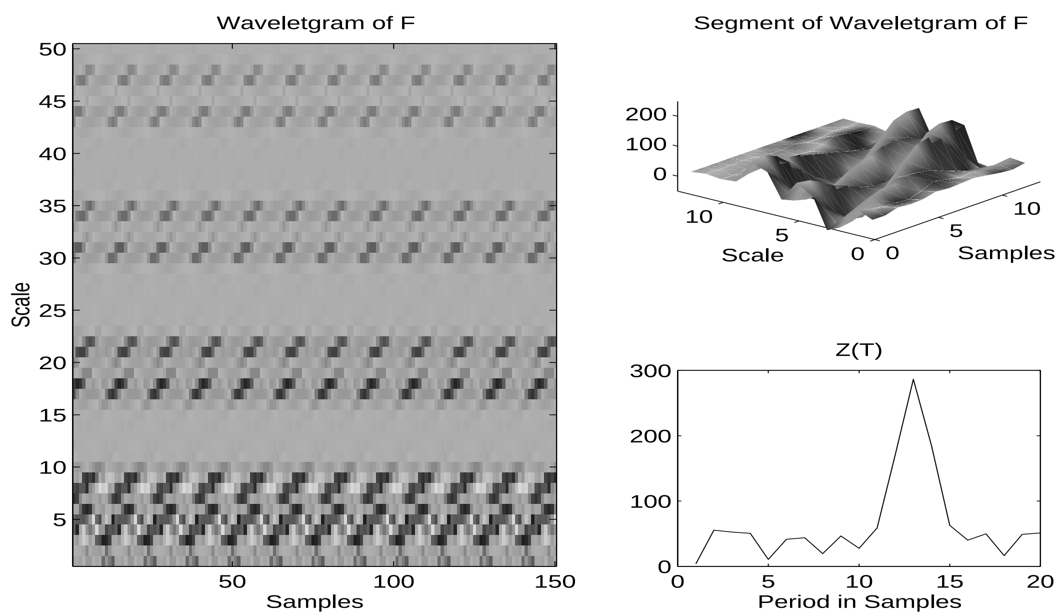

Figure 6.5. The waveletgram of the signal shown in Figure **??** sampled at 13 samples per period, and the function $Z_F^{R,Q}(T)$ indicating the periodicity of 13 samples.

Chapter 7

Implementation

In order to apply the results of the preceding chapters to the real (digital) world, we need to discretize our results.

Let us assume that we sampled a signal f and obtained the sequence $\{f[n]\}_{n\in\mathbf{Z}}$. Let ψ be a piecewise constant wavelet of degree 1. To avoid confusing notation, we let

$$\overline{\psi} = (\ldots, \overline{\psi}\,[-1]\,, \overline{\psi}\,[0]\,, \overline{\psi}\,[1]\,, \ldots)$$

be the vector representing ψ, i.e., $\overline{\psi}\,[k] = \psi(k) = c_k$ for $k \in \mathbf{Z}$.

We shall replace our continuous wavelet transform

$$W_f^\psi(b,a) = a^{-1/p} \int_{\mathbf{R}} f(t)\psi(\frac{t-b}{a})\,dt$$

with the following discretized version

$$W_f^\psi[n,m] = m^{-1/p} \sum_{k\in\mathbf{Z}} f[k]\psi(\frac{k-n}{m}) = m^{-1/p} \sum_{k\in\mathbf{Z}} f[k]\overline{\psi}\left[\left\lfloor \frac{k-n}{m} \right\rfloor\right], \qquad (7.1)$$

$m \in \mathbf{Z}^+$ and $n \in \mathbf{Z}$. $\lfloor x \rfloor$ denotes the largest integer less or equal x. The second equality of (**??**) is a consequence of the fact that ψ is a piecewise constant wavelet of degree 1.

We can easily rewrite (**??**) in the more convenient form:

$$\begin{aligned}
W_f^\psi[n,m] &= m^{-1/p} \sum_{k\in\mathbf{Z}} f[k]\overline{\psi}\left[\left\lfloor \frac{k-n}{m} \right\rfloor\right] \\
&= m^{-1/p} \begin{pmatrix} & \vdots & \vdots & & \vdots & \vdots \\ + & f[n] & \overline{\psi}\,[0] & +\ldots+ & f[n+m-1] & \overline{\psi}\,[0] \\ + & f[n+m] & \overline{\psi}\,[1] & +\ldots+ & f[n+2m-1] & \overline{\psi}\,[1] \\ + & f[n+2m] & \overline{\psi}\,[2] & +\ldots+ & f[n+3m-1] & \overline{\psi}\,[2] \\ & \vdots & \vdots & & \vdots & \vdots \end{pmatrix} \\
&= m^{-1/p} \sum_{r\in\mathbf{Z}} \left(\sum_{l=0}^{m-1} f[n+mr+l] \right) \overline{\psi}\,[r]\,. \qquad (7.2)
\end{aligned}$$

To serve as an example, we shall prove a discrete version of Proposition **??**.

PROPOSITION 7.1. *Let ψ be a piecewise constant wavelet of degree 1, and let $\{f[n]\}_{n\in\mathbf{Z}}$ be a T–periodic sequence, $T\in\mathbf{Z}^+$, i.e., $f[n+T]=f[n]$ for all $n\in\mathbf{Z}$. Then $m^{1/p}W_f^\psi[n,m]$ is T–periodic in n and T–periodic in m.*

Proof. The T–periodicity in n follows directly from (**??**) and the fact that $f[n+T]=f[n]$ for all $n\in\mathbf{Z}$. Further, setting $c=\sum_{l=0}^{T-1}f[l]$, we obtain

$$\begin{aligned}
(m+T)^{1/p}W_f^\psi[n,m+T] &= \sum_{r\in\mathbf{Z}}\left(\sum_{l=0}^{m+T-1}f[n+(m+T)r+l]\right)\overline{\psi}[r]\\
&= \sum_{r\in\mathbf{Z}}\left(\sum_{l=0}^{m+T-1}f[n+mr+l]\right)\overline{\psi}[r]\\
&= m^{1/p}W_f^\psi[n,m]+\sum_{r\in\mathbf{Z}}\left(\sum_{l=m}^{m+T-1}f[n+mr+l]\right)\overline{\psi}[r]\\
&= m^{1/p}W_f^\psi[n,m]+c\sum_{r\in\mathbf{Z}}\overline{\psi}[r]\\
&= m^{1/p}W_f^\psi[n,m].
\end{aligned}$$

□

To analyze a signal through a "continuous" wavelet transform is expensive, since we need to calculate a large number of coefficients $W_f^\psi[n,m]$. This causes redundancy and robustness to noise. For large m, the elementary operations needed to calculate $W_f^\psi[n,m]$ are of order m. The restriction to piecewise constant wavelets gives rise to a recursive procedure to obtain these coefficients. This reduces the number of calculations needed significantly. In fact, if ψ is supported on $[0,N]$, to obtain $W_f^\psi[n,m]$ from $W_f^\psi[n-1,m]$ or $W_f^\psi[n,m-1]$ requires only N multiplications regardless of how large m and hence the support of $\overline{\psi}[\lfloor\frac{\cdot-n}{m}\rfloor]$ is. We shall explain why.

In the remainder of this section, we shall omit the normalization factor $m^{-1/p}$. This factor is certainly independent of wavelet and signal and would be multiplied to $W_f^\psi[n,m]$ in the last step of an implementation.

Let us begin with the trivial case, obtaining $W_f^\psi[n,m]$ from $W_f^\psi[n-1,m]$. We have

$$\begin{aligned}
W_f^\psi[n,m] \quad - \quad & W_f^\psi[n-1,m]\\
= \quad & \sum_{r\in\mathbf{Z}}\left(\sum_{l=0}^{m-1}f[n+mr+l]-\sum_{l=0}^{m-1}f[n-1+mr+l]\right)\overline{\psi}[r]\\
= \quad & \sum_{r=0}^{N-1}\left(f[n+mr+m-1]-f[n-1+mr]\right)\overline{\psi}[r].
\end{aligned}$$

To obtain $W_f^\psi[n,m]$ from $W_f^\psi[n,m-1]$ for scales $m \geq N$ is best understood through Figure **??** and Figure **??**. Again, many products appearing in the summation of $W_f^\psi[n,m]$ in (**??**) contributed already to $W_f^\psi[n,m-1]$. We need only to make a few adjustments.

In Figure **??**, we write the part of the signal f that is relevant to obtain $W_f^\psi[n,m]$ in a rectangular pattern with N rows and m columns. We obtain the non–normalized coefficient $W_f^\psi[n,m]$ by multiplying the r-th row by $\overline{\psi}\,[r-1]$ for $r = 1, \ldots, N$ and by adding the results. This is illustrated in Figure **??**.

$\overline{\psi}[0]$ $f[n\text{-}mN{+}1]$	$f[n\text{-}mN{+}2]$					$f[n\text{-}m(N\text{-}1)]$
$\overline{\psi}[1]$ $f[n\text{-}m(N\text{-}1){+}1]$	$f[n\text{-}m(N\text{-}1){+}2]$					$f[n\text{-}m(N\text{-}2)]$
$\overline{\psi}[N-1]$ $f[n\text{-}m\ {+}1]$	$f[n\text{-}m{+}2]$					$f[n]$

Figure 7.1. Contributions of $f[n-mN+1], \ldots, f[n]$ to $W_f^\psi[n,m]$.

In Figure **??** we illustrate the contribution of the same segment of f to $W_f^\psi[n,m-1]$.

$f[n\text{-}mN{+}1]$	$f[n\text{-}mN{+}2]$			$\overline{\psi}[0]$		$f[n\text{-}m(N\text{-}1)]$
$f[n\text{-}m(N\text{-}1){+}1]$	$f[n\text{-}m(N\text{-}1){+}2]$	$\overline{\psi}[0]$	$\overline{\psi}[1]$			$f[n\text{-}m(N\text{-}2)]$
	$\overline{\psi}[N\text{-}3]$	$\overline{\psi}[N\text{-}2]$				
$\overline{\psi}[N\text{-}2]$ $f[n\text{-}m\ {+}1]$	$\overline{\psi}[N\text{-}1]$ $f[n\text{-}m{+}2]$					$f[n]$

Figure 7.2. Contributions of $f[n-mN+1], \ldots, f[n]$ to $W_f^\psi[n,m-1]$.

The difference $W_f^{\psi}[n,m] - W_f^{\psi}[n,m-1]$ is easily calculated; it is

$$
\begin{aligned}
W_f^{\psi}[n,m] - W_f^{\psi}[n,m-1] &= \overline{\psi}[0] \sum_{l=0}^{N-1} f[n-mN+l] \\
&+ (\overline{\psi}[1] - \overline{\psi}[0]) \sum_{l=1}^{N-1} f[n-m(N-1)+l] \\
&+ \dots \\
&+ (\overline{\psi}[N-1] - \overline{\psi}[N-2]) f[n].
\end{aligned}
$$

Implementing this procedure, we would use the vector $(\overline{\psi}[0], \overline{\psi}[1] - \overline{\psi}[0], \dots, \overline{\psi}[N-1] - \overline{\psi}[N-2])$, in order to reduce redundant calculations.

Chapter 8

Piecewise Constant Wavelets and Frames

In this chapter, we shall address the question, whether the discrete wavelet transformation obtained in Chapter **??** is frame related.

Before doing this, we shall recall the following basic definition.

DEFINITION 8.1. *A family of functions $\{\varphi_i\}_{i\in I}$ in a Hilbert space H is a frame, if there exist $A > 0$ and $B < \infty$ such that for all $f \in H$*

$$A \|f\|_H^2 \leq \sum_{i\in I} |\langle f, \varphi_i\rangle|^2 \leq B \|f\|_H^2 . \tag{8.1}$$

If A is chosen maximal and B is chosen minimal such that (**??**) *holds, we call A the lower and B the upper framebound.*

This is clearly equivalent to the fact that the linear map

$$\begin{array}{rccc} L: & H & \longrightarrow & l^2(I) \\ & f & \longmapsto & \{\langle f, \varphi_i\rangle\}_{i\in I} \end{array}$$

is norm bounded above and below, and hence is invertible on its range. Each element $f \in H$ is therefore fully represented by the coefficients $\{\langle f, \varphi_i\rangle\}_{i\in I}$ and can be reconstructed from the coefficients by inverting the operator L.

8.1 Wavelet Frames for $l^2(\mathbf{Z})$

Considering a discrete signal as an element of $l^2(\mathbf{Z})$ we are interested whether the analysis vectors used in our discretized wavelet transform form a frame for $H = l^2(\mathbf{Z})$. Here, our analysis frame elements are integer dilates and translates of one vector, given by the values taken by a given piecewise constant wavelet of degree 1. Alternatively, they can be seen as sampled versions of a piecewise constant wavelet.

We sample piecewise constant wavelets in the following way (see also Chapter **??**): Let $\psi \in L^2(\mathbf{R})$ be a piecewise constant wavelet of degree 1. Let

$$\overline{\psi} = (\ldots, \overline{\psi}[-1], \overline{\psi}[0], \overline{\psi}[1], \ldots)$$

be the vector representing ψ, i.e., $\overline{\psi}[k] = \psi(k)$ for $k \in \mathbf{Z}$.

Discretizing $\psi_{m,n}(\cdot) = \psi(\frac{\cdot - n}{m})$ for $(m,n) \in \mathbf{Z}^+ \times \mathbf{Z}$ we get

$$\overline{\psi_{m,n}}[\cdot] = \overline{\psi}\left[\left\lfloor \frac{\cdot - n}{m} \right\rfloor\right],$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than x.

We define

$$\overline{\psi}_{m,n}[\cdot] = m^{-\frac{s}{2}}\overline{\psi}\left[\left\lfloor \frac{\cdot - n}{m} \right\rfloor\right].$$

We are interested in classifying piecewise constant wavelets $\psi \in L^2(\mathbf{R})$ such that for a given $s \in \mathbf{R}^+$ the family $\left\{\overline{\psi}_{m,n}\right\}_{m\in\mathbf{Z}^+, n\in\mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$. We shall see that the normalization factor $m^{-\frac{3}{2}}$ will play a special role.

Remark 8.2. Our procedure can be related to the quasi affine frames discussed in [**?**, **?**]. In their work, Ron and Shen analyzed the coarse part of a signal with the set of analyzing functions $\{2^{-m}\psi(\frac{t-n}{2^m})\}_{n\in Z, m\in Z^+}$. The scaling factor 2^{-m} is necessary, since we are expanding the dyadic wavelet set $\{2^{-\frac{m}{2}}\psi(\frac{t-2^m n}{2^m})\}_{n\in Z, m\in Z^+}$. In our approach, we are expending the wavelet set further, i.e., we are using the set $\{m^{-\frac{3}{2}}\psi(\frac{t-n}{m})\}_{n\in Z, m\in Z^+}$.

In the following, let the Dirichlet functions d_m be defined by

$$d_m(\gamma) = \sum_{l=0}^{m-1} e^{-2\pi i l\gamma}.$$

Note that then

$$|d_m(\gamma)|^2 = \left(\frac{\sin(\pi m\gamma)}{\sin(\pi\gamma)}\right)^2.$$

We obtain the following theorem characterizing $\overline{\psi} \in l^2(\mathbf{Z})$ such that $\left\{\overline{\psi}_{m,n}\right\}_{m\in\mathbf{Z}^+, n\in\mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$.

Theorem 8.3. *Let $\psi \in L^2(\mathbf{R})$ be any piecewise constant wavelet. The following are equivalent:*

i. The family $\left\{\overline{\psi}_{m,n}\right\}_{m\in\mathbf{Z}^+, n\in\mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$.

ii. There exists $A > 0$ and $B < \infty$ such that

$$A \le \sum_{m\in\mathbf{Z}^+} m^{-s}|d_m(\gamma)\widehat{\overline{\psi}}(m\gamma)|^2 \le B \tag{8.2}$$

for almost all $\gamma \in [0,1]$.

In this case

$$A = \operatorname{essinf}_{\gamma\in\mathbf{T}} \left\{ \sum_{m\in\mathbf{Z}^+} m^{-s} |d_m(\gamma)\widehat{\overline{\psi}}(m\gamma)|^2 \right\}$$

is the lower, and

$$B = \left\| \sum_{m\in\mathbf{Z}^+} m^{-s} |d_m(\gamma)\widehat{\overline{\psi}}(m\gamma)|^2 \right\|_{L^\infty(\mathbf{T})}$$

is the upper framebound.

Proof. Let us first calculate the Fourier transform of $\overline{\psi}_{m,n}$:

$$\begin{aligned}
\widehat{\overline{\psi}_{m,n}}(\gamma) &= \sum_{k\in\mathbf{Z}} \overline{\psi}_{m,n}[k] e^{-2\pi i k\gamma} \\
&= \sum_{k\in\mathbf{Z}} m^{-\frac{s}{2}} \overline{\psi}\left[\left\lfloor \frac{\cdot - n}{m} \right\rfloor\right] e^{-2\pi i k\gamma} \\
&= \sum_{k\in\mathbf{Z}} m^{-\frac{s}{2}} \overline{\psi}\left[\left\lfloor \frac{\cdot}{m} \right\rfloor\right] e^{-2\pi i (k+n)\gamma} \\
&= m^{-\frac{s}{2}} e^{-2\pi i n\gamma} \left(\begin{array}{cccccc}
 & \vdots & \vdots & & \vdots & \vdots \\
+ & \overline{\psi}\,[-1] & e^{-2\pi i(-m)\gamma} & +\ldots+ & \overline{\psi}\,[-1] & e^{-2\pi i(-1)\gamma} \\
+ & \overline{\psi}\,[0] & e^{-2\pi i 0\gamma} & +\ldots+ & \overline{\psi}\,[0] & e^{-2\pi i(m-1)\gamma} \\
+ & \overline{\psi}\,[1] & e^{-2\pi i m\gamma} & +\ldots+ & \overline{\psi}\,[1] & e^{-2\pi i(2m-1)\gamma} \\
 & \vdots & \vdots & & \vdots & \vdots
\end{array} \right) \\
&= m^{-\frac{s}{2}} e^{-2\pi i n\gamma} \left(\sum_{l=0}^{m-1} e^{-2\pi i l\gamma} \right) \left(\sum_{k\in\mathbf{Z}} \overline{\psi}\,[k]\, e^{-2\pi i m k\gamma} \right) \\
&= m^{-\frac{s}{2}} e^{-2\pi i n\gamma} d_m(\gamma) \widehat{\overline{\psi}}(m\gamma).
\end{aligned}$$

By Plancherel's theorem we have

$$\langle f, \overline{\psi}_{m,n} \rangle_{l^2(\mathbf{Z})} = \langle \widehat{f}, \widehat{\overline{\psi}_{m,n}} \rangle_{L^2(\mathbf{T})}$$

and hence

$$\begin{aligned}
\sum_{\substack{(m,n)\in\\ \mathbf{Z}^+\times\mathbf{Z}}} |\langle f,\overline{\psi}_{m,n}\rangle_{l^2(\mathbf{Z})}|^2 &= \sum_{\substack{(m,n)\in\\ \mathbf{Z}^+\times\mathbf{Z}}} |\langle \widehat{f},\widehat{\overline{\psi}_{m,n}}\rangle_{L^2(\mathbf{T})}|^2 \\
&= \sum_{\substack{(m,n)\in\\ \mathbf{Z}^+\times\mathbf{Z}}} |\int_{\mathbf{T}} \widehat{f}(\gamma) m^{-\frac{s}{2}} e^{-2\pi i n\gamma} d_m(\gamma)\widehat{\overline{\psi}}(m\gamma)\,d\gamma|^2 \\
&= \sum_{\substack{(m,n)\in\\ \mathbf{Z}^+\times\mathbf{Z}}} m^{-s}|\int_{\mathbf{T}} \left(\widehat{f}(\gamma)d_m(\gamma)\widehat{\overline{\psi}}(m\gamma)\right) e^{-2\pi i n\gamma}\,d\gamma|^2 \\
&= \sum_{m\in\mathbf{Z}^+} m^{-s}\int_{\mathbf{T}} |\widehat{f}(\gamma)d_m(\gamma)\widehat{\overline{\psi}}(m\gamma)|^2\,d\gamma \qquad (8.3)\\
&= \int_{\mathbf{T}} |\widehat{f}(\gamma)|^2 \sum_{m\in\mathbf{Z}^+} m^{-s}|d_m(\gamma)\widehat{\overline{\psi}}(m\gamma)|^2\,d\gamma.
\end{aligned}$$

Note that (**??**) follows by applying Plancherel's theorem to the function $\widehat{f}(\cdot)d_m(\cdot)\widehat{\overline{\psi}}(m\,\cdot)$. This is justified, since both, *i* and *ii*, imply that $\widehat{f}(\cdot)d_m(\cdot)\widehat{\overline{\psi}}(m\,\cdot)\in L^2(\mathbf{T})$. To interchange summation and integration in (**??**), we are applying the monotone convergence theorem. The result follows immediately. □

This theorem gives us a criterion, (**??**), to check whether $\{\overline{\psi}_{m,n}\}_{(m,n)\in\mathbf{Z}^+\times\mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$. This criterion is easily checked if we restrict ourselves to wavelets ψ with compact support, i.e., $\overline{\psi}$ satisfy the property that $\overline{\psi}[k]=0$ for $k\neq 0,1,\ldots,N-1$. This compactness condition is satisfied in all applications.

THEOREM 8.4. *Let $\overline{\psi}\in l^2(\mathbf{Z})$ satisfy the condition that $\overline{\psi}[k]=0$ for $k\neq 0,1,\ldots,N-1$. The following are equivalent:*
i. The family $\{\overline{\psi}_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$.
ii. The polynomial $1+z+z^2+\ldots+z^n$ does not divide $\overline{\psi}\,[0]+\overline{\psi}\,[1]\,z+\overline{\psi}\,[2]\,z^2+\ldots+\overline{\psi}\,[N-1]\,z^{N-1}$ for all $n\leq N-1$ and either $\sum\overline{\psi}\,[k]=0$ and $s=3$ or $\sum\overline{\psi}\,[k]\neq 0$ and $s>3$.

This theorem follows from Theorem **??** and the following lemmata.

Note that the tophat wavelet mentioned in Example **??** as well as the wavelet $\frac{1}{\sqrt{70}}(1,-4,6,-4,1)$, which is mentioned in Example **??**, do satisfy the condition *ii* of Theorem **??**.

LEMMA 8.5. *Let $\overline{\psi}$ satisfy the property that $\overline{\psi}[k]=0$ for $k\neq 0,1,\ldots,N-1$. The following are equivalent:*
i. The function G_s satisfies

$$G_s(\gamma)=\sum_{m=1}^{\infty} m^{-s}|d_m(\gamma)\widehat{\overline{\psi}}(m\gamma)|^2\neq 0$$

for all $\gamma\in(0,1)$.

ii. The polynomial $1+z+z^2+\ldots+z^n$ *does not divide* $\overline{\psi}[0]+\overline{\psi}[1]z+\overline{\psi}[2]z^2+\ldots+\overline{\psi}[N-1]z^{N-1}$ *for all* $n \leq N-1$.

