Generalized Haar Wavelets and Frames

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ABSTRACT

Generalized Haar wavelets were introduced in connection with the problem of detecting specific periodic components in noisy signals.[1,2] John Benedetto and I showed that the non–normalized continuous wavelet transform of a periodic function taken with respect to a generalized Haar wavelet is periodic in time as well as in scale, and that generalized Haar wavelets are the only bounded functions with this property.

In this paper, I shall discuss generalized Haar wavelets in a discrete setting. I shall present a characterization of all generalized Haar wavelets which have the property that our discretized version of the continuous wavelet transform is a topological isomorphism onto its range. This is equivalent to the fact that the set of analysis vectors used constitute a frame for $l^2(\mathbf{Z})$. A similar result is obtained for $l^2(\mathbf{Z}^d)$. Generalized Haar wavelets allow a fast computation of a discretized version of the continuous wavelet transform of a function, as I shall show.

I shall present examples of generalized Haar wavelets and calculate the corresponding frame bounds and analysis filter banks.

Keywords: Wavelets, frames, $l^2(\mathbf{Z}^d)$, generalized Haar wavelets, periodicity detection

1. INTRODUCTION

The wavelet transformation discussed in this paper is a discretization of the continuous wavelet transformation and is obtained via

$$W_\psi^s f[m,n] := m^{-\frac{s}{2}} \sum_{k\in\mathbf{Z}^d} f[k]\psi(\frac{k-n}{m}) \approx m^{-\frac{s}{2}} \int_{\mathbf{R}^d} f(t)\psi(\frac{t-n}{m})\, dt = W_\psi^s f(m,n),$$

where the normalization parameter $s>0$ is fixed. For $(m,n)\in\mathbf{Z}^+\times\mathbf{Z}$ and integer arguments, we define

$$\overline{\psi_{m,n}}[\cdot] = m^{-\frac{s}{2}}\psi(\frac{\cdot - n}{m}) \tag{1}$$

and get

$$W_\psi^s f[m,n] = \sum_{k\in\mathbf{Z}^d} f[k]\overline{\psi_{m,n}}[k] = \langle f, \overline{\psi_{m,n}}\rangle_{l^2(\mathbf{Z})}.$$

For sake of completeness, let us recall the definition of a frame.

Definition 1.1. *A family of functions $\{\varphi_i\}_{i\in I}$ in a Hilbert space H is a frame, if there exist $A>0$ and $B<\infty$ such that for all $f\in H$*

$$A\,\|f\|_H^2 \le \sum_{i\in I} |\langle f,\varphi_i\rangle|^2 \le B\,\|f\|_H^2\,. \tag{2}$$

If A is chosen maximal and B is chosen minimal such that (2) *holds, A is called the lower and B the upper framebound.*

Clearly, $\{\varphi_i\}_{i\in I}$ being a frame is equivalent to the fact that the linear map

$$L: H \longrightarrow l^2(I), \quad f \longmapsto \{\langle f,\varphi_i\rangle\}_{i\in I}$$

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is norm bounded above and below, and hence is invertible on its range. Each element $f \in H$ is therefore fully represented by the coefficients $\{\langle f, \varphi_i \rangle\}_{i \in I}$ and a stable reconstruction of f from these coefficients is possible.

Gerneralized Haar wavelets (of degree 1) are defined to be square integrable functions which are constant on the intervals $[n, n+1),\ n \in \mathbf{Z}$. If ψ is a generalized Haar wavelet, we have

$$\overline{\psi_{m,n}}[\cdot] = m^{-\frac{s}{2}} \psi(\frac{\cdot - n}{m}) = m^{-\frac{s}{2}} \overline{\psi_{1,0}} \left[\left\lfloor \frac{\cdot - n}{m} \right\rfloor \right],$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than x.

In this case, $\{\overline{\psi}_{m,n}\}$ are integer translates and dilates of the vector $\overline{\psi} = \overline{\psi}_{1,0}$. In this paper, we shall restrict ourselves to generalized Haar wavelets in the described discrete setting. We shall omit the bar and write $\psi = \overline{\psi} \in l^2(\mathbf{Z})$.

In Section 2 we shall classify all generalized Haar wavelets ψ such that $\{\psi_{m,n}\}$ is a frame for $H = l^2(\mathbf{Z})$ for a given normalization parameter s. The main result of Section 2 is generalized to the multi dimensional setting $H = l^2(\mathbf{Z}^d)$ in Section 3.

The inherent redundancy in discretized versions of the continuous wavelet transformation offers some robustness to noise, but requires more calculations than needed to calculate a dyadic wavelet transform. In Section 4, we shall present a fast algorithm which reduces the number of calculations that are needed to compute the wavelet transform of a signal in $l^2(\mathbf{Z})$ significantly, assuming that a generalized Haar wavelet is used.

