Note on B-splines, wavelet scaling functions, and Gabor frames

Karlheinz Gröchenig, Augustus J. E. M. Janssen, Senior Member, IEEE, Norbert Kaiblinger, Götz E. Pfander

Abstract—**Let g be a continuous, compactly supported function on $\mathbb{R}$ such that the integer translates of g constitute a partition of unity. We show that the Gabor system (g,a,b), with window g and time-shift and frequency-shift parameters $a,b>0$ has no lower frame bound larger than 0 if $b=2,3,\dots$ and $a>0$. In particular, (g,a,b) is not a Gabor frame if g is a continuous, compactly supported wavelet scaling function and if $b=2,3,\dots$ and $a>0$. We exemplify our result for the case that $g=B_1$, the triangle function supported by $[-1,1]$, by showing pictures of the canonical dual corresponding to (g,a,b) when $ab=1/4$ and b crosses the lines $N=2,3,\dots$.**

Keywords—**B–splines, Gabor frame, partition of unity, Ron–Shen condition, wavelet scaling function.**

I. Introduction

Let $a>0$, $b>0$ and $g\in L^2(\mathbb{R})$. We call the function system

$$\left(g_{na,mb}\right)_{n,m\in\mathbb{Z}} \equiv (g,a,b) \tag{1}$$

a Gabor frame if there are $A>0$, $B<\infty$ such that for all $f\in L^2(\mathbb{R})$

$$A\|f\|^2 \le \sum_{n,m} \left|\langle f, g_{na,mb}\rangle\right|^2 \le B\|f\|^2 . \tag{2}$$

Here $g_{x,y}$ denotes for $x,y\in\mathbb{R}$ the time-frequency shifted function

$$g_{x,y}(t) = e^{2\pi i y t} g(t-x), \quad t\in\mathbb{R} . \tag{3}$$

The numbers A and B that appear in (2) are called lower and upper frame bound respectively. It is well-known that (g,a,b) can be a Gabor frame only if $ab\le 1$; also, if $ab=1$ and (g,a,b) is a Gabor frame, then g cannot be continuous and compactly supported. We refer for basic information about (Gabor) frames to [1, Sections 3.4, 3.5, 4.1, 4.2]; a comprehensive and recent treatment of Gabor systems and frames can be found in [2, Chapters 5-9, 11-13]. We shall use here the following criterion, due to Ron and Shen [3], for being a Gabor frame, see [2, p. 117, Proposition 6.3.4]; for convenience we restrict ourselves to continuous, compactly supported windows g. The system (g,a,b) is a Gabor frame with frame bounds $A>0$, $B<\infty$ if and only if for all $c\in l^2(\mathbb{Z})$ and all $t\in\mathbb{R}$

$$A\|c\|^2 \le \frac{1}{b} \sum_{n=-\infty}^{\infty} \Big| \sum_{k=-\infty}^{\infty} g(t-na-k/b)c_k \Big|^2 \le B\|c\|^2 . \tag{4}$$

For $g\in L^2(\mathbb{R})$ we denote by F_g the set

$$F_g = \left\{ (a,b) \mid a>0,\, b>0,\, (g,a,b) \text{ is a Gabor frame} \right\}. \tag{5}$$

It is often quite hard to determine the set F_g for a given window $g\in L^2(\mathbb{R})$. In certain cases, for instance when one has restrictions on the supporting set of g, the Ron-Shen criterion can be of great help in telling whether (a,b) belongs to F_g. In [4, Section 3] a considerable effort has been made to determine F_g for the case that $g=B_0=\chi_{[0,1)}$; the result is a complicated subset of $\left\{ (a,b) \mid a>0,\, b>0,\, ab\le 1 \right\}$ where (ir)rationality of ab plays a key role. Furthermore, only in a few cases of well-behaved windows g it has been shown that $F_g = \left\{ (a,b) \mid a>0,\, b>0,\, ab<1 \right\}$; among these g are the Gaussians [5, 6], hyperbolic secants [7] and two sided exponentials [8, Section 5].

