Polarization Based Phase Retrieval for Time-Frequency Structured Measurements

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Abstract—**We consider phaseless measurements in the case when the measurement frame is a Gabor frame, that is, the frame coefficients are of the form of masked Fourier coefficients where the masks are time shifts of the Gabor window. This makes measurements meaningful for applications, but at the same time preserves the flexibility of the frame-theoretic approach. We are going to present a recovery algorithm which requires a sufficiently small number of measurements; it is based on the idea of polarization, which is proposed by Alexeev, Bandeira, Fickus and Mixon [1], [4].**

I. INTRODUCTION

In many areas of imaging science, optical detectors can only record the squared modulus of the Fraunhofer or Fresnel diffraction pattern of the radiation that is scattered from the object at hand. In such setting, the phase of the optical wave reaching the detector is lost. So, it is needed to reconstruct a signal from intensity measurements, frequently from the magnitudes of its Fourier coefficients, only. The problem of recovering a signal from intensity measurements is called *phase retrieval.*

For simplicity we consider one-dimensional signals with finite length only. Formally, we seek to recover a signal $x \in \mathbb{C}^M$ from measurements of the form

$$b_j = |\langle x, \varphi_j \rangle|, \ \varphi_j \in \Phi,$$

where $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ is a frame in $\mathbb{C}^M$.

Clearly, x can only be reconstructed up to a global phase. Indeed, for every $\omega \in \mathbb{C}$ with $|\omega| = 1$, the signals x and ωx produce the same set of intensity measurements.

Not every frame provides an injective measurement procedure. Even in the case when Φ is known to give injective measurements, the problem of reconstructing x from $\{b_j\}_{j=1}^N$ is NP-hard in general [12].

A number of polynomial-time numerical algorithms that work for very specific choices of Φ are proposed in the literature. But for many of these algorithms there are no recovery and stability guarantees proven, the measurements $|\langle x, \varphi \rangle|$, $\varphi \in \Phi$, cannot be physically implemented in practice, or the number of measurements needed is unreasonably large. Until recently, efficient algorithms with $O(M^2)$ measurements were known only, see for example [3]. The situation changed significantly when Candès, Strohmer and

Voroninski proposed PhaseLift which uses semidefinite programming to stably reconstruct a signal x from $N = O(M)$ measurements of the form $\{|\langle x, \varphi_j \rangle|^2\}_{j=1}^N$, where all φ_j are independent Gaussian random vectors [6].

In our work, we consider the case when the frame Φ is a Gabor frame, that is, a time-frequency structured frame. The main motivation for this is that in this case, the frame coefficients are of the form of masked Fourier coefficients, where the masks are shifts of the possibly randomly chosen Gabor window. This makes the measurements implementable in many diffraction imaging problems.

Our paper is organised as follows. In Section II we describe the general idea of the polarization method [1], [4]. We also discuss some relevant properties of expander graphs. In Section III we describe time-frequency structured measurements, our algorithm, and discuss stability issues.

II. PHASE RETRIEVAL WITH POLARIZATION

First, we are going to describe the so-called polarization approach, proposed by Alexeev, Bandeira, Fickus and Mixon [1], [4]. The main difference between PhaseLift and polarization is that polarization uses structured measurement vectors. If a phase retrieval application allows one to design measurement vectors with the required structure, then the phase can be recovered rather quickly. A number of numerical simulations which compare polarization to PhaseLift show that polarization is much faster (i.e., seconds versus hours), but PhaseLift offers more stable estimates of the signal. Stability of the polarization method still remains to be established.

Suppose we take phaseless measurements of $x \in \mathbb{C}^M$ with a frame $\Phi_V = \{\varphi_i\}_{i \in V} \subset \mathbb{C}^M$, where V is a finite set. Having $|\langle x, \varphi_i \rangle|$ for every $i \in V$, it suffices to determine the relative phase between pairs of frame coefficients. Indeed, if we choose a nonzero frame coefficient $|\langle x, \varphi_i \rangle|$ and assume that $c_i = \langle x, \varphi_i \rangle = |\langle x, \varphi_i \rangle|$, then any $\langle x, \varphi_j \rangle \neq 0$ has a well-defined relative phase