Generalized Haar wavelets were applied to the periodicity detection problem, since they have the property that the non–normalized continuous wavelet transform of a periodic function is periodic in time and in scale. This property is preserved in our discrete setting.

Proposition 1.2. *Let ψ be a generalized Haar wavelet of degree 1, and let $\{f[n]\}_{n \in \mathbf{Z}}$ be a T–periodic sequence, $T \in \mathbf{Z}^+$, i.e., $f[n+T] = f[n]$ for all $n \in \mathbf{Z}$. Then $m^{s/2} W^s_\psi f[m,n]$ is T–periodic in n and T–periodic in m.*

The proof of this proposition, as well as the proofs of all results stated in this paper, is omitted, but will be published at a later time.[3]

2. WAVELET FRAMES FOR $l^2(\mathbf{Z})$

We shall classify all generalized Haar wavelets $\psi \in l^2(\mathbf{Z})$ such that for a given $s \in \mathbf{R}^+$ the family $\{\psi_{m,n}\}_{m \in \mathbf{Z}^+, n \in \mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$.

The Dirichlet functions d_m is defined by $d_m(\gamma) = \sum_{l=0}^{m-1} e^{-2\pi i l \gamma}$. Note that then $|d_m(\gamma)|^2 = \left(\frac{\sin(\pi m \gamma)}{\sin(\pi \gamma)}\right)^2$. For any $\psi \in l^2(\mathbf{Z})$ and $s > 0$ we define

$$\Psi^s : \mathbf{T} \longrightarrow \mathbf{R}^+ \cup \{\infty\}, \quad \gamma \longmapsto \sum_{m \in \mathbf{Z}^+} m^{-s} |d_m(\gamma) \widehat{\psi}(m\gamma)|^2 \ \text{ a.e.}$$

We obtain the following theorem characterizing $\psi \in l^2(\mathbf{Z})$ such that $\{\psi_{m,n}\}_{m \in \mathbf{Z}^+, n \in \mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$.

Theorem 2.1. *For $\psi \in l^2(\mathbf{Z})$, the following are equivalent:*

i. The family $\{\psi_{m,n}\}_{m \in \mathbf{Z}^+, n \in \mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$.

ii. There exists $A > 0$ and $B < \infty$ such that

$$A \le \Psi^s(\gamma) \le B$$

for almost all $\gamma \in \mathbf{T}$.

In this case, the lower and upper framebounds are given by $A = \operatorname{essinf}_{\gamma \in \mathbf{T}} \Psi^s(\gamma)$ and $B = \|\Psi^s\|_{L^\infty(\mathbf{T})}$.

This theorem gives us a criterion, to check whether $\{\psi_{m,n}\}_{(m,n) \in \mathbf{Z}^+ \times \mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$. This criterion is easily verified if we restrict ourselves to wavelets ψ with compact support, i.e., ψ satisfy the property that $\psi[k] = 0$ for $k \neq 0, 1, \ldots, N-1$. This compactness condition is generally satisfied in applications.

Theorem 2.2. *Let $\psi \in l^2(\mathbf{Z})$ satisfy the condition that $\psi[k]=0$ for $k \neq 0,1,\ldots,N-1$. The following are equivalent:*

i. The family $\{\psi_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$.

ii. The polynomial $1+z+z^2+\ldots+z^n$ does not divide $H(z)=\psi[0]+\psi[1]z+\psi[2]z^2+\ldots+\psi[N-1]z^{N-1}$ for all $n \leq N-1$ and either $\sum\psi[k]=0$ and $s=3$ or $\sum\psi[k]\neq 0$ and $s>3$.

Remark 2.3. Our construction can be related to the quasi affine frames.[4,5] In their work, Ron and Shen analyzed the coarse part of a signal with the set of analyzing functions $\{2^{-m}\psi(\frac{t-n}{2^m})\}_{n\in Z,m\in Z^+}$. The scaling factor 2^{-m} is necessary, since we are expanding the dyadic wavelet set $\{2^{-\frac{m}{2}}\psi(\frac{t-2^m n}{2^m})\}_{n\in Z,m\in Z^+}$. In our approach, we are expanding the wavelet set further, i.e., we are using $s=3$ and $\{m^{-\frac{3}{2}}\psi(\frac{t-n}{m})\}_{n\in Z,m\in Z^+}$.

Remark 2.4. We can use Theorem 2.2 to obtain frames for any separable Hilbert space. As example, we shall construct frames for the Paley–Wiener spaces[6,7]

$$PW(\Omega)=\{f\in L^2(\mathbf{R}):\ \operatorname{supp}\mathcal{F}(f)\subseteq[-\Omega,\Omega]\},$$

with $\Omega>0$. Paley–Wiener spaces are closed subspaces of $L^2(\mathbf{R})$, and, hence, Hilbert spaces with the inner product $\langle\cdot,\cdot\rangle_{L^2(\mathbf{R})}$. Without loss of generality, we shall consider only the case $\Omega=\frac{1}{2}$.

