On the invertibility of "rectangular" bi-infinite matrices and applications in time–frequency analysis

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ABSTRACT

Finite dimensional matrices having more columns than rows have no left inverses while those having more rows than columns have no right inverses. We give generalizations of these simple facts to bi–infinite matrices and use those to obtain density results for p–frames of time–frequency molecules in modulation spaces and identifiability results for operators with bandlimited Kohn–Nirenberg symbols.

1. INTRODUCTION

Matrices in $\mathbb{C}^{m\times n}$ are not invertible if $m \neq n$. To generalize this basic fact from linear algebra to bi–infinte matrices, we first associate the quadratic shape of $M \in \mathbb{C}^{m\times n}$, $m = n$, to bi-infinite matrices decaying away from their diagonals, more precisely, by matrices $M = (m_{j'j})_{j',j\in\mathbb{Z}^d}$ with $m_{j'j}$ small for $\big|\|j'\|_\infty - \|j\|_\infty\big|$ large. The rectangular shape of $M \in \mathbb{C}^{m\times n}$, $m < n$, is then taken to correspond to bi-infinite matrices decaying off wedges which are situated between two slanted diagonals of slope less than one and which are open to the left and to the right. In short, for $\lambda > 1$, we assume $m_{j'j}$ small for $\lambda\|j'\|_\infty - \|j\|_\infty$ positive and large. To this case, we associate the symbol $\blacktriangleright\!\blacktriangleleft$. Similarly, $M \in \mathbb{C}^{m\times n}$, $m > n$, corresponds to bi-infinite matrices that are the adjoints of the $\blacktriangleright\!\blacktriangleleft$ matrices described above. That is, the case ⧗ is described by: for $\lambda < 1$, we assume $m_{j'j}$ small for $-\lambda\|j'\|_\infty + \|j\|_\infty$ positive and large. In both cases, $\lambda \neq 1$ corresponds to $\frac{n}{m} \neq 1$ in the theory of finite dimensional matrices.

We consider bi–infinite matrices that act on weighted l^p spaces, $1 \leq p \leq \infty$. To illustrate our main result we first resort to its simplest case.

Theorem 1.1. *Let* $M = (m_{j'j}) : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ *and* $w : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ *satisfies* $w(x) = o(x^{-1-\delta})$, $\delta > 0$.

1. *If* $|m_{j'j}| < w(\lambda|j'| - |j|)$ *for* $\lambda|j'| - |j| > 0$ *and* $\lambda > 1$, *then* M *has no bounded left inverses.*
2. *If* $|m_{j'j}| < w(-\lambda|j'| + |j|)$ *for* $-\lambda|j'| + |j| > 0$ *and* $\lambda < 1$, *then* M *has no bounded right inverses.*

Note that slanted matrices as covered in [1] and in the wavelets literature [2, 3, 4, 5], decay off slanted diagonals, that is, $|m_{j',j}|$ small if $\|\lambda j' - j\|_\infty$ large. Since $\|\lambda j' - j\|_\infty \geq \big|\lambda\|j'\|_\infty - \|j\|_\infty\big|$, the results in Section 2 apply in the setting of slanted matrices as well.

After stating and proving our main result as Theorem 2.1 in Section 2, we illustrate its usefulness in Section 3 by applying it in the area of time–frequency analysis. First, Theorem 2.1 is used to obtain elementary proofs of density theorems for Banach frames of Gabor systems and of time–frequency molecules in so-called modulation spaces [6, 7]. Second, we discuss how special cases of Theorem 2.1 have been used to give necessary conditions on the identifiability of pseudodifferential operators which are characterized by a bandlimitation of the operators' Kohn–Nirenberg symbols [8, 9, 10]. The background on time–frequency analysis that is used throughout Section 3 is given in Section 3.1.

2. NON–INVERTIBILITY OF "RECTANGULAR" BI-INFINITE MATRICES

Let $l^p_s(\mathbb{Z}^d)$, $1 \leq p \leq \infty$, $s \in \mathbb{R}$, be the weighted l^p-space with norm $\|\{x_j\}\|_{l^p_s} = \|\,\{(1 + \|j\|_\infty)^s\, x_j\}\,\|_{l^p}$, where $\|\{x_j\}\|_p = \left(\sum_j |x_j|^p\right)^{\frac{1}{p}}$ and $\|\{x_j\}\|_\infty = \sup_j |x_j|$.

Theorem 2.1. *Let* $1 \leq p_1, p_2, q_1, q_2 \leq \infty$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $\frac{1}{p_2} + \frac{1}{q_2} = 1$, $r_1, r_2, s_1, s_2 \in \mathbb{R}$, *and* $M = (m_{j'j}) : l^{p_1}_{s_1}(\mathbb{Z}^d) \to l^{p_2}_{s_2}(\mathbb{Z}^d)$.

1. *If there exists a* $\delta \geq 0$ *with* $r_1 - s_1 + \delta > 0$ *and* $\frac{d}{p_2} + r_1 + r_2 - s_1 + s_2 + \delta > 0$, *and if there exists* $\lambda > 1$, $K_0 > 0$, *and a function* $w : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ *with* $w(x) = o\left(x^{-(\frac{1}{q_1} + \frac{1}{p_2})d - r_1 - r_2 + s_1 - s_2 - \delta}\right)$ *and*

$$|m_{j'j}| \leq w(\lambda\|j'\|_\infty - \|j\|_\infty)\ (1 + \|j\|_\infty)^{r_1}\ (1 + \|j'\|_\infty)^{r_2}, \quad \lambda\|j'\|_\infty - \|j\|_\infty > K_0,$$

   then M *has no bounded left inverses.*

2. *If there exists a* $\delta \geq 0$ *with* $r_2 - s_2 + \delta > 0$ *and* $\frac{d}{p_1} + r_1 + r_2 + s_1 - s_2 + \delta > 0$ *and if there exists* $0 < \lambda < 1$, $K_0 > 0$ *and a function* $w : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ *with* $w(x) = o\left(x^{-(\frac{1}{p_1}+\frac{1}{q_2})d-r_1-r_2-s_1+s_2+\delta}\right)$ *and*

$$|m_{j'j}| \leq w(-\lambda\|j'\|_\infty + \|j\|_\infty)\,(1+\|j\|_\infty)^{r_1}\,(1+\|j'\|_\infty)^{r_2}, \quad -\lambda\|j'\|_\infty + \|j\|_\infty > K_0,$$

$\lambda, K_0 > 0$, *then* M *has no bounded right inverses.*

Clearly, Theorem 1.1 is Theorem 2.1 for $r_1 = r_2 = s_1 = s_2 = 0$, $p_1 = q_1 = p_2 = q_2 = 2$, and $d = 1$. Theorem 2.1 is a direct consequence of

LEMMA 2.2. *Let* $1 \leq p_1, q_1, p_2 \leq \infty$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$, *and* $M = (m_{j'j}) : l^{p_1}(\mathbb{Z}^d) \to l^{p_2}(\mathbb{Z}^d)$. *If there exists a function* $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *with* $w(x) = o\left(x^{-(\frac{1}{q_1}+\frac{1}{p_2})d-r_1-r_2-\delta}\right)$ *satisfying*

$$|m_{j'j}| \leq w(\lambda\|j'\|_\infty - \|j\|_\infty)\,(1+\|j\|_\infty)^{r_1}\,(1+\|j'\|_\infty)^{r_2}, \quad \lambda\|j'\|_\infty - \|j\|_\infty > K_0\,,$$

for some constants $\lambda, K_0, r_1, r_2, \delta$, *with* $\lambda, K_0 > 1$, $\delta \geq 0$, $r_1 + \delta > 0$, *and* $\frac{d}{p_2} + r_1 + r_2 + \delta > 0$, *then* M *has no bounded left inverses.*

Proof. We begin with the case $p_1 > 1$, $p_2 < \infty$ and show that if $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfies $w(x) = o\left(x^{-(\frac{1}{q_1}+\frac{1}{p_2})d-r_1-r_2-\delta}\right)$, $\delta \geq 0$, $r_1 + \delta > 0$ and $\frac{d}{p_2} + r_1 + r_2 + \delta > 0$, then

$$A_{K_1} = K_1^{p_2 r_1} \sum_{K \geq K_1} K^{p_2 r_2 + d - 1} \left(\sum_{k \geq K} k^{d-1}\, w(k)^{q_1} \right)^{\frac{p_2}{q_1}} \to 0 \text{ as } K_1 \to \infty. \tag{1}$$

