Perturbation Stability of Various Coherent Riesz Families

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ABSTRACT

We compare three types of coherent Riesz families (Gabor systems, Wilson bases, and wavelets) with respect to their perturbation stability under convolution with elements of a family of typical channel functions. This problem is of key relevance in the design of modulation signal sets for digital communication over time–invariant channels. Upper and lower bounds on the orthogonal perturbation are formulated in terms of spectral spread and temporal support of the prototype, and by the approximate design of worst case convolution kernels. Among the considered bases, the Weyl–Heisenberg structure which generates Gabor systems turns out to be optimal.

Keywords: Perturbation stability, coherent families, Gabor systems, Wilson bases, wavelets, channel function

1. INTRODUCTION

A coherent function system is built from a finite number of prototype functions by the group action of unitary operators such as translation, modulation and/or scaling. The inherent structure of such systems leads to computationally efficient design and implementation of frames or Riesz bases. The most prominent coherent function systems are wavelet and Gabor systems. Both structures are potential candidates in the two fundamental applications of modern digital communication:

- *Source coding (signal compression)*: The coherent function system conveyes the transform step which aims at decorrelating the data prior to quantization. In near-to-lossless compression completeness is a must, hence the function system is required to be a frame.
- *Channel coding (signal transmission)*: The channel input signal is synthesized as a linear combination of certain basis functions whose coefficients are bearing the information. Here, injectivity of this synthesis mapping is crucial, therefore one actually wants to use a Riesz basis for some closed subspace of the underlying Hilbert space (on which the channel acts as a linear operator).

In both applications, the performance is reflected by an operator diagonalization problem; the operator corresponds either to the correlation of the source or to the action of the channel, respectively. Exact diagonalization is unrealistic because the a priori knowledge of the underlying operator is incomplete, and even if we had this prior knowledge, the resulting eigenbases are unstructured and do not satisfy practical side constraints (such as finite support).

We shall concentrate on channel coding. As bases, we consider shift-invariant Riesz systems $g_{k,l}$ defined by

$$g_{k,l}(x) = g_l(x-ak), \qquad k \in \mathbb{Z}\,, \quad l = 0,1,\ldots,N-1\,, \tag{1}$$

where $a > 0$ is the time shift, each g_l has support of length at most a (because of the latency constraints), and the family has one of the following specific structures:

- *Gabor* or *Weyl–Heisenberg* systems[1] correspond to a rectangular tiling of the time–frequency plane, the g_l are modulated versions of a prototype function g_0:

  $$g_l(x) = g_0(x)e^{2\pi ibl x}\,.$$

  Note that in order to have existence of Riesz families, one necessarily has $b \geq 1/a$.

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- The real-valued *Wilson* bases[2] have a structure related to but different from the Weyl–Heisenberg systems:

$$\begin{aligned}
g_0(x) &= g(x)\,, \\
g_m^{(1)}(x) &= g(x)\,\sqrt{2}\cos(2\pi\tfrac{2m}{a}x)\,, \qquad & g_m^{(2)}(x) &= g(x-\tfrac{a}{2})\,\sqrt{2}\cos(2\pi\tfrac{2m-1}{a}x)\,, \\
g_m^{(3)}(x) &= g(x)\,\sqrt{2}\sin(2\pi\tfrac{2m-1}{a}x)\,, \qquad & g_m^{(4)}(x) &= g(x-\tfrac{a}{2})\,\sqrt{2}\sin(2\pi\tfrac{2m}{a}x)\,, \\
& & m &= 1,\dots,M \ \ (\text{i.e., } N = 4M{+}1)\,.
\end{aligned}$$

- The popular dyadic *wavelet* bases[3]:

$$g_m^{(n)}(x) = 2^{m/2} g_0\big(2^m(x-n\frac{a}{2^m})\big)\,, \qquad m = 0,1,\dots,M\,, \quad n = 0,1,\dots,2^m-1 \quad (\text{i.e., } N = 2^{M+1}-1)\,.$$

The transmission signal is given by a doubly-indexed series

$$f(x) = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{N-1} c_{k,l}\, g_l(x-ak)\,,$$

where $c_{k,l}$ are the information bearing complex-valued coefficients. In digital communication applications these coefficients are elements of a finite alphabet ("QAM Constellation"), but for our purpose it is more appropriate to assume a Hilbert space setting, i.e., $\{c_{k,l}\} \in \boldsymbol{\ell}^2$.