Defining $h_k(\cdot)=\frac{\sin(\pi(\cdot-k))}{\pi(\cdot-k)}\in PW(\frac{1}{2})$ we obtain an orthonormal basis $\{h_k\}_{k\in\mathbf{Z}}$ of $PW(\frac{1}{2})$. Furthermore $f(k)=\langle f,h_k\rangle$ for $f\in PW(\frac{1}{2})$ and $k\in\mathbf{Z}$, and, hence, $\{f(k)\}_{k\in\mathbf{Z}}\in l^2(\mathbf{Z})$. The classical sampling theorem implies

$$f=\sum_{k\in\mathbf{Z}} f(k)h_k.$$

Let $\psi\in l^2(\mathbf{Z})$ be chosen such that $\{\psi_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$ with frame bounds A and B. Define

$$\varphi_{m,n}=\sum_{k\in\mathbf{Z}}\psi_{m,n}(k)h_k$$

for $m\in\mathbf{Z}^+, n\in\mathbf{Z}$. The function $\varphi_{m,n}\in PW(\frac{1}{2})$ is well-defined, since $\psi_{m,n}\in l^2(\mathbf{Z})$ for all $m\in\mathbf{Z}^+$, $n\in\mathbf{Z}$, and since $\{h_k\}_{k\in\mathbf{Z}}$ is an orthonormal set. For $f\in PW(\frac{1}{2})$, $m\in\mathbf{Z}^+$, and $n\in\mathbf{Z}$, we compute

$$\langle f,\varphi_{m,n}\rangle_{L^2(\mathbf{R})}=\int_{\mathbf{R}} f(t)\overline{\sum_{k\in\mathbf{Z}}\psi_{m,n}(k)h_k(t)}\,dt=\sum_{k\in\mathbf{Z}}\overline{\psi_{m,n}(k)}\int_{\mathbf{R}} f(t)\overline{h_k(t)}\,dt=\langle f|_{\mathbf{Z}},\psi_{m,n}\rangle_{l^2(\mathbf{Z})}.$$

This results in

$$A\|f\|^2_{L^2(\mathbf{R})}=A\|f|_{\mathbf{Z}}\|^2_{l^2(\mathbf{Z})}\leq\sum_{m\in\mathbf{Z}^+,n\in\mathbf{Z}}|\langle f|_{\mathbf{Z}},\psi_{m,n}\rangle_{l^2(\mathbf{Z})}|^2=\sum_{m\in\mathbf{Z}^+,n\in\mathbf{Z}}|\langle f,\varphi_{m,n}\rangle_{L^2(\mathbf{R})}|^2\leq B\|f[\cdot]\|^2_{l^2(\mathbf{Z})}=B\|f\|^2_{L^2(\mathbf{R})}$$

and, hence, $\{\varphi_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}}$ is a frame for $PW(\frac{1}{2})$.

We shall conclude this section by discussing three Examples.

Example 2.5. Let ψ be the Haar wavelet, i.e., $\psi[m]=\frac{1}{\sqrt{2}}(\delta_0[m]-\delta_1[m])$ for $m\in\mathbf{Z}$. For $s=3$, ψ fulfills the hypothesis of Theorem 2.2, and, hence, $\{\psi_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$. To obtain the corresponding framebounds of this frame, we examine the function Ψ^3, which is given by

$$\Psi^3(\gamma)=2\sum_{m\in\mathbf{Z}^+} m^{-3}\frac{\sin^4(\pi m\gamma)}{\sin^2(\pi\gamma)},$$

and is shown in Figure 1.C. As lower framebound we obtain

$$A=\lim_{\gamma\to 0^+}\Psi^3(\gamma)=\frac{2}{\pi^2}\int_0^\infty\frac{\sin^4(\pi x)}{x^3}\,dx=2\ln(2)\approx 1.386,$$