We set $\widetilde{w}(x) = \sup_{y \leq x} w(y) \in o\left(x^{-(\frac{1}{q_1}+\frac{1}{p_2})d-r_1-r_2-\delta}\right)$ and $v \in C_0(\mathbb{R}^+)$ with $\widetilde{w}(x) \leq v(x)\, x^{-(\frac{1}{q_1}+\frac{1}{p_2})d-r_1-r_2-\delta}$. Then

$$\begin{aligned}
\sum_{K \geq K_1+2} K^{p_2 r_2+d-1} \left(\sum_{k \geq K} k^{d-1}\, w(k)^{q_1} \right)^{\frac{p_2}{q_1}} &\leq \sum_{K \geq K_1+1} K^{p_2 r_2+d-1} \left(\sum_{k \geq K+1} k^{d-1}\, \widetilde{w}(k)^{q_1} \right)^{\frac{p_2}{q_1}} \\
&\leq \int_{K_1}^\infty x^{p_2 r_2+d-1} \left(\int_x^\infty y^{d-1}\, \widetilde{w}(y)^{q_1}\, dy \right)^{\frac{p_2}{q_1}} dx \\
&\leq \int_{K_1}^\infty x^{p_2 r_2+d-1} \left(\int_x^\infty v(y)^{q_1} y^{-1-\frac{q_1}{p_2}d-q_1 r_2-q_1 r_1-q_1\delta}\, dy \right)^{\frac{p_2}{q_1}} dx \\
&\leq \frac{\|v|_{[K_1,\infty)}\|_\infty^{p_2}}{\frac{q_1}{p_2}d + q_1 r_2 + q_1 r_1 + q_1\delta} \int_{K_1}^\infty x^{p_2 r_2+d-1}\, x^{-d-p_2 r_2-p_2 r_1-p_2\delta}\, dx \\
&\leq \frac{\|v|_{[K_1,\infty)}\|_\infty^{p_2}}{(r_1+\delta)(q_1 d + p_2 q_1 r_2 + p_2 q_1 r_1 + p_2 q_1 \delta)} K_1^{-p_2 r_1 - p_2\delta} = o(K_1^{-p_2 r_1}),
\end{aligned}$$

since $\|v|_{[K_1,\infty)}\|_\infty \to 0$ as $K_1 \to \infty$ and (1) follows.

To show that $\inf_{x\in l_0(\mathbb{Z}^d)}\{\frac{\|Mx\|_{l^{p_2}}}{\|x\|_{l^{p_1}}}\} = 0$, we fix $\epsilon > 0$ and note that (1) provides us with a $K_1 > K_0$ satisfying $A_{K_1} \leq (2^d d)^{-\frac{p_2}{q_1}-1} 2^{-p_2 r_2} \left(\frac{\lambda-1}{\lambda}\right)^{p_2 r_1} \epsilon^{p_2}$.

Set $N = \left\lceil \frac{\lambda(K_1+1)}{\lambda-1} \right\rceil$ and $\widetilde{N} = \lceil \frac{N}{\lambda} \rceil + K_1$. Then $\frac{\lambda(K_1+1)}{\lambda-1} \leq N \leq \frac{\lambda(K_1+2)}{\lambda-1}$ implies $\lambda N \geq \lambda K_1 + \lambda + N$ and $N \geq K_1 + \frac{N}{\lambda} + 1 > K_1 + \lceil \frac{N}{\lambda} \rceil = \widetilde{N}$. Therefore, $(2\widetilde{N}+1)^d < (2N+1)^d$ and the matrix $\widetilde{M} = (m_{j'j})_{\|j'\|_\infty \leq \widetilde{N}, \|j\| \leq N} : \mathbb{C}^{(2N+1)^d} \longrightarrow \mathbb{C}^{(2\widetilde{N}+1)^d}$ has a nontrivial kernel. We now choose $\widetilde{x} \in \mathbb{C}^{(2N+1)^d}$ with $\|\widetilde{x}\|_{p_1} = 1$ and $\widetilde{M}\widetilde{x} = 0$ and define $x \in l_0(\mathbb{Z}^2)$ according to $x_j = \widetilde{x}_j$ if $\|j\|_\infty \leq N$ and $x_j = 0$ otherwise.

By construction, we have $\|x\|_{l^{p_1}} = 1$, and $(Mx)_{j'} = 0$ for $\|j'\|_\infty \leq \widetilde{N}$. To estimate $(Mx)_{j'}$ for $\|j'\|_\infty > \widetilde{N}$, we fix $K > K_1$ and one of the $2d(2(\lceil \frac{N}{\lambda} \rceil + K))^{d-1}$ indices $j' \in \mathbb{Z}^d$ with $\|j'\|_\infty = \lceil \frac{N}{\lambda} \rceil + K$. We have $\|\lambda j'\|_\infty \geq N + K\lambda$ and $\lambda\|j'\|_\infty - \|j\|_\infty \geq K\lambda \geq K$ for all $j \in \mathbb{Z}^d$ with $\|j\|_\infty \leq N$. Therefore

$$\begin{aligned}
|(Mx)_{j'}|^{q_1} &= \Big| \sum_{\|j\|_\infty \leq N} m_{j'j} x_j \Big|^{q_1} \leq \|x\|_{p_1}^{q_1} \sum_{\|j\|_\infty \leq N} |m_{j'j}|^{q_1} \\
&\leq (1+\|j'\|_\infty)^{q_1 r_2} \sum_{\|j\|_\infty \leq N} (1+\|j\|_\infty)^{q_1 r_1} w(\lambda\|j'\|_\infty - \|j\|_\infty)^{q_1} \\
&\leq (1+\|j'\|_\infty)^{q_1 r_2} (N+1)^{q_1 r_1} \sum_{\|j\|_\infty \geq K} w(\|j\|_\infty)^{q_1} \\
&= 2^d d\, (1+\|j'\|_\infty)^{q_1 r_2} (N+1)^{q_1 r_1} \sum_{k \geq K} k^{d-1} w(k)^{q_1}.
\end{aligned}$$

Finally, we compute

$$\|Mx\|_{l^{p_2}}^{p_2} = \sum_{j'\in\mathbb{Z}^d} |(Mx)_{j'}|^{p_2} = \sum_{\|j'\|_\infty \geq \lceil \frac{N}{\lambda}\rceil + K_1} |(Mx)_{j'}|^{p_2}$$

$$\begin{aligned}
&\leq (2^d d)^{\frac{p_2}{q_1}} \sum_{\|j'\|_\infty \geq \lceil \frac{N}{\lambda}\rceil + K_1} (1+\|j'\|_\infty)^{p_2 r_2} (N+1)^{p_2 r_1} \left(\sum_{k\geq \|j'\|_\infty} k^{d-1}\, w(k)^{q_1} \right)^{\frac{p_2}{q_1}} \\
&\leq (2^d d)^{\frac{p_2}{q_1}} (N+1)^{p_2 r_1} \sum_{K \geq \lceil \frac{N}{\lambda}\rceil + K_1} 2d(2K)^{d-1} (K+1)^{p_2 r_2} \left(\sum_{k\geq K} k^{d-1}\, w(k)^{q_1} \right)^{\frac{p_2}{q_1}} \\
&\leq (2^d d)^{\frac{p_2}{q_1}+1} 2^{p_2 r_2} \left(\frac{\lambda(K_1+2)}{\lambda-1} + 1 \right)^{p_2 r_1} \sum_{K \geq \lceil \frac{N}{\lambda}\rceil + K_1} K^{p_2 r_2 + d - 1} \left(\sum_{k\geq K} k^{d-1}\, w(k)^{q_1} \right)^{\frac{p_2}{q_1}} \\
&\leq (2^d d)^{\frac{p_2}{q_1}+1} 2^{p_2 r_2} \left(\frac{\lambda}{\lambda-1} \right)^{p_2 r_1} (K_1+3)^{p_2 r_1} \sum_{K \geq \lceil \frac{N}{\lambda}\rceil + K_1} K^{p_2 r_2 + d - 1} \left(\sum_{k\geq K} k^{d-1}\, w(k)^{q_1} \right)^{\frac{p_2}{q_1}} \\
&\leq \epsilon^{p_2},
\end{aligned}$$

that is, $\|Mx\|_{l^{p_2}} \leq \epsilon$. Since ϵ was chosen arbitrarily and $\|x\|_{l^{p_1}} = 1$, we have $\inf_{x\in l_0(\mathbb{Z}^2)}\{\frac{\|Mx\|_{l^{p_2}}}{\|x\|_{l^{p_1}}}\} = 0$ and M is not bounded below and has no bounded left inverses.