$$\omega_{ij} = \left(\frac{\langle x, \varphi_i \rangle}{|\langle x, \varphi_i \rangle|} \right)^{-1} \frac{\langle x, \varphi_j \rangle}{|\langle x, \varphi_j \rangle|} = \frac{\overline{\langle x, \varphi_i \rangle} \langle x, \varphi_j \rangle}{|\langle x, \varphi_i \rangle| |\langle x, \varphi_j \rangle|},$$

and we can set

$$c_j = \omega_{ij} |\langle x, \varphi_j \rangle|.$$

In the case when $\langle x, \varphi_j \rangle = 0$, we set $c_j = 0$, $\omega_{ij} = 1$. The original signal x can then be identified up to a global phase by using a dual frame $\{\tilde{\varphi}_j\}_{j \in V}$ of a frame $\{\varphi_j\}_{j \in V}$. Indeed, we have:

$$\sum_{j \in V} c_j \tilde{\varphi}_j = \left(\frac{\langle x, \varphi_i \rangle}{|\langle x, \varphi_i \rangle|}\right)^{-1} x.$$

To obtain the relative phase between frame coefficients, the following polarization identity is useful:

Lemma II.1. [1] *For any $i, j \in V$, if $\langle x, \varphi_i \rangle \neq 0$ and $\langle x, \varphi_j \rangle \neq 0$, the following holds:*

$$\omega_{ij} = \frac{1}{3|\langle x, \varphi_i \rangle||\langle x, \varphi_j \rangle|} \sum_{k=0}^{2} \omega^k |\langle x, \varphi_i + \omega^k \varphi_j \rangle|^2.$$

Let us introduce the graph $G = (V, E)$ with a small edge set E representing relative phases to be chosen later. If we obtain phaseless measurements with respect to $\Phi_V \cup \Phi_E$, where $\Phi_E = \{\varphi_i + \omega^k \varphi_j\}_{k \in \{0,1,2\}, (i,j) \in E}$, then we can use Lemma II.1 to determine ω_{ij} in case $\langle x, \varphi_i \rangle \neq 0$ and $\langle x, \varphi_j \rangle \neq 0$. If for some $j \in V$, $\langle x, \phi_j \rangle = 0$ then relative phases involving j are not defined. Thus, phase information cannot be propagated through this vertex and this has the effect of removing the vertex from the graph.

Else, we can propagate phases from one vertex to another (see Figure 1). Indeed, if we know $c_i \neq 0$, for some $i \in V$, then for any j, such that $(i, j) \in E$ and $|\langle x, \phi_j \rangle| \neq 0$, we have $c_j = \omega_{ij} \frac{c_i}{|c_i|} |\langle x, \phi_j \rangle|$, and we can repeat this step iteratively. Hence, to reconstruct x up to a global phase, it is sufficient that after removing all "zero" vertices the graph G has a connected component whose vertex set corresponds to a subframe of Φ.
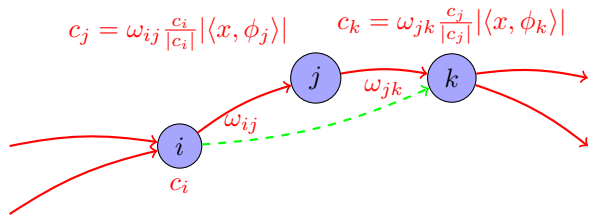

$$c_j = \omega_{ij} \frac{c_i}{|c_i|} |\langle x, \phi_j \rangle| \qquad c_k = \omega_{jk} \frac{c_j}{|c_j|} |\langle x, \phi_k \rangle|$$

Fig. 1: The phase propagation process.

The algorithm proposed in [1] hinges on the following:
(i) We require Φ_V to be full spark (see [2]), that is, Φ_V has the property that every subcollection of M frame elements spans $\mathbb{C}^M$. There are two reasons to use full spark frames. First, this implies that any $x \neq 0$ is orthogonal to at most $M-1$ vectors from Φ_V and thus at most $M-1$ vertices will be deleted from the graph G. Secondly, any subcollection of M vectors from Φ_V form a frame in $\mathbb{C}^M$ and thus x can be reconstructed from any M frame coefficients.
(ii) If Φ is full spark, It is sufficient that G is a graph with the following connectivity property: deleting any $M - 1$ vertices results in a connected component of size at least M. To this end, we are going to use a well-studied family of sparse graphs known as *expander graphs*.