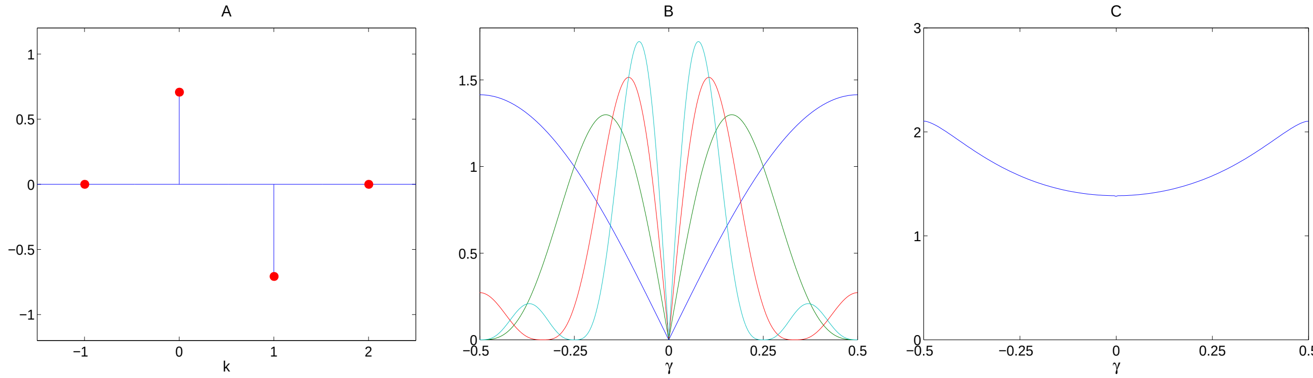

Figure 1. A: The Haar wavelet. B: The Haar wavelet analysis filterbank for $m = 1, 2, 3, 4$. C: $\Psi^3(\gamma)$ for the Haar wavelet ψ.

and as upper framebound we have

$$B = \Psi^3\left(\frac{1}{2}\right) = 2\sum_{\substack{m\in Z^+\\ m \text{ odd}}} m^{-3} = \frac{14\zeta(3)}{8} \approx 2.104.$$

Example 2.6. We can associate the vector $\varphi[m] = \delta_0[m]$ to the Haar scaling function. Theorem 2.2 asserts that $\{\varphi_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$ if $s > 3$. For $s = 4$, the function

$$\Psi^4(\gamma) = \sum_{m\in\mathbf{Z}^+} m^{-4}\frac{\sin^2(\pi m\gamma)}{\sin^2(\pi\gamma)}$$

is associated with φ. This function is shown in Figure 2.C. As framebounds we obtain

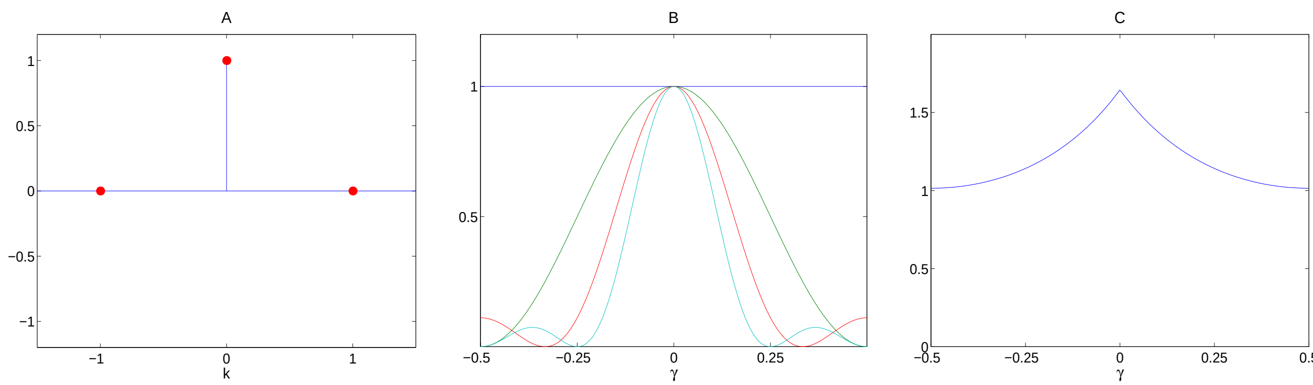

Figure 2. A: The vector associated with the Haar scaling function. B: The Haar scaling function analysis filterbank for $m = 1, 2, 3, 4$. C: $\Psi^4(\gamma)$ for the Haar scaling function φ.

$$B = \lim_{\gamma\to 0^+} \Psi^4(\gamma) = \sum_{m\in\mathbf{Z}^+} m^{-2} = \frac{\pi^2}{6} \approx 1.6449 \text{ and } A = \Psi^4\left(\frac{1}{2}\right) = \sum_{\substack{m\in Z^+\\ m \text{ odd}}} m^{-4} = \frac{\pi^4}{96} = 1.0147.$$

EXAMPLE 2.7. Let ψ be the wavelet defined by $\psi[m] = \frac{1}{\sqrt{70}}(\delta_0[m] - 4\delta_1[m] + 6\delta_2[m] - 4\delta_3[m] + \delta 4[m])$ for $m \in \mathbf{Z}$. For $s = 3$, we know that $\{\psi_{m,n}\}_{m\in\mathbf{Z}^+, n\in\mathbf{Z}}$ is a frame for $l^2(\mathbf{Z})$. We obtain

$$\Psi^3(\gamma) = \sum_{m\in\mathbf{Z}^+} m^{-3} \frac{2^8}{70} \frac{\sin^{10}(\pi m\gamma)}{\sin^2(\pi\gamma)},$$

which is shown in Figure 3.C. As lower framebound we obtain $A = 0.9905$ and as upper framebound we have $B = 3.8466$