The cases $p_1 = 1$ and/or $p_2 = \infty$ follow similarly. $\square$

Proof of Theorem 2.1.

Part 1. Let $M = (m_{j'j}) : l^{p_1}_{s_1}(\mathbb{Z}^d) \to l^{p_2}_{s_2}(\mathbb{Z}^d)$ satisfy the hypothesis of Theorem 2.1, *part 1*. Suppose that, nonetheless, $M = (m_{j'j}) : l^{p_1}_{s_1}(\mathbb{Z}^d) \to l^{p_2}_{s_2}(\mathbb{Z}^d)$ has a bounded left inverse. This clearly implies that

$$\widetilde{M} = (\widetilde{m}_{j'j}) = \big(\ m_{j'j}\,(1+\|j'\|_\infty)^{s_2}\,(1+\|j\|_\infty)^{-s_1}\ \big) : l^{p_1}(\mathbb{Z}^d) \to l^{p_2}(\mathbb{Z}^d)$$

has a bounded left inverse which contradicts Theorem 2.2, since for $\lambda\|j'\|_\infty - \|j\|_\infty > K_0$, we have

$$\begin{aligned}
|\widetilde{m}_{j'j}| &= \left| m_{j'j}(1+\|j'\|_\infty)^{s_2}\,(1+\|j\|_\infty)^{-s_1}) \right| \\
&\leq w(\lambda\|j'\|_\infty - \|j\|_\infty)\ (1+\|j\|_\infty)^{r_1-s_1}\ (1+\|j'\|_\infty)^{r_2+s_2}
\end{aligned}$$

with $\delta \geq 0$, $r_1 - s_1 + \delta > 0$, $\frac{d}{p_2} + r_1 + r_2 - s_1 + s_2 + \delta > 0$, and
$w(x) = o\left(x^{-(\frac{1}{q_1}+\frac{1}{p_2})d - r_1 - r_2 + s_1 - s_2 - \delta} \right)$.

Part 2. The matrix $M : l^{p_1}_{s_1}(\mathbb{Z}^d) \to l^{p_2}_{s_2}(\mathbb{Z}^d)$ has a bounded right inverse if and only if its adjoint $M^* : l^{p_2}_{s_2}(\mathbb{Z}^d) \to l^{p_1}_{s_1}(\mathbb{Z}^d)$ has a bounded left inverse. The conditions on M in Theorem 2.1, *part 2* are equivalent to the conditions on M^* in Theorem 2.1, *part 1.* The result follows. $\square$

3. APPLICATIONS

Before stating applications of Theorem 2.1, we give a brief account of the concepts from time–frequency analysis that appear in this section. For additional background on time–frequency analysis and, in particular, Gabor frames, see [11].

3.1. Time–frequency analysis and Gabor frames

The Fourier transform of a function $f \in L^1(\mathbb{R}^d)$, is given by $\widehat{f}(\gamma) = \int f(x)e^{-2\pi i x\cdot\gamma}\,dx$, $\gamma \in \widehat{\mathbb{R}}^d$, where $\widehat{\mathbb{R}}^d$ is the dual group of $\mathbb{R}^d$, and which, aside of notation, equals $\mathbb{R}^d$. The Fourier transform can be extended to act unitarily on $L^2(\mathbb{R}^d)$ and isomorphically on the dual space of Schwarz class functions $\mathcal{S}(\mathbb{R}^d)$, that is, on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d) \supset \mathcal{S}(\mathbb{R}^d)$.

The *translation operators* $T_y : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d)$, $y \in \mathbb{R}^d$, is given by $(T_y f)x = f(x{-}y)$, $x \in \mathbb{R}^d$, and the *modulation operator* $M_\xi : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d)$ is given by $(M_\xi f)x = e^{2\pi i x\xi} f(x)$, $x \in \mathbb{R}^d$. Both extend isomorphically to $\mathcal{S}'(\mathbb{R}^d)$, and so do their compositions, the so-called *time–frequency shifts* $\pi(z) = \pi(y,\xi) = T_y M_\xi$, $z = (y,\xi) \in \mathbb{R}^d{\times}\widehat{\mathbb{R}}^d$. Note that the adjoint operator $\pi(z)^*$ of $\pi(z) = \pi(y,\xi)$ is $\pi(z)^* = e^{2\pi i y\xi}\pi(-z)$.

The *short–time Fourier transform* $V_g f$ of $f \in L^2(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d)$ with respect to a window function $g \in L^2(\mathbb{R}^d) \setminus \{0\}$ is

$$V_g f(z) = \langle f, \pi(z)g\rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x-y)}\, e^{-2\pi i (x-y)\cdot\xi}\,dx, \quad z = (y,\xi) \in \mathbb{R}^d{\times}\widehat{\mathbb{R}}^d.$$

We have $V_g f \in L^2(\mathbb{R}^d{\times}\widehat{\mathbb{R}}^d)$ and $\|V_g f\|_{L^2} = \|f\|_{L^2}\|g\|_{L^2}$.

A central goal in Gabor analysis is to find $g \in L^2(\mathbb{R}^d)$ and *full rank lattices* $\Lambda = A\mathbb{Z}^{2d} \subset \mathbb{R}^d{\times}\widehat{\mathbb{R}}^d$, $A \in \mathbb{R}^{2d\times 2d}$ full rank, which allow the discretization of the formula $\|V_g f\|_{L^2} = \|f\|_{L^2}\|g\|_{L^2}$ in the following sense: for which $g \in L^2(\mathbb{R}^d)$ and full rank lattices Λ exists $A, B > 0$ with

$$A\|f\|^2_{L^2} \le \sum_{z\in\Lambda} |V_g f(z)|^2 \le B\|f\|^2_{L^2}, \quad f \in L^2(\mathbb{R}^d)\,. \tag{2}$$

If (2) is satisfied, then $(g,\Lambda) = \{\pi(z)g\}_{z\in\Lambda}$ is called Gabor frame for the Hilbert space $L^2(\mathbb{R}^d)$. More recently, the question above has been considered for general sequences Γ in $\mathbb{R}^d{\times}\widehat{\mathbb{R}}^d$ in place of full rank lattice Λ [12, 13, 14].

To generalize (2) to Banach spaces, we adopt the definition of p-frames from [15].

Definition 3.1. *The Banach space valued sequence $\{g_j\}_{j\in\mathbb{Z}^d} \subseteq X'$, $d \in \mathbb{N}$, is an l^p_s–frame for the Banach space X, $1 \leq p \leq \infty$, $s \in \mathbb{R}$, if the analysis operator $C_{\mathcal{F}} : X \longrightarrow l^p_s(\mathbb{Z}^d)$, $\quad f \mapsto \{\langle f, g_j\rangle\}_j$ is bounded and bounded below, that is, if there exists $A, B > 0$ with*

$$A\|f\|_X \leq \|\{\langle f, g_j\rangle\}\|_{l^p_s} \leq B\|f\|_X, \quad f \in X\,. \tag{3}$$

Note that in the Hilbert space case $X = L^2(\mathbb{R}^d)$ and $l^p_s(\mathbb{Z}^{2d}) = l^2(\mathbb{Z}^{2d})$, (2) implies that $C_{\mathcal{F}}$ has a bounded left inverse, while in the Banach space case (3) does not provide us with a left inverse. Therefore, the existence of a bounded left inverse for $C_{\mathcal{F}}$ is included in the definition of the standard generalization of frames to Banach spaces [16, 17, 18].

Analogously to Definition 3.1, we include a generalization of Riesz bases in the Banach space setting.

