Uncertainty in time–frequency representations on finite Abelian groups and applications

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ABSTRACT

Classical and recent results on uncertainty principles for functions on finite Abelian groups relate the cardinality of the support of a function to the cardinality of the support of its Fourier transforms. We obtain corresponding results relating the support sizes of functions and their *short–time Fourier transforms*. We use our findings to construct a class of equal norm tight Gabor frames that are maximally robust to erasures. Also, we discuss consequences of our findings to the theory of recovering and storing signals with sparse time–frequency representations.

1. INTRODUCTION

Uncertainty principles establish restrictions on how well localized the Fourier transform of a well localized function can be and vice versa [DS89, Grö03, FS97]. In the case of a function defined on a finite Abelian group, localization is generally expressed through the cardinality of the support of the function. Due to its relevance for compressed sensing and, in particular, for the recovery of lossy signals under the assumption of restricted spectral content [CRT06], the uncertainty principle for functions on finite Abelian groups has recently drawn renewed interest.

In this realm, a classical result on uncertainty states that the product of the number of nonzero entries in a vector representing a nontrivial function on an Abelian group and the number of nonzero entries in its Fourier transform is not smaller than the order of the group [DS89, MÖP04]. This result can be improved for any nontrivial Abelian group [Mes06]. For example, for groups of prime order, the sum of the number of nonzero entries in a vector and the number of nonzero entries in its Fourier transform exceeds the order of the group [Tao05].

The objective of this paper is to establish corresponding results for joint time–frequency representations, that is, to obtain restrictions on the minimal cardinality of the support of joint time–frequency representations of functions defined on finite Abelian groups. For example, let us consider the simplest time–frequency representation of a function, namely the one that is given by the tensor product of a function and its Fourier transform. In this case, the classical uncertainty principle for nontrivial functions on finite Abelian groups states that the cardinality of the support of this tensor is at least the order of the group.

In this paper though our focus lies on time–frequency representations given by short–time Fourier transforms. It is easy to see that, again, the cardinality of the support of any short–time Fourier

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transform of a nontrivial function defined on a finite Abelian group is bounded below by the order of the group. As seen below, we can improve this bound by using the subgroup structure of the groups and/or by allowing only well-chosen window functions. For example, we establish in Theorem 4.4 that for any group of prime order and for almost every window function on the group, the sum of the cardinality of the support of the analyzed function and the cardinality of the support of its short–time Fourier transform exceeds the square of the order of the group.

In addition to the above, we give applications of our results to the theory of so-called Gabor frames and the theory of sparse signal recovery. For instance, the results on the cardinality of the support of short–time Fourier transforms can be translated into criteria for the recovery of encoded signals from a channel with erasures.

The paper is organized as follows. In Section 2 we give a brief but self-contained account of the Fourier transformation and of the short–time Fourier transformation for functions defined on finite Abelian groups. Section 3 reviews uncertainty principles that relate the cardinality of the support of functions with the cardinality of the support of their Fourier transforms. In addition, we provide numerical evidence on the achieved support set pairs for the Fourier transformation on groups of order less than or equal to 16.

Section 4 is devoted to uncertainty inherent in the short–time Fourier transformation. There, a discussion of general results is followed by results for functions defined on cyclic groups of prime order. Results on other finite Abelian groups are given. These are based on the subgroup structure of the underlying group as were recent improvements to the classical uncertainty result for Fourier transforms obtained in [Mes06]. We conclude our discussion of the cardinality of the support set of short–time Fourier transforms with a question on the possible cardinalities of the support of short–time Fourier transforms with respect to a optimally chosen window functions. Our results are complemented by numerical experiments.

In Section 5 we give applications of the results of Section 4 to communications engineering. In Section 5.1 we discuss the identification/measurement problem for time–varying operators/channels. Also, we consider channel coding for the transmission of information through channels with erasures. In addition, we show the existence of equal norm tight frames of Gabor type. In Section 5.2 we briefly discuss connections of our work to the recovery of signals which have a sparse representation in a given dictionary.