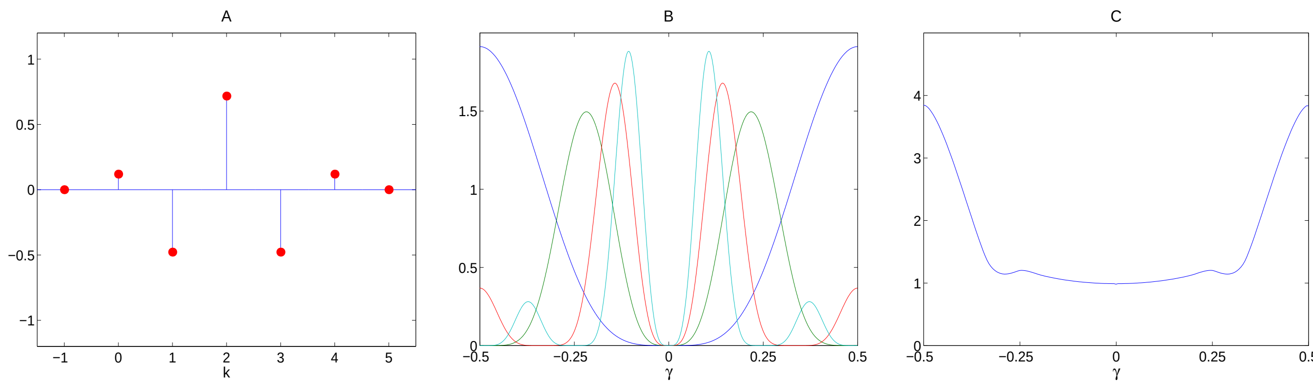

Figure 3. A: The wavelet of Example 2.7 . B: The corresponding analysis filterbank for $m = 1, 2, 3, 4$. C: $\Psi^3(\gamma)$ for the wavelet discussed in Example 2.7.

3. WAVELET FRAMES FOR $l^2(\mathbf{Z}^d)$

We shall now discuss the Hilbert space of interest is the space of square summable multidimensional discrete signals $H = l^2(\mathbf{Z}^d)$. Our goal is to characterize vectors $\psi \in l^2(\mathbf{Z}^d)$ such that, for some normalization constant s, its translates and "dilates" form a frame for $H = l^2(\mathbf{Z}^d)$. Alternatively, the frame elements can be seen as sampled versions of a generalized Haar wavelet $\psi \in L^2(\mathbf{R}^d)$.

For $\psi \in l^2(\mathbf{Z}^d)$ we define, for $n \in \mathbf{Z}^d$, $m \in \mathbf{Z}^+$,

$$\psi_{m,n}[k] = m^{-\frac{s}{2}} \psi\left[\left\lfloor \frac{k_1 - n_1}{m} \right\rfloor, \ldots, \left\lfloor \frac{k_1 - n_d}{m} \right\rfloor\right], \ k \in \mathbf{Z}^d.$$

For $m > 0$, we define m-dimensional Dirichlet functions by $\mathbf{d_m}(\gamma) = d_m(\gamma_1) \cdot \ldots \cdot d_m(\gamma_d)$ and

$$\Psi^s(\cdot) = \sum_{m=1}^{\infty} m^{-s} |\mathbf{d_m}(\cdot)\widehat{\psi}(m\,\cdot)|^2, \quad \text{for} \quad \gamma \in \mathbf{T}^d. \tag{3}$$

Theorem 2.1 generalizes to higher dimensions in the following fashion:

THEOREM 3.1. *For $\psi \in l^2(\mathbf{Z}^d)$, the following are equivalent:*

i. The family $\{\psi_{m,n}\}_{m\in\mathbf{Z}^+, n\in\mathbf{Z}^d}$ is a frame for $l^2(\mathbf{Z}^d)$.

ii. There exists $A > 0$ and $B < \infty$ such that

$$A \le \Psi^s(\gamma) \le B$$

for almost all $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbf{T}^d$.

The framebounds are obtained in exactly the same matter as in Theorem 2.1.

Our next objective is to characterize generalized Haar wavelets satisfying the criterion in Theorem 3.1. The restriction to generalized Haar wavelets with compact support does not allow a generalization to higher dimensions of Theorem 2.1. The reason for this is that it is not easy to control the zero set of the trigonometric polynomial $\widehat{\psi}$ appearing in (3) Hence, we shall not be able to give a full characterization of generalized Haar wavelets with compact support in $\mathbf{R}^d$ such that $\{\psi_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}^d}$ is a frame for $l^2(\mathbf{Z}^d)$. Nevertheless, some necessary conditions for $\{\psi_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}^d}$ being a frame for $l^2(\mathbf{Z}^d)$ are known.[3]

I shall now discuss a few generalized Haar wavelets for $d = 2$, beginning with two which have the property that $\{\psi_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}^d}$ is not a frame for $l^2(\mathbf{Z}^d)$ for all $s > 0$.