Definition 3.2. *A sequence $\{g_j\}_{j\in\mathbb{Z}^d} \subseteq X$, $d \in \mathbb{N}$ is called l^p_s–Riesz basis in the Banach space X, $1 \leq p \leq \infty$, $s \in \mathbb{R}$, if the synthesis operator $D_{\{g_j\}_j} : l^p_s(\mathbb{Z}^{2d}) \longrightarrow X$, $\quad \{c_j\}_j \mapsto \sum_j c_j g_j$ is bounded and bounded below, that is, if there is $A, B > 0$ with*

$$A\|\{c_j\}_j\|_{l^p_s} \leq \|\sum_j c_j g_j\|_X \leq B\|\{c_j\}_j\|_{l^p_s}, \quad \{c_j\}_j \in l^p_s(\mathbb{Z}^d).$$

The Banach spaces of interest here are the so–called modulation spaces [19, 20, 21]. Clearly, $V_g f(z) = \langle f, \pi(z)g\rangle$, $z \in \mathbb{R}^d\times\widehat{\mathbb{R}}^d$ is well defined whenever $g \in \mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{S}'(\mathbb{R}^d)$ (or vice versa). This together with $\|V_g f\|_{L^2} = \|g\|_{L^2}\|f\|_{L^2}$ in the L^2–theory motivates the following. We let $g = \mathfrak{g} \in \mathcal{S}(\mathbb{R}^d)$ be an L^2–normalized Gaussian, that is, $\mathfrak{g}(x) = 2^{\frac{d}{4}}\, e^{-\pi\|x\|_2^2}$, $x \in \mathbb{R}^d$, and define the *modulation space* $M^p_s(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 \leq p \leq \infty$, by

$$M^p_s(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) :\ V_{\mathfrak{g}} f \in L^p_s(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)\}$$

with Banach space norm

$$\|f\|_{M^p_s} = \|V_{\mathfrak{g}} f\|_{L^p_s} = \left(\int \big|\,(1+\|z\|)^s\, V_{\mathfrak{g}} f(z)\,\big|^p\, dz\right)^{\frac{1}{p}} < \infty\,, \quad 1 \leq p < \infty,$$

and the usual adjustment for $p = \infty$.

Example 3.3. For $\lambda < 1$, $(\mathfrak{g}, \lambda\mathbb{Z}^{2d})$ is an l^2–frame for $L^2(\mathbb{R}^d)$ [22, 23]. Since $\mathfrak{g} \in \mathcal{S}(\mathbb{R}^d) \subset M^1_t(\mathbb{R}^d)$ for all $t \geq 0$, Theorem 20 in [14] implies that in this case $(\mathfrak{g}, \lambda\mathbb{Z}^{2d})$ is an l^p_s–frames for $M^p_s(\mathbb{R}^d)$ for $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. The Wexler-Raz identity implies that for $\lambda > 1$, $(\mathfrak{g},\, \lambda\mathbb{Z}^{2d})$ is an l^2–Riesz basis in $L^2(\mathbb{R}^d)$. Hence, $D_{(\mathfrak{g},\lambda\mathbb{Z}^{2d})} :\ l^2(\mathbb{Z}^{2d}) \longrightarrow L^2(\mathbb{R}^d)$ has a bounded left inverse of the form $C_{(\widetilde{\mathfrak{g}},\lambda\mathbb{Z}^{2d})}$ where the so–called dual function $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ satisfies $\widetilde{\mathfrak{g}} \in \mathcal{S}(\mathbb{R}^d)$ [24]. The operator $C_{(\widetilde{\mathfrak{g}},\lambda\mathbb{Z}^{2d})}$ is a bounded operator mapping $M^p_s(\mathbb{R}^d)$ to $l^p_s(\mathbb{Z}^{2d})$. This implies that $D_{(\mathfrak{g},\lambda\mathbb{Z}^{2d})}$ has a left inverse and $(\mathfrak{g},\, \lambda\mathbb{Z}^{2d})$ is an l^p_s–Riesz basis in $M^p_s(\mathbb{R}^d)$ for $s \in \mathbb{R}$ and $1 \leq p \leq \infty$.

3.2. Density results for Gabor l^p_s–frames in modulation spaces

One of the central results in Gabor analysis is the fact that (g, Λ), $g \in L^2(\mathbb{R}^d)$, cannot be a frame for $L^2(\mathbb{R}^d)$ if the measure of a fundamental domain of the full rank lattice Λ is larger than 1 [25, 26, 27]. Generalizations of this result to general sequences Γ in $\mathbb{R}^d{\times}\widehat{\mathbb{R}}^d$ require an alternative definition of density [12, 28, 29].

Definition 3.4. *Let $Q_R = [-R, R]^{2d} \subseteq \mathbb{R}^d{\times}\widehat{\mathbb{R}}^d$ and let Γ be a sequence of points in $\mathbb{R}^d{\times}\widehat{\mathbb{R}}^d$. Then*

$$D^-(\Gamma) = \liminf_{R\to\infty} \inf_{z\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \frac{|\Gamma \cap Q_R{+}z|}{(2R)^{2d}} \quad \textit{and} \quad D^+(\Gamma) = \limsup_{R\to\infty} \sup_{z\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \frac{|\Gamma \cap Q_R{+}z|}{(2R)^{2d}}$$

are called lower and upper Beurling density of Γ. *If $D^+(\Gamma) = D^-(\Gamma)$, then Γ is said to have* uniform density $D(\Gamma) = D^+(\Gamma) = D^-(\Gamma)$.

Remark 3.5. The density of a sequence Γ does not equal the density of its range set. For example, the density of the sequence $\{\ldots, -2, -2, -1, -1, 0, 0, 1, 1, 2, 2, 3, 3, \ldots\}$ in $\mathbb{R}$ is 2, while the density of the range of the sequence, namely of $\mathbb{Z}$, is 1.

In [30], it was shown that if (g, Γ), $g \in L^2(\mathbb{R}^d)$, $\Gamma \subseteq \mathbb{R}^d{\times}\widehat{\mathbb{R}}^d$, is an l^2–frame for $L^2(\mathbb{R}^d) = M^2_0(\mathbb{R}^d)$, then $1 \leq D^-(\Gamma) \leq D^+(\Gamma) < \infty$, a result that has recently been refined by Theorem 3 and Theorem 5 in [13]. For l^p_s–frames for $M^p_s(\mathbb{R}^d)$, Theorem 2.1 implies

Theorem 3.6. *Let $1 \leq p \leq \infty$, $s \in \mathbb{R}$, and $g \in M^\infty_{2d}$ if $s < 0$ and $p \neq \infty$ and $g \in M^\infty_{2d+\delta}$, $\delta > s, 0$ else. If (g, Γ) is an l^p_s–frame for $M^p_s(\mathbb{R}^d)$, then $D^+(\Gamma) \geq 1$.*

Proof. Let Γ be given with $D^+(\Gamma) < 1$. We choose $\lambda > 1$ with $1 > \lambda^{-4d} > D^+(\Gamma)$ and $R_0 > 0$ with

$$|\Gamma \cap Q_R| < \sup_{z\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |\Gamma \cap Q_R{+}z| < \lambda^{-4d}(2R)^{2d}, \quad R > R_0.$$

Since $D^+(\Gamma) < \infty$, the sequence Γ has no accumulation points and we can enumerate the sequence Γ by $\mathbb{Z}^{2d}$ so that $\|\gamma_{j'}\|_\infty \leq \|\gamma_{j''}\|_\infty$ implies $\|j'\|_\infty \leq \|j''\|_\infty$ for $j', j'' \in \mathbb{Z}^{2d}$. This gives,

$$\gamma_{j'} \notin Q_R \quad \text{if} \quad (\, 2\|j'\|_\infty - 1\,)^{2d} = (\, 2(\|j'\|_\infty - 1) + 1\,)^{2d} \geq \lambda^{-4d}(2R)^{2d}, \quad R > R_0,$$

$$\begin{tikzcd} M^p_s(\mathbb{R}^d) \arrow[r, "C_{(g,\Gamma)}"] & l^p_s(\mathbb{Z}^{2d}) \\ l^p_s(\mathbb{Z}^{2d}) \arrow[u, "D_{(\mathfrak{g},\,\lambda\mathbb{Z}^{2d})}"] \arrow[ur, "M"'] & \end{tikzcd} \qquad \begin{tikzcd} \sum c_j \pi(\lambda j)\mathfrak{g} \arrow[r, "C_{(g,\Gamma)}"] & \{\sum c_j \langle \pi(\lambda j), \pi(\gamma_{j'})g\rangle\}_{j'} \\ \{c_j\}_j \arrow[u, "D_{(\mathfrak{g},\,\lambda\mathbb{Z}^{2d})}"] \arrow[ur, "M"'] & \end{tikzcd}$$