In this correspondence we ask the question whether for certain standard windows g from approximation theory and wavelet theory the set F_g consists of all (a,b) with $a>0$, $b>0$ and $ab<1$ as well. Unlike the example $g=B_0$ given above, the windows of this type are smooth and well decaying, which implies that the sets F_g are open sets [9]. We shall show in Section II that for any continuous, compactly supported g satisfying the partition-of-unity identity

$$\sum_{k=-\infty}^{\infty} g(t-k) = 1 , \quad t\in\mathbb{R}, \tag{6}$$

no lower frame bound $A>0$ for the Gabor system (g,a,b) exists when $a>0$ and $b=2,3,\dots$. Condition (6) can be shown to hold for large classes of windows, in particular it is satisfied for some commonly used windows in signal processing, such as the raised cosine

$$\mathrm{RC}(t) = \begin{cases} (1+\cos\pi t)/2, & \text{when } |t|\le 1, \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

and the trapezoidal function, for $0<\delta<1/2$ defined as

$$\mathrm{T}(t) = \begin{cases} 1, & |t| \le 1/2-\delta, \\ (2\delta)^{-1}(1/2+\delta-|t|), & \big||t|-1/2\big| < \delta, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

K. Gröchenig is with the Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009 (e-mail: groch@math.uconn.edu).

A. J. E. M. Janssen is with the Philips Research Laboratories, WY-81, 5656 AA Eindhoven, The Netherlands (e-mail: a.j.e.m.janssen@philips.com).

N. Kaiblinger is with the Department of Mathematics, University of Vienna, Strudlhofgasse 4, A-1090 Vienna, Austria (e-mail: norbert.kaiblinger@univie.ac.at).

G. E. Pfander is with the School of Engineering and Science, International University Bremen, Campus Ring 1, D-28759 Bremen, Germany (e-mail: g.pfander@iu-bremen.de).

Further, condition (6) on g is satisfied when g is a compactly supported, continuous scaling function from the theory of wavelets [1, Remark 5 on pp. 144–145 and Note 9 on p. 165], or when g is a B-spline with knots at the integers [10]. Hence, in these cases the set F_g consists apparently of countably many open sets, separated from one another by the horizontal lines $b = N = 2, 3, \ldots$.

This result came as a big surprise to us, because it is generally assumed that any "nice" window g and "reasonable" choice of $a, b > 0$ yield Gabor frames. Our observation demonstrates that even for window functions that are perfectly natural in approximation theory and wavelet theory, a and b have to be chosen extremely carefully.

In Section III we consider the linear B-spline or triangle function $g(t) = B_1(t) = \max(0, 1 - |t|)$ as an example. We show pictures of the canonical dual function corresponding to the Gabor system (g, a, b) with $ab = 1/4$ and b close to 2 and 3. It is conjectured that F_g consists of all (a, b) with $0 < a < 2$, $b > 0$, $ab < 1$ and $b \neq 2, 3, \ldots$. The pictures in Section III support this conjecture.

II. Proof of the main result

We assume that g is continuous and compactly supported (below we comment on weakening those conditions), and that g satisfies (6). Also, we let $a > 0$ and $b = 2, 3, \ldots$. We shall show that the Gabor system (g, a, b) has no positive lower frame bound (the Ron-Shen criterion (4) implies that (g, a, b) has a finite upper frame bound). To that end we shall display $c^K \in l^2(\mathbb{Z})$, $K = 1, 2, \ldots$, such that for all $t \in \mathbb{R}$

$$\frac{1}{\|c^K\|^2} \sum_{n=-\infty}^{\infty} \Big| \sum_{k=-\infty}^{\infty} g(t - na - k/N)\, c_k^K \Big|^2 \to 0, \quad K \to \infty\,. \tag{9}$$

Let $b = N = 2, 3, \ldots$ and set

$$c_k = e^{2\pi i k r/N}, \quad k \in \mathbb{Z}, \tag{10}$$

where $r = 1, \ldots, N-1$. Then for all $t \in \mathbb{R}$, $n \in \mathbb{Z}$, we have

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} g(t - na - k/N)\, c_k \\ &= \sum_{k=0}^{N-1} \sum_{l=-\infty}^{\infty} g\big(t - na - \tfrac{k}{N} - l\big) e^{2\pi i r (\frac{k}{N} + l)} \\ &= \sum_{k=0}^{N-1} e^{2\pi i k r/N} \; = 0\,. \end{aligned} \tag{11}$$

For $K = 1, 2, \ldots$ we define the sequence $c^K \in l^2(\mathbb{Z})$ by

$$c_k^K = \begin{cases} c_k\,, & |k| \le K\,, \\ 0\,, & |k| > K\,, \end{cases} \quad k \in \mathbb{Z}\,, \tag{12}$$

and the subsets of indices

$$\begin{aligned} V_k &= \big\{ l \in \mathbb{Z} \mid g(t - na - \tfrac{k}{N} - l) \neq 0 \big\}\,, \\ W_k &= \big\{ l \in \mathbb{Z} \mid c_{k+lN}^K \neq 0 \big\}\,. \end{aligned} \tag{13}$$