Example 3.2. Let $\psi[1,0] = 1/\sqrt{2}$, $\psi[0,1] = -1/\sqrt{2}$, and $\psi[n] = 0$ for $n \neq [1,0],[0,1]$. This wavelet is displayed in Figure 4.A. In this case $\sqrt{2}\widehat{\psi}(\gamma_1,\gamma_2) = e^{-2\pi i\gamma_1} - e^{-2\pi i\gamma_2}$ for $(\gamma_1,\gamma_2) \in \widehat{\mathbf{R}}^2$ and therefore $\widehat{\psi}(\gamma_1,\gamma_1) = 0$ for $\gamma_1 \in \widehat{\mathbf{R}}$. In particular $\widehat{\psi}(\frac{m}{2},\frac{m}{2}) = 0$ for all $m \in \mathbf{Z}^+$ and so $\Psi^s(\frac{1}{2},\frac{1}{2}) = 0$ for all $s \in \mathbf{R}^+$. Hence $\{\psi_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}^d}$ does not possess a lower framebound. In this case, Ψ^5 is shown in Figure 4.B.

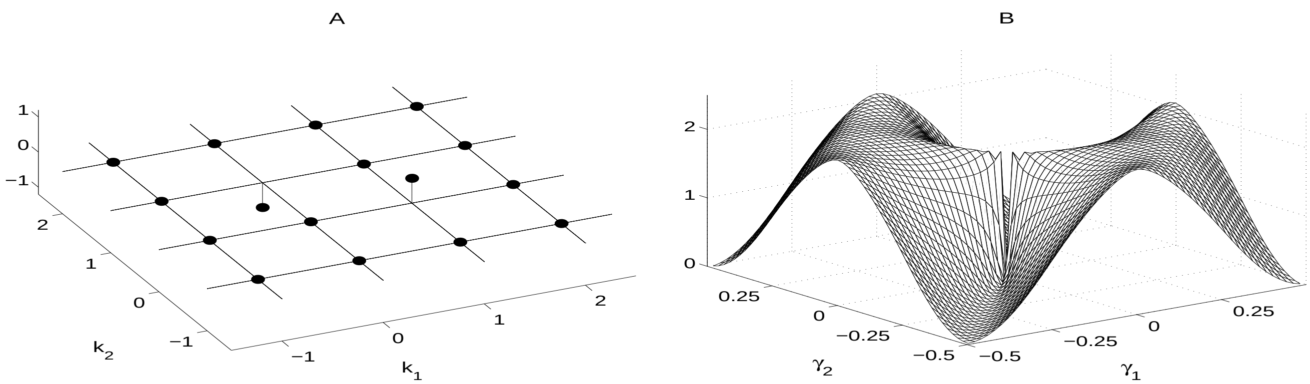

Figure 4. A: The wavelet discussed in Example 3.2. B: Ψ^5 of Example 3.2.

Example 3.3. Let $\psi[0,0] = 2/\sqrt{7}$, $\psi[0,1] = \psi[1,0] = \psi[0,-1] = \psi[-1,0] = 1/\sqrt{7}$ and $\psi[n] = 0$ for $n \neq [0,0],[1,0],[0,1],[-1,0],[0,-1]$. (See Figure 5.A.) A short calculation shows that

$$\sqrt{7}\widehat{\psi}(\gamma_1,\gamma_2) = |e^{-2\pi i\gamma_1} - (-1)|^2 + |e^{-2\pi i\gamma_2} - (-1)|^2 - 2$$

for $(\gamma_1,\gamma_2) \in \widehat{\mathbf{R}}^2$. Hence $\widehat{\psi}(\frac{1}{3},\frac{1}{3})$ and $\widehat{\psi}(\frac{2}{3},\frac{2}{3}) = 0$, and therefore $\Psi^s(\frac{1}{3},\frac{1}{3}) = 0$ for all $s \in R^+$. Hence $\{\psi_{m,n}\}_{m\in\mathbf{Z}^+,n\in\mathbf{Z}^d}$ is not a frame for $l^2(\mathbf{Z}^d)$. Ψ^6 in this case is pictured in Figure 5.B.