Proof. $i \Longrightarrow ii$. We shall first show that the condition, $p_n(z)=1+z+z^2+\ldots+z^n$ does not divide $\overline{\psi}[0]+\overline{\psi}[1]z+\overline{\psi}[2]z^2+\ldots+\overline{\psi}[N-1]z^{N-1}$ for all $n \leq N-1$, is necessary.

Suppose

$$p_{n_0}(z)q(z)=\overline{\psi}[0]+\overline{\psi}[1]z+\overline{\psi}[2]z^2+\ldots+\overline{\psi}[N-1]z^{N-1}$$

for some integer n_0 and some polynomial q. Then

$$G_s\left(\frac{1}{n_0}\right) = \sum_{m=1}^{\infty} m^{-s}\left(\frac{\sin(\pi m\frac{1}{n_0})}{\sin(\pi\frac{1}{n_0})}\right)^2\left|p_{n_0}(e^{-2\pi i\frac{m}{n_0}})\right|^2\left|q(e^{-2\pi i\frac{m}{n_0}})\right|^2.$$

If n_0 divides m, we have

$$\frac{\sin(\pi m\frac{1}{n_0})}{\sin(\pi\frac{1}{n_0})}=0.$$

Otherwise, $e^{-2\pi i\frac{m}{n_0}}$ is a nontrivial ${n_0}^{th}$ root of 1 and hence

$$p_{n_0}(e^{-2\pi i\frac{m}{n_0}})=0.$$

Therefore $G_s(\frac{1}{n_0})=0$, and necessity has been shown.

$ii \Longrightarrow i$. Assume there exists a $\gamma_0 \in (0,1)$ such that $G_s(\gamma_0)=0$.

CASE 1. $\gamma_0 \notin \mathbf{Q}$

Since $\sin(\pi m\gamma_0) \neq 0$ for all $m \geq 1$ we must have $\widehat{\widehat{\psi}}(m\gamma_0)=0$ for all $m \geq 1$. The function $\widehat{\widehat{\psi}}$ has finitely many zeros $\gamma_0,\ldots,\gamma_s$ in $(0,1)$. Since $\widehat{\widehat{\psi}}$ is $1-$periodic, there exists for every $m \geq 1$ a γ_{k_m} and an integer l_m such that

$$\gamma_{k_m}+l_m=m\gamma_0.$$

Picking γ_i such that for $m_1 \neq m_2$, we have

$$m_1\gamma_0-l_{m_1}=\gamma_i=m_2\gamma_0-l_{m_2}.$$

Solving for γ_0, we get

$$\gamma_0=\frac{l_{m_1}-l_{m_2}}{m_1-m_2},$$

contradicting $\gamma_0 \notin \mathbf{Q}$!

CASE 2. $\gamma_0 \in \mathbf{Q}$

Let $0 \neq p < q \in \mathbf{Z}^+$ such that $\gamma_0=\frac{p}{q}$ and $(p,q)=1$.

$G_s(\gamma_0) = 0$ implies that for all $m \in \mathbf{Z}^+$ either $\sin\left(\pi \frac{mp}{q}\right) = 0$ or $\widehat{\overline{\psi}}\left(\frac{mp}{q}\right) = 0$. But $\sin\left(\pi \frac{mp}{q}\right) \neq 0$ for $m < q$, since otherwise we would have for some $m < q$, $\frac{mp}{q} = l \in \mathbf{Z}^+$. $\frac{p}{q} = \frac{l}{m}$ then contradicts $(p,q) = 1$ since $m < q$.

We have

$$\left\{\frac{mp}{q}, m = 1, \ldots, q-1\right\} = \left\{\frac{1}{q}, \ldots, \frac{q-1}{q}\right\}_{\text{modulus } 1},$$

since if $\frac{m_1 p}{q} = \frac{m_2 p}{q} - l$ with $1 \leq m_1 < m_2 \leq q-1$, we obtain $\frac{l}{m_1 - m_2} = \frac{p}{q}$ contradicting $(p,q) = 1$ since $m_1 - m_2 < q$.

Hence we have $\widehat{\overline{\psi}}(\frac{m}{q}) = 0$ for $m = 1, 2, \ldots, q-1$ and therefore $e^{-2\pi i \frac{m}{q}}, m = 1, \ldots, q-1$, are zeros of $\overline{\psi}[0] + \overline{\psi}[1] z + \overline{\psi}[2] z^2 + \ldots + \overline{\psi}[N-1] z^{N-1}$, i.e., $1 + z + z^2 + \ldots + z^{q-1}$ divides $\overline{\psi}[0] + \overline{\psi}[1] z + \overline{\psi}[2] z^2 + \ldots + \overline{\psi}[N-1] z^{N-1}$. $\square$

LEMMA 8.6. *Let $\overline{\psi}$ satisfy the property that $\overline{\psi}[k] = 0$ for $k \neq 0, 1, \ldots, N-1$ and $\sum \overline{\psi}[k] = 0$. The following are equivalent:*

i. The family $\left\{\overline{\psi}_{m,n}\right\}_{m \in \mathbf{Z}^+, n \in \mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$.

ii. The polynomial $1 + z + z^2 + \ldots + z^n$ does not divide $\overline{\psi}[0] + \overline{\psi}[1] z + \overline{\psi}[2] z^2 + \ldots + \overline{\psi}[N-1] z^{N-1}$ for all $n \leq N-1$ and $s = 3$.

Proof. By hypothesis we know that the polynomial

$$\overline{\psi}[0] + \overline{\psi}[1] z + \ldots + \overline{\psi}[N-1] z^{N-1}$$

has a zero at 1. Factoring out $1 - z$ we obtain for some $q \in \mathbf{C}[z]$

$$(1-z) q(z) = \sum_{k=1} \overline{\psi}[k] z^k.$$

Setting $q(z) = \sum_{k=0}^{N-1} c_k z^k$ (in fact $c_k = \sum_{l=0}^{k} \overline{\psi}[l]$) and evaluating on the torus, we obtain

$$\begin{aligned}
\widehat{\overline{\psi}}(\gamma) &= (1 - e^{-2\pi i \gamma}) \sum_{k=0}^{N-1} c_k e^{-2\pi i k \gamma} \\
&= e^{-\pi i \gamma}(e^{\pi i \gamma} - e^{-\pi i \gamma}) \sum_{k=0}^{N-1} c_k e^{-2\pi i k \gamma} \\
&= e^{-\pi i \gamma} 2i \sin(\pi \gamma) \sum_{k=0}^{N-1} c_k e^{-2\pi i k \gamma}.
\end{aligned}$$

Setting

$$p(\gamma) = \left|\sum_{k=0}^{N-1} c_k e^{-2\pi i k \gamma}\right|^2$$

we have

$$|\widehat{\overline{\psi}}(\gamma)|^2 = 4\sin^2(\pi\gamma)p(\gamma).$$

$ii \Longrightarrow i$. We assume that $1+z+z^2+\ldots+z^n$ does not divide $\overline{\psi}\,[0]+\overline{\psi}\,[1]\,z+\overline{\psi}\,[2]\,z^2+\ldots+\overline{\psi}\,[N-1]\,z^{N-1}$ for all $n \leq N-1$ and $s=3$.

Since

$$|d_m(\gamma)|^2 = \left(\frac{\sin(\pi m\gamma)}{\sin(\pi\gamma)}\right)^2,$$

we obtain

$$\begin{aligned} G_3(\gamma) &= \sum_{m=1}^{\infty}\frac{1}{m^3}\frac{4\sin^4(\pi m\gamma)}{\sin^2(\pi\gamma)}p(m\gamma) \\ &= 4\frac{\gamma^2}{\sin^2(\pi\gamma)}\gamma\sum_{m=1}^{\infty}\frac{\sin^4(\pi m\gamma)}{(m\gamma)^3}p(m\gamma). \end{aligned}$$

The function

$$f(x) = \frac{\sin^4(\pi x)}{x^3}p(x),$$

$x \in \mathbf{R}^+$ is continuous, bounded, of order $O(x^{-3})$, $x \to \infty$, and Riemann integrable. Hence we have

$$\lim_{\gamma\to 0}\gamma\sum_{m=1}^{\infty}\frac{\sin^4(\pi m\gamma)}{(m\gamma)^3}p(m\gamma) = \int_0^{\infty} f(x)\,dx. \tag{8.4}$$

Since

$$\lim_{\gamma\to 0} 4\frac{\gamma^2}{\sin^2(\pi\gamma)} = \frac{4}{\pi^2}$$

we have

$$\lim_{\gamma\to 0} G_3(\gamma) = \frac{4}{\pi^2}\int_0^{\infty} f(x)\,dx. \tag{8.5}$$

Further, we have

$$\begin{aligned} G_3(1-\gamma) &= \sum_{m=1}^{\infty}\frac{1}{m^3}\frac{4\sin^4(\pi m(1-\gamma))}{\sin^2(\pi(1-\gamma))}p(m(1-\gamma)) \\ &= \sum_{m=1}^{\infty}\frac{1}{m^3}\frac{4\sin^4(\pi m\gamma)}{\sin^2(\pi\gamma)}p(-m\gamma)) \\ &= 4\frac{\gamma^2}{\sin^2(\pi\gamma)}\gamma\sum_{m=1}^{\infty}\frac{\sin^4(\pi m\gamma)}{(m\gamma)^3}p(-m\gamma). \end{aligned}$$

Similarly to (**??**), (**??**) we have

$$\lim_{\gamma\to 1} G_3(\gamma) = \lim_{\gamma\to 0} G_3(1-\gamma) = \frac{4}{\pi^2}\int_0^\infty \frac{\sin^4(\pi x)}{(x)^3} p(-x)\,dx. \tag{8.6}$$

Since G_3 is continuous on $(0,1)$, we can now extend it continuously to $[0,1]$, hence, G_3 is bounded.

For this direction of the proof it remains to show that G_3 is bounded away from 0. Our hypothesis together with Lemma **??** implies that G_3 is nonzero on $(0,1)$. Clearly $\overline{\psi} \not\equiv 0$ and hence

$$\int_0^\infty \frac{\sin^2(\pi x)}{(x)^3} |\widehat{\overline{\psi}}(\pm x)|^2\,dx > 0,$$

and by (**??**) and (**??**), $G_3(0), G_3(1) > 0$.

The continuity of G_3 on $[0,1]$ implies that G_3 is bounded away from zero. We observed earlier that G_3 is bounded above, hence we can apply Lemma **??** and obtain that $\{\overline{\psi}_{m,n}\}_{m\in\mathbf{Z}^+, n\in\mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$.

i $\Longrightarrow$ *ii*. The condition $1+z+z^2+\ldots+z^n$ does not divide $\overline{\psi}\,[0]+\overline{\psi}\,[1]\,z+\overline{\psi}\,[2]\,z^2+\ldots+\overline{\psi}\,[N-1]\,z^{N-1}$ for all $n\le N-1$ is necessary since else, we could find $\gamma_0\in(0,1)$ such that $G_s(\gamma_0)=0$

The necessity of choosing $s=3$ follows from the following argument: Choosing $s=3+\epsilon$, $-3<\epsilon\le 1$, $\epsilon\neq 0$, we obtain for $\gamma\in(0,1)$

$$G_{3+\epsilon}(\gamma) \;=\; 4\gamma^\epsilon \frac{\gamma^2}{\sin^2(\pi\gamma)}\gamma\sum_{m=1}^{\infty}\frac{\sin^4(\pi m\gamma)}{(m\gamma)^{3+\epsilon}}p(m\gamma). \tag{8.7}$$

Similarly to (**??**), we choose

$$f(x) = \frac{\sin^4(\pi x)}{x^{3+\epsilon}},$$

and obtain, since the factor γ^ϵ dominates the right hand side of (**??**),

$$\lim_{\gamma\to 0} G_s(\gamma) = \begin{cases} 0, & \epsilon > 0 \\ \infty, & \epsilon < 0 \end{cases} \tag{8.8}$$

Further observe that $G_{3+\epsilon_1} \ge G_{3+\epsilon_2}$ for general $\epsilon_1 \le \epsilon_2$. Hence (**??**) holds for all $\epsilon\neq 0$. This completes the proof of the necessity of our conditions. □

LEMMA 8.7. *Let $\overline{\psi}$ satisfy the property that $\overline{\psi}[k]\neq 0$ for $k\neq 0,1,\ldots,N-1$ and $\sum_{k=0}^{N-1}\overline{\psi}\,[k]\neq 0$. The following are equivalent:*

i. The family $\{\overline{\psi}_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$.

ii. The polynomial $1+z+z^2+\ldots+z^n$ does not divide $\overline{\psi}\,[0]+\overline{\psi}\,[1]\,z+\overline{\psi}\,[2]\,z^2+\ldots+\overline{\psi}\,[N-1]\,z^{N-1}$ for all $n\le N-1$ and $s>3$.

Proof. *ii* $\Longrightarrow$ *i*. We assume $1+z+z^2+\ldots+z^n$ does not divide $\overline{\psi}\,[0]+\overline{\psi}\,[1]\,z+\overline{\psi}\,[2]\,z^2+\ldots+\overline{\psi}\,[N-1]\,z^{N-1}$ for all $n \leq N-1$ and $s > 3$.

Since $\widehat{\overline{\psi}}$ is a 1–periodic continuous function, we can find an upper bound W for $|\widehat{\overline{\psi}}|$. We have

$$m^{-s}|d_m(\gamma)\widehat{\overline{\psi}}(m\gamma)|^2 \leq W^2 m^{-s+2},$$

and, since $s-2>1$,

$$G_s(\gamma) = \sum_{m=1}^{\infty} m^{-s}|d_m(\gamma)\widehat{\overline{\psi}}(m\gamma)|^2$$

is defined through a uniformly converging series of continuous functions and is therefore continuous on $[0,1]$, and, hence, bounded above. Further

$$0 < |\sum_{k=0}^{N-1} \overline{\psi}\,[k]\,|^2 = G_s(0) = G_s(1) < \infty$$

and by hypothesis and Lemma (**??**) we have $G_s \neq 0$ also on $[0,1]$. Hence G_s bounded away from 0.

i $\Longrightarrow$ *ii*. In order to prove the necessity of $s > 3$, we suppose the opposite. We obtain, applying Fatou's Lemma to (**??**),

$$\begin{aligned} \lim_{\gamma\to 0} G_s(\gamma) &= \lim_{\gamma\to 0}\sum_{m=1}^{\infty} m^{-s}|d_m(\gamma)\widehat{\overline{\psi}}(m\gamma)|^2 \\ &\geq \sum_{m=1}^{\infty}\lim_{\gamma\to 0} m^{-s}|d_m(\gamma)\widehat{\overline{\psi}}(m\gamma)|^2 \\ &= |\sum_{k=0}^{N-1}\overline{\psi}\,[k]\,|^2 \sum_{m=1}^{\infty}\frac{1}{m^s}m^2 \\ &= \infty, \end{aligned} \tag{8.9}$$

and hence G_s has no upper bound. □

Remark 8.8. We can use Theorem **??** to obtain frames for some vector spaces of functions defined on $\mathbf{R}$. For example, we shall construct frames for the Paley–Wiener spaces

$$PW_\Omega(\mathbf{R}) = \{f \in L^2(\mathbf{R}):\ \operatorname{supp}\mathcal{F}(f) \subseteq [-\Omega,\Omega]\},$$

with $\Omega > 0$ [**?**, **?**]. Paley–Wiener spaces are closed subspaces of $L^2(\mathbf{R})$, and, hence, Hilbert spaces with the innerproduct $\langle\cdot,\cdot\rangle_{L^2(\mathbf{R})}$. In order to simplify notation, we shall only consider the case $\Omega = \frac{1}{2}$. This is done without loss of generality.

Defining $h_k(\cdot) = \frac{\sin(\pi(\cdot - k))}{\pi(\cdot - k)} \in PW_{\frac{1}{2}}(\mathbf{R})$ we obtain an orthonormal basis $\{h_k\}_{k\in\mathbf{Z}}$ of $PW_{\frac{1}{2}}(\mathbf{R})$. Furthermore $f(k) = \langle f, h_k\rangle$ for $f \in PW_{\frac{1}{2}}(\mathbf{R})$ and $k \in \mathbf{Z}$, and, hence, $\{f(k)\}_{k\in\mathbf{Z}} \in l^2(\mathbf{Z})$. The classical sampling theorem implies

$$f = \sum_{k\in\mathbf{Z}} f(k) h_k.$$

Let $\overline{\psi} \in l^2(\mathbf{Z})$ be chosen such that $\{\overline{\psi}_{m,n}\}_{m\in\mathbf{Z}^+, n\in\mathbf{Z}}$ is a frame with frame bounds A and B. Define

$$\varphi_{m,n} = \sum_{k\in\mathbf{Z}} \overline{\psi}_{m,n}(k) h_k$$

for $m \in \mathbf{Z}^+, n \in \mathbf{Z}$. The function $\varphi_{m,n} \in PW_{\frac{1}{2}}(\mathbf{R})$ is well-defined, since $\overline{\psi}_{m,n} \in l^2(\mathbf{Z})$ for all $m \in \mathbf{Z}^+$, $n \in \mathbf{Z}$, and since $\{h_k\}_{k\in\mathbf{Z}}$ is an orthonormal set. For $f \in PW_{\frac{1}{2}}(\mathbf{R})$, $m \in \mathbf{Z}^+$, and $n \in \mathbf{Z}$, we compute

$$\begin{aligned}
\langle f, \varphi_{m,n}\rangle_{L^2(\mathbf{R})} &= \int_{\mathbf{R}} f(t) \overline{\sum_{k\in\mathbf{Z}} \overline{\psi}_{m,n}(k) h_k(t)}\, dt \\
&= \sum_{k\in\mathbf{Z}} \overline{\overline{\psi}_{m,n}}(k) \int_{\mathbf{R}} f(t)\, dt \overline{h_k(t)} \\
&= \langle f[\cdot], \overline{\psi}_{m,n}\rangle_{l^2(\mathbf{Z})}.
\end{aligned}$$

This results in

$$\begin{aligned}
A \left\| f \right\|^2_{L^2(\mathbf{R})} = A \left\| f[\cdot] \right\|^2_{l^2(\mathbf{Z})} &\leq \sum_{m\in\mathbf{Z}^+, n\in\mathbf{Z}} \left| \langle f[\cdot], \overline{\psi}_{m,n}\rangle_{l^2(\mathbf{Z})} \right|^2 \\
&= \sum_{m\in\mathbf{Z}^+, n\in\mathbf{Z}} \left| \langle f, \varphi_{m,n}\rangle_{L^2(\mathbf{R})} \right|^2 \\
&= \sum_{m\in\mathbf{Z}^+, n\in\mathbf{Z}} \left| \langle f[\cdot], \overline{\psi}_{m,n}\rangle_{l^2(\mathbf{Z})} \right|^2 \\
&\leq B \left\| f[\cdot] \right\|^2_{l^2(\mathbf{Z})} = B \left\| f \right\|^2_{L^2(\mathbf{R})},
\end{aligned}$$

and, hence, $\{\varphi_{m,n}\}_{m\in\mathbf{Z}^+, n\in\mathbf{Z}}$ is a frame for $PW_{\frac{1}{2}}(\mathbf{R})$.

8.2 Examples

EXAMPLE 8.9. Let ψ be the Haar wavelet, i.e., $\overline{\psi}[m] = \frac{1}{2}(\delta_0[m] - \delta_1[m])$ for $m \in \mathbf{Z}$. For $s = 3$, $\overline{\psi}$ fulfills the hypothesis of Theorem **??**, and, hence, $\{\overline{\psi}_{m,n}\}_{m\in\mathbf{Z}^+, n\in\mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$. To find the framebounds of this frame, we shall study the corresponding function G_3, which is given by

$$G_3(\gamma) = \sum_{m\in\mathbf{Z}^+} m^{-3} \frac{\sin^4(\pi m\gamma)}{\sin^2(\pi\gamma)},$$

and is shown in Figure **??**.

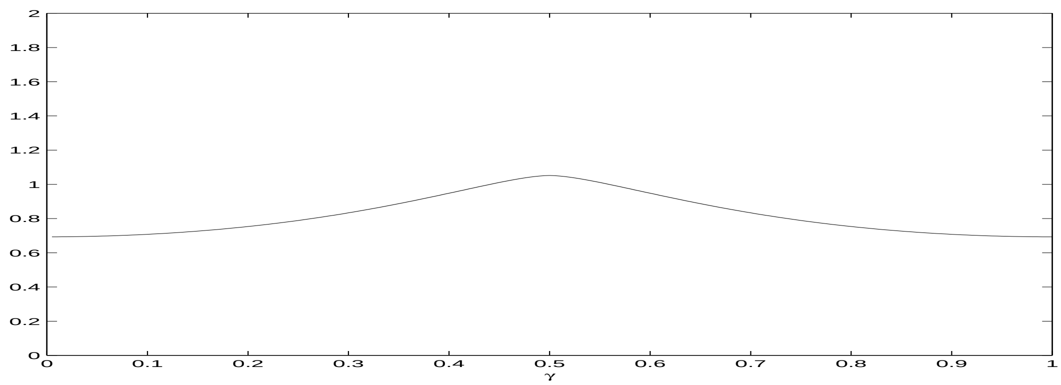

Figure 8.1. $G_3(\gamma)$ for the Haar wavelet ψ.

As lower framebound we obtain

$$A = \lim_{\gamma \to 0^+} G_3(\gamma) = \frac{1}{\pi^2} \int_0^\infty \frac{\sin^4(\pi x)}{x^3}\, dx = \ln(2) \approx 0.6931,$$

and as upper framebound we have

$$B = G_3\left(\frac{1}{2}\right) = \sum_{\substack{m \in Z^+ \\ m \text{ odd}}} m^{-3} = \frac{7\zeta(3)}{8} \approx 1.0518.$$

EXAMPLE 8.10. The Haar scaling function φ has as associated vector $\overline{\varphi}[m] = \delta_0[m]$. Theorem **??** asserts that $\left\{\overline{\varphi}_{m,n}\right\}_{m \in \mathbf{Z}^+, n \in \mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$ if $s > 3$. For $s = 4$, the function

$$G_4(\gamma) = \sum_{m \in \mathbf{Z}^+} m^{-4} \frac{\sin^2(\pi m \gamma)}{\sin^2(\pi \gamma)}$$

is associated to $\overline{\varphi}$. This function is shown in Figure **??**.