2. BACKGROUND AND NOTATION

For any finite set A we set $\mathbb{C}^A = \{f : A \longrightarrow \mathbb{C}\}$. For $|A| = |B| = n$, $\mathbb{C}^A \cong \mathbb{C}^B \cong \mathbb{C}^n$ as vector spaces, where $|A|$ denotes the cardinality of the set A. For $M \in \mathbb{C}^{m \times n}$ and $A \subseteq \{0, 1, \ldots, n-1\}$ and $B \subseteq \{0, 1, \ldots, m-1\}$, we let $M_{A,B}$ denote the $|B| \times |A|$–submatrix of M with columns and rows enumerated from the index sets B and A. For $f \in \mathbb{C}^A$, we use the now customary notation $\|f\|_0 = |\text{supp } f|$ where $\text{supp } f = \{a \in A : f(a) \neq 0\}$. Clearly, $\|\cdot\|_0$ is not a norm.

Throughout this paper, G denotes a finite Abelian group. The dual group of characters $\widehat{G}$ of G is the set of homomorphisms $\xi \in \mathbb{C}^G$ which map G into the multiplicative group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ [Ben97, Kat76, Ter99]. The set $\widehat{G}$ is an Abelian group under pointwise multiplication and, as is customary, we shall write this commutative group operation additively. Note that $G \cong \widehat{G}$ as groups

and Pontryagin duality implies that $\widehat{\widehat{G}}$ can be canonically identified with G, a fact which is emphasized by writing $\langle \xi, x \rangle$ for $\xi(x)$.

The Fourier transform $\mathcal{F}f = \widehat{f} \in \mathbb{C}^{\widehat{G}}$ of $f \in \mathbb{C}^G$ is given by $\widehat{f}(\xi) = \sum_{x \in G} f(x) \overline{\langle \xi, x \rangle}$, $\xi \in \widehat{G}$. The inversion formula for the Fourier transformation is $f(x) = \frac{1}{|G|} \sum_{\xi \in \widehat{G}} \widehat{f}(\xi) \langle \xi, x \rangle$, $x \in G$. It implies that $\|f\|_2^2 = \frac{1}{|G|} \sum_{\xi \in \widehat{G}} |\widehat{f}(\xi)|^2 = \frac{1}{|G|} \|\widehat{f}\|_2^2$, where $\|f\|_2 := (\sum_{x \in G} |f(x)|^2)^{\frac{1}{2}}$. Further, this fact together with $\|\xi\|_2 = |G|^{\frac{1}{2}}$ for all $\xi \in \widehat{G}$ implies that the normalized characters $|G|^{-\frac{1}{2}}\xi$, $\xi \in \widehat{G}$ form an orthonormal basis for $\mathbb{C}^G$, and $\sum_x \langle \xi, x \rangle = 0$ if $\xi \neq 0$ and $\sum_\xi \langle \xi, x \rangle = 0$ if $x \neq 0$.