Then the equality $\sum_k g(t - na - \frac{k}{N})\, c_k^K = 0$ holds for all $t \in \mathbb{R}$, $n \in \mathbb{Z}$ whenever either

$$V_k \subset W_k\,, \quad k = 0, \ldots, N-1\,, \tag{14}$$

or

$$V_k \cap W_k = \varnothing\,, \quad k = 0, \ldots, N-1\,. \tag{15}$$

Conversely, for $t \in \mathbb{R}$, $n \in \mathbb{Z}$ both (14) and (15) fail to be true if and only if $W_k \setminus V_k \neq \varnothing$, i.e., if there are $j_1, j_2 \in \mathbb{Z}$ such that

$$g(t - na - \tfrac{j_1}{N}) \neq 0 \neq g(t - na - \tfrac{j_2}{N}), \quad c_{j_1}^K = 0 \neq c_{j_2}^K\,. \tag{16}$$

Let $|\mathrm{I}|$ be the length of a supporting interval I of g. Then the number of $n \in \mathbb{Z}$ such that both (14) and (15) fail to hold is at most $2(|\mathrm{I}|/a + 1)$ for any $t \in \mathbb{R}$. Hence

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} \Big| \sum_{k=-\infty}^{\infty} g(t - na - k/N) c_k^K \Big|^2 \\ &\le 2(|\mathrm{I}|/a + 1) \sum_{k=-\infty}^{\infty} |g(t - na - k/N)|^2 \\ &\le 2(|\mathrm{I}|/a + 1)(MN(|\mathrm{I}| + 1))^2 \end{aligned} \tag{17}$$

where M is an upper bound for $|g|$. Since (17) is independent of K and since $\|c^K\| = (2K+1)^{1/2}$, we obtain (9) for any $t \in \mathbb{R}$.

Remark. The property (11) with c_k given in (10) can be phrased as $(Zg)(s, r/N) = 0$ for $s \in \mathbb{R}$, $r = 1, \ldots, N-1$. Here

$$(Zg)(s, \nu) = N^{-1/2} \sum_{k=-\infty}^{\infty} g\Big(\frac{s-k}{N}\Big) e^{2\pi i k \nu}, \quad s, \nu \in \mathbb{R}\,, \tag{18}$$

is a Zak transform of g, see [11, Section 1.5]. Using Gabor frame operator theory in the Zak transform domain, one can get the main result under considerably weaker conditions on g than the ones made above.

III. Example

In this section we consider the choice

$$g(t) = B_1(t) = \max(0, 1 - |t|)\,, \quad t \in \mathbb{R}, \tag{19}$$

and we display the canonical dual $\gamma_{a,b} = S_{a,b}^{-1}\, g$, see [2, Section 7.6] for the role of the canonical dual in Gabor analysis, for some values of $a > 0$, $b > 0$, with $ab = 1/4$ and b near 2 or 3 (Fig. 1). Here $S_{a,b}$ is the frame operator corresponding to (g, a, b), which is invertible when (g, a, b) is a frame. One easily sees from the Ron-Shen criterion in (4) or [2, Thm. 6.4.1] that (g, a, b) is a frame when $a < 2 \le 1/b$. For the other cases that are shown in the figure, we have only numerical evidence that the lower frame bound A for (g, a, b) is positive. The dual windows were approximated by considering sampled Gabor systems for $l^2(\mathbb{Z})$ and their dual systems, see [12] for details, with sample rates taken so high that further increasing them produced no visible changes in the figure anymore. As we see, the $\gamma_{a,b}$ so obtained turns from a well-behaved function for the values $b = 1.5$, 2.5 into a quite irregularly behaved one when b approaches 2 or 3.

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Karlheinz Gröchenig received his Ph.D. degree in mathematics from the University of Vienna, Austria, in 1985. In 1988 he joined the Department of Mathematics at The University of Connecticut where he is now Full Professor. His research interest is in applied harmonic analysis with a focus on wavelet theory, time-frequency analysis, and sampling theory. Dr. Gröchenig has written a book on the "Foundations of Time-Frequency Analysis" and more than 70 research articles.

A. (Augustus) J. E. M. Janssen (1953) received the Eng. degree and Ph.D. degree in mathematics from the Eindhoven University of Technology, Eindhoven, The Netherlands in October 1976 and June 1979, respectively.