As framebounds we obtain

$$B = \lim_{\gamma \to 0^+} G_4(\gamma) = \sum_{m \in \mathbf{Z}^+} m^{-2} = \frac{\pi^2}{6} \approx 1.6449$$

and

$$A = G_4\left(\frac{1}{2}\right) = \sum_{\substack{m \in Z^+ \\ m \text{ odd}}} m^{-4} = \frac{\pi^4}{96} = 1.0147.$$

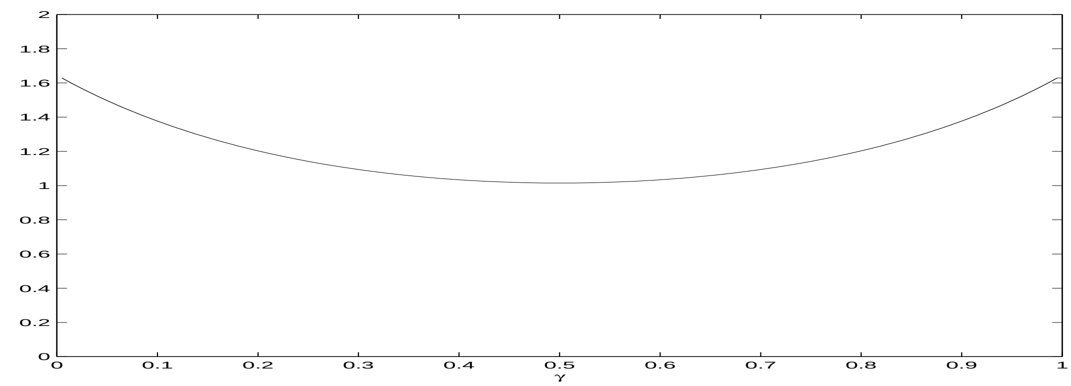

Figure 8.2. $G_4(\gamma)$ for the Haar scaling function φ.

Example 8.11. Let ψ be the wavelet mentioned in Example **??**, i.e., $\overline{\psi}[m] = \frac{1}{\sqrt{70}}(\delta_0[m] - 4\delta_1[m] + 6\delta_2[m] - 4\delta_3[m] + \delta 4[m])$ for $m \in \mathbf{Z}$. For $s = 3$, $\overline{\psi}$, we obtain, using Theorem **??**, that $\left\{\overline{\psi}_{m,n}\right\}_{m\in\mathbf{Z}^+, n\in\mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$. We obtain

$$G_3(\gamma) = \sum_{m\in\mathbf{Z}^+} m^{-3}\frac{2^8}{70}\frac{\sin^{10}(\pi m\gamma)}{\sin^2(\pi\gamma)},$$

which is shown in Figure **??**.

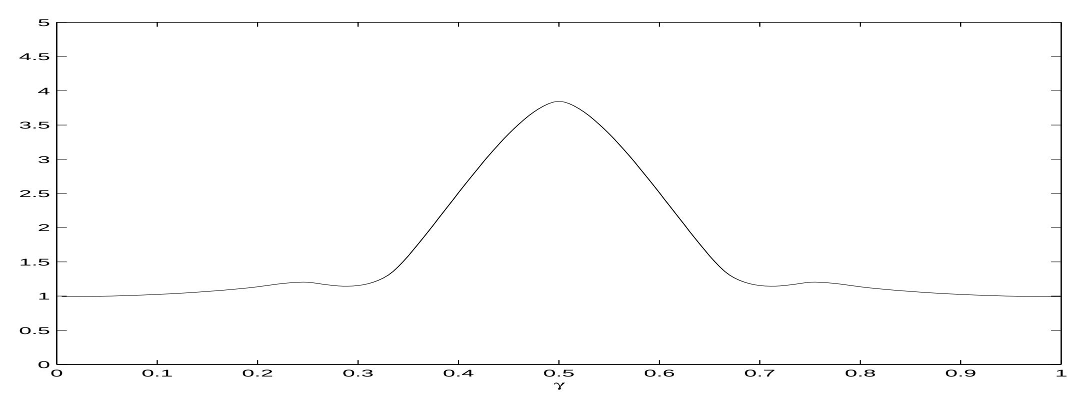

Figure 8.3. $G_3(\gamma)$ for the wavelet obtained in Example **??**.

As lower framebound we obtain

$$\begin{aligned} A &= \lim_{\gamma\to 0^+} G_3(\gamma) = \frac{2^8}{70\pi^2}\int_0^\infty \frac{\sin^{10}(\pi x)}{x^3}\,dx = \frac{2^8(160\ln(4) - 81\ln(9) - 5\ln(25))}{1960\pi^2} \\ &\approx 0.9905, \end{aligned}$$

and as upper framebound we have

$$B = G_3\left(\frac{1}{2}\right) = \frac{2^8}{70}\sum_{\substack{m\in Z^+\\ m \text{ odd}}} m^{-3} = \frac{2^5\zeta(3)}{10} \approx 3.8466.$$

8.3 Wavelet Frames for $l^2(\mathbf{Z}^d)$

We shall proceed similarly to the approach in Section **??**. The Hilbert space we are interested in is the space of square summable multidimensional discrete signals $H = l^2(\mathbf{Z}^d)$. Our goal is to characterize vectors $\overline{\psi} \in l^2(\mathbf{Z}^d)$ such that, for some normalization factor, its translates and "dilates" form a frame for $H = l^2(\mathbf{Z}^d)$. Again, the frame elements can be seen as sampled versions of a piecewise constant wavelet $\psi \in L^2(\mathbf{R}^d)$.

For $\overline{\psi} \in l^2(\mathbf{Z}^d)$ we define, for $n \in \mathbf{Z}^d$, $m \in \mathbf{Z}^+$,

$$\overline{\psi}_{m,n}[k] = m^{-\frac{s}{2}}\overline{\psi}\left[\left\lfloor\frac{k_1-n_1}{m}\right\rfloor,\ldots,\left\lfloor\frac{k_1-n_d}{m}\right\rfloor\right],\ k\in\mathbf{Z}^d.$$

For $m > 0$, we define m-dimensional Dirichlet functions by

$$\mathbf{d_m}(\gamma) = d_m(\gamma_1)\cdot\ldots\cdot d_m(\gamma_d)$$

Theorem **??** generalizes to higher dimensions in the following fashion:

THEOREM 8.12. *For $\overline{\psi} \in l^2(\mathbf{Z}^d)$, the following are equivalent:*

i. The family $\left\{\overline{\psi}_{m,n}\right\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}^d}$ is a frame for $l^2(\mathbf{Z}^d)$.

ii. There exists $A > 0$ and $B < \infty$ such that

$$A \le \sum_{m\in\mathbf{Z}^+} m^{-s}|\mathbf{d_m}(\gamma)\widehat{\overline{\psi}}(m\gamma)|^2 \le B \tag{8.10}$$

for almost all $\gamma = (\gamma_1,\ldots,\gamma_d) \in [0,1]^d$.

*The framebounds can be obtained in exactly the same matter as in the one dimensional case (Theorem **??**).*

Proof. We can generalize the proof of Theorem **??**. Calculating the Fourier transform of $\overline{\psi}_{m,n}$, we obtain

$$\begin{aligned}
\widehat{\overline{\psi}_{m,n}}(\gamma) &= \sum_{k\in\mathbf{Z}^d} m^{-\frac{s}{2}}\overline{\psi}\left[\left\lfloor\frac{k_1-n_1}{m}\right\rfloor,\ldots,\left\lfloor\frac{k_d-n_d}{m}\right\rfloor\right] e^{-2\pi i\langle k,\gamma\rangle} \\
&= \sum_{k\in\mathbf{Z}^d} m^{-\frac{s}{2}}\overline{\psi}\left[\left\lfloor\frac{k_1}{m}\right\rfloor,\ldots,\left\lfloor\frac{k_d}{m}\right\rfloor\right] e^{-2\pi i\langle k+n,\gamma\rangle} \\
&= m^{-\frac{s}{2}} e^{-2\pi i\langle n,\gamma\rangle}\left(\sum_{0\le l<m} e^{-2\pi i\langle l,\gamma\rangle}\right)\left(\sum_{k\in\mathbf{Z}^d}\overline{\psi}\,[k]\, e^{-2\pi i m\langle k,\gamma\rangle}\right) \quad (8.11)\\
&= m^{-\frac{s}{2}} e^{-2\pi i\langle n,\gamma\rangle}\mathbf{d_m}(\gamma)\widehat{\overline{\psi}}(m\gamma).
\end{aligned}$$

We obtain (**??**) by observing that $\overline{\psi}\,[r]$ appears as the coefficient of $e^{-2\pi i\langle mr+l,\cdot\rangle}$, $0\le l<m$.

Again, we have

$$\sum_{(m,n)\in\mathbf{Z}^+\times\,\mathbf{Z}^d} |\langle f,\overline{\psi}_{m,n}\rangle_{l^2(\mathbf{Z}^+\times\,\mathbf{Z})}|^2 = \int_{\mathbf{T}^d} |\widehat{f}(\gamma)|^2 \sum_{m\in\mathbf{Z}^+} m^{-s}|\mathbf{d_m}(\gamma)\widehat{\overline{\psi}}(m\gamma)|^2\, d\gamma.$$

□

Our next objective is to characterize piecewise constant wavelets satisfying the criterion in Theorem **??**. The restriction to piecewise constant wavelets with compact support does not allow a generalization to higher dimensions of Lemma **??**. The reason for this is that it is not easy to control the zero sets of the trigonometric polynomial $\widehat{\overline{\psi}}$ appearing in

$$G_s(\cdot) = \sum_{m=1}^{\infty} m^{-s}|\mathbf{d_m}(\cdot)\widehat{\overline{\psi}}(m\,\cdot)|^2.$$

Hence, we shall not be able to give a full characterization of piecewise constant wavelets with compact support in $\mathbf{R}^d$ such that $\{\overline{\psi}_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}^d}$ is a frame for $l^2(\mathbf{Z}^d)$.

Nevertheless, we can state some necessary conditions for $\{\overline{\psi}_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}^d}$ to be a frame for $l^2(\mathbf{Z}^d)$. Before stating this result, let us recall that $0\in\mathbf{T}^d$ represents the equivalence class of all "corners" of $\mathbf{T}^d$.

THEOREM 8.13. *Let $\overline{\psi}\in l^2(\mathbf{Z}^d)$ be such that $\overline{\psi}\,[k]\neq 0$ only for $0\le k\le N-1$ and such that $\{\overline{\psi}_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}^d}$ is a frame for $l^2(\mathbf{Z}^d)$ with normalization factor m^{-s}. Then:*

a. $\widehat{\overline{\psi}}(\gamma)\neq 0$ for all $\gamma\in\partial\mathbf{T}^d-\{0\}$.

b. If $\widehat{\overline{\psi}}(0)=\sum_{k\in\mathbf{Z}^d}\overline{\psi}[k]=0$ then necessarily $s=2d+1$, and if $\sum_{k\in\mathbf{Z}^d}\overline{\psi}[k]\neq 0$, then necessarily $s>2d+1$.

Proof. We shall first show b. For the case that $\widehat{\overline{\psi}}(0) \neq 0$ we proceed as in Lemma **??**. If $s \leq 2d+1$, then setting $t = (t, \ldots, t)$ we compute

$$\begin{aligned}
\lim_{t\to 0} G_s(t) &= \lim_{t\to 0} \sum_{m=1}^{\infty} m^{-s} |\mathbf{d_m}(t) \widehat{\overline{\psi}}(mt)|^2 \\
&\geq \sum_{m=1}^{\infty} \lim_{t\to 0} m^{-s} |\mathbf{d_m}(t) \widehat{\overline{\psi}}(mt)|^2 \\
&= |\widehat{\overline{\psi}}(0)|^2 \sum_{m=1}^{\infty} \frac{1}{m^s} m^{2d} \\
&= \infty,
\end{aligned}$$

and hence G_s has no upper bound.

Let us now assume that $\widehat{\overline{\psi}}(0) = 0$. Since by hypothesis $\overline{\psi} \neq 0$, we can find some $\gamma = (\gamma_1, \ldots, \gamma_d) \in (\mathbf{T}^d)^o$ such that $\widehat{\overline{\psi}}(\gamma) \neq 0$.

As in the proof of Lemma **??** we shall rewrite G_s, and we shall study $G_s(t\gamma)$, for $t > 0$. We have

$$\begin{aligned}
G_s(t\gamma) &= \sum_{m=1}^{\infty} m^{-s} |\mathbf{d_m}(t\gamma) \widehat{\overline{\psi}}(mt\gamma)|^2 \\
&= \left(\prod_{i=1,\ldots,d} \frac{(t\gamma_i)^2}{\sin^2(\pi t \gamma_i)} \right) t \sum_{m=1}^{\infty} \frac{1}{m^{s-2d-1}} \left(\prod_{i=1,\ldots,d} \frac{\sin^2(\pi m t \gamma_i)}{(mt\gamma_i)^2} \right) \frac{|\widehat{\overline{\psi}}(mt\gamma)|^2}{tm}.
\end{aligned}$$

We shall apply the same trick as in the proof of Lemma **??** to the function

$$f(x) = \prod_{i=1,\ldots,d} \frac{\sin^2(\pi x \gamma_i)}{(x\gamma_i)^2} \frac{|\widehat{\overline{\psi}}(x\gamma)|^2}{x}.$$

Note that this function is again integrable, since the function we obtain by restricting $\widehat{\overline{\psi}}$ to the half line $t\gamma$, $t \in R^+$, vanishes at 0. Clearly, for $s = 2d+1$ we do not encounter problems while, by the same argument as in (**??**), for any other s we either violate the upper or lower framebound of our frame $\{\overline{\psi}_{m,n}\}_{m\in\mathbf{Z}^+, n\in\mathbf{Z}^d}$.

It remains to prove part a. To do this, let us assume that there exists a $\gamma = (\gamma_1, \ldots, \gamma_{q-1}, 0, \gamma_{q+1}) \neq 0$ such that $\widehat{\overline{\psi}}(\gamma) = 0$.

Let $\mathcal{F}$ be the set of indices such that $\gamma_i = 0$ for $i \in \mathcal{F}$, $l = |\mathcal{F}| \leq d-1$. Define γ_t by $(\gamma_t)_i = t$ if $i \in \mathcal{F}$ and $(\gamma_t)_i = \gamma_i$ if $i \notin \mathcal{F}$. Hence for $t \in \mathbf{T}^o$ we have

$\gamma_t \in (\mathbf{T}^d)^o$, $\gamma_t \stackrel{t\to 0}{\longrightarrow} \gamma$, and

$$\begin{aligned} G_s(\gamma_t) &= \frac{t^{2l}}{\sin^{2l}(\pi t)} \left(\prod_{i \notin F} \frac{\gamma_i^2}{\sin^2(\pi \gamma_i)} \right) t^2 \\ & \cdot \sum_{m=1}^{\infty} \frac{1}{m^{s-2l+2}} \frac{\sin^{2l}(\pi m \gamma)}{(mt)^{2l}} \left(\prod_{i \notin F} \frac{\sin^2(\pi m \gamma_i)}{(m\gamma_i)^2} \right) \frac{|\widehat{\overline{\psi}}(mt\gamma)|^2}{(tm)^2}. \end{aligned}$$

Setting

$$f(x) = \left(\frac{\sin(\pi x)}{x} \right)^{2l}$$

we have that

$$\lim_{\gamma \to 0} t \sum_{m=0}^{\infty} \left(\frac{\sin(\pi m t)}{mt} \right)^{2l} \int f$$

and we obtain for small t the following bound:

$$t \sum_{m=1}^{\infty} \frac{1}{m^{s-2l-2}} \frac{\sin^{2l}(\pi m \gamma)}{(mt)^{2l}} \left(\prod_{i \notin F} \frac{\sin^2(\pi m \gamma_i)}{(m\gamma_i)^2} \right) \frac{|\widehat{\overline{\psi}}(mt\gamma)|^2}{(tm)^2} < \left(\int f + 1 \right) \prod_{i \notin F} \frac{1}{\gamma_i^2} C,$$

where C is an upper bound of

$$\frac{|\widehat{\overline{\psi}}(mt\gamma)|^2}{(tm)^2}.$$

Further we are using the fact, that $s \geq 2d+1$ and $l \leq d-1$ imply $s-2l-2 \geq 1$.

Hence,

$$\lim_{\gamma \to 0} G_s(\gamma_t) = 0,$$

contradicting the existence of a lower bound of G_s. $\square$

Corollary 8.14. *Let $\overline{\psi} \in l^2(\mathbf{Z}^d)$ be such that $\overline{\psi}[k] \neq 0$ only for $0 \leq k \leq N-1$, and $\{\overline{\psi}_{m,n}\}_{m \in \mathbf{Z}^+, n \in \mathbf{Z}^d}$ is a frame for $l^2(\mathbf{Z}^d)$, and such that $\overline{\psi}$ is separable in the following sense: there exist non empty index sets E, F with $E \cup F = \{1, \ldots, d\}$ and $E \cap F = \emptyset$, such that $\overline{w} = f_E f_F$, and where f_E only depends on x_i, $i \in E$ and f_F only depends on x_i, $i \in F$. Then $\widehat{\overline{\psi}}(0) \neq 0$.*

Proof. Assume $\overline{w} = f_E f_F$, where E, F, f_E, f_F are chosen as described above, and assume $\widehat{\overline{\psi}}(0) = 0$. Then $\widehat{\overline{\psi}} = \widehat{f_E}\widehat{f_F}$ and either $\widehat{f_E}(0) = 0$ or $\widehat{f_F}(0) = 0$. Without loss of generality, let $\widehat{f_E}(0) = 0$. But then $\widehat{\overline{\psi}}(\alpha) = 0$, where $\alpha_i = 0$ if $i \in E$ and $\alpha_i = \frac{1}{2}$ if $i \in F$. This contradicts Theorem **??**, part a. $\square$

Corollary **??** implies, that for $s = 2d+1$, we cannot produce a single piecewise constant wavelets such that $\{\overline{\psi}_{m,n}\}_{m \in \mathbf{Z}^+, n \in \mathbf{Z}^d}$ is a frame for $l^2(\mathbf{Z}^d)$ through forming simple tensors of lower dimensional wavelets. A similar problem is well known for dyadic wavelets.

8.4 Examples

We shall discuss a few piecewise constant wavelets for $d = 2$, starting with some which have the property that $\{\overline{\psi}_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}^d}$ is not a frame for $l^2(\mathbf{Z}^d)$ for all possible choices of s.

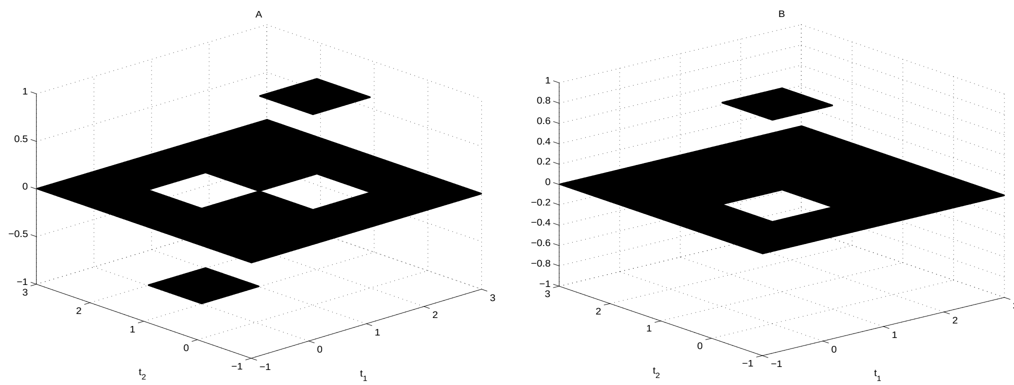

Figure 8.4. A: The wavelet described in Example **??**. B: The wavelet described in Example **??**.

Example 8.15. Let $\overline{w}[1,0] = 1$, $\overline{w}[0,1] = -1$, and $\overline{w}[n] = 0$ for $n \neq [1,0],[0,1]$. This wavelet is displayed in Figure **??**.A. In this case $\widehat{\overline{w}}(\gamma_1,\gamma_2) = e^{-2\pi i\gamma_1} - e^{-2\pi i\gamma_2}$ for $(\gamma_1,\gamma_2) \in \widehat{\mathbf{R}}^2$ and therefore $\widehat{\overline{w}}(\gamma_1,\gamma_1) = 0$ for $\gamma_1 \in \widehat{\mathbf{R}}$. In particular $\widehat{\overline{w}}(\frac{m}{2},\frac{m}{2}) = 0$ for all $m \in \mathbf{Z}^+$ and so $G_s(\frac{1}{2},\frac{1}{2}) = 0$ for all $s \in \mathbf{R}^+$. Hence $\{\overline{\psi}_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}^d}$ does not possess a lower framebound. G_5 in this case is shown in Figure **??**.

Example 8.16. Let $\overline{w}[0,0] = 2$, $\overline{w}[0,1] = \overline{w}[1,0] = \overline{w}[0,-1] = \overline{w}[-1,0] = 1$ and $\overline{w}[n] = 0$ for $n \neq [0,0],[1,0],[0,1],[-1,0],[0,-1]$. (See Figure **??**.B.) A short calculation shows that

$$\widehat{\overline{w}}(\gamma_1,\gamma_2) = |e^{-2\pi i\gamma_1} - (-1)|^2 + |e^{-2\pi i\gamma_2} - (-1)|^2 - 2$$

for $(\gamma_1,\gamma_2) \in \widehat{\mathbf{R}}^2$. Hence $\widehat{\overline{w}}(\frac{1}{3},\frac{1}{3})$ and $\widehat{\overline{w}}(\frac{2}{3},\frac{2}{3}) = 0$, and therefore $G_s(\frac{1}{3},\frac{1}{3}) = 0$ for all $s \in R^+$. Hence $\{\overline{\psi}_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}^d}$ is not a frame for $l^2(\mathbf{Z}^d)$. G_6 in this case is shown in Figure **??**.

Example 8.17. Let $\overline{w}[0,0] = -3$, $\overline{w}[0,1] = \overline{w}[1,0] = \overline{w}[1,1] = 1$ and $\overline{w}[n] = 0$ for $n \neq [0,0],[1,0],[0,1],[1,1]$. This wavelet is shown in Figure **??**.A. Numerical experiments imply that, for $s = 5$, $\{\overline{\psi}_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}^d}$ is a frame for $l^2(\mathbf{Z}^d)$. As approximate lower framebound we obtain $A = 3.87$ and as approximate upper framebound, we obtain $B = 22.53$. The resulting function G_5 is supplied in Figure **??**.