For $n \in \mathbb{N}$ and $\omega = e^{2\pi i/n}$, the *discrete Fourier matrix* $W_{\mathbb{Z}_n}$ of the cyclic group $\mathbb{Z}_n$ is defined by $W_{\mathbb{Z}_n} = (\omega^{rs})_{r,s=0}^{n-1}$. Identifying $\mathbb{C}^{\mathbb{Z}_n}$ with $\mathbb{C}^n$, we have $\widehat{f} = W_{\mathbb{Z}_n} f$. An arbitrary finite Abelian group G can be represented as a direct product of cyclic groups $\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \ldots \times \mathbb{Z}_{d_m}$ where $d_1, \ldots, d_m$ can be chosen to be powers of prime numbers. A character in the dual group $\widehat{G}$ is then given by $\langle (\xi_1, \xi_2, \ldots, \xi_m), (x_1, x_2, \ldots, x_m) \rangle = \langle \xi_1, x_1 \rangle \langle \xi_2, x_2 \rangle \ldots \langle \xi_m, x_m \rangle$, where $(\xi_1, \xi_2, \ldots, \xi_m) \in \widehat{\mathbb{Z}}_{d_1} \times \widehat{\mathbb{Z}}_{d_2} \times \ldots \times \widehat{\mathbb{Z}}_{d_m} \cong \widehat{G}$. The discrete Fourier matrix W_G for $G = \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \ldots \times \mathbb{Z}_{d_m}$ is the Kronecker product of the Fourier matrices for the groups $\mathbb{Z}_{d_1}, \mathbb{Z}_{d_2}, \ldots, \mathbb{Z}_{d_m}$, that is, $W_G = W_{d_1} \otimes W_{d_2} \otimes \ldots \otimes W_{d_m}$. For example, we have

$$W_{\mathbb{Z}_4} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \quad \text{and} \quad W_{\mathbb{Z}_2 \times \mathbb{Z}_2} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Note that for appropriately chosen bijections $S_1 : \{0, 1, \ldots, |G|-1\} \longrightarrow G$ and $S_2 : \{0, 1, \ldots, |G|-1\} \longrightarrow \widehat{G}$ we have $\widehat{f} \circ S_2 = W_G(f \circ S_1)$ for $f \in \mathbb{C}^G$.

The translation operator T_x, $x \in G$, is the unitary operator on $\mathbb{C}^G$ given by $T_x f(y) = f(y-x)$, $y \in G$. Similarly, the modulation operator M_ξ, $\xi \in \widehat{G}$, is the unitary operator defined by $M_\xi f = f \cdot \xi$, where here and in the following $f \cdot g$ denotes the pointwise product of $f, g \in \mathbb{C}^G$. We have $\widehat{M_\xi f} = T_\xi \widehat{f}$. We refer to the unitary operators $\pi(\lambda) = M_\xi \circ T_x$ for $\lambda = (x, \xi) \in G \times \widehat{G}$ as time–frequency shift operators.

The short–time Fourier transformation $V_g : \mathbb{C}^G \longrightarrow \mathbb{C}^{G \times \widehat{G}}$ with respect to the window $g \in \mathbb{C}^G \setminus \{0\}$ is given by [FK98, FKL07, Grö01, Grö03]

$$V_g f(x, \xi) = \langle f, \pi(x, \xi) g \rangle = \sum_{y \in G} f(y) \overline{g(y-x) \langle \xi, y \rangle}, f \in \mathbb{C}^G, (x, \xi) \in G \times \widehat{G}.$$

The inversion formula for the short–time Fourier transform is

$$f(y) = \frac{1}{|G| \|g\|_2^2} \sum_{(x, \xi) \in G \times \widehat{G}} V_g f(x, \xi) \, g(y-x) \langle \xi, y \rangle, \quad y \in G, \tag{1}$$

that is, f can be composed of time–frequency shifted copies of any given $g \in \mathbb{C}^G \setminus \{0\}$. Further, $\|V_g f\|_2 = \sqrt{|G|} \|f\|_2 \|g\|_2$. The so-called Gabor system $\{\pi(x, \xi) g\}_{(x, \xi) \in G \times \widehat{G}}$ is clearly not an orthonormal basis if $|G| > 1$ since it consists of $|G|^2$ vectors in a $|G|$ dimensional space.