Figure 1. Sketch of the proof of Theorem 3.6. We choose $\lambda > 1$ so that $(\mathfrak{g},\,\lambda\mathbb{Z}^{2d})$ is an l^p_s–Riesz basis in $M^p_s(\mathbb{R}^d)$, so $D_{(\mathfrak{g},\,\lambda\mathbb{Z}^{2d})}$ is bounded below. Theorem 2.1 applies to $M = C_{(g,\Gamma)} \circ D_{(\mathfrak{g},\,\lambda\mathbb{Z}^{2d})}$, showing that M is not bounded below. This implies that $C_{(g,\Gamma)}$ is not bounded below and has no bounded left inverses.

and, therefore,

$$\gamma_{j'} \notin Q_{\lambda^2\|j'\|_\infty - \frac{\lambda^2}{2}} \quad \text{for} \quad \lambda^2\|j'\|_\infty - \tfrac{\lambda^2}{2} > R_0. \tag{4}$$

We have

$$C_{(g,\Gamma)} \circ D_{(\mathfrak{g},\,\lambda\mathbb{Z}^{2d})} : l^p_s(\mathbb{Z}^{2d}) \longrightarrow l^p_s(\mathbb{Z}^{2d}), \quad \{c_j\}_j \mapsto \left\{ \sum_j c_j \langle \pi(\lambda j)h, \pi(\gamma_{j'})g\rangle \right\} = M\{c_j\}_j,$$

with $M = (m_{j'j})$ and $|m_{j'j}| = |\langle \pi(\lambda j)h, \pi(\gamma_{j'})g\rangle| = |V_g h(\gamma_{j'} - \lambda j)|$.

Note that (4) implies

$$\|\gamma_{j'} - \lambda j\|_\infty \geq \lambda^2\|j'\|_\infty - \tfrac{\lambda^2}{2} - \|\lambda j\|_\infty = \lambda\left(\lambda\|j'\|_\infty - \|j\|_\infty - \tfrac{\lambda}{2}\right),$$

and so

$$|m_{j'j}| = |\langle \pi(\lambda j)\mathfrak{g}, \pi(\gamma_{j'})g\rangle| = |V_g\mathfrak{g}(\gamma_{j'} - \lambda j)| \leq w(\lambda\|j'\|_\infty - \|j\|_\infty)$$

where

$$w(\|z\|) = (1 + \|z\|)^{-2d-\delta} \sup_{\widetilde{z}} \left((1 + \|\widetilde{z}\|)^{2d+\delta}\, |V_g\mathfrak{g}(\widetilde{z})|\right), \quad z \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

A direct application of Theorem 2.1 implies that $C_{(g,\Gamma)} \circ D_{(\mathfrak{g},\,\lambda\mathbb{Z}^{2d})}$ is not bounded below. Since $D_{(\mathfrak{g},\,\lambda\mathbb{Z}^{2d})}$ is bounded below, we conclude that $C_{(g,\Gamma)}$ is not bounded below which completes the proof. $\square$

Note that the last lines in the proof of Theorem 3.6 can be modified to apply to time–frequency molecules which we shall consider in the following. We say that a sequence $\{g_{j'}\}_{j'}$ of functions consist of at $\Gamma = \{\gamma_{j'}\}_{j'}$ (v, r_1, r_2)*–localized time–frequency molecules* if

$$|V_{\mathfrak{g}} g_{j'}(z)| \leq (1 + \|z\|_\infty)^{r_1} (1 + \|j'\|_\infty)^{r_2} w(\|z - \gamma_{j'}\|_\infty), \quad w = o(x^{-v}). \tag{5}$$

If (5) is satisfied for $r_1 = r_2 = 0$, then we simply speak of at Γ v–localized time–frequency molecules. Note that if $\{g_{j'}\}_{j'} \subseteq (M^p_s(\mathbb{R}^d))'$ is (v, r_1, r_2)–localized, then by definition $\{g_{j'}\}_{j'} \subseteq M^\infty_{v-r_1}(\mathbb{R}^d)$, and, consequently, if $v - r_1 > 2d$ we have $\{g_{j'}\}_{j'} \subseteq M^1(\mathbb{R}^d)$, a fact which we take into consideration when stating the hypothesis of Theorem 3.7 and Theorem 3.8

Related concepts of localization were introduced in [1, 14, 12, 13], partly to obtain density results and partly to describe the time–frequency localization of dual frames of irregular Gabor frames (see also Remark 3.10).

Theorem 3.7. *If $\{g_{j'}\}_{j'} \subseteq (M^p_s(\mathbb{R}^d))' \cap M^\infty_{v-r_1}$, $1 \leq p \leq \infty$, $s \in \mathbb{R}$ is an l^p_s–frame for $M^p_s(\mathbb{R}^d)$ which is (v, r_1, r_2)–localized at $\Gamma = \{\gamma_{j'}\}_{j'}$, with $\delta - s$, $v - r_1 - r_2 - 2d - \delta$, $r_1 + \frac{2d}{p} + \delta > 0$ and $\delta \geq 0$, then $D^+(\Gamma) \geq 1$.*

Note that Theorem 9 in [13] states that if $\{g_{j'}\}$ is an l^2–frame for $L^2(\mathbb{R}^d)$ which consists of at Γ $d + \delta$–localized time–frequency molecules, $\delta > 0$, then actually $1 \leq D^-(\Gamma)$. Below, we show that components of the proof of Theorem 2.2 can be used to obtain some of the density results given above with $D^+(\Gamma)$ being replaced by $D^-(\Gamma)$.

Theorem 3.8. *If $\{g_{j'}\}_{j'} \subseteq M^1(\mathbb{R}^d)$ is an l^p–frame for $M^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, which is $2d + \delta$–localized at $\Gamma = \{\gamma_{j'}\}_{j'}$ with $D^+(\Gamma) < \infty$ and $\delta > 0$, then $D^-(\Gamma) \geq 1$.*

Proof. Suppose that $\{g_{j'}\}_{j'}$ is an l^p_s–frame for $M^p(\mathbb{R}^d)$ which is $2d + \delta$–localized at $\Gamma = \{\gamma_{j'}\}_{j'}$, $D^-(\Gamma) < 1$. For z_0, α_3 chosen below, we shall consider the Gabor system $\{\pi(\alpha_3^{-1} j + z_0)\, \mathfrak{g}\}_{j \in \mathbb{Z}^{2d}}$ which is an l^p–Riesz basis for $M^p(\mathbb{R}^d)$. We shall show that $\{g_{j'}\}$ is not an l^p–frame by arguing that

$$\inf_{x \in l^p(\mathbb{Z}^d)} \frac{\|C_{\{g_{j'}\}} \circ D_{\{\pi(\alpha_3^{-1} j + z_0)\mathfrak{g}\}} x\|_{l^p}}{\|x\|_{l^p}} = 0.$$

To this end, fix $\epsilon > 0$. We first assume $1 < p < \infty$.