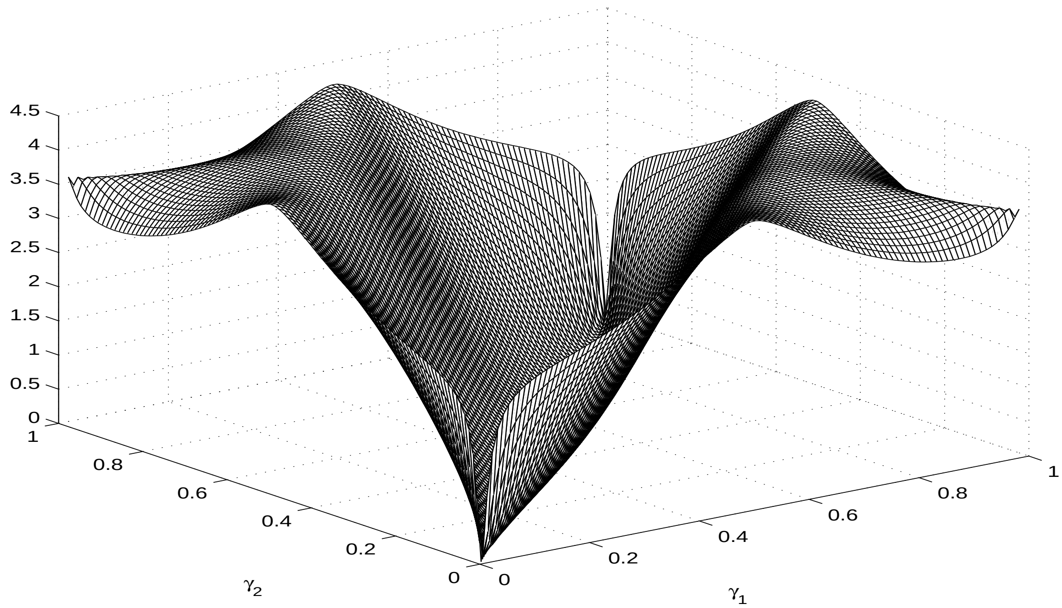

Figure 8.5. G_5 in Example **??**.

EXAMPLE 8.18. Let $\overline{w}[0,0] = -1$, $\overline{w}[1,0] = -2, \overline{w}[0,1] = 2, \overline{w}[1,1] = 4$ and $\overline{w}[n] = 0$ for $n \neq [0,0],[1,0],[0,1],[1,1]$. Figure **??**.B shows this wavelet. For $s = 6$, we obtain as numerical approximation $A = 1.02$ as lower framebound, and $B = 82.10$ as upper framebound. This implies that $\{\overline{\psi}_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}^d}$ is a frame for $l^2(\mathbf{Z}^d)$. The function G_6 is supplied in Figure **??**.

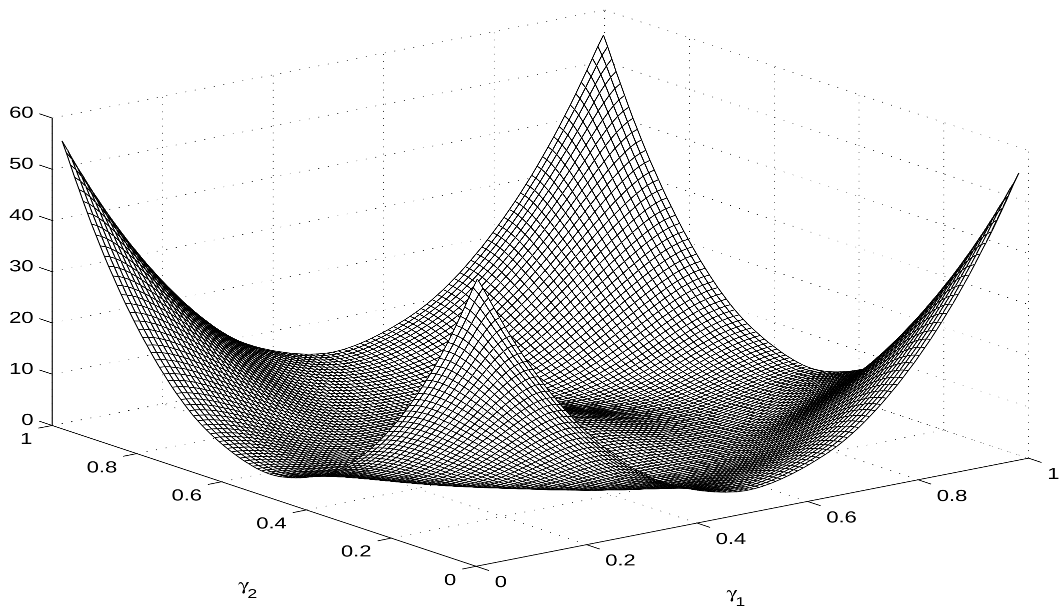

Figure 8.6. G_6 in Example **??**.

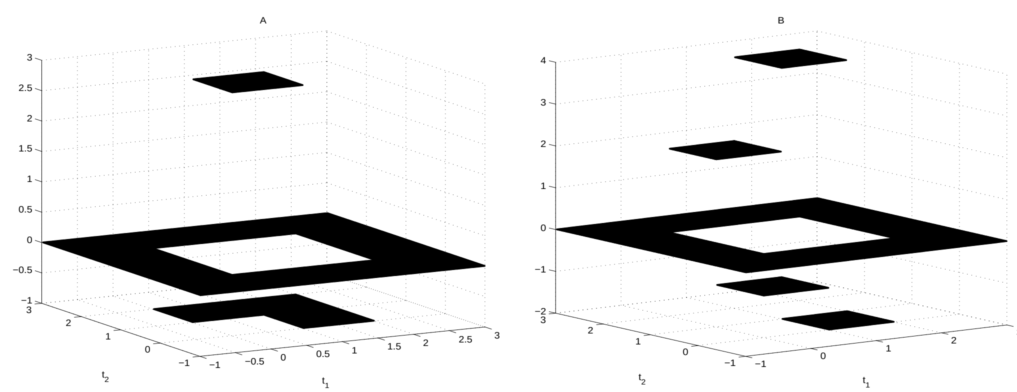

Figure 8.7. A: The wavelet described in Example **??**. B: The wavelet described in Example **??**.

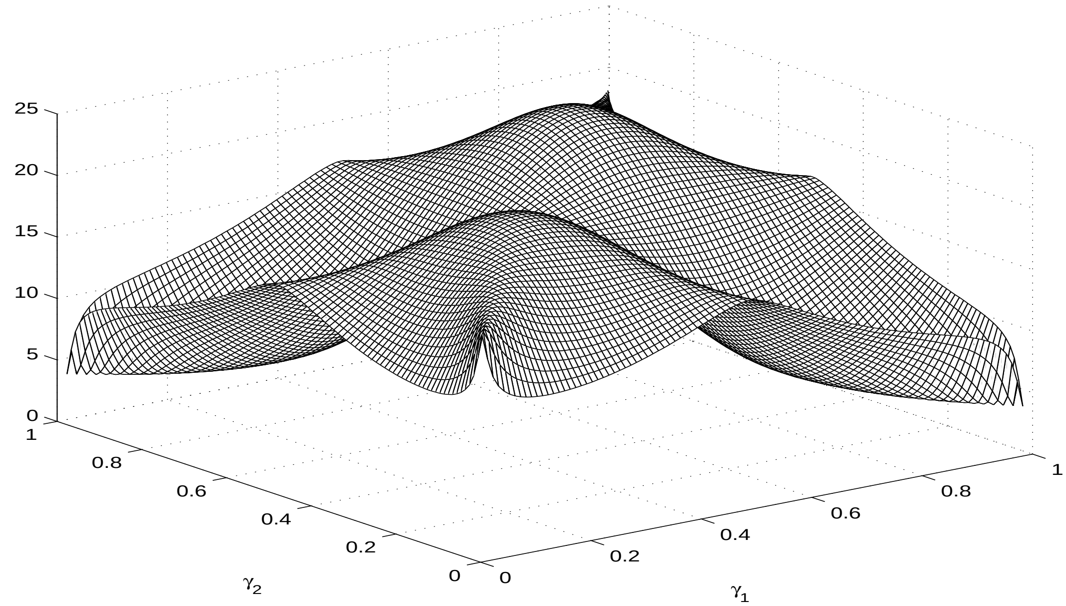

Figure 8.8. G_5 in Example **??**.

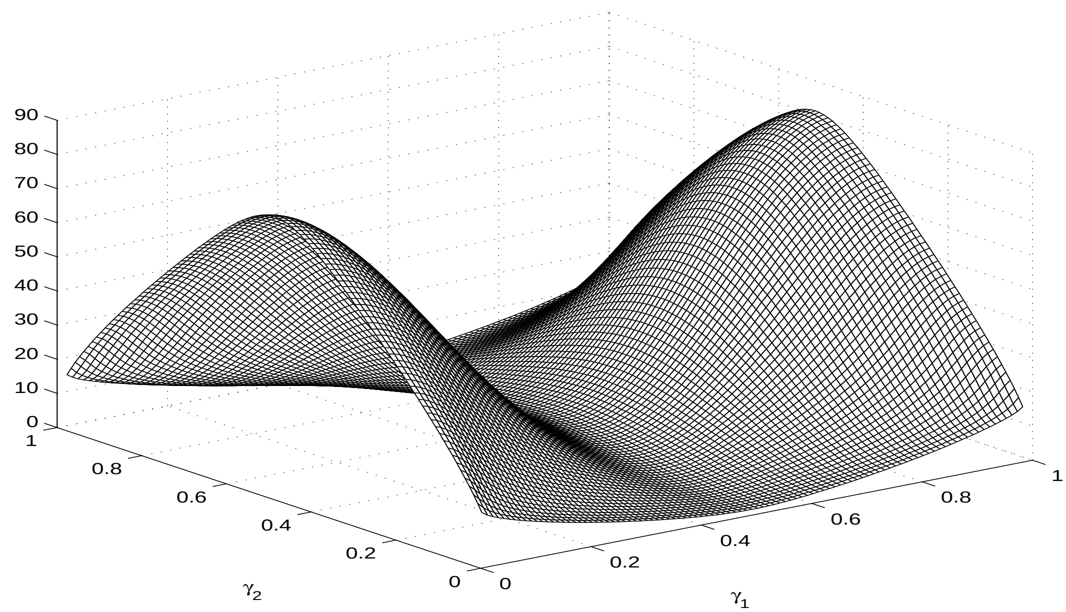

Figure 8.9. G_6 in Example **??**.

Chapter 9

Classification of wavelets with periodic waveletgrams

9.1 The Fundamental Theorem

The following theorem completely classifies all wavelets which have the property that the non–normalized wavelet transform of any periodic function $f \in L^\infty(\mathbf{R})$ is periodic in scale. Recall that continuous wavelet transforms of periodic functions are always periodic in time. In this section, we shall use a non–normalized wavelet tranformation.

THEOREM 9.1. *Let $\psi \in L^1(\mathbf{R})$. The following are equivalent:*
i. $W_f^\psi(b,a) = \int_{\mathbf{R}} f(t)\psi(\frac{t-b}{a})\,dt$ is 1–periodic in a for all $f \in L^\infty(\mathbf{T})$. (P)
ii. $\widehat{\psi}(0) = 0$ and ψ has the form

$$\psi(\cdot) = \sum_{n\in\mathbf{Z}} e_n \mathbf{1}_{(n,n+1)}(\cdot) + \sum_{n\in\mathbf{Z}} b_n \ln|\cdot - n|,$$

where $\{b_n\}$, $\{e_n - e_{n-1}\} \in l^1(\mathbf{Z})$ and $\{e_n - \pi i \sum_{k\leq n} b_k\} \in l^2(\mathbf{Z})$.

REMARK 9.2. Theorem **??**, as well as most results and remarks in this chapter, has a trivial generalization to S–periodic functions. If $\psi \in L^1(\mathbf{R})$ has the property that $W_f^\psi(b,a)$ is T–periodic in a for all $f \in L^\infty(\mathbf{T}_S)$, then $\tilde{\psi}(\cdot) = \psi(\frac{S}{T}\cdot)$ has property (P). To check this, we let $f \in L^\infty(\mathbf{T})$, and, hence, $\tilde{f}(\cdot) = f(\frac{\cdot}{S}) \in L^\infty(\mathbf{T}_S)$, and compute

$$\begin{aligned}
W_f^{\tilde{\psi}}(b,a+1) - W_f^{\tilde{\psi}}(b,a) &= \int_{\mathbf{R}} \tilde{\psi}(\frac{t-b}{a+1})f(t)\,dt - \int_{\mathbf{R}} \tilde{\psi}(\frac{t-b}{a})f(t)\,dt \\
&= \int_{\mathbf{R}} \psi(\frac{St-Sb}{Ta+T})f(t)\,dt - \int_{\mathbf{R}} \psi(\frac{St-Sb}{Ta})f(t)\,dt \\
&= \int_{\mathbf{R}} \psi(\frac{t-Sb}{Ta+T})f(\frac{t}{S})\,dt - \int_{\mathbf{R}} \psi(\frac{t-Sb}{Ta})f(\frac{t}{S})\,dt \\
&= W_{\tilde{f}}^\psi(Sb,Ta+T) - W_{\tilde{f}}^\psi(Sb,Ta) \\
&= 0.
\end{aligned}$$

Hence $\tilde{\psi}$ has the form

$$\tilde{\psi}(\cdot) = \sum_{n\in\mathbf{Z}} e_n \mathbf{1}_{(n,n+1)}(\cdot) + \sum_{n\in\mathbf{Z}} b_n \ln|\cdot - n|,$$

where $\{b_n\}$, $\{e_n - e_{n-1}\} \in l^1(\mathbf{Z})$ and $\{e_n - \pi i \sum_{k\le n} b_k\} \in l^2(\mathbf{Z})$. Consequently, ψ has the form

$$\psi(\cdot) = \sum_{n\in\mathbf{Z}} e_n \mathbf{1}_{(\frac{Tn}{S}, \frac{Tn+T}{S})}(\cdot) + \sum_{n\in\mathbf{Z}} b_n \ln\left|\frac{T\ \cdot - Sn}{S}\right|,$$

where $\{b_n\}$, $\{e_n - e_{n-1}\} \in l^1(\mathbf{Z})$ and $\{e_n - \pi i \sum_{k\le n} b_k\} \in l^2(\mathbf{Z})$.

To prove Theorem **??**, we need to establish various lemmata (Lemma **??**, Lemma **??**, Lemma **??**, Lemma **??**, Lemma **??**, and Lemma **??**).

Lemma 9.3. *Let $\psi \in L^1(\mathbf{R})$. The following are equivalent:*
i. $W_f^\psi(b,a) = \int_{\mathbf{R}} f(t)\psi(\frac{t-b}{a})\, dt$ is 1–periodic in a for all $f \in L^\infty(\mathbf{T})$.
ii. There exists a continuous function φ on $\mathbf{R}$, 1–periodic on $\widehat{\mathbf{R}}^+$ and 1–periodic on $\widehat{\mathbf{R}}^-$, such that

$$\widehat{\psi}(\gamma) = \frac{\varphi(\gamma)}{\gamma}$$

for $\gamma \in \widehat{\mathbf{R}} \setminus \{0\}$ and $\widehat{\psi}(0) = 0$.

Proof. $i \Longrightarrow ii$. Let us assume $W_f(b,a) = \int_{\mathbf{R}} f(t)\psi(\frac{t-b}{a})\, dt$ is $T = 1$ periodic in a for all $f \in L^\infty(\mathbf{T})$.

Then, for fixed $b \in \mathbf{R}$ and $a \in \mathbf{R}^+$, we obtain

$$\begin{aligned}
0 &= W_f^\psi(b,a) - W_f^\psi(b,a+1) \\
&= \int_{\mathbf{R}} f(t)\psi(\frac{t-b}{a})\, dt - \int_{\mathbf{R}} f(t)\psi(\frac{t-b}{a+1})\, dt \\
&= \int_{\mathbf{R}} f(t)(\psi(\frac{t-b}{a}) - \psi(\frac{t-b}{a+1}))\, dt \\
&= \sum_{n\in\mathbf{Z}} \int_0^1 f(t-n)(\psi(\frac{t-n-b}{a}) - \psi(\frac{t-n-b}{a+1}))\, dt \\
&= \sum_{n\in\mathbf{Z}} \int_0^1 f(t)(\psi(\frac{t-n-b}{a}) - \psi(\frac{t-n-b}{a+1}))\, dt \qquad (9.1) \\
&= \int_0^1 \sum_{n\in\mathbf{Z}} f(t)(\psi(\frac{t-n-b}{a}) - \psi(\frac{t-n-b}{a+1}))\, dt \\
&= \int_0^1 f(t) \sum_{n\in\mathbf{Z}} (\psi(\frac{t-n-b}{a}) - \psi(\frac{t-n-b}{a+1}))\, dt. \qquad (9.2)
\end{aligned}$$

We can interchange the sum and the integral in (**??**) using the Bounded Convergence Theorem on the partial sums, since for all l

$$\begin{aligned} |S_l(t)| &= |\sum_{|n|\leq l}(\psi(\frac{t-n-b}{a}) - \psi(\frac{t-n-b}{a+1}))| \\ &\leq \sum_{n\in\mathbf{Z}}(|\psi(\frac{t-n-b}{a})| + |\psi(\frac{t-n-b}{a+1})|), \end{aligned}$$

where the latter expression is in $L^1(\mathbf{T})$ since $\psi \in L^1(\mathbf{R})$.

Our calculation shows that if W_f^ψ is periodic in scale for all $f \in L^\infty(\mathbf{T})$ we obtain

$$S(t) = \sum_{n\in\mathbf{Z}}(\psi(\frac{t-n-b}{a}) - \psi(\frac{t-n-b}{a+1}))\,dt = 0.$$

For all $m \in \mathbf{Z}$ we have

$$\begin{aligned} 0 &= \widehat{S}[m] \\ &= \int_0^1 S(t)e^{-2\pi imt}\,dt \\ &= \int_0^1 \sum_{n\in\mathbf{Z}}(\psi(\frac{t-n-b}{a}) - \psi(\frac{t-n-b}{a+1}))e^{-2\pi imt}\,dt \\ &= \sum_{n\in\mathbf{Z}}\int_0^1 (\psi(\frac{t-n-b}{a}) - \psi(\frac{t-n-b}{a+1}))e^{-2\pi imt}\,dt \\ &= \sum_{n\in\mathbf{Z}}(\int_0^1 \psi(\frac{t-n-b}{a})e^{-2\pi imt}\,dt - \int_0^1 \psi(\frac{t-n-b}{a+1})e^{-2\pi imt}\,dt) \\ &= \sum_{n\in\mathbf{Z}}(\int_{\frac{0-b-n}{a}}^{\frac{1-b-n}{a}} \psi(u)e^{-2\pi im(au+n+b)}a\,du - \int_{\frac{0-b-n}{a+1}}^{\frac{1-b-n}{a+1}} \psi(u)e^{-2\pi im((a+1)u+n+b)}(a+1)\,du) \\ &= \sum_{n\in\mathbf{Z}}(\int_{\frac{0-b-n}{a}}^{\frac{1-b-n}{a}} \psi(u)e^{-2\pi im(au+b)}a\,du - \int_{\frac{0-b-n}{a+1}}^{\frac{1-b-n}{a+1}} \psi(u)e^{-2\pi im((a+1)u+b)}(a+1)\,du) \\ &= \int_{\mathbf{R}} \psi(u)e^{-2\pi im(au+b)}a\,du - \int_{\mathbf{R}} \psi(u)e^{-2\pi im((a+1)u+b)}(a+1)\,du \\ &= e^{-2\pi imb}(a\int_{\mathbf{R}} \psi(u)e^{-2\pi imau}du - (a+1)\int_{\mathbf{R}} \psi(u)e^{-2\pi im(a+1)u}du) \\ &= e^{-2\pi imb}(a\widehat{\psi}(ma) - (a+1)\widehat{\psi}(m(a+1))). \end{aligned} \tag{9.3}$$

Interchanging sum and integral in (**??**) is justified by the bounded convergence theorem.

This calculation implies that we have for all $a \in \mathbf{R}^+$ and all $m \in \mathbf{Z}$

$$a\widehat{\psi}(ma) - (a+1)\widehat{\psi}(m(a+1)) = 0,$$

and in particular $\widehat{\psi}(0) = 0$. If we let

$$\varphi(\gamma) = \gamma\widehat{\psi}(\gamma)$$

for $\gamma \in \widehat{\mathbf{R}}$, and take $m = 1$, we obtain for $\gamma \in \widehat{\mathbf{R}}^+$ that

$$\varphi(\gamma) - \varphi(\gamma+1) = \gamma\widehat{\psi}(\gamma) - (\gamma+1)\widehat{\psi}(\gamma+1) = 0.$$

Hence, $\varphi(\gamma) = \varphi(\gamma+1)$ for all $\gamma \in \widehat{\mathbf{R}}^+$. For $\gamma \in \widehat{\mathbf{R}}^-$ and $a = -\gamma > 0$, we set $m = -1$ and obtain

$$\begin{aligned}
\varphi(\gamma) - \varphi(\gamma-1) &= \gamma\widehat{\psi}(\gamma) - (\gamma-1)\widehat{\psi}(\gamma-1) \\
&= \gamma\widehat{\psi}((-1)(-\gamma)) - (\gamma-1)\widehat{\psi}((-1)(-\gamma+1) \\
&= (-1)(-\gamma\widehat{\psi}((-1)(-\gamma)) - (-\gamma+1)\widehat{\psi}((-1)(-\gamma+1) \\
&= (-1)(a\widehat{\psi}(ma) - (a+1)\widehat{\psi}(m(a+1)) \\
&= 0.
\end{aligned}$$

Hence, $\varphi(\gamma) = \varphi(\gamma-1)$ for all $\gamma \in \widehat{\mathbf{R}}^-$ and *ii* holds.

ii $\Longrightarrow$ *i*. Conversely, if statement *ii* holds, we have for $m > 0$

$$\begin{aligned}
a\widehat{\psi}(ma) - (a+1)\widehat{\psi}(m(a+1)) &= a\frac{\varphi(ma)}{ma} - (a+1)\frac{\varphi(m(a+1))}{m(a+1)} \\
&= \frac{1}{m}(\varphi(ma) - \varphi(ma+m)) \\
&= 0,
\end{aligned}$$

since $m, am > 0$ and by the 1–periodicity of φ on $\widehat{\mathbf{R}}^+$. Similarly, for $m < 0$,

$$a\widehat{\psi}(ma) - (a+1)\widehat{\psi}(m(a+1)) = \frac{1}{m}(\varphi(ma) - \varphi(ma+m)) = 0,$$

since now $m, am < 0$ and the 1–periodicity of φ on $\widehat{\mathbf{R}}^-$ applies.

Also $\widehat{S}[0] = 0$ since $\widehat{\psi}(0) = 0$.