For $g \in \mathbb{C}^G$ and $x \in G$, we define the $|G| \times |G|$–diagonal matrix

$$D_{x,g} = \begin{pmatrix} g(S_1(0) + x) & & & 0 \\ & g(S_1(1) + x) & & \\ & & \ddots & \\ 0 & & & g(S_1(|G|-1) + x) \end{pmatrix}.$$

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Then, the $|G|\times|G|^2$–*full Gabor system matrix* with respect to g is given by

$$A_{G,g} = (D_{S_1(0),g}W_G \,|\, D_{S_1(1),g}W_G \,|\, \cdots \,|\, D_{S_1(|G|-1),g}W_G)^*, \tag{2}$$

where M^* denotes the adjoint of the matrix M. For example, for $G = \mathbb{Z}_4$, and $g = (1,2,3,4)^T$,

$$A_{\mathbb{Z}_4,(1,2,3,4)^T} := \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 2 & 2i & -2 & -2i & 3 & 3i & -3 & -3i & 4 & 4i & -4 & -4i & 1 & i & -1 & -i \\ 3 & -3 & 3 & -3 & 4 & -4 & 4 & -4 & 1 & -1 & 1 & -1 & 2 & -2 & 2 & -2 \\ 4 & -4i & -4 & 4i & 1 & -i & -1 & i & 2 & -2i & -2 & 2i & 3 & -3i & -3 & 3i \end{pmatrix}^*.$$

Similarly, for the group $G = \mathbb{Z}_2^2$ we have

$$A_{\mathbb{Z}_2^2,(1,2,3,4)^T} := \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 2 & -2 & 2 & -2 & 1 & -1 & 1 & -1 & 4 & -4 & 4 & -4 & 3 & -3 & 3 & -3 \\ 3 & 3 & -3 & -3 & 4 & 4 & -4 & -4 & 1 & 1 & -1 & -1 & 2 & 2 & -2 & -2 \\ 4 & -4 & -4 & 4 & 3 & -3 & -3 & 3 & 2 & -2 & -2 & 2 & 1 & -1 & -1 & 1 \end{pmatrix}^*. \tag{3}$$

Using the enumeration $S : \{0,1,\ldots,|G|^2-1\} \longrightarrow G\times\widehat{G}$ that is given by the lexicographic order induced by S_1 and S_2 described above on $G\times\widehat{G}$, we have $V_g f \circ S = A_{G,g}f$. Therefore, we shall refer to $A_{G,g}$ as short–time Fourier transform matrix with respect to the window g. Clearly, the rows of $A_{G,g}$ represent the vectors in the Gabor system $\{\pi(\lambda)g\}_{\lambda\in G\times\widehat{G}}$, and formula (1) implies that $A_{G,g}^* A_{G,g}$ is a multiple of the identity matrix.

3. UNCERTAINTY PRINCIPLES FOR THE FOURIER TRANSFORM ON FINITE ABELIAN GROUPS

The following uncertainty theorem for functions defined on finite Abelian groups is the natural starting point for our discussion [DS89].

THEOREM 3.1. *Let* $f \in \mathbb{C}^G\backslash\{0\}$, *then* $\|f\|_0 \cdot \|\widehat{f}\|_0 \geq |G|$.

A complementary result characterizes those f for which the bound in Theorem 3.1 is sharp [DS89, Smi90, MÖP04]. Namely, if k divides $|G|$, then there exists $f \in \mathbb{C}^G$ with $\|f\|_0 = k$ and $\|\widehat{f}\|_0 = \frac{|G|}{k}$. Further, if $\|f\|_0\|\widehat{f}\|_0 = |G|$ and $\operatorname{supp} f$ contains the identity element, then $\operatorname{supp} f$ is a subgroup of G. A generalization of Theorem 3.1 to non Abelian groups is given in [Mes92] and those f achieving the respective lower bounds are described in [Kan07].

Theorem 3.1 implies the weaker inequality $\|f\|_0 + \|\widehat{f}\|_0 \geq 2\sqrt{|G|}$ for $f \in \mathbb{C}^G \setminus \{0\}$. If G is a cyclic group of prime order, then this inequality and also Theorem 3.1 can be improved significantly [Fre04, Tao05].

THEOREM 3.2. *Let* $G = \mathbb{Z}_p$ *with* p *prime. Then* $\|f\|_0 + \|\widehat{f}\|_0 \geq |G| + 1$ *holds for all* $f \in \mathbb{C}^G\backslash\{0\}$.