After transmission over a physical communication channel, the received signal can be split up into a linearly transformed version of the transmitted signal and statistically independent additive noise n, so we obtain

$$r(x) = (K\,f)(x) + n(x)\,.$$

We assume throughout this paper that the channel distortion corresponds to a translation invariant system, i.e.,

$$(K\,f)(x) = (K_h\,f)(x) = (h{*}f)(x) = \int_{\mathbb{R}} h(x-y)\,f(y)\,dy$$

for some $h \in \boldsymbol{L}^2(\mathbb{R})$. It should be emphasized, however, that strict translation invariance is always an approximation whose validity has to be checked for the critical time scale in question. In the present context the critical scale is the length of the (finite support) prototype function g_0 which is (by the latency constraints for speech communication) short enough that K can well be considered as a convolution. Since h and thus K_h are not fixed, but will vary from case to case, we consider the following ensemble of possible impulse responses:

$$\mathcal{H} = \big\{h \in \boldsymbol{L}^2(\mathbb{R}) : \operatorname{supp} h \subseteq [-\tfrac{x_0}{2}, +\tfrac{x_0}{2}]\,, \int_{\mathbb{R}} |h(x)|^2\,x\,dx = 0\,, \|\widehat{h}\|_{\boldsymbol{L}^\infty} = \sup|\widehat{h}(\xi)| = 1\big\}\,. \tag{2}$$

The three conditions imposed on h seem realistic for the following reasons:

- The receiver does not know when the transmission starts, so he has to fix the time $t = 0$ in some way. Since this is equivalent to choosing some translate of h, we may as well fix h to have vanishing first moment.
- Although h does not have compact support, we may cut it off at some point and treat the influence of the remaining part as noise.
- Consequently, we have $h \in \boldsymbol{L}^1(\mathbb{R})$, so $\widehat{h} \in \boldsymbol{L}^\infty(\widehat{\mathbb{R}})$, and we may normalize h in some arbitrary way by assuming an appropriate amplifier.

Outline of the paper

In the following section, we introduce the concept of orthogonal perturbation, and derive upper and lower bounds on this quantity for a given function under a class of channel operators. These bounds are formulated in terms of the spectral variance and the temporal support of the prototype function. The lower bound is obtained by the approximate design of a worst case operator via an interpolation procedure.

In Section 3, we compare the three above-mentioned structures of coherent Riesz bases using these upper and lower bounds. The numerical parameters we use are chosen to be compatible with the digital subscriber loop setup.

For shortness' sake, we omit practically all proofs and refer the interested reader to an upcoming publication[4] for technical details.

Notation

For the Fourier transformation, we use the normalization

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i \xi x}\, dx \quad \text{for } \xi \in \widehat{\mathbb{R}} = \mathbb{R}.$$

Consequently, we can define the inverse Fourier transformation via

$$\overset{\vee}{g}(x) = \int_{\widehat{\mathbb{R}}} g(\xi)\, e^{2\pi i \xi x}\, d\xi$$

to obtain $(\widehat{f})^{\vee} = f$.

We define the translation operator by

$$(\tau_y\, f)(x) = f(x{-}y)\,,$$

it has the property

$$(\tau_\eta\, \widehat{\varphi})^{\vee} = e^{2\pi i \eta x}\, \varphi\,.$$

2. ORTHOGONAL PERTURBATIONS

As mentioned in the introduction, an optimal function system $\{g_{k,l}\}$ would consist of eigenfunctions of K_h. Since this is impossible to achieve for all h, we aim for approximate eigenfunctions and use the orthogonal perturbation of the g_l by K_h as a measure of stability, i.e.,

$$d_{g,h} = \left\| K_h g - P_{<g>}(K_h g) \right\|_{\boldsymbol{L}^2}\,,$$

where $P_{<g>}$ is the orthogonal projection onto the span of g, given by $P_{<g>}(K_h g) = \frac{\langle K_h g, g\rangle}{\langle g, g\rangle}\, g$ (cf., Figure 1).

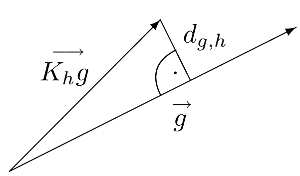

Figure 1.