Therefore $\widehat{S}[m] = 0$ for all $m \in \mathbf{Z}$. By the uniqueness theorem for Fourier transformations we have $S = 0$ in $L^1(\mathbf{T})$. The periodicity in scale follows, since

$$W_f^\psi(b,a) - W_f^\psi(b,a+1) = \int_0^1 f(t)S(t)\,dt = 0.$$

Therefore *i* holds. $\square$

REMARK 9.4.
a. This result implies that if ψ satisfies property (P), then $\widehat{\psi}$ has the form:

$$\widehat{\psi}(\gamma) = \frac{\varphi_1(\gamma)}{\gamma} + \mathrm{H}(\gamma)\frac{\varphi_2(\gamma)}{\gamma},$$

where φ_1 and φ_2 are 1–periodic on all of $\widehat{\mathbf{R}}$, and H denotes the Heaviside function, i.e., $\mathrm{H}(\gamma) = \mathbf{1}_{(0,\infty)}$.
b. If ψ has property (P), then $\widehat{\psi}(\gamma) = O(\frac{1}{\gamma})$ and $\widehat{\psi}(\gamma) \neq o(\frac{1}{\gamma})$ as $|\gamma| \to \infty$.
c. No absolutely continuous function ψ can have property (P), since in this case $\widehat{\psi}(\gamma) = o(\frac{1}{\gamma})$, $|\gamma| \to \infty$.
d. Proposition **??** is a corollary of this result.

From the proof of Lemma **??**, in particular from (**??**), we can deduce the following corollary:

COROLLARY 9.5. *Let $\psi \in L^1(\mathbf{R})$. For fixed $f \in L^\infty(\mathbf{T})$, W_f^ψ is 1–periodic in scale if and only if, for all $a \in \mathbf{R}^+$ and all $b \in \mathbf{R}$,*

$$\int_0^1 f(t) \sum_{n\in\mathbf{Z}} (\psi(\frac{t-n-b}{a}) - \psi(\frac{t-n-b}{a+1}))\, dt = 0.$$

REMARK 9.6. This corollary can be helpful, if we are interested in picking up one specific periodic component $f \in L^\infty(\mathbf{T})$ in a signal that carries other periodic components besides f. The fact that for $g \neq f \in L^\infty(\mathbf{T})$ the wavelet transform of g might not be periodic implies that the components of g in a signal get blurred in the waveletgram. This can be helpful to distinguish periodic signals of different shapes.

Lemma **??** and Lemma **??** will be used to prove Lemma **??**:

LEMMA 9.7. *For all $0 < \epsilon < 1$ and $\gamma \in \widehat{\mathbf{R}}$ we have*

$$\left| \int_\epsilon^{\frac{1}{\epsilon}} \frac{\sin(2\pi t\gamma)}{t}\, dt \right| \leq 5\pi.$$

Proof. Let us assume, without loss of generality, that $\gamma > 0$. For $k \in \mathbf{N}_0$ let

$$C_k = (-1)^k \int_k^{k+1} \frac{\sin(\pi x)}{x}\, dx$$

Clearly,

$$\sum_{k=0}^{\infty} (-1)^k C_k = \int_0^\infty \frac{\sin(\pi x)}{x}\, dx = \frac{\pi}{2},$$

and

$$0 < \ldots < C_{k+1} < C_k < \ldots < C_0 = \int_0^1 \frac{\sin(\pi x)}{x}\, dx < \int_0^1 \pi\, dx = \pi.$$

We obtain for any positive integer r and any integer $s > r$

$$0 < \sum_{l=r}^{s-1}(C_{2l} - C_{2l+1}) = \sum_{l=2r}^{2s-1}(-1)^l C_l < C_{2r} < \pi.$$

For γ and ϵ fixed, choose r such that $r < \gamma\epsilon < r+1$ and s such that $s < \frac{\gamma}{\epsilon} < s+1$. Then

$$\begin{aligned}
\left|\int_{\epsilon}^{\frac{1}{\epsilon}} \frac{\sin(2\pi t\gamma)}{t}\, dt\right| &= \left|\int_{2\epsilon\gamma}^{\frac{2\gamma}{\epsilon}} \frac{\sin(\pi x)}{x}\, dx\right| \\
&\leq \left|\int_{\epsilon\gamma}^{2(r+1)} \frac{\sin(\pi x)}{x}\, dx\right| + \left|\int_{2(r+1)}^{2s} \frac{\sin(\pi x)}{x}\, dx\right| \\
&\quad + \left|\int_{2s}^{\frac{\gamma}{\epsilon}} \frac{\sin(\pi x)}{x}\, dx\right| \\
&\leq \int_{2r}^{2(r+1)} \left|\frac{\sin(\pi x)}{x}\right| dx + \sum_{l=2(r+1)}^{2s-1} (-1)^l C_l \\
&\quad + \int_{2s}^{2(s+1)} \left|\frac{\sin(\pi x)}{x}\right| dx \\
&\leq C_{2r} + C_{2r+1} + \pi + C_{2s} + C_{2s+1} \\
&\leq 5\pi.
\end{aligned}$$

□

LEMMA 9.8. *Let $\psi \in L^1(\mathbf{R})$ be such that for all $\gamma \in \widehat{\mathbf{R}}^+$, resp. for all $\gamma \in \widehat{\mathbf{R}}^-$,*

$$\widehat{\psi}(\gamma) = \frac{\varphi(\gamma)}{\gamma}$$

with φ 1–periodic on $\widehat{\mathbf{R}}$. Then $\varphi \in A(\mathbf{T})$ and hence, the Fourier Series $S\big(\varphi\big)(\gamma) = \sum_{n\in\mathbf{Z}} b_n e^{-2\pi n\gamma}$, $\gamma \in \widehat{\mathbf{R}}$, converges to φ absolutely and uniformly, i.e., $\{b_n\} \in l^1(\mathbf{Z})$.

Before proving this Lemma, we shall recall a few basic facts from Harmonic Analyis. Let w denote the Fejér Kernel, i.e., for $x \in \mathbf{R}$,

$$w(x) = \frac{1}{2\pi}\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right)^2.$$

Clearly, $w \in L^1(\mathbf{R}) \cap C^1(\mathbf{R})$ and $w' \in L^1(\mathbf{R})$, since

$$w'(x) = \sin(\frac{x}{2})\left[\frac{\cos\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2} - \frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2}\right],$$

and therefore $w'(x) = O(\frac{1}{x^2})$, $|x| \to \infty$. The Fourier transform of w is

$$\widehat{w}(\gamma) = \max\{0, 1 - |\gamma|\},$$

$\gamma \in \widehat{\mathbf{R}}$.

The de la Vallée – Poussin Kernel v is then defined by:

$$v(x) = 4w(2x) - w(x),$$

, $x \in \mathbf{R}$. As before, $v \in L^1(\mathbf{R}) \cap C^1(\mathbf{R})$ and $v' \in L^1(\mathbf{R})$. Clearly,

$$\widehat{v}(\gamma) = 2\widehat{w}(\frac{\gamma}{2}) - \widehat{w}(\gamma),$$

and therefore $\widehat{v}(\gamma) = 1$ for $\gamma \in [-1, 1]$ and $\widehat{v}(\gamma) = 0$ for $\gamma \in [-2, 2]^c$. Note that since $v, v' \in L^1(\mathbf{R})$ and

$$\widehat{v'}(\gamma) = i\gamma\widehat{v}(\gamma),$$

$\gamma \in \widehat{\mathbf{R}}$, we have $\gamma\widehat{v}(\gamma) \in A(\widehat{\mathbf{R}})$.

Proof. Fix $\gamma_0 \in [1, 2)$, resp., $\gamma_0 \in [-2, 1)$. Define

$$v_{\gamma_0}(x) = \frac{1}{8}v(\frac{x}{8})e^{2\pi i\gamma_0 x},$$

$x \in \mathbf{R}$. Then ,clearly,

$$\widehat{v_{\gamma_0}}(\gamma) = \widehat{v}(8(\gamma - \gamma_0)),$$

$\gamma \in \widehat{\mathbf{R}}$, $\widehat{v_{\gamma_0}}(\gamma) = 1$ for $\gamma \in [\gamma_0 - \frac{1}{8}, \gamma_0 + \frac{1}{8}]$, and $\text{supp}(\widehat{v_{\gamma_0}}) \subset [\gamma_0 - \frac{1}{4}, \gamma_0 + \frac{1}{4}] \subset [\frac{3}{4}, \frac{9}{4}] \subset \widehat{\mathbf{R}}^+$, resp., $\text{supp}(\widehat{v_{\gamma_0}}) \subset \widehat{\mathbf{R}}^-$. As before, $\gamma\widehat{v_{\gamma_0}}(\gamma) \in A(\widehat{\mathbf{R}})$.

Since $\widehat{\psi} \in A(\widehat{\mathbf{R}})$ we have $\widehat{v_{\gamma_0}}(\gamma)\varphi(\gamma) = \gamma\widehat{v_{\gamma_0}}(\gamma)\widehat{\psi}(\gamma) \in A(\widehat{\mathbf{R}})$. Therefore, by a Theorem of Wiener ([**?**], [**?**], page 202, [**?**], page 56), $\widehat{v_{\gamma_0}}\varphi \in A(\mathbf{T})$. Since $\varphi = \widehat{v_{\gamma_0}}\varphi$ in a neighborhood of γ_0, we have $\varphi \in A_{loc(\gamma_0)}(\mathbf{T})$. This result holds for any $\gamma_0 \in [1, 2]$, resp., $[-2, -1)$, and hence $\varphi \in A_{loc}(\mathbf{T})$. By Wiener's local membership theorem we have $\varphi \in A(\mathbf{T})$[**?**].

Therefore, we can write $\varphi(\gamma) = \sum_{n\in\mathbf{Z}} b_n e^{-2\pi n\gamma}$ with $\{b_n\} \in l^1(\mathbf{Z})$ and the result is proven. □

Recall that $\mathcal{F} : L^2(\mathbf{R}) \longrightarrow L^2(\mathbf{R})$ denotes the $L^2(\mathbf{R})$-Fourier transformation.

Lemma 9.9. *Let $\psi \in L^2(\mathbf{R})$ be such that for all $\gamma \in \widehat{\mathbf{R}} \setminus \{0\}$*

$$\mathcal{F}(\psi)(\gamma) = \frac{\varphi(\gamma)}{\gamma},$$

where $\varphi \in A(\mathbf{T})$. Then ψ has the form

$$\psi(\cdot) = \sum_{n\in\mathbf{Z}} \pi i a_n \ \text{sign}(\cdot - n) = \sum_{n\in\mathbf{Z}} c_n \mathbf{1}_{(n,n+1)}(\cdot),$$

where

$$\varphi(\gamma) = \sum_{n\in\mathbf{Z}} a_n e^{-2\pi i n\gamma},$$

and

$$c_n = 2\pi i \sum_{k\leq n} a_n,$$

where $\{c_n\} \in l^2(\mathbf{Z})$*. The convergence is pointwise for* $t \notin \mathbf{Z}$*, as well as in* $L^2(\mathbf{R})$*.*

Thus, for $c_n = 2\pi i \sum_{k\leq n} a_n$ *we have the* $\mathcal{F}$*–pairing,*

$$\sum_{n\in\mathbf{Z}} c_n \mathbf{1}_{(n,n+1)}(t) \longleftrightarrow \frac{1}{\gamma} \sum_{n\in\mathbf{Z}} a_n e^{-2\pi i n\gamma}. \tag{9.4}$$

Proof. Clearly, $\widehat{\psi} \in L^2(\widehat{\mathbf{R}})$ since $\psi \in L^2(\mathbf{R})$. Thus, we can apply the L^2–inversion formula

$$\psi(t) = \lim_{N\to\infty} \int_{-N}^{N} \widehat{\psi}(\gamma) e^{2\pi i\gamma t}\, d\gamma,$$

with convergence of this limit in $L^2(\mathbf{R})$. We have

$$\varphi(\gamma) = \sum_{n\in\mathbf{Z}} a_n e^{-2\pi i n\gamma}$$

with $\{a_n\} \in l^1(\mathbf{Z})$ since $\varphi \in A(\mathbf{T})$. We obtain

$$
\begin{aligned}
\psi(t) &= \lim_{N\to\infty} \int_{-N}^{N} \widehat{\psi}(\gamma) e^{2\pi i \gamma t} d\gamma \\
&= \lim_{N\to\infty} \int_{-N}^{N} \frac{1}{\gamma} \left(\sum_{n\in\mathbf{Z}} a_n e^{-2\pi i n \gamma} \right) e^{2\pi i \gamma t}\, d\gamma \\
&= \lim_{N\to\infty} \int_{\frac{1}{N}\le|\gamma|\le N} \frac{1}{\gamma} \left(\sum_{n\in\mathbf{Z}} a_n e^{-2\pi i n \gamma} \right) e^{2\pi i \gamma t}\, d\gamma \\
&= \lim_{N\to\infty} \int_{\frac{1}{N}\le|\gamma|\le N} \sum_{n\in\mathbf{Z}} a_n \frac{e^{2\pi i (t-n)\gamma}}{\gamma}\, d\gamma \\
&= \lim_{N\to\infty} \sum_{n\in\mathbf{Z}} a_n \int_{\frac{1}{N}\le|\gamma|\le N} \frac{e^{2\pi i (t-n)\gamma}}{\gamma}\, d\gamma \qquad (9.5) \\
&= \lim_{N\to\infty} \sum_{n\in\mathbf{Z}} a_n \int_{\frac{1}{N}\le|\gamma|\le N} \left(\frac{\cos(2\pi(t-n)\gamma)}{\gamma} + i\, \frac{\sin(2\pi(t-n)\gamma)}{\gamma} \right)\, d\gamma \\
&= i \lim_{N\to\infty} \sum_{n\in\mathbf{Z}} a_n \int_{\frac{1}{N}\le|\gamma|\le N} \frac{\sin(2\pi(t-n)\gamma)}{\gamma}\, d\gamma \\
&= i \sum_{n\in\mathbf{Z}} a_n \lim_{N\to\infty} \int_{\frac{1}{N}\le|\gamma|\le N} \frac{\sin(2\pi(t-n)\gamma)}{\gamma}\, d\gamma \qquad (9.6) \\
&= i \sum_{n\in\mathbf{Z}} a_n \left\{ \begin{array}{ccc} \pi & \text{for} & t > n \\ 0 & \text{for} & t = n \\ -\pi & \text{for} & t < n \end{array} \right\}.
\end{aligned}
$$

Note that (**??**) is true since $\{a_n\} \in l^1(\mathbf{Z})$ (Lemma **??**), and (**??**) holds due to Lemma **??**.

Further note that for $t \in (k, k+1)$ we have

$$
\begin{aligned}
\psi(t) &= \sum_{n\in\mathbf{Z}} \pi i a_n \operatorname{sign}(t-n) \\
&= \sum_{n\le k} \pi i a_n - \sum_{n\ge k+1} \pi i a_n \\
&= \sum_{n\le k} \pi i a_n + \sum_{n\in\mathbf{Z}} \pi i a_n - \sum_{n\ge k+1} \pi i a_n \\
&= 2\pi i \sum_{n\le k} a_n \\
&= c_k.
\end{aligned}
$$

Finally, we have $\|\{c_n\}\|_{l^2(\mathbf{Z})} = \|\psi\|_{L^2(\mathbf{R})} < \infty$ and therefore $\{c_n\} \in l^2(\mathbf{Z})$. $\square$

Remark 9.10. The result can be proven in a seemingly more elegant way using tempered distributional calculus.

The following theorem is a corollary to Lemma **??**.

Theorem 9.11. *Let $\psi \in L^1(\mathbf{R})$ be such that for $\gamma \in \widehat{\mathbf{R}}$*

$$\widehat{\psi}(\gamma) = \frac{\varphi(\gamma)}{\gamma}$$

where φ is 1–periodic on $\widehat{\mathbf{R}}$ and $\widehat{\psi}(0) = 0$. Then ψ is a piecewise constant wavelet of degree 1. In fact,

$$\psi(\cdot) = \sum_{n \in \mathbf{Z}} \pi i a_n \ \mathrm{sign}(\cdot - n) = \sum_{n \in \mathbf{Z}} c_n \mathbf{1}_{(n,n+1)}(\cdot)$$

where

$$\varphi(\gamma) = \sum_{n \in \mathbf{Z}} a_n e^{-2\pi i n \gamma},$$

$\gamma \in \widehat{\mathbf{R}}$, and

$$c_n = 2\pi i \sum_{k \leq n} a_k.$$

Proof. Lemma **??** implies that since $\psi \in L^1(\mathbf{R})$, $\widehat{\psi}(0) = 0$, and for $\gamma \in \widehat{\mathbf{R}}$

$$\widehat{\psi}(\gamma) = \frac{\varphi(\gamma)}{\gamma}$$

we have $\varphi \in A(\mathbf{T})$. Then Lemma **??** implies the result. Note that since $\psi \in L^1(\mathbf{R})$

$$\sum_{n \in \mathbf{Z}} |c_n| = \|\psi\|_{L^1(\mathbf{R})} < \infty,$$

and therefore $\{c_n\} \in l^1(\mathbf{Z})$. □

To obtain the main result, we need two more lemmata:

Lemma 9.12. *Let $\psi \in L^2(\mathbf{R})$ be such that for $\gamma \in \widehat{\mathbf{R}}$*

$$\mathcal{F}(\psi)(\gamma) = \mathrm{H}(\gamma)\frac{\varphi(\gamma)}{\gamma},$$

where $\varphi \in A(\mathbf{T})$. Then there exists $\{d_n\} \in l^2(\mathbf{Z})$ and $\{b_n\} \in l^1(\mathbf{Z})$ such that $\sum_{n \in \mathbf{Z}} b_n = 0$ and

$$\psi(\cdot) = \frac{1}{2}\sum_{n \in \mathbf{Z}} d_n \mathbf{1}_{(n,n+1)}(\cdot) + \sum_{n \in \mathbf{Z}} b_n \ln|\cdot - n|$$

with pointwise convergence for $t \notin \mathbf{Z}$, as well as convergence in $L^2(\mathbf{R})$.

Thus, if $d_n = 2\pi i \sum_{k\leq n} b_k$, *we have the* $\mathcal{F}$*–pairing*

$$\sum_{n\in\mathbf{Z}} d_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbf{Z}} b_n \ln|t-n| \longleftrightarrow \mathrm{H}(\gamma)\frac{1}{\gamma}\sum_{n\in\mathbf{Z}} b_n e^{-2\pi i n\gamma}. \qquad (9.7)$$

Proof. Let $\Theta(\gamma) = \frac{\varphi(\gamma)}{\gamma}$, $\varphi(\gamma) = \sum_{n\in\mathbf{Z}} b_n e^{-2\pi i n\gamma}$. Clearly $\Theta \in L^2(\widehat{\mathbf{R}})$. Let

$$\mathcal{H} : L^2(\mathbf{R}) \longrightarrow L^2(\mathbf{R})$$

denote the Hilbert transformation.

Lemma **??** implies

$$\mathcal{F}^{-1}(\Theta) = \sum_{n\in\mathbf{Z}} d_n \mathbf{1}_{(n,n+1)},$$

where

$$d_n = 2\pi i \sum_{k\leq n} b_k$$

and $\{d_n\} \in l^2(\mathbf{Z})$. Define

$$g = \frac{1}{2}\mathcal{F}^{-1}(\Theta) - \frac{1}{2i}\mathcal{H}(\mathcal{F}^{-1}(\Theta)).$$

Clearly, $g \in L^2(\mathbf{R})$, and

$$\begin{aligned}
\mathcal{F}(g) &= \frac{1}{2}\Theta - \frac{1}{2i}\mathcal{F}(\mathcal{F}^{-1}(-i \text{ sign } \mathcal{F}(\mathcal{F}^{-1}(\Theta)))) \\
&= \frac{1}{2}\Theta + \frac{1}{2}\,\mathrm{sign}\,\Theta \\
&= \mathrm{H}\Theta.
\end{aligned}$$

Hence

$$g = \mathcal{F}^{-1}(\mathrm{H}\Theta).$$

Further, for $t \notin \mathbf{Z}$

$$
\begin{aligned}
\mathcal{H}(\mathcal{F}^{-1}(\Theta))(t) &= \lim_{\epsilon\to 0} \frac{1}{\pi} \int_{|t-u|\geq\epsilon} \frac{\mathcal{F}^{-1}(\Theta)(u)}{t-u}\, du \\
&= \lim_{\epsilon\to 0} \frac{1}{\pi} \int_{|t-u|\geq\epsilon} \frac{\sum_{n\in\mathbf{Z}} d_n \mathbf{1}_{(n,n+1)}(u)}{t-u}\, du \\
&= -\lim_{\epsilon\to 0} \frac{1}{\pi} \int_{|x|\geq\epsilon} \frac{\sum_{n\in\mathbf{Z}} d_n \mathbf{1}_{(n,n+1)}(t-x)}{x}\, dx \\
&= -\lim_{\epsilon\to 0} \frac{1}{\pi} \int_{|x|\geq\epsilon} \frac{\sum_{n\in\mathbf{Z}} d_n \mathbf{1}_{(t-n-1,t-n)}(x)}{x}\, dx \\
&= -\frac{1}{\pi} \sum_{n\in\mathbf{Z}} d_n \int_{t-n-1}^{t-n} \frac{1}{x}\, dx \\
&= -\frac{1}{\pi} \sum_{n\in\mathbf{Z}} d_n (\ln|t-n| - \ln|t-n-1|) \\
&= -\frac{1}{\pi} \sum_{n\in\mathbf{Z}} (d_n - d_{n-1}) \ln|t-n| \\
&= -\frac{1}{\pi} \sum_{n\in\mathbf{Z}} (2\pi i \sum_{k<n} a_k - 2\pi i \sum_{k<n-1} a_k) \ln|t-n| \\
&= -\frac{1}{\pi} \sum_{n\in\mathbf{Z}} 2\pi i b_n \ln|t-n| \\
&= -2i \sum_{n\in\mathbf{Z}} b_n \ln|t-n|.
\end{aligned}
$$

Therefore, we have for $t \notin \mathbf{Z}$

$$
\begin{aligned}
g(t) &= \frac{1}{2}\mathcal{F}^{-1}(\Theta)(t) - \frac{1}{2i}\mathcal{H}(\mathcal{F}^{-1}(\Theta))(t) \\
&= \frac{1}{2}\sum_{n\in\mathbf{Z}} d_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbf{Z}} b_n \ln|t-n|.
\end{aligned}
$$

$\square$

LEMMA 9.13. *Let $\psi \in L^1(\mathbf{R})$ have the property that $\widehat{\psi}(0) = 0$ and for $t \notin \mathbf{Z}$,*

$$\psi(t) = \sum_{n\in\mathbf{Z}} e_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbf{Z}} b_n \ln|t-n|$$

where $\{a_n\}$, $\{b_n\} \in l^1(\mathbf{Z})$, and $\{c_n\} \in l^2(\mathbf{Z})$ such that

$$a_n = \frac{1}{2\pi i}(e_n - e_{n-1}) - \frac{1}{2}b_n,$$

and
$$c_n = e_n - \pi i \sum_{k\le n} b_k.$$
Then $\psi \in L^2(\mathbf{R})$.