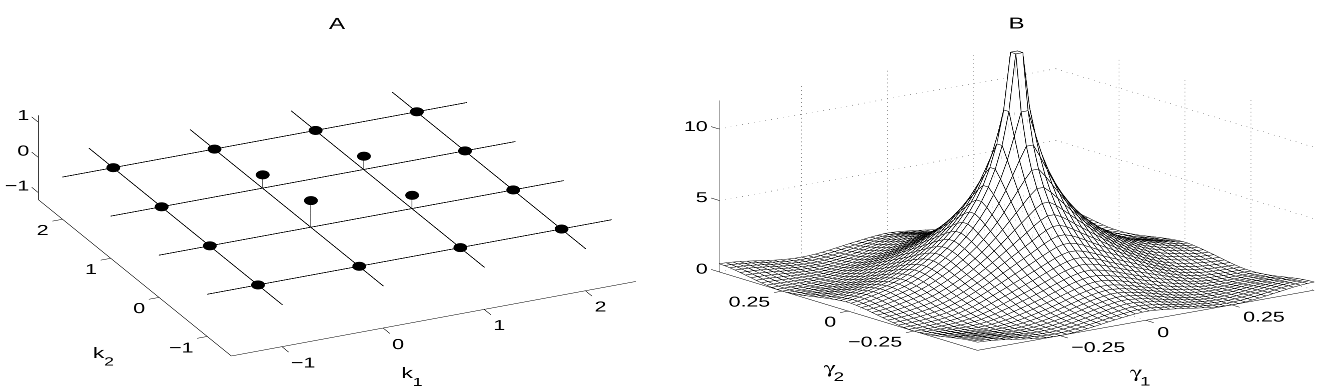

Figure 5. A: The wavelet discussed in Example 3.3. B: Ψ^6 of Example 3.3.

EXAMPLE 3.4. Let $\psi[0,0] = -3/\sqrt{12}$, $\psi[0,1] = \psi[1,0] = \psi[1,1] = 1/\sqrt{12}$ and $\psi[n] = 0$ for $n \neq [0,0], [1,0], [0,1], [1,1]$. This wavelet is shown in Figure 6.A. Numerical experiments imply that, for $s = 5$, $\{\psi_{m,n}\}_{m\in\mathbf{Z}^+, n\in\mathbf{Z}^d}$ is a frame for $l^2(\mathbf{Z}^d)$. As approximate lower framebound we obtain $A = 0.32$ and as approximate upper framebound, we obtain $B = 1.88$. The resulting function Ψ^5 is supplied in Figure 6.B.

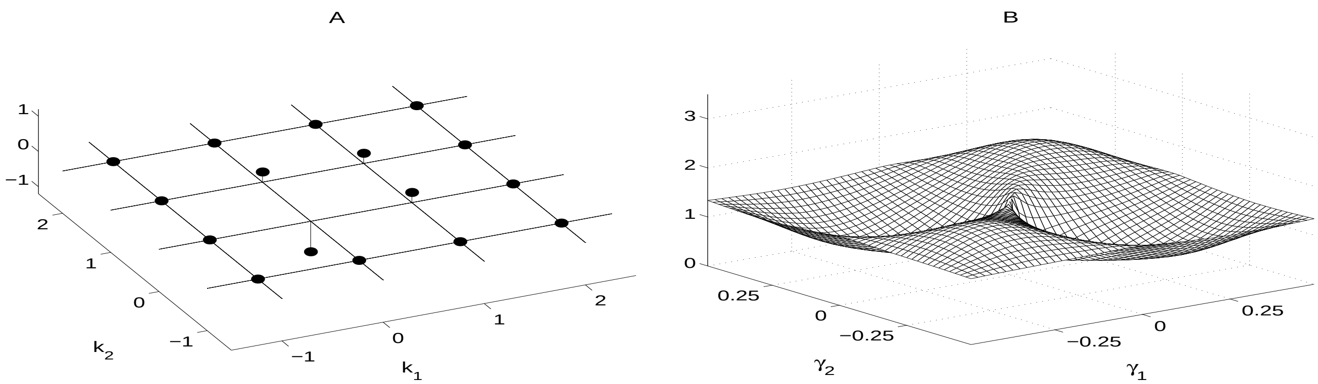

Figure 6. A: The wavelet discussed in Example 3.4. B: Ψ^5 of Example 3.4.

EXAMPLE 3.5. Let $\psi[0,0] = -1/5$, $\psi[1,0] = -2/5$, $\psi[0,1] = 2/5$, $\psi[1,1] = 4/5$ and $\psi[n] = 0$ for $n \neq [0,0], [1,0], [0,1], [1,1]$. Figure 7.A shows this wavelet. For $s = 6$, we obtain as numerical approximation $A = 0.04$ as lower framebound, and $B = 3.28$ as upper framebound. This implies that $\{\psi_{m,n}\}_{m\in\mathbf{Z}^+, n\in\mathbf{Z}^d}$ is a frame for $l^2(\mathbf{Z}^d)$. The function Ψ^6 is supplied in Figure 7.B.