From 1979 to 1981, he was a Bateman Research Instructor at the Mathematics Department of California Institute of Technology, Pasadena, USA. In 1981 he joined the Philips Research Laboratories, Eindhoven, where his principal responsibility is to provide high level mathematical service and consultancy in mathematical analysis. His research interest is in Fourier analysis with emphasis on time-frequency analysis, in particular Gabor analysis. His current research interests include the analysis of optical point spread functions and the retrieval of optical aberrations from intensity measurements using Nijboer-Zernike theory.

Dr. Janssen has published over 100 papers in the fields of signal analysis, mathematical analysis, Wigner distribution and Gabor analysis, information theory, electron microscopy, optics. Furthermore, he has published 39 internal reports, and he holds 6 US-patents. He received the prize for the best contribution to the Mathematical Entertainments column of the Mathematical Intelligencer in 1987 and the EURASIP's 1988 Award for the Best Paper of the Year in Signal Processing.

Norbert Kaiblinger received his Ph.D. degree in mathematics from the University of Vienna, Austria in 1999. Currently, granted by an Erwin Schrödinger fellowship of the Austrian Science Fund FWF, he is with the Georgia Institute of Technology, Atlanta.

Götz E. Pfander received his Ph.D. degree in mathematics from the University of Maryland in 1999. Since 2002 he is Assistant Professor of Mathematics at the International University Bremen, Germany. Dr. Pfander's interests include abstract harmonic analysis, wavelet and Gabor theory, and signal processing.

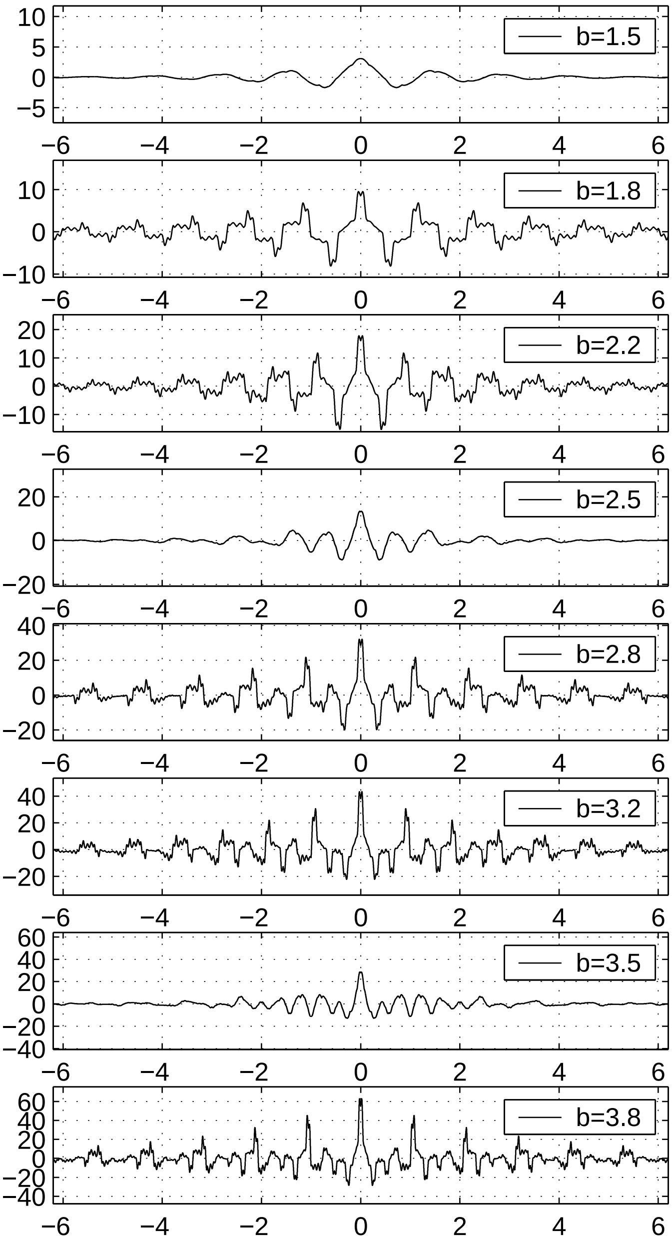

Fig. 1. The canonical Gabor dual $\gamma_{a,b} = S_{a,b}^{-1} g$ of the triangle function $g = B_1$, see (19), for $ab = 1/4$, $b = 1.5, 1.8, 2.2, 2.5, 2.8, 3.2, 3.5, 3.8$.