Proof. We shall prove this technical result in four steps.
STEP 1. *There exists* C_1 *such that* $|e_n| \le C_1$ *for all* $n \in \mathbf{Z}$.

By hypothesis we have
$$a_n = \frac{1}{2\pi i}(e_n - e_{n-1}) - \frac{1}{2} b_n,$$
and we obtain for all $N \in \mathbf{Z}^+$
$$\begin{aligned} 2\pi i \sum_{1\le n\le N} a_n &= \sum_{1\le n\le N} (e_n - e_{n-1}) - \pi i \sum_{1\le n\le N} b_n \\ &= e_N - e_{1-1} - \pi i \sum_{1\le n\le N} b_n. \end{aligned}$$
Therefore for all $N \in \mathbf{Z}^+$, since $\{a_n\}$ and $\{b_n\} \in l^1(\mathbf{Z})$, we have for all $N \in \mathbf{Z}^+$
$$\begin{aligned} |e_N - e_0| &= |\pi i \sum_{1\le n\le N} b_n + 2\pi i \sum_{1\le n\le N} a_n| \\ &\le \pi(\sum_{1\le n\le N} |b_n| + 2 \sum_{1\le n\le N} |a_n|) \\ &\le \pi(\|\{b_n\}\|_{l^1(\mathbf{Z})} + 2\,\|\{a_n\}\|_{l^1(\mathbf{Z})}). \end{aligned}$$
Setting
$$C_1 = \pi(\|\{b_n\}\|_{l^1(\mathbf{Z})} + 2\,\|\{a_n\}\|_{l^1(\mathbf{Z})}) + |e_0|$$
we obtain for positive N
$$|e_N| \le C_1.$$
Clearly, the same bound holds for negative N and $|e_0|$. ■

For $n \in \mathbf{Z}$ define
$$g_n(t) = \sum_{k\ne n} b_k \ln|t-k| = \psi(t) - b_n \ln|t-n| - e_{n-1}\mathbf{1}_{(n-1,n)}(t) - e_n \mathbf{1}_{(n,n+1)}(t),$$
$t \in [n-\frac{1}{2}, n+\frac{1}{2}]$, where the sum converges pointwise.
STEP 2. *There exists* C_2 *such that* $|g_n'| \le C_2$ *on* $[n-\frac{1}{2}, n+\frac{1}{2}]$ *for all* $n \in \mathbf{Z}$.

Fix $n \in \mathbf{Z}$. For $N \ne n$ let
$$S_N(t) = b_N \ln|t-N|.$$

Clearly S_N is continuously differentiable on the interval $[n-\frac{1}{2}, n+\frac{1}{2}]$ and

$$|S_N{}'(t)| = |b_N \frac{1}{t-N}| \leq |b_N \frac{1}{\frac{1}{2}}| = 2|b_N|.$$

Since $\{b_n\} \in l^1(\mathbf{Z})$, we can define

$$h_n(t) = \sum_{k \neq n} b_k \frac{1}{t-N}$$

with uniform convergence on $[n-\frac{1}{2}, n+\frac{1}{2}]$. Hence, g_n is continuously differentiable on $[n-\frac{1}{2}, n+\frac{1}{2}]$ and $g_n' = h_n$. Letting $C_2 = 2\,\|\{b_n\}\|_{l^1(\mathbf{Z})}$ we have

$$|g_n'(t)| \leq C_2$$

for $t \in [n-\frac{1}{2}, n+\frac{1}{2}]$. ∎

STEP 3. *There exist $N > 0$ and C_6 such that $|g_n| \leq C_6$ on $[n-\frac{1}{2}, n+\frac{1}{2}]$ for all n for which $|n| \geq N$.*

There exists N such that for all $n \geq N$ there exists a $t_n \in [n-\frac{1}{2}, n-\frac{1}{4}]$ such that $\psi(t_n) \leq 1$. This statement holds since if there were infinitely many n_k with $\psi(t) \geq 1$ for all $t \in [n_k - \frac{1}{2}, n_k - \frac{1}{4}]$ we would obtain

$$\int |\psi| \geq \sum_{k \in \mathbf{N}} \int_{n_k - \frac{1}{2}}^{n_k - \frac{1}{4}} 1 = \infty,$$

which contradicts the hypothesis that $\psi \in L^1(\mathbf{R})$.

Let $|b_n| \leq C_4$ for all $n \in \mathbf{Z}$, and set $C_5 = 1 + C_4 \ln 4 + 2C_1$. We obtain

$$\begin{aligned} |g_n(t_n)| &= |\psi(t_n) - b_n \ln|t_n - n| - e_{n-1}\mathbf{1}_{(n-1,n)}(t_n) - e_n \mathbf{1}_{(n,n+1)}(t_n)| \\ &\leq 1 + |b_n| \ln 4 + |e_{n-1}| + |e_n| \\ &\leq 1 + C_4 \ln 4 + 2C_1 \\ &= C_5. \end{aligned}$$

For $t \in [n-\frac{1}{2}, n+\frac{1}{2}]$ and setting $C_6 = C_2 + C_5$ we have for some $\xi_n \in [\min\{t, t_n\}, \max\{t, t_n\}]$

$$\begin{aligned} |g_n(t)| &\leq |g_n(t) - g_n(t_n)| + |g_n(t_n)| \\ &\leq |(t - t_n) g_n'(\xi_n)| + |g_n(t_n)| \\ &\leq C_6. \end{aligned}$$

∎

STEP 4. $\left\{\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\psi|^2\right\} \in l^1(\mathbf{Z})$ *and therefore* $\psi \in L^2(\mathbf{R})$.

For $n \in \mathbf{Z}$ define for $t \in [n - \frac{1}{2}, n + \frac{1}{2}]$

$$\widetilde{g_n}(t) = \sum_{k \neq n} b_k \ln|t-k| + e_{n-1}\mathbf{1}_{(n-1,n)}(t) + e_n \mathbf{1}_{(n,n+1)}(t) = \psi(t) - b_n \ln|t-n|.$$

Hence, for $|n| \geq N$, we obtain

$$|\widetilde{g_n}(t)| \leq C_1 + C_6 \text{ for } t \in [n - \frac{1}{2}, n + \frac{1}{2}].$$

We shall first show $\left\{\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\widetilde{g_n}|\right\} \in l^1(\mathbf{Z})$. Note that

$$\begin{aligned}
\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |b_n \ln|t-n||\, dt &= |b_n| \int_{-\frac{1}{2}}^{\frac{1}{2}} |\ln|t||\, dt \\
&= 2|b_n| \int_0^{\frac{1}{2}} |\ln|t||\, dt \\
&= |b_n|(\ln 2 + 1),
\end{aligned}$$

and therefore $\left\{\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |b_n \ln|t-n||\, dt\right\} \in l^1(\mathbf{Z})$. Since

$$\begin{aligned}
\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\widetilde{g_n}(t)|\, dt &= \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\psi(t) - b_n \ln|t-n||\, dt \\
&\leq \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\psi(t)|\, dt + \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |b_n \ln|t-n||\, dt
\end{aligned}$$

and $\left\{\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\psi|\right\} \in l^1(\mathbf{Z})$, we obtain $\left\{\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\widetilde{g_n}|\right\} \in l^1(\mathbf{Z})$.

The following calculation will conclude our proof:

$$\begin{aligned}
\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\psi(t)|^2\, dt &= \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\widetilde{g_n}(t) + b_n \ln|t-n||^2\, dt \\
&\leq \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\widetilde{g_n}(t)|^2 + 2|\widetilde{g_n}(t) b_n \ln|t-n|| + |b_n \ln|t-n||^2\, dt \\
&\leq (C_1 + C_6) \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\widetilde{g_n}(t)|\, dt \\
&\quad + 2(C_1 + C_6) \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |b_n \ln|t-n|| + C_4 |b_n| \int_{-\frac{1}{2}}^{\frac{1}{2}} |\ln|t||^2\, dt
\end{aligned}$$

The elements on the right hand side form an $l^1(\mathbf{Z})$ sequence, and therefore $\left\{\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} |\psi|^2\right\} \in l^1(\mathbf{Z})$ and $\psi \in L^2(\mathbf{R})$. ■

$\square$

Now we can procede to state and proof the main result, which was earlier stated as Theorem **??**.

THEOREM 9.14. *Let $\psi \in L^1(\mathbf{R})$. The following are equivalent:*
i. $W_f^{\psi}(b,a) = \int_{\mathbf{R}} f(t)\psi(\frac{t-b}{a})\,dt$ *is 1–periodic in a for all $f \in L^\infty(\mathbf{T})$.*
ii. $\widehat{\psi}(0) = 0$ *and ψ has the form*

$$\psi(\cdot) = \sum_{n\in\mathbf{Z}} e_n \mathbf{1}_{(n,n+1)}(\cdot) + \sum_{n\in\mathbf{Z}} b_n \ln|\cdot - n|,$$

where $\{b_n\}$, $\{e_n - e_{n-1}\} \in l^1(\mathbf{Z})$, and $\{e_n - \pi i \sum_{k\leq n} b_k\} \in l^2(\mathbf{Z})$.

Proof. $i \Longrightarrow ii$. We apply Lemma **??** and obtain

$$\widehat{\psi}(\gamma) = \frac{\varphi_1(\gamma)}{\gamma} + \mathrm{H}(\gamma)\frac{\varphi_2(\gamma)}{\gamma}$$

and $\widehat{\psi}(0) = 0$, where φ_1 and φ_2 are 1–periodic. By Lemma **??** we have φ_1 and $\varphi_1 + \varphi_2 \in A(\mathbf{T})$, and, hence, φ_1 and $\varphi_2 \in A(\mathbf{T})$. Let us denote

$$\varphi_1(\gamma) = \sum_{n\in\mathbf{Z}} a_n e^{-2\pi i n\gamma} \quad \text{and} \quad \varphi_2(\gamma) = \sum_{n\in\mathbf{Z}} b_n e^{-2\pi i n\gamma}.$$

Since $\widehat{\psi}(0) = 0$, we have $\widehat{\psi}$, $\frac{\varphi_1(\gamma)}{\gamma}$, and $\mathrm{H}(\gamma)\frac{\varphi_2(\gamma)}{\gamma}$ are bounded and of order $O(\frac{1}{\gamma})$, $|\gamma| \to \infty$, and are therefore in $L^2(\widehat{\mathbf{R}})$. We can calculate $\psi = \mathcal{F}^{-1}(\widehat{\psi})$, using the linearity of $\mathcal{F}^{-1}$, Lemma **??**, and Lemma **??**,

$$\begin{aligned}
\psi(t) &= \mathcal{F}^{-1}(\widehat{\psi})(t) \\
&= \mathcal{F}^{-1}(\frac{\varphi_1(\gamma)}{\gamma})(t) + \mathcal{F}^{-1}(\mathrm{H}(\gamma)\frac{\varphi_2(\gamma)}{\gamma})(t) \\
&= \sum_{n\in\mathbf{Z}} c_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbf{Z}} \frac{1}{2} d_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbf{Z}} b_n \ln|t-n| \\
&= \sum_{n\in\mathbf{Z}} (c_n + \frac{1}{2} d_n)\mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbf{Z}} b_n \ln|t-n|,
\end{aligned}$$

where $d_n = 2\pi i \sum_{k\leq n} b_k$ and $c_n = 2\pi i \sum_{k\leq n} a_k$. Let $e_n = c_n + \frac{1}{2}d_n$.

Clearly, $\{b_n\} \in l^1(\mathbf{Z})$ and $\{a_n\} \in l^1(\mathbf{Z})$, where

$$a_n = \frac{1}{2\pi i}(c_n - c_{n-1}) = \frac{1}{2\pi i}(e_n - \frac{1}{2}d_n - e_{n-1} + \frac{1}{2}d_{n-1}) = \frac{1}{2\pi i}(e_n - e_{n-1}) - \frac{1}{2}b_n.$$

Therefore, $\{e_n - e_{n-1}\} \in l^1(\mathbf{Z})$.

Further, since $\sum_{n\in\mathbf{Z}} c_n \mathbf{1}_{(n,n+1)} = \mathcal{F}^{-1}(\frac{\varphi_1(\gamma)}{\gamma}) \in L^2(\mathbf{R})$, we have $\{c_n\} \in l^2(\mathbf{Z})$, where

$$c_n = 2\pi i \sum_{k\leq n} a_k = \sum_{k\leq n} ((e_k - e_{k-1}) - \pi i b_k) = e_n - \pi i \sum_{k\leq n} b_k.$$

Therefore *ii* holds.

ii $\Longrightarrow$ *i*. Let $\psi \in L^1(\mathbf{R})$ be of the form

$$\psi(t) = \sum_{n\in\mathbf{Z}} e_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbf{Z}} b_n \ln|t-n|$$

with $\{b_n\}$ and $\{e_n - e_{n-1}\} \in l^1(\mathbf{Z})$, and $\{e_n - \pi i \sum_{k\le n} b_k\} \in l^2(\mathbf{Z})$. Let

$$a_n = \frac{1}{2\pi i}(e_n - e_{n-1}) - \frac{1}{2}b_n$$

and

$$c_n = e_n - \pi i \sum_{k\le n} b_k.$$

Define, for $n \in \mathbf{Z}$,

$$d_n = 2\pi i \sum_{k\le n} b_k.$$

Then

$$c_n = e_n - \frac{1}{2}d_n$$

and

$$a_n = \frac{1}{2\pi i}(c_n - c_{n-1}).$$

By Lemma **??** we obtain $\psi \in L^2(\mathbf{R})$.

Let

$$\psi_1(t) = \sum_{n\in\mathbf{Z}} c_n \mathbf{1}_{(n,n+1)}(t).$$

Clearly, $\psi_1 \in L^2(\mathbf{R})$ since $\{c_n\} \in l^2(\mathbf{Z})$ and $\|\psi_1\|_{L^2(\mathbf{R})} = \|\{c_n\}\|_{l^2(\mathbf{Z})}$. Further, let

$$\psi_2(t) = \frac{1}{2}\sum_{n\in\mathbf{Z}} d_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbf{Z}} b_n \ln|t-n|.$$

Since $\psi \in L^2(\mathbf{R})$ and $\psi_1 \in L^2(\mathbf{R})$ we have $\psi_2 = \psi - \psi_1 \in L^2(\mathbf{R})$.

We can apply Lemma **??** to ψ_1 and Lemma **??** to ψ_2, and conclude that $\mathcal{F}(\psi_1)$ and $\mathcal{F}(\psi_2)$ are 1–periodic on $\widehat{\mathbf{R}}^+$ and 1–periodic on $\widehat{\mathbf{R}}^-$.

Hence,

$$\widehat{\psi} = \mathcal{F}(\psi) = \mathcal{F}(\psi_1 + \psi_2) = \mathcal{F}(\psi_1) + \mathcal{F}(\psi_2)$$

is 1–periodic on $\widehat{\mathbf{R}}^+$ and 1–periodic on $\widehat{\mathbf{R}}^-$.

Since $\widehat{\psi}(0) = 0$ by hypothesis, we can apply Lemma **??**, and *i* follows. □

Remark 9.15. It is easy to give a formal proof for the direction *i* $\Longrightarrow$ *ii* in the previous theorem. For this we need to use Proposition **??**, as well as the following calculation for $\psi \in L^1(\mathbf{R})$ defined by

$$\psi(t) = \sum_{n\in\mathbf{Z}} b_n \ln|t-n|,$$

where $\sum_{n\in\mathbf{Z}} b_n = 0$:

$$\begin{aligned}
\psi(\frac{t-b}{a}) &= \sum_{n\in\mathbf{Z}} b_n \ln\left|\frac{t-b}{a} - n\right| \\
&= \sum_{n\in\mathbf{Z}} b_n \ln\left|\frac{t-b-na}{a}\right| \\
&= \sum_{n\in\mathbf{Z}} b_n \left(\ln|t-b-na| - \ln|a|\right) \\
&= \sum_{n\in\mathbf{Z}} b_n \ln|t-b-na| .
\end{aligned}$$

Therefore,

$$\begin{aligned}
W_f^\psi(b,a+1) &= \int \left(\sum_{n\in\mathbf{Z}} b_n \ln|t-b-n(a+1)|\right) f(t)\,dt \\
&= \sum_{n\in\mathbf{Z}} b_n \int \ln|t-b-n(a+1)|\, f(t)\,dt \\
&= \sum_{n\in\mathbf{Z}} b_n \int \ln|t-b-na|\, f(t+n)\,dt \\
&= \sum_{n\in\mathbf{Z}} b_n \int \ln|t-b-na|\, f(t)\,dt \\
&= \int \left(\sum_{n\in\mathbf{Z}} b_n \ln|t-b-na|\right) f(t)\,dt \\
&= W_f^\psi(b,a).
\end{aligned}$$

Remark 9.16. Note that there is no redundancy in the three conditions
a. $\{b_n\} \in l^1(\mathbf{Z})$
b.$\{e_n - e_{n-1}\} \in l^1(\mathbf{Z})$
c.$\{e_n - \pi i \sum_{k\le n} b_k\} \in l^2(\mathbf{Z})$.

To see this, first let $\{e_n\} = \{0\}$ and $b_n = \frac{1}{n^2+1}$ for $n \in \mathbf{Z}$. These sequences satisfy a and b but not c. The sequences, defined by $\{b_n\} = \{0\}$ and $e_n = \frac{1}{n}$ for n positive and odd and $e_n = 0$ otherwise, fulfill conditions a and c, but not condition b. Last, $\{e_n\} = \{0\}$ and $b_n = (-1)^n\frac{1}{n}$ for n positive and $b_n = 0$ otherwise, define sequences satisfying b and c, but not a.

Remark 9.17. For background on representation theory see [?, ?, ?]. Let $GL(L^p(\mathbf{R}))$ denote the group of invertible bounded linear operators on $L^p(\mathbf{R})$ which are continuous in the strong operator topology. Let G be the $ax+b$ group. Recall that G is a locally compact, non-abelian Lie group, topologically isomorphic to $\mathbf{R} \times \mathbf{R}^+$, with group operation $(b,a) \circ (b',a') = (b + ab', aa')$. The linear representation U_p of G on $L^p(\mathbf{R})$ is given by

$$\begin{array}{rcl} U_p : G & \rightarrow & GL(L^p(\mathbf{R})) \\ (b,a) & \mapsto & U_p(b,a) \end{array}$$

and

$$U_p(b,a)(\psi)(\cdot) = a^{-\frac{1}{p}} \psi(\frac{\cdot - b}{a}).$$

We define the subgroup G^+ of G by $G^+ = \{(b,a) \in G, a > 0\}$, and let U_p^+ be the linear representation obtained by restricting U_p to G^+.

The left transformation of a function ψ with respect to U_p^+ is, up to a modulation, the continuous wavelet transformation defined in Chapter **??**, see (**??**). In fact, if f is in the dual space of $L^p(\mathbf{R})$, i.e., $f \in L^p(\mathbf{R})'$, $1 \leq p < \infty$, and $(b,a) \in G^+$, we have

$$W_{\overline{f}}^{\psi}(b,a) = \langle U_p^+(b,a)(\psi), f \rangle.$$

U_p^+ is a reducible linear representation on $L^p(\mathbf{R})$ for $1 \leq p < \infty$, in fact, $V_p = \{\psi \in L^p(\mathbf{R}) : \operatorname{supp} \widehat{\psi} \subset \widehat{\mathbf{R}}^+\}$ is a closed linear subspace of $L^p(\mathbf{R})$ for $1 \leq p < \infty$ with the property that $U_p^+(b,a)(\psi) \in V_p$ for all $\psi \in V_p$ and all $(b,a) \in G^+$. It can be shown that U_p^+ is an irreducible linear representation on V_p for $1 \leq p < \infty$. V_2 is referred to as Hardy space.

Nevertheless, U_p is an irreducible linear representation on $L^p(\mathbf{R})$ for $1 \leq p < \infty$. For $p = 1$, U_1 is irreducible on the closed linear subspace $\tilde{V}_1 = \{\psi \in L^1(\mathbf{R}) : \widehat{\psi}(0) = 0\}$ of $L^1(\mathbf{R})$. The left transformation of U_1 differs from the wavelet transformation in Theorem **??** only in the scaling domain we use, i.e., using $\mathbf{R} \setminus \{0\}$ instead of $\mathbf{R}^+$. But in this case, the analogous of property (P) forces that $\widehat{\psi}(\gamma) = \frac{\varphi(\gamma)}{\gamma}$, where φ is periodic on $\widehat{\mathbf{R}}$ (Proof of Lemma **??**). Theorem **??** implies that ψ has in this case the more canonical form

$$\psi(t) = \sum_{n \in \mathbf{Z}} c_n \mathbf{1}_{(n,n+1)},$$

where $\{c_n\} \in l^1(\mathbf{Z})$.

We obtain a more canonical result after adjusting the underlying group representation in a way, such that the underlying group representation is irreducible. This leads to the question, whether we can formulate a representation theoretical theorem which would generalize Theorem **??**.

This idea is not supported by the following observation. The representation $U_1^+ : G^+ \longrightarrow GL(V_1^+)$ is irreducible, but any $\psi \in V_1^+$ that satisfies property (P) in this setting, has a Fourier transform of the form $\widehat{\psi}(\gamma) = H(\gamma)\frac{\varphi(\gamma)}{\gamma}$, where φ is periodic on $\widehat{\mathbf{R}}$. Hence, ψ is not of the form $\psi(t) = \sum_{n\in\mathbf{Z}} c_n \mathbf{1}_{(n,n+1)}$, but of the form

$$\psi(t) = \sum_{n\in\mathbf{Z}} e_n \mathbf{1}_{(n,n+1)}(t) + \sum_{n\in\mathbf{Z}} b_n \ln|t-n|,$$

where $\{b_n\}, \in l^1(\mathbf{Z})$ and $\{d_n\} = \{2\pi i \sum_{k\leq n} b_k\} \in l^2(\mathbf{Z})$. See Lemma **??**.

9.2 Examples

We shall construct two wavelets $\psi \in L^1(\mathbf{R})$ which are not piecewise constant, but which have the property that for all $f \in L^\infty(\mathbf{T})$, W_f^ψ is 1–periodic in scale.
EXAMPLE 9.18. Consider

$$\psi(t) = \ln|t| - \ln|t-1| = \ln\left|\frac{t}{t-1}\right|,$$

$t \neq 0, 1$. This function clearly satisfies *ii* of Theorem **??**, but note that $\psi \notin L^1(\mathbf{R})$. In fact, we can apply the sum criterion for integrals, since ψ is positive and decreasing on $[2,\infty)$, and calculate

$$\begin{aligned}
\sum_{n=2}^{N}(\ln|n| - \ln|n-1|) &= \ln 2 - \ln 1 + \ln 3 - \ln 2 + ... + \ln N - \ln(N-1) \\
&= \ln N \longrightarrow \infty \quad \text{as} \quad N \longrightarrow \infty.
\end{aligned}$$

Therefore,

$$\int |\psi(t)|\, dt \geq \int_1^\infty |\psi(t)|\, dt = \int_1^\infty \psi(t)\, dt \geq \sum_{n=2}^{\infty} \psi(n) = \infty,$$

and $\psi \notin L^1(\mathbf{R})$.