Since $D^+(\Gamma) < \infty$, there exists $\alpha_1 \geq 1$ and $\widetilde{R_0} \geq 1$ with $\infty > \alpha_1^{2d} > D^+(\Gamma) \geq 0$ and

$$|\Gamma \cap Q_R + z| \leq \alpha_1^{2d}\, (2R)^{2d}\,, \quad z \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,\ R \geq \widetilde{R_0}.$$

Further, we can pick $\alpha_2, \alpha_3 > \frac{1}{2}$ with $D^-(\Gamma) < \alpha_2^{2d} < \alpha_3^{2d} < 1$, and $n_0 \in \mathbb{N}$ with

$$\alpha_2 + \alpha_1 \left(\left(1 + \tfrac{1}{n_0}\right)^{2d} - 1 \right)^{-2d} < \alpha_3 \left(1 - \tfrac{1}{2n_0}\right)^{2d}.$$

We now choose a monotonically decreasing $w(x) = o(x^{-2d-\delta})$ with $|V_{\mathfrak{g}} g_{j'}(z)| \leq w(\|z - \gamma_{j'}\|_\infty)$. As demonstrated in the proof of Theorem 2.2, $w = o(x^{-2d-\delta})$, $\delta > 0$, allows us to pick $\widetilde{K}_2$ such that for all $K_2 \geq \widetilde{K}_2$

$$(2^{2d}2d)^{\frac{p}{q}+1} \sum_{K \geq K_2} K^{2d-1} \left(\sum_{k \geq \frac{\alpha_3}{2\alpha_1} K} k^{2d-1}\, w(k)^q \right)^{\frac{p}{q}} < \epsilon^p\,.$$

Also, there exists R_0, $N_0 = \lceil \alpha_3 R_0 \rceil$, such that

- there exists $z_0 \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ with $|Q_{R_0} + z_0 \cap \Gamma| \leq \alpha_2^{2d} (2R_0)^{2d}$;
- $R_0 \geq \widetilde{R_0}\, n_0$; $\quad N_0 \geq n_0, \frac{\alpha_1}{\alpha_2} \widetilde{R_0}$;
- $(5 \frac{\alpha_1}{\alpha_3} R_0)^{2d}\, w\left(\frac{R_0}{n_0} - 2\right) < \epsilon$;
- $K_1 = N_0 - 1 - \lceil \alpha_2 N_0 \rceil > 1$;
- $K_2 = 2\left(\frac{\alpha_1}{\alpha_3} N_0 - \lceil \alpha_2 N_0 \rceil\right) \geq \widetilde{K}_2, K_1\,.$

The sequence Γ has no accumulation point since $D^+(\Gamma) < \infty$ which implies that we can choose an enumeration of the sequence Γ by $\mathbb{Z}^{2d}$ with $\|j'\|_\infty \leq \|j''\|_\infty$ if $\|\gamma_{j'} - z_0\|_\infty \leq \|\gamma_{j''} - z_0\|_\infty$, $j', j'' \in \mathbb{Z}^{2d}$. As mentioned earlier, we set $\mathfrak{g}_j = \pi\left(\alpha_3^{-1} j + z_0\right) \mathfrak{g}$ for $j \in \mathbb{Z}^{2d}$, and $M = (m_{j'j}) = (\langle g_{j'}, \mathfrak{g}_j \rangle)$.

The matrix $\widetilde{M} = (m_{j'j})_{\|j'\|_\infty \leq N_0 - 1,\, \|j\| \leq N_0} : \mathbb{C}^{(2N_0+1)^d} \to \mathbb{C}^{(2N_0-1)^d}$ has a nontrivial kernel, so we may choose $\widetilde{x} \in \mathbb{C}^{(2N_0+1)^d}$ with $\|\widetilde{x}\|_p = 1$ and $\widetilde{M}\widetilde{x} = 0$ and define $x \in l_0(\mathbb{Z}^2)$ according to $x_j = \widetilde{x}_j$ if $\|j\|_\infty \leq N_0$ and $x_j = 0$ otherwise.

To estimate the contributions of $|(Mx)_{j'}|$ for $j' \in \mathbb{Z}^{2d}$ to $\|Mx\|_{l^p}$, we consider three cases.

Case 1. $\|j'\|_\infty \leq \lceil \alpha_2 N_0 \rceil + K_1 = N_0 - 1.$ This implies $(Mx)_{j'} = 0$ by construction.

Case 2. $\lceil \alpha_2 N_0 \rceil + K_1 < \|j'\|_\infty \le \lceil \alpha_2 N_0 \rceil + K_2$. Observe that the set $Q_{R_0+\frac{R_0}{n_0}}+z_0 \setminus Q_{R_0}+z_0$ consists of a finite number of hypercubes of width $\frac{R_0}{n_0} \ge \widetilde{R_0}$, so we can estimate

$$
\begin{aligned}
|Q_{R_0+\frac{R_0}{n_0}}+z_0 \cap \Gamma| &\le \alpha_2^{2d}(2R_0)^{2d} + \alpha_1^{2d}\left(\left(2\left(R_0+\tfrac{R_0}{n_0}\right)\right)^{2d} - (2R_0)^{2d}\right) \\
&\le (2R_0)^{2d}\left(\alpha_2^{2d} + \alpha_1^{2d}\left(\left(1+\tfrac{1}{n_0}\right)^{2d} - 1\right)\right) \\
&\le (2\alpha_3^{-1}N_0)^{2d}\alpha_3^{2d}\left(1-\tfrac{1}{2n_0}\right)^{2d} \\
&\le \left(2N_0 - \tfrac{2N_0}{2n_0}\right)^{2d} \le (2N_0-1)^{2d}
\end{aligned}
$$

Hence, for any j' with $\|j'\|_\infty \ge N_0 = \lceil \alpha_2 N_0 \rceil + K_1 + 1$, we have $\gamma_j' \notin Q_{R_0+\frac{R_0}{n_0}}+z_0$ and, therefore, for $\|j\|_\infty \le N_0 = \lceil \alpha_3 R_0 \rceil$ we have

$$\|\alpha_3^{-1}j+z_0-\gamma_{j'}\|_\infty = \|(\gamma_{j'}-z_0)-\alpha_3^{-1}j\|_\infty \ge R_0+\tfrac{R_0}{n_0}-\alpha_3^{-1}\lceil \alpha_3 R_0\rceil \ge \tfrac{R_0}{n_0}-\alpha_3^{-1} \ge \tfrac{R_0}{n_0}-2,$$

and, therefore,

$$|m_{j'j}| = |\langle g_{j'}, \mathfrak{g}_j\rangle| = |V_{\mathfrak{g}} g_{j'}(\alpha_3^{-1}j+z_0)| \le w\left(\|\alpha_3^{-1}j+z_0-\gamma_{j'}\|_\infty\right) \le w\left(\tfrac{R_0}{n_0}-2\right).$$

This gives

$$
\begin{aligned}
&\|Mx|_{\{j' \colon \lceil \alpha_2 N_0\rceil + K_1 < \|j'\|_\infty \le \lceil \alpha_2 N_0 \rceil + K_2\}}\|_p^p \\
&\quad = \sum_{\lceil \alpha_2 N_0\rceil + K_1 < \|j'\|_\infty \le \lceil \alpha_2 N_0 \rceil + K_2} \Big|\sum_{\|j\|_\infty \le N_0} m_{j'j}x_j\Big|^p \\
&\quad \le \sum_{\lceil \alpha_2 N_0\rceil + K_1 < \|j'\|_\infty \le \lceil \alpha_2 N_0 \rceil + K_2} \left(\sum_{\|j\|_\infty \le N_0} |m_{j'j}|^q\right)^{\frac{p}{q}} \|\widetilde{x}\|_p^p \\
&\quad \le w\left(\tfrac{R_0}{n_0}-2\right)^p \sum_{\lceil \alpha_2 N_0\rceil + K_1 < \|j'\|_\infty \le \lceil \alpha_2 N_0 \rceil + K_2} (2N_0+1)^{2d\frac{p}{q}} \sum_{\|j\|_\infty \le N_0} \big|x_j\big|^p \\
&\quad \le w\left(\tfrac{R_0}{n_0}-2\right)^p (2\cdot 2\tfrac{\alpha_1}{\alpha_3}N_0+1)^{2d}(2N_0+1)^{2d\frac{p}{q}} \\
&\quad \le w\left(\tfrac{R_0}{n_0}-2\right)^p (5\tfrac{\alpha_1}{\alpha_3}R_0)^{2d(1+\frac{p}{q})} \le \epsilon^p
\end{aligned} \tag{6}
$$

Case 3. $\lceil \alpha_2 N_0 \rceil + K_2 < \|j'\|_\infty$. For such j', we set $N = \|j'\|_\infty$ and obtain $\alpha_1^{-1}(N-\frac{1}{2}) \ge \alpha_1^{-1}(\lceil \alpha_2 N_0\rceil + K_2 + 1 - \frac{1}{2}) \ge \frac{\alpha_2}{\alpha_1}N_0 \ge \widetilde{R_0}$, and, hence,

$$|\Gamma \cap Q_{\alpha_1^{-1}(N-\frac{1}{2})} + z_0| \le \alpha_1^{2d}(2\alpha_1^{-1}(N-\tfrac{1}{2}))^{2d} = (2N-1)^{2d}.$$