Assuming $\langle g, g\rangle = \|g\|^2 = 1$, we obtain by the Pythagorean theorem

$$d_{g,h}^2 = \|K_h g\|^2 - |\langle K_h g, g\rangle|^2\,. \tag{3}$$

Since the convolution $K_h g = h * g$ corresponds to multiplication in the Fourier domain, $d_{g,h}$ can be related to the frequency localization of g, as the following lemma shows.

Lemma 2.1. *Let g, $h \in \boldsymbol{L}^2(\mathbb{R})$ with $\|g\|_{\boldsymbol{L}^2} = 1$. Then*

$$d_{g,h}^2 = \mathrm{V}\left\{\widehat{h}(\Xi)\right\}, \tag{4}$$

where Ξ is a random variable with probability density $|\widehat{g}|^2$, i.e.,

$$\mathrm{V}\left\{\widehat{h}(\Xi)\right\} = \int_{\widehat{\mathbb{R}}} \left|\widehat{h}(\xi) - \mathrm{E}\{\widehat{h}(\Xi)\}\right|^2 |\widehat{g}(\xi)|^2\, d\xi$$

with expected value

$$\mathrm{E}\{\widehat{h}(\Xi)\} = \int_{\widehat{\mathbb{R}}} \widehat{h}(\xi)\, |\widehat{g}(\xi)|^2\, d\xi\,.$$

Upper bound

Using the identity (4), we can find an upper bound for the orthogonal perturbation $d_{g,h}$ for all $h \in \mathcal{H}$. For simplicity, we define

$$d_g = \sup_{h \in \mathcal{H}} d_{g,h} \,.$$

PROPOSITION 2.2. *For $g \in \boldsymbol{L}^2(\mathbb{R})$ with $\|g\|_{\boldsymbol{L}^2} = 1$, we have*

$$d_g^2 \le (\pi x_0)^2 \sigma^2_{|\widehat{g}|^2} \,,$$

where $\sigma^2_{|\widehat{g}|^2}$ is the variance of $|\widehat{g}|^2$, i.e.,

$$\sigma^2_{|\widehat{g}|^2} = \int_{\widehat{\mathbb{R}}} (\xi - \mu)^2 \, |\widehat{g}(\xi)|^2 \, d\xi \qquad \text{with} \qquad \mu = \mu_{|\widehat{g}|^2} = \int_{\widehat{\mathbb{R}}} \xi \, |\widehat{g}(\xi)|^2 \, d\xi \,.$$

REMARK. The upper bound in Proposition 2.2 does not make sense whenever the decay of $|\widehat{g}|$ is too slow (e.g., if g is not continuous). In that case, we can obtain a more conservative (though less elegant) bound by using the following measure of spectral spread instead of the variance:

$$\widetilde{\sigma}^2_{|\widehat{g}|^2} = \left(\int_{-F}^{-F/3} |\widehat{g}(\xi - \xi_c)|^2 \, d\xi \right) \left(\int_{+F/3}^{+F} |\widehat{g}(\xi - \xi_c)|^2 \, d\xi \right),$$

where F and ξ_c characterize the ε–essential support of $\widehat{g}$ in the sense that

$$\int_{-\infty}^{-F} |\widehat{g}(\xi - \xi_c)|^2 \, d\xi = \int_{+F}^{+\infty} |\widehat{g}(\xi - \xi_c)|^2 \, d\xi = \tfrac{\varepsilon}{2}$$

for an appropriate $\varepsilon > 0$.

Lower bound

On the other hand, one must expect that signals which are not well localized on the frequency side potentially undergo a relatively strong orthogonal perturbation. Clearly, for a given convolution operator there might be arbitrarily bad localized functions g which are exact eigenfunctions of this specific operator, so $d_{g,h} = 0$ for this particular h — but for practical purposes, we require a family of basis functions that are stable under the action of all $h \in \mathcal{H}$. Therefore, to be able to show that certain families are inadequate, we want to determine a lower bound for d_g. To this end, we shall use the following kind of uncertainty principle obtained by Slepian, Pollak, and Landau.[5–7]