Let us now correct this problem and construct $\psi \in L^1(\mathbf{R})$ with $\{b_n\} \neq \{0\}$. Let

$$\psi(t) = \sum_{|n|\geq 2} \ln\left|\frac{n}{n+1}\right| \mathbf{1}_{(n,n+1)} + \ln|t| - \ln|t-1|.$$

Then $b_0 = 1$, $b_1 = -1$, $b_n = 0$ for $n \neq 0, -1$ and $\{b_n\} \in l^1(\mathbf{Z})$. Let $e_n = \ln\left|\frac{n}{n+1}\right|$ for $|n| \geq 2$. Further

$$\begin{aligned}
a_n &= e_n - e_{n-1} = \left(\ln\left|\frac{n}{n+1}\right| - \ln\left|\frac{n-1}{n}\right|\right) \\
&= -\ln\left|\frac{(n+1)(n-1)}{n^2}\right| \\
&= -\ln\left|1 - \frac{1}{n^2}\right|
\end{aligned}$$

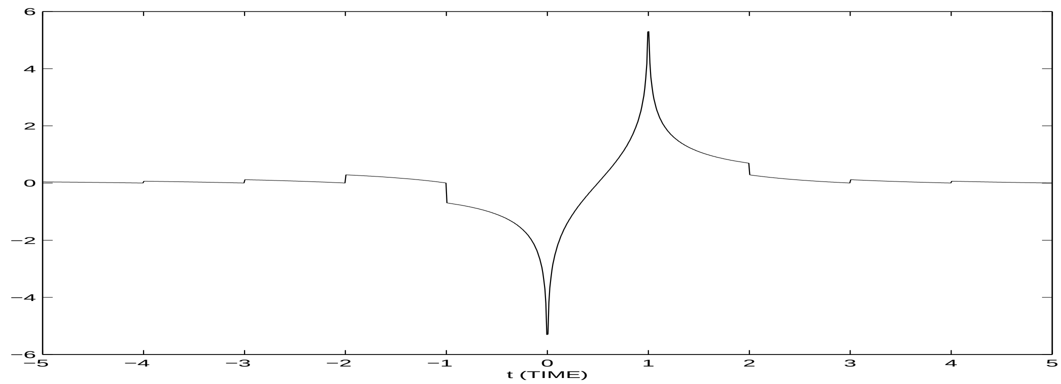

Figure 9.1. Example **??** of a wavelet ψ satisfying Theorem **??**.

for $|n| \geq 2$ and, hence, $\{e_n - e_{n-1}\} \in l^1(\mathbf{Z})$.

For $|n| \geq 3$ we have

$$e_n - \pi i \sum_{k \leq n} b_k = e_n.$$

Therefore, $\{b_n\}$, and $\{e_n\}$ fulfill the necessary requirements. Further, since $\int_{\mathbf{R}} \psi(t)\, dt = 0$, ψ satisfies condition *ii* of Theorem **??**.

It remains to show that $\psi \in L^1(\mathbf{R})$. For this, observe that on the positive part of the real axis

$$\begin{aligned}
\int_2^{\infty} |\psi(t)|\, dt &\leq \lim_{N\to\infty} \sum_{n=2}^{N} \psi(n) \\
&= \lim_{N\to\infty} \sum_{n=2}^{N} (\ln \left|\frac{n}{n+1}\right| + \ln|n| - \ln|n-1|) \\
&= \lim_{N\to\infty} \sum_{n=2}^{N} (\ln \left|\frac{n}{n+1}\right| - \ln \left|\frac{n-1}{n}\right|) \\
&= \lim_{N\to\infty} -\ln\left|\frac{1}{2}\right| + \ln\left|\frac{N}{N+1}\right| \\
&= \ln 2.
\end{aligned}$$

A similar calculation holds for the negative part of the real axis and, therefore, Theorem **??** applies and W_f^{ψ} is 1–periodic in scale for all $f \in L^{\infty}(\mathbf{T})$.

Example 9.19. We shall construct $\psi \in L^1(\mathbf{R})$ satisfying condition *ii* of Theorem **??** and which has the property that in its representation given in Theorem **??**, part *i*, $\{e_n\} = \{0\}$.

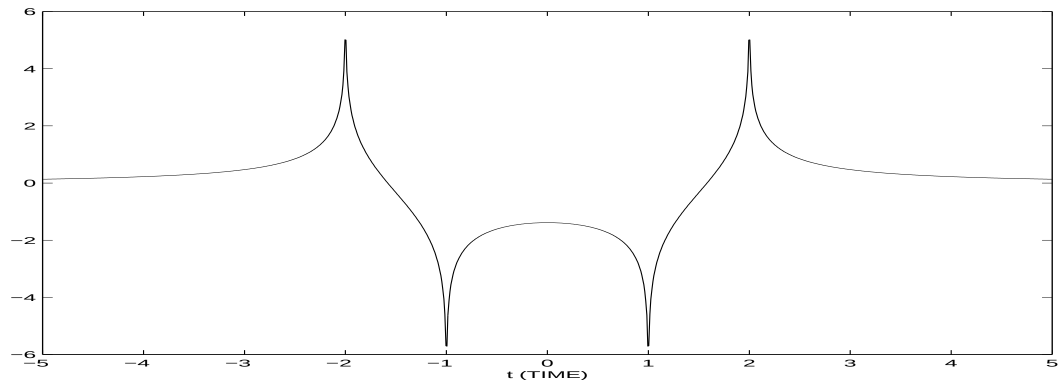

Figure 9.2. Example **??** of a wavelet ψ satisfying Theorem **??**.

Let

$$\begin{aligned}\psi(t) &= \ln|t+1| - \ln|t+2| + \ln|t-1| - \ln|t-2| \\ &= \ln\left|\frac{(t+1)(t-1)}{(t+2)(t-2)}\right| \\ &= \ln\left|\frac{t^2-1}{t^2-4}\right|.\end{aligned}$$

ψ is monotonically decreasing for $|t| \to \infty$. Hence, we can apply the sum criteria to show $\psi \in L^1(\mathbf{R})$. We have

$$\begin{aligned}\sum_{3\le|n|\le N} |\psi(n)| &= 2\sum_{3\le n\le N} |\psi(n)| \\ &= 2\sum_{3\le n\le N} -(\ln(n+1) - \ln(n+2)) \\ &\quad +2\sum_{3\le n\le N} (\ln(n-1) - \ln(n-2)) \\ &= 2(\ln 4 - \ln(N+2) - \ln 1 + \ln(N-1)) \\ &= 2\ln 4 - 2\ln\left(\frac{N+2}{N-1}\right).\end{aligned}$$

Hence

$$\lim_{N\to\infty} \sum_{3\le|n|\le N} |\psi(n)| = 2\ln 4.$$

9.3 Generalization of the fundamental theorem to $\mathbf{R}^d$

In order to generalize Theorem **??** to higher dimensions, we need to clarify some terms. For example, there is more than one way to define a periodic function on $\mathbf{R}^d$. The basic techniques applied in the one-dimensional case generalize to the following setting. Here, $\{e_i\}$ will denote the Euclidean basis of $\mathbf{R}^d$. Confusion with the coefficient sequence $\{e_n\}_{n\in\mathbf{Z}}$, which we obtained in the previous section, should not arise. A d-quadrant $Q \subset \mathbf{R}^d$ is a maximal connected subset of $\mathbf{R}^d \setminus S$, where $S = \{x \in \mathbf{R}^d : p(x) = x_1 \cdot \ldots \cdot x_d = 0\}$. See Chapter **??** for more of the notation used in this section.

Definition 9.20. *A function f defined on $D \subseteq \mathbf{R}^d$ is T–periodic, $T = (T_1, \ldots, T_d) \in \mathbf{R}^d$, if $f(x + T_i e_i) = f(x)$ for all $x \in D$ where $x + T_i e_i \in D$ for all $i = 1, ..., d$.*

Hence, f is T–periodic on each d–quadrant, $T = (T_1, \ldots, T_d) \in \mathbf{R}^d$, if $f(x + T_i e_i) = f(x)$ for all x and $x + T_i e_i$ being in the same d–quadrant.

The wavelet transformation we shall use, has domain $\mathbf{R}^d \times (\mathbf{R}^+)^d$, is not normalized, and is defined by:

$$W_\psi f(b, a) = \int_{\mathbf{R}^d} f(t)\psi(\frac{t-b}{a})\, dt.$$

Note, that the vector $\frac{t-b}{a}$ has been defined in Chapter **??**.

In Remark **??** we shall discuss the necessity to scale in each coordinate of $\mathbf{R}^d$ separately. Using only the scaling domain R^+, which is embedded in $(R^+)^d$ by means of $a \to (a, \ldots, a)$, we cannot achieve a complete classification theorem similar to Theorem **??**.

We shall again classify all wavelets with the property that the non–normalized wavelet transform of any periodic function $f \in L^\infty(\mathbf{R}^d)$ is periodic in scale. As in the one-dimensional case, continuous wavelet transforms of periodic functions are always periodic in "time".

Note that, defying convention, we are continuing to use $t = (t_1, \ldots, t_d) \in \mathbf{R}^d$ as main variable.

Let $M = \{m \in \mathbf{Z}^d : m_i \in \{+1, -1\}$ for $i = 1, \ldots, d\}$.

Theorem 9.21. *Let $\psi \in L^1(\mathbf{R}^d)$. The following are equivalent:*
i. $W_\psi f(b, a) = \int_{\mathbf{R}^d} f(t)\psi(\frac{t-b}{a})\, dt$ is 1–periodic in a for all $f \in L^\infty(\mathbf{T}^d)$.
ii. $\widehat{\psi}(k) = 0$ for $k \in \mathbf{Z}^d \cap S$ and ψ has the form:

$$\psi(t) = \sum_{m\in M} \sum_{n\in\mathbf{Z}^d} c_n^m \prod_{\substack{i \\ m_i=1}} \mathbf{1}_{(n_i,n_i+1)}(t_i) \prod_{\substack{i \\ m_i=-1}} \ln|\frac{t_i - n_i}{t_i - n_i - 1}|,$$

$t = (t_1, \ldots, t_d) \in \mathbf{R}^d$, where $\{c_n^m\}_n \in l^2(\mathbf{Z}^d)$ such that there exist $\mathbf{Z}^d$ sequences

$\{a_n^m\}_n \in l^1(\mathbf{Z}^d)$ *with the property that for all* $k \in \mathbf{Z}^d$ *and all* $m \in M$,

$$c_k^m = \sum_{p(n-k) \geq 0} a_n^m.$$

The proof of this theorem is very similar to the proof of Theorem **??**, and, hence, we shall skip many details.

LEMMA 9.22. *Let* $\psi \in L^1(\mathbf{R}^d)$. *The following are equivalent:*
i. $W_\psi f(b,a) = \int_{\mathbf{R}^d} f(t)\psi(\frac{t-b}{a})\,dt$ *is 1–periodic in* a *for all* $f \in L^\infty(\mathbf{T^d})$.
ii. There exists a continuous function φ, *1–periodic on each* d*–quadrant, such that*

$$\widehat{\psi}(\gamma) = \frac{\varphi(\gamma)}{p(\gamma)} \quad \textit{for} \quad \gamma \in \widehat{\mathbf{R}^d} \setminus S$$

and $\widehat{\psi}(m) = 0$ *for* $m \in \mathbf{Z}^d \cap S$.

Proof. $i \Longrightarrow ii$. We assume that $W_f(b,a) = \int_{\mathbf{R}^d} f(t)\psi(\frac{t-b}{a})\,dt$ is 1–periodic in a for all $f \in L^\infty(\mathbf{T^d})$.

The condition $W_f(b,a) = W_f(b, a+e_k)$ for $k = 1,\ldots,d$ and all $f \in L^\infty(\mathbf{T}^d)$ implies that for $k = 1,\ldots,d$

$$S_k(t) = \sum_{n \in \mathbf{Z}} (\psi(\frac{t-n-b}{a}) - \psi(\frac{t-n-b}{a+e_k}))\,dt = 0.$$

The main calculation (**??**) in the proof of Lemma **??** can be easily generalized. Hence, for all $m \in \mathbf{Z}^d$ and $k = 1,\ldots,d$

$$0 = \widehat{S_k}[m] = e^{-2\pi i \langle m,b \rangle}(p(a)\widehat{\psi}(m \star a) - p(a+e_k)\widehat{\psi}(m \star (a+e_k)).,$$

$a \in (\mathbf{R}^+)^d$. For $m \in \mathbf{Z}^d \cap S$, let k_0 be such that $m_{k_0} = 0$, and let $a = 1 - e_{k_0}$. Then

$$0 = (p(a)\widehat{\psi}(m \star a) - p(a+e_k)\widehat{\psi}(m \star (a+e_k)) = -\widehat{\psi}(m),$$

and the second part of ii is shown.

As before, we define $\varphi(\gamma) = p(\gamma)\widehat{\psi}(\gamma)$ for $gamma \in \widehat{\mathbf{R}}^d$.

In order to show that φ is 1–periodic on each d–quadrant, observe that for $\gamma \in \widehat{\mathbf{R}}^d - S$, γ is in the same d–quadrant as $\gamma + \operatorname{sign}(\gamma_k)e_k$ for each $k = 1,\ldots,d$. In fact, the assertion that $\varphi(\gamma) = \varphi(\gamma + \operatorname{sign}(\gamma_k)e_k)$ for all $\gamma \in \widehat{\mathbf{R}}^d \setminus S$ and all $k = 1,\ldots,d$ is equivalent to the fact that φ is 1–periodic in each d–quadrant.

If $\gamma \in \widehat{\mathbf{R}}^d - S$, $a = |\gamma|$ and $m = \operatorname{sign}(\gamma)$, then we obtain

$$\varphi(\gamma) = \varphi(\operatorname{sign}(\gamma)| \star \gamma|) = \varphi(\operatorname{sign}(\gamma) \star (|\gamma| + e_k)) = \varphi(\gamma + \operatorname{sign}(\gamma_k)e_k)$$

for each $k = 1,\ldots,d$, and the proof that $i \Longrightarrow ii$ is complete.

$ii \Longrightarrow i$. If ii holds we have for $a \in (R^+)^d$, $m \in \mathbf{Z}^d \setminus S$, and $k = 1, \ldots, d$ that

$$\begin{aligned}
\left|\widehat{S_k}[m]\right| &= \left|p(a)\widehat{\psi}(m \star a) - p(a+e_k)\widehat{\psi}(m \star (a+e_k))\right| \\
&= \left|p(a)\frac{\varphi(m \star a)}{p(m \star a)} - p(a+e_k)\frac{\varphi(m \star (a+e_k))}{p(m \star (a+e_k)}\right| \\
&= \left|\frac{1}{p(m)}(\varphi(m \star a) - \varphi(m \star a + m))\right| \\
&= 0
\end{aligned}$$

since m and $m + a \star m$ are always in the same d–quadrant. Also $\widehat{S_k}[m] = 0$ for $m \in \mathbf{Z}^d \cap S$ since $\widehat{\psi}(m) = 0$ for $m \in \mathbf{Z}^d \cap S$ by hypothesis.

Therefore $\widehat{S_k}[m] = 0$ for all $m \in \mathbf{Z}$ and all $k = 1, \ldots, d$. The periodicity in scale follows in the same way as in the proof of Theorem **??**. □

REMARK 9.23. Our d-dimensional classification Theorem (Theorem **??**) implies that if the wavelet ψ has a specific form, then the non–normalized wavelet transform of any periodic signal is periodic in scale $(\mathbf{R}^+)^d$. Certainly, if we restrict the scaling domain to $\mathbf{R}^+$, which is embedded in $(\mathbf{R}^+)^d$ by means of $a \mapsto (a, \ldots, a)$, we obtain wavelet transforms defined on $\mathbf{R}^d \times \mathbf{R}^+$ which are again periodic in scale.

Nevertheless we cannot classify all wavelets such that the wavelet transform defined on $\mathbf{R}^d \times \mathbf{R}^+$ of any periodic functions on R^d is in turn periodic in scale.

To see this, we shall refer to the proof of Lemma **??**. Observe that the periodicity condition of wavelet transforms on $\mathbf{R}^d \times \mathbf{R}^+$, now only implies that for all $m \in \mathbf{Z}^d$ we have

$$\begin{aligned}
a^d\widehat{\psi}\left(a\begin{pmatrix} m_1 \\ \vdots \\ m_1 \end{pmatrix}\right) &= (a+1)^d\widehat{\psi}\left((a+1)\begin{pmatrix} m_1 \\ \vdots \\ m_1 \end{pmatrix}\right) \\
&= (a+1)^d\widehat{\psi}\left(a\begin{pmatrix} m_1 \\ \vdots \\ m_1 \end{pmatrix} + \begin{pmatrix} m_1 \\ \vdots \\ m_1 \end{pmatrix}\right)
\end{aligned}$$

This condition is not sufficiently strong to imply that ψ has a representation similar to the one obtained in Theorem **??**.ii.

In fact, choosing as scaling domain $(\mathbf{R}^+)^d$, we obtained that $p\widehat{\psi}$ is periodic in each d–quadrant. So if we know $\widehat{\psi}$ on $[0,1]^d$, we actually know $\widehat{\psi}$ on all of $(\mathbf{R}^+)^d$. Choosing as scaling domain $\mathbf{R}^+$, we only know that the values of $\widehat{\psi}$ on $[0,1]^d$ determine the values on the thick diagonal lines in Figure **??**. The collection of all non horizontal and non vertical lines in Figure **??** show the areas on which the values of $\widehat{\psi}$ are determined by values taken closer to the origin in the waveletgram.

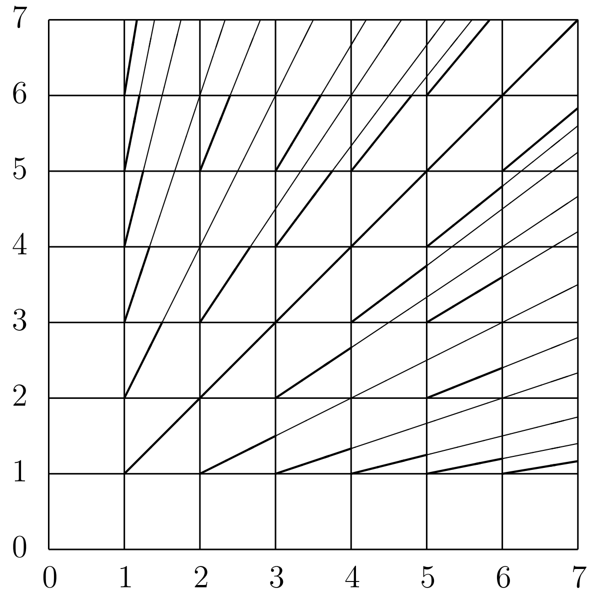

Figure 9.3. Predetermined lines in the first 2–quadrant of the domain of the Fourier transform of ψ, as explained in Remark **??**.

The following lemmata will be used in the proof of Theorem **??**.

LEMMA 9.24. *Let $\psi \in L^1(\mathbf{R}^d)$ be such that for all γ in the d–quadrant Q of $\widehat{\mathbf{R}}^d$ we have*

$$\widehat{\psi}(\gamma) = \frac{\varphi(\gamma)}{\gamma},$$

where φ is 1–periodic on $\widehat{\mathbf{R}}$. Then $\varphi \in A(\mathbf{T}^d)$ and, hence, the Fourier series $S(\varphi)(\gamma) = \sum_{n\in\mathbf{Z}^d} b_n e^{-2\pi i\langle n,\gamma\rangle}$ converges to φ absolutely and uniformly, i.e., $\{b_n\} \in l^1(\mathbf{Z})$.

Here, $\mathcal{F}$ denotes the $L^2(\mathbf{R}^d)$-Fourier transformation.

LEMMA 9.25. *Let $\psi \in L^2(\mathbf{R}^d)$ be such that, for all $\gamma \in \widehat{\mathbf{R}^d}$,*

$$\mathcal{F}(\psi)(\gamma) = \frac{\varphi(\gamma)}{p(\gamma)}$$

where $\varphi \in A(\mathbf{T}^d)$. Then

$$\psi(\cdot) = (i\pi)^d \sum_{n\in\mathbf{Z}^d} a_n p(\mathrm{sign}(\cdot - n)) = \sum_{n\in\mathbf{Z}^d} c_n \mathbf{1}_{(n,n+1)}(\cdot),$$

where

$$\varphi(\gamma) = \sum_{n\in\mathbf{Z}^d} a_n e^{-2\pi i\langle n,\gamma\rangle}$$

and

$$c_n = 2(\pi i)^d \sum_{p(n-k)\geq 0} a_n,$$

and where $\{c_n\} \in l^2(\mathbf{Z})$. The convergence is pointwise for $t \notin \mathbf{Z^d}$, as well as in $L^2(\mathbf{R}^d)$.

Thus, we have the $\mathcal{F}$–pairing

$$\sum_{n\in\mathbf{Z}^d} c_n \mathbf{1}_{(n,n+1)}(t) \longleftrightarrow \frac{1}{p(\gamma)} \sum_{n\in\mathbf{Z}^d} a_n e^{-2\pi i\langle n,\gamma\rangle} \tag{9.8}$$

Proof. For $N \in \mathbf{Z}^+$, let $A_N = \{\gamma \in \widehat{\mathbf{R}}^d : \|\gamma\|_\infty < N$ and $B_N = \{\gamma \in A_N : d(\gamma, S) > 1/N\}$. Since $\widehat{\psi} \in L^2(\widehat{\mathbf{R}^d})$ we can apply the L^2–inversion formula to obtain

$$\psi(t) = \lim_{N\to\infty} \int_{A_N} \widehat{\psi}(\gamma) e^{2\pi i\langle t,\gamma\rangle}\, d\gamma,$$

with convergence of this limit in $L^2(\mathbf{R}^d)$. Similar to the proof of Lemma **??**, we obtain

$$\begin{aligned}
\psi(t) &= \lim_{N\to\infty} \int_{B_N} \frac{1}{p(\gamma)} \left(\sum_{n\in\mathbf{Z^d}} a_n e^{-2\pi i n\gamma} \right) e^{2\pi i\gamma t}\, d\gamma \\
&= \lim_{N\to\infty} \sum_{n\in\mathbf{Z^d}} a_n \int_{B_N} \frac{e^{2\pi i\langle t-n,\gamma\rangle}}{p(\gamma)}\, d\gamma \\
&= \lim_{N\to\infty} \sum_{n\in\mathbf{Z^d}} a_n \prod_{k=1}^{d} \int_{\frac{1}{N}\le|\gamma_k|\le N} \frac{e^{2\pi i(t_k-n_k)\gamma_k}}{\gamma_k}\, d\gamma_k \\
&= i^d \sum_{n\in\mathbf{Z}^d} a_n \prod_{k=1}^{d} \left\{ \begin{array}{ccc} \pi & \text{for} & t_k > n_k \\ 0 & \text{for} & t_k = n_k \\ -\pi & \text{for} & t_k < n_k \end{array} \right\} \\
&= (i\pi)^d \sum_{n\in\mathbf{Z}^d} a_n p(\operatorname{sign}(t-n)).
\end{aligned}$$

We can change the order of the integral, the limit, and the sum by the same arguments presented in the proof of Lemma **??**.