This implies $\gamma_{j'} \notin Q_{\alpha_1^{-1}(\|j'\|_\infty - \frac{1}{2})} + z_0$. Similarly as in *Case 2.*, we fix j', K with $\|j'\|_\infty = \lceil \alpha_2 N_0 \rceil + K$, $K > K_2$, and conclude that for $\|j\|_\infty \leq N_0$,

$$\begin{aligned}
\|\alpha_3^{-1} j + z_0 - \gamma_{j'}\|_\infty &= \|(\gamma_{j'} - z_0) - \alpha_3^{-1} j\|_\infty \geq \alpha_1^{-1}(\|j'\|_\infty - \tfrac{1}{2}) - \alpha_3^{-1}\|j\|_\infty \\
&\geq \frac{\alpha_3}{\alpha_1}\|j'\|_\infty - \|j\|_\infty - \frac{\alpha_3}{2\alpha_1} \\
&\geq \frac{\alpha_3}{\alpha_1}\lceil \alpha_2 N_0 \rceil + 2\frac{\alpha_3}{2\alpha_1}K - N_0 - \frac{\alpha_3}{2\alpha_1} \\
&\geq \frac{\alpha_3}{2\alpha_1}\left(K - 2\left(\frac{\alpha_1}{\alpha_3}N_0 - \lceil \alpha_2 N_0 \rceil\right) - 1\right) + \frac{\alpha_3}{2\alpha_1}K \geq \frac{\alpha_3}{2\alpha_1}K.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|(Mx)_{j'}|^q &= \Big|\sum_{\|j\|_\infty \leq N_0} m_{j'j} x_j\Big|^q \leq \|x\|_p^q \sum_{\|j\|_\infty \leq N_0} |m_{j'j}|^q \\
&\leq \sum_{\|j\|_\infty \leq N_0} w\left(\frac{\alpha_3}{\alpha_1}\|j'\|_\infty - \|j\|_\infty - \frac{\alpha_3}{2\alpha_1}\right)^q \\
&\leq \sum_{\|j\|_\infty \geq \frac{\alpha_3}{2\alpha_1}K} w(\|j\|_\infty)^q = \sum_{k \geq \frac{\alpha_3}{2\alpha_1}K} 2(2d)(2k)^{2d-1} w(k)^q \\
&= 2^{2d} 2d \sum_{k \geq \frac{\alpha_3}{2\alpha_1}K} k^{2d-1} w(k)^q.
\end{aligned}$$

Finally, we compute

$$\begin{aligned}
\sum_{\|j'\|_\infty > \lceil \alpha_2 N_0 \rceil + K_2} |(Mx)_{j'}|^p &\leq (2^{2d}2d)^{\frac{p}{q}} \sum_{\|j'\|_\infty \geq \lceil \alpha_2 N_0 \rceil + K_2} \left(\sum_{k \geq \frac{\alpha_3}{2\alpha_1}\|j'\|_\infty} k^{2d-1} w(k)^q\right)^{\frac{p}{q}} \\
&\leq (2^{2d}2d)^{\frac{p}{q}} \sum_{K \geq \lceil \alpha_2 N_0 \rceil + K_2} 2(2d)(2K)^{2d-1} \left(\sum_{k \geq \frac{\alpha_3}{2\alpha_1}K} k^{2d-1}\, w(k)^q\right)^{\frac{p}{q}} \\
&\leq (2^{2d}2d)^{\frac{p}{q}+1} \sum_{K \geq \lceil \alpha_2 N_0 \rceil + K_2} K^{2d-1} \left(\sum_{k \geq \frac{\alpha_3}{2\alpha_1}K} k^{2d-1}\, w(k)^{q_2}\right)^{\frac{p}{q}} \leq \epsilon^p \quad (7)
\end{aligned}$$

by hypothesis. Clearly, (6) and (7) give $\|Mx\|_{l^p} \leq 2^{\frac{1}{p}}\epsilon$ which completes the proof for $1 < p < \infty$. The cases $p = 1$ and $p = \infty$ follow similarly. $\square$

REMARK 3.9. If $\{g_j\} = (g, \Gamma)$ and the analysis operator $C_{(g,\Gamma)}$ is bounded, then $D^+(\Gamma) < \infty$ follows [30]. If $\{g_j\}$ are only assumed to be Γ localized time–frequency molecules, then boundedness of $C_{\{g_j\}}$ does not imply $D^+(\Gamma) < \infty$. For example, consider $\{g_j\} = \{\frac{1}{k!}\mathfrak{g}\}_{k\in\mathbb{N}}$.

REMARK 3.10. Theorem 9 in [13] implies that time–frequency molecules $\{g_j\}$ which are v–localized at $\Gamma = \{\gamma_j\}$, $v > d$, and which generate an l^2–frame for $L^2(\mathbb{R})$ satisfy $1 \leq D^-(\Gamma) \leq D^+(\Gamma)$. Further, Theorem 22 in [14] states that under the same hypothesis but $v > 2d + s$ implies that being an l^2–frame for $L^2(\mathbb{R}^d)$ is equivalent to being an l^p_s–frame for $M^p_s(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$ and all $s \geq 0$. This result alone does not imply Theorem 3.7 nor Theorem 3.8 as they only assume that $\{g_j\}$ is an l^p_s–frame for $M^p_s(\mathbb{R}^d)$ for some p and s. Under stronger conditions, [1] fills this gap. Namely, Theorem 3.1 and Example 3.1 in [1] show that if $v > (2d+1)^2 + 2d$ and $\{g_j\}$ is an at $\Gamma = \{\gamma_j\}$ v–localized l^p–frame for $M^p(\mathbb{R}^d)$ for one p, $1 \leq p \leq \infty$, then $\{g_j\}$ is an l^p frame for $M^p(\mathbb{R}^d)$ for all p and therefore for the well studied case $p = 2$ [13]. This implies Theorem 3.8 for $v > (2d+1)^2 + 2d$.

3.3. Identification of operators with bandlimited Kohn–Nirenberg symbols

A central goal in applied sciences is to identify a partially known operators H from a single input–output pair (g, Hg). We refer to an operator class $\mathcal{H}$ as identifiable, if there exists an element g in the domain of all $H \in \mathcal{H}$ that induces a map $\Phi_g : \mathcal{H} \longrightarrow Y, \quad H \mapsto Hg$ which is bounded and bounded below as map between Banach spaces.

In [8, 9], special cases of Theorem 2.1 played a crucial role in showing that classes of pseudodifferential operators with Kohn–Nirenberg symbol bandlimited to a rectangular domain $[-\frac{a}{2}, \frac{a}{2}]\times[-\frac{b}{2}, \frac{b}{2}]$ are not identifiable if $ab > 1$. The bandlimitation of a Kohn–Nirenberg symbol to a rectangular domain $[-\frac{a}{2}, \frac{a}{2}]\times[-\frac{b}{2}, \frac{b}{2}]$ can be expressed by a corresponding support condition on the operators so-called *spreading function* η_H [1]. Consequently, we consider operators $H : D \longrightarrow M^p_s(\mathbb{R})$, $D \subseteq M^\infty(\mathbb{R})$, included in

$$\mathcal{H}^p_s([-\tfrac{a}{2}, \tfrac{a}{2}]\times[-\tfrac{b}{2}, \tfrac{b}{2}]) = \left\{ H = \int_{[-\frac{a}{2},\frac{a}{2}]\times[-\frac{b}{2},\frac{b}{2}]} \eta_H(z)\pi(z)\, dz, \quad \eta_H \in M^p_s(\mathbb{R}\times\widehat{\mathbb{R}}) \right\} \tag{8}$$

and with norm $\|H\|_{\mathcal{H}^p_s} = \|\eta_H\|_{M^p_s}$. The integral in (8) is defined weakly using $\langle Hf, h\rangle = \langle \eta_H, V_h f\rangle$ [2] [9]. In [8] it was shown that

[1] In fact, the spreading function of an operator is the symplectic Fourier transform of the operator's Kohn–Nirenberg symbol [8, 10].