Let G be a d-regular graph with adjacency matrix $A = A(G)$. Being real and symmetric, $A(G)$ has n real eigenvalues. Let us denote them by $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and let $\lambda = \max(|\lambda_2|, |\lambda_n|)$. The value $\mathrm{spg}(G) = \frac{d - \lambda}{d}$ is known as the *spectral gap* of G. As the following lemma shows, a big $\mathrm{spg}(G)$ ensures good connectivity properties of graph G [1], [7]:

Lemma II.2. *Let G be a d-regular graph with $|V| = n$. For all $\varepsilon \leq \frac{\mathrm{spg}(G)}{6}$, removing any εn vertices from G results in a connected component of size at least $\left(1 - \frac{2\varepsilon}{\mathrm{spg}(G)}\right) n$.*

III. POLARIZATION FOR TIME-FREQUENCY STRUCTURED MEASUREMENTS

Let us first define the following two unitary operators on $\mathbb{C}^M$ (for more information see [11]):
The *time shift operator* $T_k : \mathbb{C}^M \to \mathbb{C}^M$, $k \in \mathbb{Z}_M$ is given by

$$T_k x = T_k \{x(0), x(1), \ldots, x(M-1)\} = \{x(m-k)\}_{m \in \mathbb{Z}_M},$$

and the *frequency shift operator* $M_\ell : \mathbb{C}^M \to \mathbb{C}^M$ satisfies

$$M_\ell x = \{e^{2\pi i \ell m / M} x(m)\}_{m \in \mathbb{Z}_M}.$$

We also define *time-frequency shift operators* as composition $\pi(k, \ell) = M_\ell T_k$, for $k, \ell \in \mathbb{Z}_M$.

Definition III.1. *For a* window $\zeta \in \mathbb{C}^M \setminus \{0\}$ *and* $\Lambda \subseteq \mathbb{Z}_M \times \mathbb{Z}_M$, *the set of vectors*

$$(\zeta, \Lambda) = \{\pi(k, \ell)\zeta\}_{(k, \ell) \in \Lambda}$$

is called Gabor system. *A Gabor system which is a frame is referred to as* Gabor frame.

The following result was shown for M prime in [8] and for M composite in [9]:

Theorem III.2. *Let M be a positive integer and let Λ be a subgroup of $\mathbb{Z}_M \times \mathbb{Z}_M$ with $|\Lambda| \geq M$. Then, for almost all $\zeta \in \mathbb{S}^{M-1} \subset \mathbb{C}^M$, (ζ, Λ) is a full spark frame.*

A. Algorithm

Let $x \in \mathbb{C}^M$ be the signal that is to be reconstructed. Consider a subgroup $\Lambda = F \times \mathbb{Z}_M \subset \mathbb{Z}_M \times \mathbb{Z}_M$, such that $|F| = K$ is fixed (that is, we consider all frequency shifts and only a constant number of time shifts). We assume that measurements are of the form $\{|\langle x, \phi \rangle|, \quad \phi \in \Phi_V\}$, where $\Phi_V = (\zeta, \Lambda)$ is a Gabor frame and $\zeta = \{e^{2\pi i y_m}\}_{m \in \mathbb{Z}_M}$, where $y_m \in [0, 1)$ are independent uniformly distributed random variables. Theorem III.2 implies that with probability 1 the frame $\Phi_V = \{\pi(\lambda)\zeta\}_{\lambda \in \Lambda}$ is full spark, and thus for any vector $x \in \mathbb{C}^M$, the number of zero measurements among $\{|\langle x, \pi(\lambda)g \rangle|\}_{\lambda \in \Lambda}$ is at most $M - 1$.

For each $\lambda \in \Lambda$, $\lambda = (k, \ell)$, $\ell \in \mathbb{Z}_M$, $k \in F$ we have:

$$|\langle x, \pi(\lambda)\zeta \rangle| = \left| \mathcal{F}\left(x \odot T_k \bar{\zeta}\right)(\ell) \right|. \qquad (1)$$

Here, $\odot$ denotes a componentwise product. As equation (1) shows, our measurements have the form of magnitudes of masked Fourier transform coefficients with the set of masks being $\{T_k \bar{\zeta}\}_{k \in F}$.