For $t \in (k, k+1)$ we have the following calculation analogous to the one dimensional case:

$$\begin{aligned}
\psi(t) &= (\pi i)^d \sum_{n\in\mathbf{Z}^d} \pi i a_n \operatorname{sign}(t-n) \\
&= (\pi i)^d \sum_{p(n-k)\ge 0} a_n + (\pi i)^d \sum_{n\in\mathbf{Z}^d} a_n - (\pi i)^d \sum_{p(n-k)<0} a_n \\
&= 2(\pi i)^d \sum_{p(n-k)\ge 0} a_n \\
&= c_k.
\end{aligned}$$

In particular,
$$\|\{c_n\}\|_{l^2(\mathbf{Z}^d)} = \|\psi\|_{L^2(\mathbf{R}^d)} < \infty,$$
and therefore $\{c_n\} \in l^2(\mathbf{Z}^d)$. $\square$

In order to handle the general case, $\widehat{\psi} = \frac{\varphi}{p}$ with φ being only periodic on each d–quadrant, we need the following Lemma.

Lemma 9.26. *Let φ be periodic in each d–quadrant. Let $M = \{m \in \mathbf{Z}^d : m_i \in \{+1, -1\}$ for $i = 1, \ldots, d\}$ and define the Heaviside functions $H_m \in L^\infty(\mathbf{R}^d)$, $m \in M$, by*
$$H_m(x) = \prod_{\substack{i \\ m_i=-1}} \operatorname{sgn}(x_i),$$
$x = (x_1, \ldots, x_d) \in \mathbf{R}^d$. There exist 1–periodic function φ_m, $m \in M$ such that
$$\varphi = \sum_{m\in M} H_m \varphi_m. \tag{9.9}$$

The set $\{H_m\}_{m\in M}$ is minimal in the following sense. There exists no set of functions $\{F_i\}_{i\in I}$, $|I| < |M| = 2^d$, such that we can decompose a function periodic in each d–quadrant into periodic function analogous to (**??**). This is obvious, since $\mathbf{R}^d$ has 2^d d–quadrants and $|M| = 2^d$.

Proof of Lemma **??**. Let us enumerate the 2^d d–quadrants of $\mathbf{R}^d$ by setting $Q_n = \{x \in \mathbf{R}^d : \operatorname{sgn}(x_i) = \operatorname{sgn}(n_i) \text{ for } i = 1, \ldots, d\}$ for $n \in M$.

We shall construct $c_m^n \in \{+1, -1\}$ for $m, n \in M$ such that $2^d \mathbf{1}_{Q_n} = \sum_{m,n\in M} c_m^n H_m$ for $n \in M$.

Assuming this, for each $n \in M$ we let $\tilde{\varphi}_n$ be the 1–periodic continuation of $\varphi \cdot \mathbf{1}_{Q_n}$ to all of $\mathbf{R}^d$. By setting $\varphi_m = 2^{-d} \sum_{n\in M} c_m^n \tilde{\varphi}_n$ we obtain
$$\begin{aligned}
\varphi &= \sum_{n\in M} \mathbf{1}_{Q_n} \tilde{\varphi}_n \\
&= \sum_{n\in M} \sum_{m\in M} 2^{-d} c_m^n H_m \tilde{\varphi}_n \\
&= \sum_{m\in M} H_m \left(\sum_{n\in M} 2^{-d} c_m^n \tilde{\varphi}_n \right) \\
&= \sum_{m\in M} H_m \varphi_m.
\end{aligned}$$

It remains to construct the required $c_m^n \in \{+1, -1\}$ for $m, n \in M$. Let us begin by setting $n = 1 = (1, \ldots, 1)$. We let $c_m^1 = 1$ for all $m \in M$. Since each H_m is constant on each d–quadrant, it suffices to show $\sum_{m\in M} H_m(k) = 2^d \delta_{1,k}$ for $k \in M$.

First observe that clearly $H_m(1) = 1$ for all $m \in M$ and hence $\sum_{m\in M} H_m(1) = 2^d$. To show $\sum_{m\in M} H_m(k) = 0$ for $k \neq 1$, we shall use induction with respect to d.

For $d = 1$ we have $H_1(-1) + H_{-1}(-1) = 1 - 1 = 0$. Suppose the result holds for $d-1$, $d > 1$. Let $\tilde{M} = \{m \in \mathbf{Z}^{d-1}, m_i \in \{+1, -1\}$ for $i = 1, \dots, d-1\}$ and for $m = (m_1, \dots, m_d) \in M$ define $\tilde{m} = (m_1, \dots, m_{d-1}) \in \tilde{M}$. Observe that

$$H_m(k) = \left\{ \begin{array}{ccc} H_{\tilde{m}}(\tilde{k}) & \text{for} & m_d = 1 \\ \operatorname{sign}(k_d) H_{\tilde{m}}(\tilde{k}) & \text{for} & m_d = -1 \end{array} . \right\}$$

and therefore

$$\begin{aligned} \sum_{m\in M} H_m(k) &= \operatorname{sign}(k_d) \sum_{\tilde{m}\in\tilde{M}} H_{\tilde{m}}(\tilde{k}) + \sum_{\tilde{m}\in\tilde{M}} H_{\tilde{m}}(\tilde{k}) \\ &= (\operatorname{sign}(k_d) + 1) \sum_{\tilde{m}\in\tilde{M}} H_{\tilde{m}}(\tilde{k}) \\ &= 0, \end{aligned}$$

since either $k_d = -1$ or $\tilde{k} \neq 1$, in which case $\sum_{\tilde{m}\in\tilde{M}} H_{\tilde{m}}(\tilde{k}) = 0$.

For the general case, $n \neq 1$, let $c_m^n = H_m(n)$ and observe that

$$\sum_{m\in M} H_m(n) H_m(k) = \sum_{m\in M} H_m(n \star k) = \delta_{1,n\star k} = \delta_{n,k}$$

since $n \star k = 1$ if and only if $n = k$. □

Now we can write $\widehat{\psi} = \sum_{m\in M} H_m \frac{\varphi_m}{p}$ where all φ_m, $m \in M$ are periodic on all of R^d. In our final lemma we calculate the inverse Fourier transforms of $H_m \frac{\varphi_m}{p}$ for $m \in M$.

LEMMA 9.27. *Let $\psi \in L^2(\mathbf{R}^d)$ such that for $\gamma \in \widehat{\mathbf{R}}^d$ and some $m \in M$*

$$\mathcal{F}(\psi)(\gamma) = H_m(\gamma) \frac{\varphi(\gamma)}{p(\gamma)}$$

with $\varphi \in A(\mathbf{T}^d)$,$\varphi(\gamma) = \sum_{n\in\mathbf{Z}^d} a_n e^{-2\pi i\langle n,\gamma\rangle}$. Letting $c_n = 2\frac{(\pi i)^d}{\pi^r} \sum_{p(n-k)\geq 0} a_n$, $r = |\{i, m_i = -1\}|$, we get $\{c_n\} \in l^2(\mathbf{Z})$ and

$$\psi(t) = \frac{i^d}{\pi^{d-r}} \sum_{n\in\mathbf{Z}^d} c_n \prod_{\substack{i \\ m_i=1}} \mathbf{1}_{(n_i,n_i+1)}(t_i) \prod_{\substack{i \\ m_i=-1}} \ln|\frac{t_i - n_i}{t_i - n_i - 1}|,$$

$t = (t_1, \dots, t_d) \in \mathbf{R}^d$. The convergence is pointwise for $t \notin \mathbf{Z^d}$, as well as in $L^2(\mathbf{R}^d)$.

That is, we have the $\mathcal{F}$–pairing:

$$\sum_{n\in\mathbf{Z}^d} c_n \prod_{\substack{i \\ m_i=1}} \mathbf{1}_{(n_i,n_i+1)}(t_i) \prod_{\substack{i \\ m_i=-1}} \ln|\frac{t_i - n_i}{t_i - n_i - 1}| \longleftrightarrow H_m(\gamma) \frac{1}{p(\gamma)} \sum_{n\in\mathbf{Z}^d} a_n e^{-2\pi i\langle n,\gamma\rangle}$$

To prove this lemma in an elementary fashion, we need some observations with respect to partial Fourier and Hilbert transformations.

For $k = 1, \ldots, d$, let
$$\mathcal{F}_k : L^2(\mathbf{R}^d) \longrightarrow L^2(\mathbf{R}^d)$$
be the partial Fourier transformation. $\mathcal{F}_k(f)$ is defined for almost every $\gamma \in \widehat{\mathbf{R}}^d$ by
$$\mathcal{F}_k(f)(\gamma) = \lim_{N\to\infty} \int_{-N}^{N} f(x_1, \ldots, x_d) e^{-2\pi i x_k \gamma_k}\, dx_k.$$
This is well defined, since for $f \in L^2(\mathbf{R}^d)$, by Tonelli's Theorem
$$|f(x_1, \ldots, x_{k-1}, \cdot, x_{k+1}, \ldots, x_d)|^2$$
is integrable for almost every $x^k = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d)$ (as a subset of $\mathbf{R}^{d-1}$) and we have
$$\begin{aligned}
\|\mathcal{F}_k(f)\|_{L(\widehat{\mathbf{R}}^d)} &= \int_{\mathbf{R}^{d-1}} \int_{\widehat{\mathbf{R}}} |\mathcal{F}_k(f)(x_1, \ldots, x_{k-1}, \gamma, x_{k+1}, \ldots, x_d)|^2\, d\gamma\, dx^k \\
&= \int_{\mathbf{R}^{d-1}} \int_{\mathbf{R}} |f(x_1, \ldots, x_d)|^2\, dx_k\, dx^k \qquad (9.10) \\
&= \|f\|^2_{L^2(\mathbf{R}^d)}
\end{aligned}$$
to obtain (**??**) we applied the Parseval–Plancherel Theorem. The calculation also showed that $\mathcal{F}_k$ is continuous for all $k = 1, \ldots, d$.

We can apply a similar argument to any continuous operator on $L^2(\mathbf{R})$ to obtain continuous operators on $L^2(\mathbf{R}^d)$. The only modification we need, is to use the continuity of the operator instead of Parseval–Plancherel to guarantee that the new operator is well defined and continuous. In particular, we shall apply this to the Hilbert transformation, by defining for $k = 1, \ldots, d$
$$\mathcal{H}_k : L^2(\mathbf{R}^d) \longrightarrow L^2(\mathbf{R}^d)$$
to be the Hilbert transformation with respect to the k–th variable.

Observe now that if $k \neq l$ we have $\mathcal{F}_k \mathcal{H}_l = \mathcal{H}_l \mathcal{F}_k$. This is trivial for separable functions in $L^2(\mathbf{R}^d)$. The set of linear combinations of such functions is dense in $L^2(\mathbf{R}^d)$, hence, by continuity, the operators commute on all of $L^2(\mathbf{R}^d)$.

For separable functions with compact support, we have clearly $\mathcal{F} = \mathcal{F}_1 \circ \ldots \circ \mathcal{F}_d$. The linear span of these functions is again dense in $L^2(\mathbf{R}^d)$, and hence $\mathcal{F} = \mathcal{F}_1 \circ \ldots \circ \mathcal{F}_d$ holds for all of $L^2(\mathbf{R}^d)$.

Proof of Lemma **??**: Fix $m \in M$. We need to obtain $\mathcal{F}^{-1}\left(H_m \frac{\varphi}{p}\right)$. In fact

$$\begin{aligned}
&\mathcal{F}^{-1}\left(H_m \frac{\varphi}{p}\right) \\
&= \mathcal{F}_d^{-1} \circ \ldots \circ \mathcal{F}_1^{-1}\left(\prod_{\substack{i \\ m_i=-1}} \operatorname{sgn}(x_i)\left(\mathcal{F}_1 \circ \ldots \circ \mathcal{F}_d\left(\mathcal{F}^{-1}\left(\frac{\varphi}{p}\right)\right)\right)\right) \\
&= \prod_{\substack{i \\ m_i=-1}}^{\circ} \mathcal{H}_i\left(\mathcal{F}^{-1}\left(\frac{\varphi}{p}\right)\right).
\end{aligned}$$

Let $\tilde{c}_k = 2(\pi\imath)^d \sum_{p(n-k)\geq 0} a_n$, $\{i_1, \ldots, i_r\} = \{i, m_i = 1\}$. For permissible $t \in \mathbf{R}^d$, we have

$$\begin{aligned}
&\mathcal{F}^{-1}\left(H_m \frac{\varphi}{p}\right)(t) \\
&= \lim_{\epsilon_{i_1}\to 0} \ldots \lim_{\epsilon_{i_r}\to 0} \frac{1}{\pi^r} \int_{|t_{i_1}-u_{i_1}|\geq \epsilon_{i_1}} \ldots \int_{|t_{i_r}-u_{i_r}|\geq \epsilon_{i_r}} \frac{\mathcal{F}^{-1}\left(\frac{\varphi}{p}\right)(u)}{\prod_{l=1}^r (t_{i_l}-u_{i_l})}\, du_{i_r} \ldots du_{i_1} \\
&= \lim_{\epsilon_{i_1}\to 0} \ldots \lim_{\epsilon_{i_r}\to 0} \frac{1}{\pi^r} \int_{|t_{i_1}-u_{i_1}|\geq \epsilon_1} \ldots \int_{|t_{i_r}-u_{i_r}|\geq \epsilon_r} \frac{\sum_{n\in\mathbf{Z}^d} \tilde{c}_n 1_{(n,n+1)}(u)}{\prod_{l=1}^r (t_{i_l}-u_{i_l})}\, du_{i_r} \ldots du_{i_1} \\
&= \frac{1}{\pi^r} \sum_{n\in\mathbf{Z}^d} \tilde{c}_n \prod_{\substack{i \\ m_i=1}} \mathbf{1}_{(n_i,n_i+1)}(t_i) \prod_{\substack{i \\ m_i=-1}} \lim_{\epsilon_i\to 0} \int_{|t_i-u_i|\geq \epsilon_i} \frac{\mathbf{1}_{(n_i,n_i+1)}(u_i)}{t_i-u_i} du_i \\
&= \sum_{n\in\mathbf{Z}^d} c_n \prod_{\substack{i \\ m_i=1}} \mathbf{1}_{(n_i,n_i+1)}(t_i) \prod_{\substack{i \\ m_i=-1}} (\ln|t_i-n_i| - \ln|t_i-n_i-1|).
\end{aligned}$$

We used the fact that $c_n = 2\frac{(\pi i)^d}{\pi^r} \sum_{p(n-k)\geq 0} a_n = \frac{1}{\pi^r}\tilde{c}_n$. □

We can now proceed to state and prove the main result.

THEOREM 9.28. *Let $\psi \in L^1(\mathbf{R}^d)$. The following are equivalent:*
i. $W_\psi f(b,a) = \int_{\mathbf{R}^d} f(t)\psi(\frac{t-b}{a})\, dt$ is 1–periodic in a for all $f \in L^\infty(\mathbf{T}^d)$.
ii. $\widehat{\psi}(k) = 0$ for $k \in \mathbf{Z}^d \cap S$ and ψ can be written in the form:

$$\psi(t) = \sum_{m\in M} \sum_{n\in\mathbf{Z}^d} c_n^m \prod_{\substack{i \\ m_i=1}} \mathbf{1}_{(n_i,n_i+1)}(t_i) \prod_{\substack{i \\ m_i=-1}} \ln\left|\frac{t_i-n_i}{t_i-n_i-1}\right|,$$

,$t = (t_1, \ldots, t_d) \in \mathbf{R}^d$, where $\{c_n^m\}_n \in l^2(\mathbf{Z}^d)$, such that there exists $\{a_n^m\}_n \in l^1(\mathbf{Z}^d)$ such that for all $k \in \mathbf{Z}^d$

$$c_k^m = \sum_{p(n-k)\geq 0} a_n^m,$$

and such that for

$$\varphi_m(\gamma) = \sum_{n\in\mathbf{Z}^d} a_n^m e^{2\pi i\langle n,\gamma\rangle}$$

$\frac{\varphi_m}{p}$ *is bounded.*

Note that the main difference between Theorem **??** and Theorem **??** are the conditions we obtain on the sequences that are involved. The conditions in Theorem **??** are significantly harder to check.

Proof. $i \Longrightarrow ii$: Lemma **??** and Lemma **??** imply that $\widehat{\psi}(k) = 0$ for $k \in \mathbf{Z}^d \cap S$ and

$$\widehat{\psi} = \sum_{m\in M} H_m \frac{\varphi_m}{p}$$

with $\varphi_m \in A(\mathbf{T}^d)$ for $m \in M$. Since $\widehat{\psi}$ is continuous, we have, by the construction in Lemma **??**, that $H_m\frac{\varphi_m}{p}$ are bounded for all $m \in M$, and hence clearly $H_m\frac{\varphi_m}{p} \in L^2(\widehat{\mathbf{R}}^d)$. Therefore, by Lemma **??** and Lemma **??** we have

$$\psi(t) = \sum_{m\in M}\sum_{n\in\mathbf{Z}^d} c_n^m \prod_{\substack{i\\ m_i=1}} \mathbf{1}_{(n_i,n_i+1)}(t_i) \prod_{\substack{i\\ m_i=-1}} \ln\left|\frac{t_i-n_i}{t_i-n_i-1}\right|$$

with $\{c_n^1\}_n = \{a_n^1\}_n \in l^1(\mathbf{Z}^d)$ and for $m \neq 1$ we have $\{c_n^m\}_n \in l^2(\mathbf{Z}^d)$. Further, $\varphi_m(\gamma) = \sum_{n\in\mathbf{Z}^d} a_n^m e^{-2\pi i\langle\gamma,n\rangle} \in A(\mathbf{T}^d)$ and for $k \in \mathbf{Z}^d$

$$c_k^m = \sum_{p(n-k)\geq 0} a_n^m.$$

Therefore ii holds.

$ii \Longrightarrow i$: Since $\varphi_m(\gamma) = \sum_{n\in\mathbf{Z}^d} a_n^m e^{-2\pi i\langle\gamma,n\rangle} \in A(\mathbf{T}^d)$, and $\frac{\varphi_m}{p}$ is bounded for all $m \in M$, we have $H_m\frac{\varphi_m}{p} \in L^2(\widehat{\mathbf{R}}^d)$ for all $m \in M$. This allows us to apply Lemma **??**, and Lemma **??** then supplies the result. □

BIBLIOGRAPHY

[BBKC+95] G. Benke, M. Bozek-Kuzmicki, D. Colella, G.M. Jacyna, and J.J. Benedetto. *Wavelet-based analysis of EEG signals for detection and localization of epileptic seizures*. Spie Orlando, 1995.

[BC95] J.J. Benedetto and D. Colella. *Wavelet analysis of spectogram seizure chirps*. Proc. SPIE, 2569:512–521, 1995.

[Ben97] J.J. Benedetto. Harmonic Analysis and Applications. CRC Press, Boca Raton, 1997.

[BP97] J.J. Benedetto and G.E. Pfander. *Wavelet detection of periodic behavior in EEG and ECoG data*. In 15th IMACS World Congress, Berlin, volume 1, 1997.

[BP98] J.J. Benedetto and G.E. Pfander. *Wavelet periodicity detection*. In Wavelet Applications in Signal and Image Processing VI, volume 3458. SPIE, 1998.

[BP99] J.J. Benedetto and G.E. Pfander. *Wavelet periodicity detection*. preprint, 1999.

[CHT95] R.A. Carmona, W.L. Hwang, and B. Torresani. *Multi-ridge detection and time-frequency reconstruction*. preprint, 1995.

[Cuf96] B.N. Cuffin. EEG *localisation accuracy improvements using realistically shaped head models*. IEEE Trans.Biomed.Eng., 43 NO.3:299–303, 1996.

[DEG+64] N. Delprat, B. Escudie, P. Guillemain, R. Kronland-Martinet, P.Tchamitchian, and B. Torresani. *Asymptotic wavelet and gabor analysis: Extraction of instantaneous frequencies*. IEEE Transactions on Information Theory, 38 NO.2:299–303, 644-664.

[FAJF96] D.J. Fletcher, A. Amir, D.L. Jewett, and G. Fein. *Improved method for computation of potentials in a realistic head shape model*. IEEE Trans.Biomed.Eng., 42 NO.11:1094–1103, 1996.

[FG89] H.G. Feichtinger and K.H. Groechenig. *Banach spaces related to integrable group representations and their atomic decompositions.* Journal of Functional Analysis, 86:307–340, 1989.

[LMR94] A.K. Louis, P. Maass, and A. Rieder. Wavelets: Theorie und Anwendungen. B.G. Teubner, New York, 1994.

[Mal98] S. Mallat. A Wavelet Tour of Signal Processing. Academic Press, San Diego, 1998.

[Ner67] U. Neri. Singular integrals: An introduction. Technical report, University of Maryland, 1967. Lecture Notes No.3.

[Nun81] P. Nunez. Electric Fields of the Brain: The Neurophysics of EEG. Oxford University Press, New York, 1981.

[RS95] A. Ron and Z. Shen. *Frames and stable bases for shift invariant subspaces of* $L^2(\mathbf{R}^d)$. Can. J. Math., 47 no.5:1051–1094, 1995.

[RS97] A. Ron and Z. Shen. *Affine systems in* $L^2(\mathbf{R}^d)$*: The analysis of the analysis operator.* Journal of Fourier Analysis, 1997.

[Rud62] W. Rudin. Fourier Analysis on Groups. John Wiley and Sons, New York, 1962.

[Sei89] K. Seip. *Some remarks on a method for detection of spectral lines in signals.* Marseille CPT-89, 1989.

[SM93] S.J. Schiff and J. Milton. *Wavelet transforms of electroencephalographic spike and seizure detection.* Spie, 1993.

[SMHW94] S.J. Schiff, J. Milton, J. Heller, and S.L. Weinstein. *Wavelet-transforms and surrogate data for electroencephalographic spike and seizure localization.* Optical Engineering, 33(3):2162–2169, 1994.

[Tay86] M.E. Taylor. Noncommutative Harmonic Analysis. American Mathematical Society, Providence, Rhode Island, 1986.

[Wie33] N. Wiener. The Fourier Integral and Certain of its Applications. Camebridge University Press, New York, 1933.