LEMMA 2.3. *Let $f \in \boldsymbol{L}^2(\mathbb{R})$ with $\operatorname{supp} f \subset [-\frac{T}{2}, +\frac{T}{2}]$. Then we have for all $\Omega > 0$ that*

$$\int_{-\Omega/2}^{+\Omega/2} |\widehat{f}(\xi)|^2 \, d\xi \le \lambda_0 \|f\|^2 \,,$$

where $\lambda_0 = \lambda_0(\Omega, T)$ is the square of the largest eigenvalue of the operator

$$\begin{aligned} O_{\Omega,T} : \quad \boldsymbol{L}^2(\mathbb{R}) \quad &\longrightarrow \boldsymbol{L}^2(\mathbb{R}) \,, \\ f \quad &\longmapsto \int_{-T/2}^{T/2} f(x) \, \frac{\sin(\pi\Omega(\cdot - x))}{\pi(\cdot - x)} \, dx \,. \end{aligned}$$

A scaling argument shows that λ_0 only depends on the product $\Omega \, T$. The eigenfunctions of $O_{\Omega,T}$ are the so-called prolate spheroidal wave functions, which have been studied extensively as solutions of the second-order differential equation eigenvalue problem[8]

$$\frac{d}{dx}\left((1 - x^2) \frac{d\psi}{dx} \right) + (\lambda - c^2 x^2) \psi = 0 \,.$$

We can obtain a somewhat weak upper bound on the operator norm of $O_{\Omega,T}$ using the following lemma.[9]

LEMMA 2.4. *(i) Let $A \subset \mathbb{R}$ and $B \subset \widehat{\mathbb{R}}$ be sets of finite measure. Define the operator $P_A : \boldsymbol{L}^2(\mathbb{R}) \to \boldsymbol{L}^2(\mathbb{R})$, $f \mapsto \chi_A f$, and the operator $Q_B : \boldsymbol{L}^2(\mathbb{R}) \to \boldsymbol{L}^2(\mathbb{R})$, $f \mapsto (\chi_B \widehat{f})^\vee = \overset{\vee}{\chi_B} * f$. Then $\|Q_B P_A\|_{\mathcal{L}(\boldsymbol{L}^2)} \leq \sqrt{m(A)\, m(B)}$, where m denotes Lebesgue measure on $\mathbb{R}$.*

(ii) For $f \in \boldsymbol{L}^2(\mathbb{R})$ with $\operatorname{supp} f \subseteq [\alpha, \alpha + T]$ for some $\alpha \in \mathbb{R}$, we have

$$\int_{-\Omega/2}^{+\Omega/2} |\widehat{f}(\xi)|^2 \, d\xi \leq \Omega T \, \|f\|_{\boldsymbol{L}^2}^2 .$$

In order to find a lower bound on d_g^2, we consider a particular family of convolution operators K_h with $h \in \mathcal{H}$.

LEMMA 2.5. *For $N \in \mathbb{N}$, there exists $h_N \in \mathcal{H}$ with*

$$\widehat{h}_N \leq \widehat{k} \quad \text{on } [-\tfrac{2N-1}{x_0}, 0] \qquad \text{and} \qquad \widehat{h}_N \geq \widehat{k} \quad \text{on } [0, +\tfrac{2N-1}{x_0}], \tag{5}$$

where $\widehat{k} : \widehat{\mathbb{R}} \to \mathbb{R}$ is given by

$$\widehat{k}(\xi) = \begin{cases} -0.9, & \text{for } \xi \leq -\frac{1}{x_0}, \\ 0.9\, x_0\, \xi, & \text{for } \xi \in [-\frac{1}{x_0}, +\frac{1}{x_0}], \\ +0.9, & \text{for } \xi \geq +\frac{1}{x_0} \end{cases}$$

(compare Figure 2).