Using Lemma II.1, we get that for any $\lambda_1, \lambda_2 \in V$, provided $|\langle x, \pi(\lambda_1)\zeta \rangle| \neq 0$ and $|\langle x, \pi(\lambda_2)\zeta \rangle| \neq 0$, $\omega_{\lambda_1 \lambda_2}$ equals

$$\frac{1}{3|\langle x, \pi(\lambda_1)\zeta \rangle||\langle x, \pi(\lambda_2)\zeta \rangle|} \sum_{k=0}^{2} \omega^k |\langle x, \pi(\lambda_1)\zeta + \omega^k \pi(\lambda_2)\zeta \rangle|^2. \quad (2)$$

Thus to find x up to a global phase, we require additional measurements of the form $|\langle x, \pi(\lambda_1)\zeta + \omega^t \pi(\lambda_2)\zeta \rangle|$, $t \in \{0, 1, 2\}$. We have

$$\pi(\lambda_1)\zeta + \omega^t \pi(\lambda_2)\zeta = \pi(k_1, \ell_1)\zeta + \omega^t \pi(k_2, \ell_2)\zeta$$
$$= p_{(\ell_2 - \ell_1)k_1 k_2}(t) \odot \pi(\lambda_1)\zeta$$

where the vector $p_{ck_1 k_2}(t) \in \mathbb{C}^M$ is defined by

$$p_{ck_1 k_2}(t)(m) = \left(1 + e^{2\pi i \left(\frac{cm}{M} + (y_{m-k_2} - y_{m-k_1}) + \frac{t}{3}\right)}\right)$$

for every $m \in \mathbb{Z}_M$ and with parameters $c_1 \in \mathbb{Z}_M$, $k_1, k_2 \in F$ and $t \in \{0, 1, 2\}$. Therefore, for reconstruction, for each fixed parameter triple (c, k_1, k_2), we need additional measurements of the form

$$\{|\langle x, p_{ck_1 k_2}(t) \odot \pi(k_1, \ell)\zeta \rangle|\}_{\ell \in \mathbb{Z}_M, \ t \in \{0,1,2\}} =$$
$$\{|\mathcal{F}\left(x \odot \bar{p}_{ck_1 k_2}(t) \odot T_{k_1} \bar{\zeta}\right)(\ell)|\}_{\ell \in \mathbb{Z}_M, \ t \in \{0,1,2\}}, \quad (3)$$

that is, the required additional measurements also have the form of masked Fourier transform coefficients that are obtainable in many diffraction imaging applications.

Let us construct the graph $G = (V, E)$ in the following way. Let us first pick a set $C \subset \mathbb{Z}_M$, such that $0 \notin C$ and $C = -C$. Then choose $V = \Lambda = F \times \mathbb{Z}_M$ and $E = \{(\lambda_1, \lambda_2) \in V \times V,$ such that $\lambda_1 = (k_1, \ell_1)$ and $\lambda_2 = (k_2, \ell_2)$ satisfy $(\ell_2 - \ell_1) \in C\}$. As follows from formula (3), for each element $c \in C$ we need $3|F|^2 M$ additional measurements that correspond to the edges of the graph G of the form

$$\{|\langle x, p_{ck_1 k_2}(t) \odot \pi(k_1, \ell)\zeta \rangle|\}_{\ell \in \mathbb{Z}_M, \ k_1, k_2 \in F, \ t \in \{0,1,2\}}.$$

Since $0 \notin C$, the graph G constructed above has no loops, and since $C = -C$, it is not directed. Also, each vertex $\lambda = (k, \ell)$ of G is adjacent to any vertex $\lambda' = (k', \ell + c)$, where $c \in C$ and $k' \in F$. Thus each vertex in G has degree $|F||C|$ and G is regular. We have the following result estimating the spectral gap of G. Its proof is similar to the one of an analogous result from [4] and is therefore omitted.
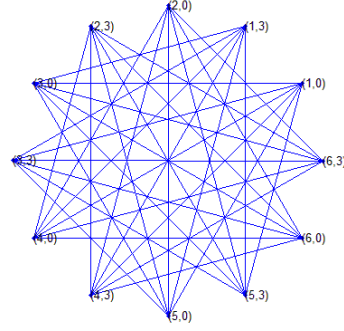

Fig. 2: An example of the graph G with $M = 6$, $F = \{0, 3\}$ and $C = \{2, 3, 4\}$.