[2] Here, $\langle\cdot,\cdot\rangle$ is taken to belinear in the first component and conjugate linear in the second.

$$
\begin{tikzcd}
\mathcal{H}_s^p(\mathbb{R}) \arrow[r, "\Phi_g"] & M_s^p(\mathbb{R}) \arrow[d, "C_{(\mathfrak{g},\,\lambda\mathbb{Z}^{2d})}"] \\
l_s^p(\mathbb{Z}^2) \arrow[u, "D_{\{P_j\}}"] \arrow[r, "M"] & l_s^p(\mathbb{Z}^2)
\end{tikzcd}
\qquad
\begin{tikzcd}
\sum_j c_j P_j \arrow[r, "\Phi_g"] & \sum_j c_j P_j g \arrow[d, "C_{(\mathfrak{g},\,\lambda\mathbb{Z}^{2d})}"] \\
\{c_j\}_j \arrow[u, "D_{\{P_j\}}"] \arrow[r, "M"] & \{\sum c_j \langle P_j g, \pi(\lambda j')\mathfrak{g}\}_{j'}
\end{tikzcd}
$$

Figure 2. Sketch of the proof of Theorem 3.13. We choose a structured operator family $\{P_j\} \subseteq \mathcal{H}_s^p$ so that the corresponding synthesis map $D_{\{P_j\}} : \{c_j\} \longrightarrow \sum c_j P_j$ has a bounded left inverse. Further, $C_{(\mathfrak{g},\,\lambda\mathbb{Z}^{2d})}$ has a bounded left inverse for $\lambda < 1$. We then use Theorem 2.1 to show that for any $g \in M^\infty(\mathbb{R})$, the composition $M = C_{(\mathfrak{g},\,\lambda\mathbb{Z}^{2d})} \circ \phi_g \circ D_{\{P_j\}}$ is not bounded below, therefore implying that $\phi_g : \mathcal{H}_s^p \longrightarrow M_s^p(\mathbb{R})$ is not bounded below as well.

Theorem 3.11. *There exists $g \in M^\infty(\mathbb{R})$ with $\Phi_g : \mathcal{H}_0^2([-\frac{a}{2}, \frac{a}{2}] \times [-\frac{b}{2}, \frac{b}{2}]) \longrightarrow M_0^2(\mathbb{R})$ bounded and bounded below if and only if $ab \leq 1$.*

Note that $H_0^1([-\frac{a}{2}, \frac{a}{2}] \times [-\frac{b}{2}, \frac{b}{2}])$ consists of Hilbert–Schmidt operators, the norm $\|\cdot\|_{\mathcal{H}_0^2}$ is equivalent to the Hilbert–Schmidt space norm, and $\|\cdot\|_{M_0^2}$ is a scalar multiple of the L^2–norm.

The main result in [9] is

Theorem 3.12. *For $ab < 1$ exists $g \in M^\infty(\mathbb{R})$ with $\Phi_g : \mathcal{H}_0^\infty([-\frac{a}{2}, \frac{a}{2}] \times [-\frac{b}{2}, \frac{b}{2}]) \longrightarrow M_0^\infty(\mathbb{R})$ bounded and bounded below, while for $ab > 1$ exists no such $g \in M^\infty(\mathbb{R})$.*

Here, we use the generality of Theorem 2.1 to obtain

Theorem 3.13. *Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. For $ab > 1$ exists no $g \in M^\infty(\mathbb{R})$ with $\Phi_g : \mathcal{H}_s^p([-\frac{a}{2}, \frac{a}{2}] \times [-\frac{b}{2}, \frac{b}{2}]) \longrightarrow M_s^p(\mathbb{R})$ bounded and bounded below.*

Sketch of proof. We assume $a = b$ and $a^2 > 1$. The general case $ab > 1$ follows similarly. The goal is to show that for any $g \in M^\infty(\mathbb{R})$ which induces a bounded operator $\Phi_g \colon \mathcal{H}_s^p([-\frac{a}{2}, \frac{a}{2}]^2) \longrightarrow M_s^p(\mathbb{R})$, this operator is not bounded below.

To see this, we pick $\lambda > 1$ with $1 < \lambda^4 < a^2$ and define a prototype operator $P \in \mathcal{H}_s^p([-\frac{a}{2}, \frac{a}{2}]^2)$ via its spreading function $\eta_P(t, \nu) = \eta(t)\,\eta(\nu)$ where η is smooth, takes values in $[0, 1]$ and satisfies $\eta(t) = 1$ for $|t - a/2| \leq a/2\lambda$ and $\eta(t) = 0$ for $|t - a/2| \geq a/2$.

The collection of functions $\{M_{\frac{\lambda}{a}j}\,\eta_P\}_{j\in\mathbb{Z}^2}$ corresponds to the operator family $\{\pi(\frac{\lambda}{a}j)\,P\pi(\frac{\lambda}{a}j)^*\}_{j\in\mathbb{Z}^2}$ [9]. Further, it forms a Riesz basis for its closed linear span in $L^2(\mathbb{R}\times\widehat{\mathbb{R}})$ and, for $c>0$ sufficiently large, the collection $\{\pi(\frac{\lambda}{a}j,\frac{1}{c}k)\,\eta_P\}_{j,k\in\mathbb{Z}^2}$ is a frame for $L^2(\mathbb{R}^2)$ [11, 31]. Arguing as in Example 3.3, we obtain a bounded left inverse of $D_{\{M_{\frac{\lambda}{a}j}\eta_P\}}:l^p_s(\mathbb{Z}^2)\longrightarrow M^p_s(\mathbb{R}\times\widehat{\mathbb{R}})$, thereby showing that $D_{\{M_{\frac{\lambda}{a}j}\eta_P\}}$ and also the corresponding operator synthesis map $D_{\{P_j\}}:l^p_s(\mathbb{Z}^2)\longrightarrow\mathcal{H}^p_s(\mathbb{R}\times\widehat{\mathbb{R}})$ with $P_j=\pi(\frac{\lambda}{a}j)\,P\pi(\frac{\lambda}{a}j)^*$, $j\in\mathbb{Z}^2$, are bounded below.

For any fixed $g\in M^\infty(\mathbb{R})$ which induces a bounded map $\Phi_g:\mathcal{H}^p_s([-\frac{a}{2},\frac{a}{2}]^2)\longrightarrow M^p_s(\mathbb{R})$ we consider the operator

$$M=(m_{jj'})=C_{(\mathfrak{g},\frac{\lambda^2}{a})}\circ\Phi_g\circ D_{\{P_j\}}\colon l^p_s(\mathbb{Z}^2)\longrightarrow l^p_s(\mathbb{Z}^2).$$

We have $\left|m_{jj'}\right|=\left|\left\langle\pi(\frac{\lambda}{a}j)P\pi(\frac{\lambda}{a}j)^*g,\ \pi(\frac{\lambda^2}{a}j')\,\mathfrak{g}\right\rangle\right|=\left|V_{\mathfrak{g}}\,P\pi(\frac{\lambda}{a}j)^*g\left(\frac{\lambda}{a}(\lambda j'-j)\right)\right|$. In [8] it is shown that smoothness and compact support of η_P implies that there exist nonnegative functions d_1 and d_2 on $\mathbb{R}$, decaying rapidly at infinity, such that for all $g\in M^\infty(\mathbb{R})$, $|Pg(x)|\le\|g\|_{M^\infty}\,d_1(x)$ and $|\widehat{Pg}(\xi)|\le\|g\|_{M^\infty}\,d_2(\xi)$. This implies that $V_{\mathfrak{g}}\,P\pi(\frac{\lambda}{a}j)^*g$ decays rapidly and independently of j, so that we can apply Theorem 2.1 to show that M is not bounded below. Since $\frac{\lambda^2}{a}<1$, Example 3.3 implies that $C_{(\mathfrak{g},\frac{\lambda^2}{a})}$ is bounded below. Also, $D_{\{P_j\}}$ is bounded below, implying that Φ_g cannot be bounded below. Since $g\in M^\infty(\mathbb{R})$ was chosen arbitrarily, this completes the proof. $\square$

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