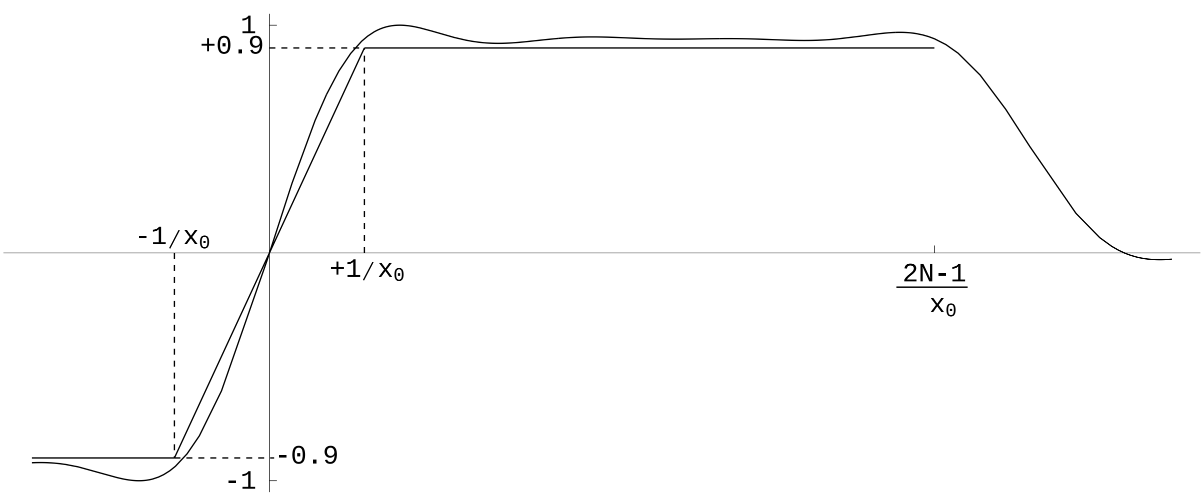

Figure 2. Graphs of $\widehat{h}_N$ and $\widehat{k}$.

Making use of these particularly bad channel functions, we obtain the following terms as lower bounds for d_g^2.

PROPOSITION 2.6. *For $g \in \boldsymbol{L}^2(\mathbb{R})$, $\|g\|_{\boldsymbol{L}^2} = 1$, with $\operatorname{supp} g \subseteq [\alpha, \alpha + T]$ for some $\alpha \in \mathbb{R}$ and $T > 0$, we have*

$$d_g^2 \geq 0.9^2 \Big(1 - \frac{4}{3}\, \frac{T}{x_0}\Big) \qquad \text{for } \frac{T}{x_0} \leq \frac{1}{2},$$

$$\text{and} \qquad d_g^2 \geq \frac{1}{12} \Big(\frac{0.9\, x_0}{T}\Big)^2 \qquad \text{for } \frac{T}{x_0} > \frac{1}{2}.$$

REMARK. To obtain a lower bound for d_g^2 in Proposition 2.6, we used the upper bound for $\|Q_B P_A\|_{\mathcal{L}(\boldsymbol{L}^2)}$ from Lemma 2.4. But for the case $B = \big[-\frac{\Omega}{2}, +\frac{\Omega}{2}\big]$ and $A = [T, T{+}\alpha]$, Lemma 2.3 provides a sharp upper bound of $\|Q_B P_A\|_{\mathcal{L}(\boldsymbol{L}^2)} = \|O_{\Omega,T}\|_{\mathcal{L}(\boldsymbol{L}^2)}$ in terms of the largest eigenvalue $\sqrt{\lambda_0(\Omega, T)}$ of the operator $O_{\Omega,T}$. Since this eigenvalue only depends on the product of Ω and T, we shall write $\lambda_0(\Omega, T) = \lambda_0(\Omega \cdot T)$. If in the proof of Proposition 2.6, we use this sharp bound, we get

$$d_g^2 \geq 0.9^2 \int_0^1 1 - \lambda_0\big((2\tfrac{1}{x_0}\sqrt{t}) \cdot T\big)\, dt \, .$$

The graph of this lower bound for d_g^2 (dashed) as well as the graph of that obtained in Proposition 2.6 (solid line) are shown in Figure 3.