Lemma III.3. *Pick $b > 36$ and suppose the entries of the characteristic vector $\mathbf{1}_B$ of a set B are independent, identically distributed Bernoulli random variables with mean $\frac{b \log M}{M}$. Take $C = B \cup (-B) \setminus \{0\}$ and construct the graph G as above. Then with overwhelming probability*

$$\mathrm{spg}(G) \geq 1 - \frac{6}{\sqrt{b}}.$$

Note that if the sets B and C are constructed as in Lemma III.3, then, with high probability, $|C| = O(\log M)$. The following result then follows from Lemmas III.3 and II.2.

Theorem III.4. *Let frames Φ_V and Φ_E be constructed as above, where $|F| = 12$ and $b = 144$. Then every signal $x \in \mathbb{C}^M$ can be reconstructed form $M + 3|F|^2 M|C| = O(M \log M)$ phaseless measurements with respect to the frame $\Phi_V \cup \Phi_E$ using the algorithm described above.*

B. Numerical results in the noisy case

In many applications, measurements are corrupted by noise, that is, they are of the form

$$\{|\mathcal{F}(x \odot T_k \bar{\zeta})(\ell) + \varepsilon_{k,l}|\}_{(k,l) \in \Lambda} \cup$$
$$\{|\mathcal{F}(x \odot T_k \bar{\zeta} \odot \bar{p}_{c,k_1,k_2})(\ell_1) + \varepsilon_{k_1,\ell_1,k_2,c}|\}_{k_1,k_2 \in F, \ell_1 \in \mathbb{Z}_M},$$

where $\varepsilon_{k,l}$, $\varepsilon_{k_1,\ell_1,k_2,c} \sim \mathcal{N}(0, \sigma)$ are independent identically distributed Gaussian variables with variance σ. For simplicity let us assume that $\varepsilon_{k_1,\ell_1,k_2,c} = \bar{\varepsilon}_{k_2,\ell_1+c,k_1,-c}$ (see also [1]).

We start our reconstruction procedure by deleting "zero" vertices, as these cannot be used for phase propagation. Formula (2) shows that the relative phase is very sensitive to perturbations when either $|\langle x, \pi(\lambda_1)\zeta \rangle|$ or $|\langle x, \pi(\lambda_2)\zeta \rangle|$ is small. As such, vertices with small corresponding vertex measurements provide unreliable information, consequently they should be deleted as well. To do so, we are going to use the following algorithm.

ALGORITHM 1 (DELETING "SMALL" VERTICES):
1) Input: graph $G = (V, E)$ with weighted vertices V, parameter α.
2) For $i = 0$ to $(1 - \alpha)|V|$
3) Find $\lambda \in V$ with the smallest value of $|\langle x, \pi(\lambda)\zeta\rangle|$.
4) Delete the vertex λ from G; end for
5) Output: Graph G with large smallest vertex weight.

After applying Algorithm 1, the graph G will have slightly fewer vertices, but the remaining edges will provide more reliable information on relative phases. Note that during the phase propagation procedure, noise is propagated with phase, and thus can accumulate and grow while passing from one vertex to another. To overcome this problem, we would like to use the information coming from different edges to a vertex to reduce the noise. To be able to perform noise reduction, we need to ensure that the graph G has a strongly connected component, that is, a connected component with big spectral gap, of size at least M. To this end, we modify G using a standard spectral clustering algorithm [10], [1]:

ALGORITHM 2 (SPECTRAL CLUSTERING):
1) Input: Graph $G = (V, E)$, parameter μ.
2) While $\mathrm{spg}(G) < \mu$ and $|V| > M + 1$
3) Set $D = \mathrm{diag}(d_1, \ldots, d_n)$, where d_i is the degree of the ith vertex.
4) Set A to be adjacency matrix of G.
5) Compute the Laplacian $L = I - D^{-1/2}AD^{-1/2}$.
6) Compute u, the eigenvector corresponding to the second smallest eigenvalue λ_2 of L.
7) For $i = 1$ to $|V|$
8) Let S_i be a set of vertices corresponding to i smallest entries of $D^{-1/2}u$.
9) Set $h_i = \frac{|E(S_i, S_i^C)|}{\min\{\sum_{v \in S_i} \deg v, \ \sum_{v \in S_i^C} \deg v\}}$; end for
10) Set $S = S_i$, s.t. h_i is minimal.
11) Set $G = G \setminus S$; end while
12) Output: Graph G with at least M vertices and big enough spectral gap.