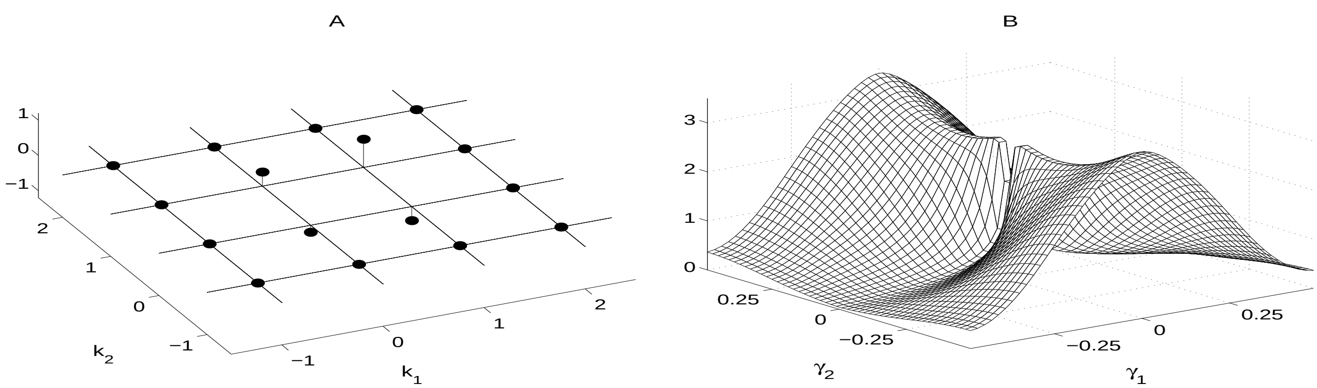

Figure 7. A: The wavelet discussed in Example 3.5. B: Ψ^6 of Example 3.5.

4. IMPLEMENTATION

To analyze a signal through a "continuous" wavelet transform is expensive, since it requires the calculation of a large number of coefficients $W^s_\psi f[m,n]$. In general, for a wavelet with support $[0,N]$, we need mN additions and $mN+1$ multiplications to calculate $W^s_\psi f[m,n]$. The restriction to generalized Haar wavelets gives rise to a recursive procedure to obtain these coefficients. This reduces the number of calculations needed significantly. In fact, if ψ is a generalized Haar wavelet which is supported on $[0,N]$, to obtain $W^s_\psi f[m,n]$ from $W^s_\psi f[m,n-1]$ or $W^s_\psi f[m-1,n]$ requires only $2N+1$ additions and $N+1$ multiplications or $\frac{N(N-1)}{2}+1$ additions and $N+3$ multiplications respectively, regardless of how large m and hence the support of $\psi[\lfloor\frac{\cdot-n}{m}\rfloor]$ is.

In order to develop the algorithm to compute $W_f^s[m,n]$ recursively, we shall write (1) in a more convenient form:

$$\begin{aligned}
W_\psi^s f[m,n] &= m^{-\frac{s}{2}} \sum_{k\in\mathbf{Z}} f[k]\psi\left[\left\lfloor \frac{k-n}{m}\right\rfloor\right] \\
&= m^{-\frac{s}{2}} \begin{pmatrix} & \vdots & \vdots & & \vdots & \vdots \\ + & f[n] & \psi[0] & +\ldots+ & f[n+m-1] & \psi[0] \\ + & f[n+m] & \psi[1] & +\ldots+ & f[n+2m-1] & \psi[1] \\ + & f[n+2m] & \psi[2] & +\ldots+ & f[n+3m-1] & \psi[2] \\ & \vdots & \vdots & & \vdots & \vdots \end{pmatrix} \\
&= m^{-\frac{s}{2}} \sum_{r\in\mathbf{Z}} \left(\sum_{l=0}^{m-1} f[n+mr+l]\right)\psi[r]
\end{aligned}$$

At this point, the fact that ψ is a generalized Haar wavelet has reduced the number of necessary operations needed to calculate $W_\psi^s f[m,n]$ to mN additions and $N+1$ multiplications.

In the remainder of this section, we shall omit the normalization factor $m^{-\frac{s}{2}}$. This factor is certainly independent of wavelet and signal and would be multiplied to $W_\psi f[m,n]$ in the last step of computing $W_\psi^s f[m,n]$. Let us begin with the trivial case of obtaining $W_\psi f[m,n]$ from $W_\psi f[m,n-1]$. We have

$$\begin{aligned}
W_\psi f[m,n] &- W_\psi f[m,n-1] \\
&= \sum_{r\in\mathbf{Z}} \left(\sum_{l=0}^{m-1} f[n+mr+l] - \sum_{l=0}^{m-1} f[n-1+mr+l]\right)\psi[r] \\
&= \sum_{r=0}^{N-1} \left(f[n+mr+m-1] - f[n-1+mr]\right)\psi[r].
\end{aligned}$$

To explain how to obtain $W_\psi f[m,n]$ from $W_\psi f[m-1,n]$ for scales $m \geq N$ is not as easy. Again, many products appearing in the summation of $W_\psi f[m,n]$ contributed already to $W_\psi f[m-1,n]$. Sorting this through, we obtain

$$\begin{aligned}
W_\psi f[m,n] - W_\psi f[m-1,n] &= \psi[0] \sum_{l=0}^{N-1} f[n-mN+l] \\
&+ (\psi[1]-\psi[0]) \sum_{l=1}^{N-1} f[n-m(N-1)+l] \\
&+ \ldots \\
&+ (\psi[N-1]-\psi[N-2]) f[n].
\end{aligned}$$

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