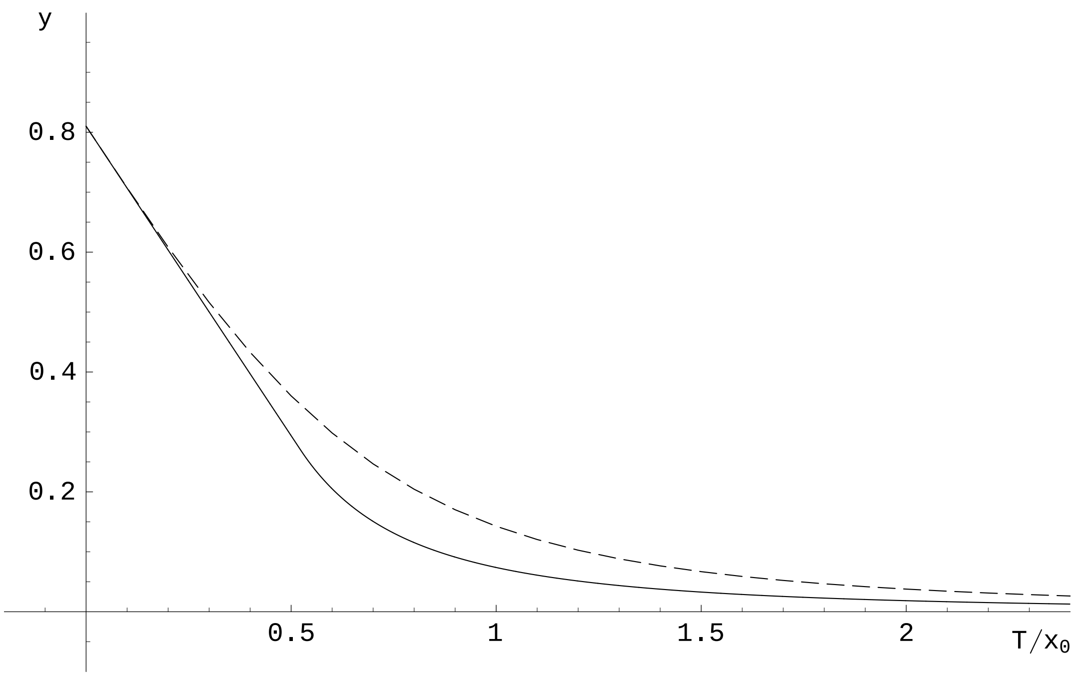

Figure 3. Lower bounds for d_g^2.

We also should note that $\sqrt{\lambda_0(\Omega \cdot T)}$ is always a simple eigenvalue of the operator $O_{\Omega,T}$. For $\Omega \cdot T < 1$ (i.e., $\frac{T}{x_0} < \frac{1}{2}$), the second largest eigenvalue is already considerably smaller. This reflects the fact that only a number of about $\Omega \cdot T$ linearly independent functions have "approximate duration" $[0, T]$ and "approximate bandwidth" $[-\frac{\Omega}{2}, +\frac{\Omega}{2}]$.[7] Consequently, we see that unless we use for g the appropriate spheroidal wave function itself, d_g^2 will be significantly bigger than the bound given above.

3. ORTHOGONAL PERTURBATIONS OF COHERENT FAMILIES

We now want to compare the three types of coherent families described at the beginning with respect to their performance under orthogonal perturbation. As for the parameters, we assume that $\operatorname{supp} g \subseteq [0,a]$ and $\operatorname{supp} h \subseteq [-\frac{x_0}{2}, +\frac{x_0}{2}]$ satisfy

$$a = 50\, x_0 \,.$$

For the number of elements N in the family, $N \geq 256$ seems realistic; in VDSL applications, $N \approx 2000$ is used.

Weyl–Heisenberg families

Recall that a Weyl–Heisenberg family is generated by fixing a basic function g_0 with $\operatorname{supp} g_0 \subseteq [0,a]$ and then letting $g_l(x) = g_0(x)\, e^{2\pi i b l x}$. Thus we have $\operatorname{supp} g_l = \operatorname{supp} g_0$ and $|\widehat{g}_l|^2 = \tau_{bl}\, |\widehat{g}_0|^2$. Since the variance is translation invariant, we have $\sigma^2_{|\widehat{g}_l|^2} = \sigma^2_{|\widehat{g}_0|^2}$ for all g_l, so the upper bound from Proposition 2.2 holds uniformly in $h \in \mathcal{H}$ and $l = 0 \ldots N{-}1$.

Using for g_0 a triangle function, a trapezoidal function, or the polynomial $x^2(x-a)^2$ (properly normalized) yields

$$d_g^2 \doteq 0.0012 \,.$$

It is worth emphasizing that the main property ensuring this uniform upper bound is the fact that within a Weyl–Heisenberg family, all $\widehat{g}_l$ share the same frequency localization.