By applying Algorithms 1 and 2, we obtain reliable relative phase data on a well-connected graph. Now, we seek an efficient method to reconstruct (up to a global constant) the phases of the vertex frame coefficients using measured relative phase data. For this purpose we use an algorithm known as *angular synchronisation* [13], [1]. The idea behind this algorithm is the following. Let A be a weighted adjacency matrix of our graph G, that is, A is given by

$$A(\lambda_i, \lambda_j) = \begin{cases} \frac{\overline{\langle x, \pi(\lambda_i)\zeta\rangle}\langle x, \pi(\lambda_j)\zeta\rangle + \varepsilon_{ij}}{|\overline{\langle x, \pi(\lambda_i)\zeta\rangle}\langle x, \pi(\lambda_j)\zeta\rangle + \varepsilon_{ij}|} & , \text{if } (\lambda_i, \lambda_j) \in E, \\ 0 & , \text{if } (\lambda_i, \lambda_j) \notin E; \end{cases}$$

Let $\omega_{\lambda_i} = \frac{\langle x, \pi(\lambda_i)\zeta\rangle}{|\langle x, \pi(\lambda_i)\zeta\rangle|}$, then $A(\lambda_i, \lambda_j)$ can be considered as an approximation of the relative phase $\omega_{\lambda_i}^{-1}\omega_{\lambda_j}$. The idea is to find a vector $\omega = \{\omega_\lambda\}_{\lambda \in \Lambda}$ that approximates the phases of vertex measurements as the minimizer of the following error quantity:

$$\sum_{\{\lambda_i, \lambda_j\} \in E} |\omega_{\lambda_j} - A(\lambda_i, \lambda_j)\omega_{\lambda_i}|^2 = \omega^*(D - \bar{A})\omega,$$

where D is a diagonal matrix of vertex degrees and $\bar{A}$ is a componentwise conjugate of A. One can show that ω is the minimizer if $D^{1/2}\omega = u$, where u is an eigenvector corresponding to the smallest eigenvalue of $L_1 = I - D^{-1/2}AD^{-1/2}$ [1], [5]. The procedure, summarized in Algorithm 3, provides a stable estimate of vertex phases provided the spectral gap of a graph is sufficiently big.

ALGORITHM 3 (ANGULAR SYNCHRONIZATION):
1) Input: $G = (V, E)$ graph with weighted edges.
2) Set $A \leftarrow$ weighted adjacency matrix.
3) Set $D = \mathrm{diag}(d_1, \ldots, d_n)$, where d_i is the degree of the ith vertex.
4) Compute $L_1 = I - D^{-1/2}AD^{-1/2}$.
5) Compute $u \leftarrow$ the eigenvector corresponding to the smallest eigenvalue α_1 of L_1.
6) Output the vector of phases of u.

Let us now summarize the above discussion in the following modification of the reconstruction algorithm:

ALGORITHM 4 (RECONSTRUCTION ALGORITHM):
1) Construct graph $G = (\Lambda, E)$, where $E = \{(k_1, \ell_1), (k_2, \ell_2)), \text{ s.t. } \ell_2 - \ell_1 \in C\}$.
2) Assign to each $\lambda \in V$ weight $|\langle x, \pi(\lambda)\zeta\rangle|$.
3) Assign to each edge $(\lambda_1, \lambda_2) \in E$ weight $\omega_{\lambda_1 \lambda_2}$.
4) Run Algorithm 1 with parameter α.
5) Run Spectral Clustering with parameter μ.
6) Run Angular Synchronization $\mapsto \{u_\lambda\}_{\lambda \in \Lambda}$.
7) Set $c_\lambda = u_\lambda |\langle x, \pi(\lambda)\zeta\rangle|, \ \lambda \in \Lambda$.
8) Reconstruct x from $\{c_\lambda\}_{\lambda \in \Lambda}$.