Wilson bases

In a Wilson basis, the Fourier transforms of the elements satisfy

$$|\widehat{g}_m^{(j)}(\xi)|^2 = \tfrac{1}{2}\left|\widehat{g}(\xi{+}\xi_0) \pm \widehat{g}(\xi{-}\xi_0)\right|^2, \tag{6}$$

in particular, for $j = 1$ we have "+" and $\xi_0 = \frac{2m}{a}$. Thus the variance of $\left|\widehat{g}_m^{(1)}\right|^2$ increases with m. Using the appropriate $h_N \in \mathcal{H}$ from Lemma 2.5 shows that the orthogonal perturbation turns bad quickly, as the following result shows (compare Figure 4).

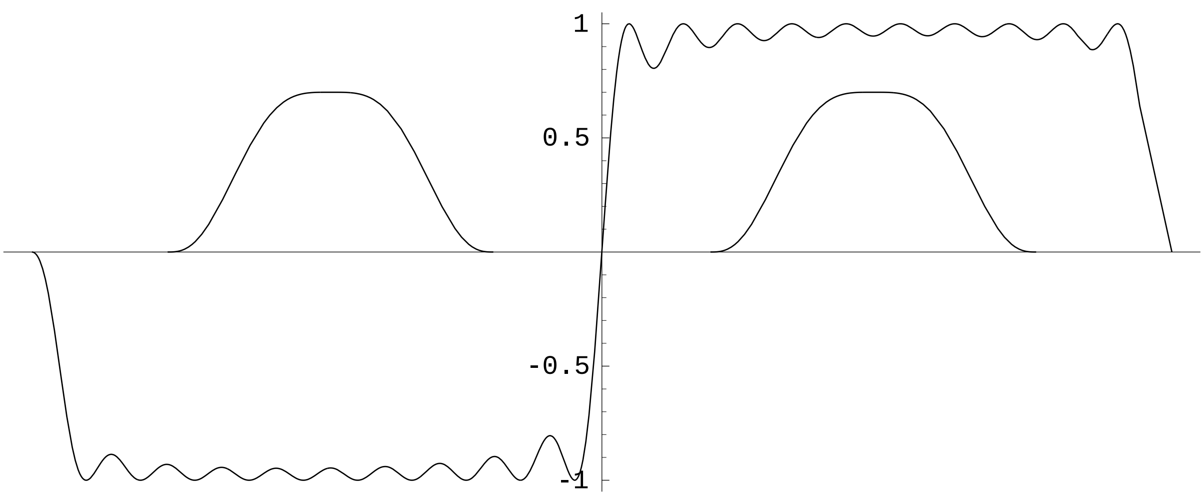

Figure 4. Graphs of $\widehat{h}_N$ and $|\widehat{g}_m^{(j)}|^2$ for large m in a Wilson basis.

THEOREM 3.1. *In a Wilson basis with at least 200 elements, there is an element g_l with*

$$d_{g_l}^2 \geq 0.16 \,.$$

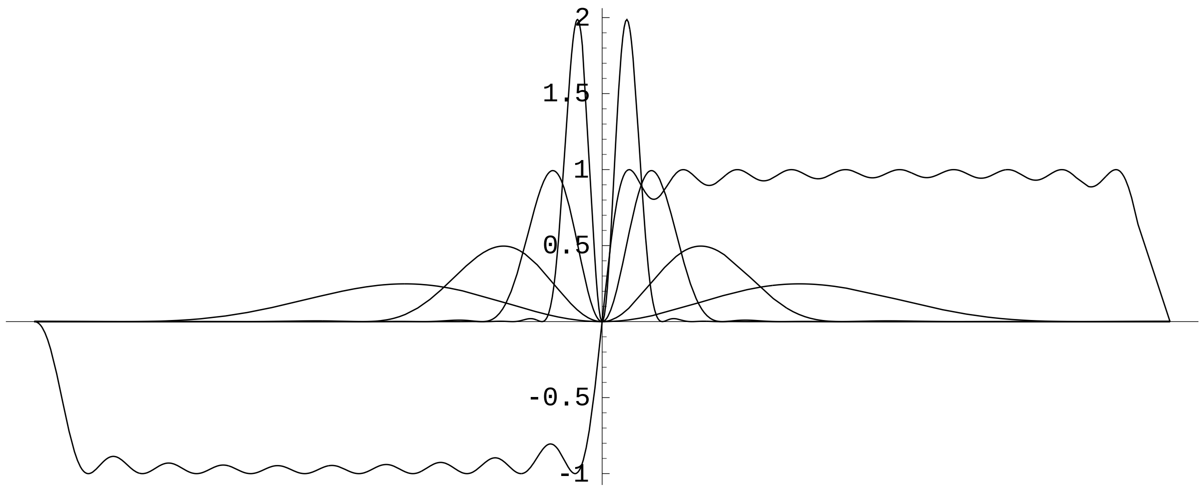

Figure 5. Graphs of $\widehat{h}_N$ and $|\widehat{g}_m^{(n)}|^2$ for various m in a wavelet basis.

Wavelet bases

In a dyadic wavelet basis, we encounter the problem that, since scaling on the time side results in reverse scaling on the frequency side, the frequency localization gets worse and worse as the indices grow (compare Figure 5). The following result gives a quantitative estimate of this effect.

Theorem 3.2. *In a dyadic wavelet family with finest scaling level $M \geq 7$, the elements $g_M^{(n)}$ on level M satisfy*

$$d^2_{g_M^{(n)}} \geq 0.81\,(1 - 67 \cdot 2^{-M})\,.$$

For $N > 128$, we need $M \geq 7$ which yields $d^2_{g_7^{(n)}} \geq 0.386$; for $N > 256$ with $M \geq 8$ we obtain $d^2_{g_8^{(n)}} \geq 0.598$.

Remark. The numerical results presented above demonstrate very clearly that the Weyl–Heisenberg systems outperform the other two types of coherent families by far. Standardized implementations of so-called multicarrier communication systems such as OFDM (orthogonal frequency division multiplex) or DMT (discrete multi-tone) are based on the Weyl–Heisenberg structure using indicator functions of different lengths at the transmitter and receiver.[1,3] There, the transmission basis is usually defined as

$$g_{k,l}(x) = \chi_{[-x_0,T]}(x-ak)\, e^{2\pi i blx}\,,$$

where $a = T+x_0$ and $b = 1/T$; i.e., a nonorthogonal Riesz basis whose span covers functions that contain a so-called cyclic prefix of length x_0. This means that within the interval $[kT-x_0, kT+T]$, one has $f(x) = f(x+T)$ for $x \in [kT-x_0, kT]$. The orthonormal basis at the receiver can be interpreted to be cutting off the cyclic prefix, since

$$\gamma_{k,l}(x) = \chi_{[0,T]}(x-ak)\, e^{2\pi i blx}\,.$$

It is straightforward to prove exact diagonalization of convolution operators with h supported in $[0, x_0]$ by this biorthogonal basis, i.e., we have

$$\big\langle K_h\, g_{k,l}\,, \gamma_{k',l'} \big\rangle = \widehat{h}\big(\tfrac{l}{T}\big)\, \delta_{k,k'}\, \delta_{l,l'}\,.$$

However, such an exact diagonalization is achieved at the cost of wasted bandwidth, since on the one hand, the space $\overline{\text{span}}\{g_{k,l}\}_{k\in\mathbb{Z},\, l=1\ldots N}$ shrinks with increasing x_0, and on the other hand, the frequency localization of the subcarriers $\overline{\text{span}}\{g_{k,l}\}_{l=1\ldots N}$ worsens. A more detailed discussion of this and other tradeoffs in the design of Weyl–Heisenberg structured signal sets for digital communication can be found in[10] and the references therein.

CONCLUSION

We have shown that among the prominent coherent function systems we discussed (Gabor bases, Wilson bases, and wavelets), the Gabor bases are best matched to a set of convolution operators with practical importance.

Based on this result, the remaining open questions concern a design tradeoff for Weyl–Heisenberg structured bases. Biorthogonal Gabor systems can be chosen to be both highly bandwidth efficient and well localized in frequency, but their diagonalization is only approximate; the cyclic prefix trick enables exact diagonalization but wastes bandwidth and leads to bad frequency localization.

ACKNOWLEDGEMENTS

The authors would like to express their thanks to K. Jetter for fruitful conversations and constructive comments.

Also, sponsoring by the German Ministry of Education and Science (BMBF) within the KOMNET program under grant 01 BP 902 is gratefully acknowledged.

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