The numerical results (see Figures 3, 4, 5, 6) indicate that the signal $\hat{x}$ reconstructed using Algorithm 4 from measurements with the noise vector ε satisfies:

$$||x - \hat{x}|| \leq \tilde{C}||\varepsilon||,$$

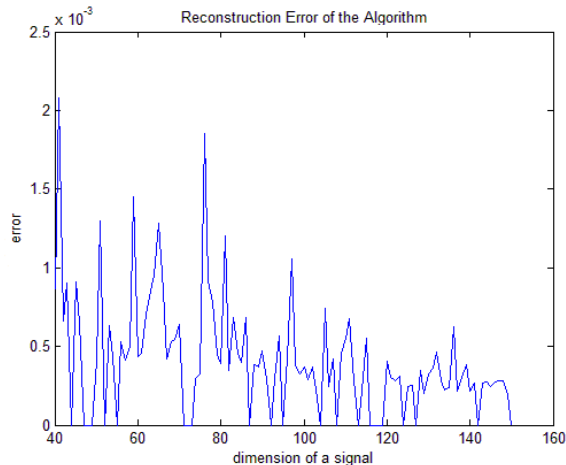where $\tilde{C}$ is a constant which might depend on $\mathrm{spg}(G)$

Fig. 3: Dependence of the error $||x - \hat{x}||$ on the dimension M (noise variance $\sigma = 10^{-3}$).

Fig. 4: Dependence of the error to noise ratio $\frac{||x-\hat{x}||}{||\varepsilon||}$ on the dimension M (noise variance $\sigma = 10^{-3}$).
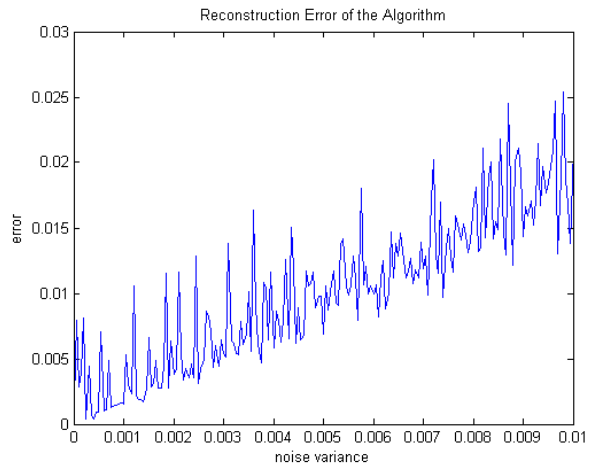

Fig. 5: Dependence of the error $||x - \hat{x}||$ on the noise variance (for a fixed dimension $M = 100$).
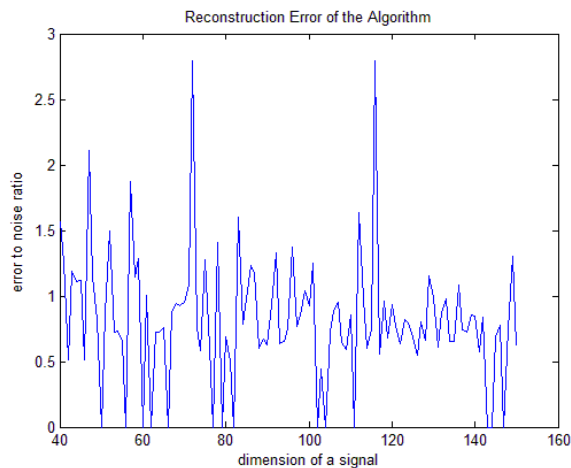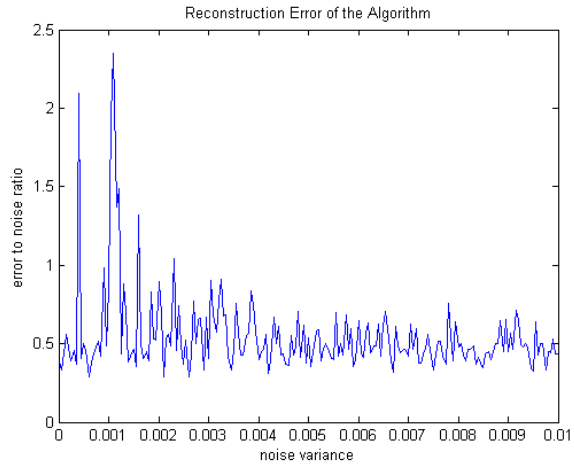

Fig. 6: Dependence of the error to noise ratio $\frac{||x-\hat{x}||}{||\varepsilon||}$ on the noise variance (for a fixed dimension $M = 100$).

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