As illustrated in [Tao05], Theorem 3.2 follows from combining Chebotarev's theorem on roots of unity which states that every minor of the Fourier transform matrix $W_{\mathbb{Z}_p}$, p prime, is nonzero [EI76, SL96, Tao05, Fre04], with

PROPOSITION 3.3. *Let* $M \in \mathbb{C}^{m\times n}$. *Then* $\|f\|_0 + \|Mf\|_0 \geq m+1$ *for all* $f \in \mathbb{C}^n$ *if and only if every minor of* M *is nonzero. Moreover, if every minor of* $M \in \mathbb{C}^{m\times n}$ *is nonzero and* k,l *are given with* $k+l \geq m+1$, *then there exists* $f \in \mathbb{C}^n$ *with* $\|f\|_0 = k$ *and* $\|Mf\|_0 = l$.

Proposition 3.3 in turn can be obtained using the following observation that will also be used in numerical experiments below.

LEMMA 3.4. *For $M \in \mathbb{C}^{m \times n}$ and $1 \leq k \leq m$, $1 \leq l \leq n$, there exists $f \in \mathbb{C}^n$ with $\|f\|_0 = k$ and $\|Mf\|_0 = l$ if and only if there exist sets $A \subseteq \{0, \ldots, n-1\}$ and $B \subseteq \{0, \ldots, m-1\}$ with $|A| = k$, $|B| = m - l$, and for all $a \in A$ and $y \in B^c$, we have*

$$\operatorname{rank} M_{A \setminus \{a\}, B} = \operatorname{rank} M_{A, B} = \operatorname{rank} M_{A, B \cup \{y\}} - 1 < |A|.$$

We refer for the proofs of Proposition 3.3 and Lemma 3.4 to [KPR07].

Theorem 3.2 improves on Theorem 3.1 but it applies only to cyclic groups of prime order since any other finite Abelian group G has proper subgroups leading to zero minors in W_G [MÖP04]. See [CR06] for estimates on the probability that for randomly chosen sets $T \subseteq G$ and $\Omega \subseteq \widehat{G}$ with $|T| + |\Omega| \leq G$ there exists $f \in \mathbb{C}^G$ with $\operatorname{supp} f = T$ and $\operatorname{supp} \widehat{f} = \Omega$.

Meshulam improved the bound in the classical uncertainty relation given in Theorem 3.1 for non-trivial finite Abelian groups of non-prime order [Mes06]. He defined for $0 < k \leq |G|$ the function

$$\theta(G, k) = \min \left\{ \|\widehat{f}\|_0 : \ f \in \mathbb{C}^G \text{ and } 0 < \|f\|_0 \leq k \right\}. \tag{4}$$

Using this notation, Theorem 3.2 implies that $\theta(\mathbb{Z}_p, k) = p + 1 - k$. The main result in [Mes06] is

THEOREM 3.5. *For $k \leq |G|$, let d_1 be the largest divisor of $|G|$ which is less than or equal to k and let d_2 be the smallest divisor of $|G|$ which is larger than or equal to k. Then*

$$\theta(G, k) \geq \frac{|G|}{d_1 d_2}(d_1 + d_2 - k). \tag{5}$$

A streamlined version of the Meshulam's proof by induction of Theorem 3.5 can be found in [KPR07] and [LM05] contains a non-inductive proof thereof.

Tao realized that Theorem 3.5 simply states that all possible support pairs $(\|f\|_0, \|\widehat{f}\|_0)$ lie in the convex hull of the points $(|H|, |G/H|)$, where H ranges over all subgroups of G [Mes06]. The results from [MÖP04] mentioned below Theorem 3.1 imply that the vertex points $(|H|, |G|/|H|)$ are attained, but little more is known about the set $\{(\|f\|_0, \|\widehat{f}\|_0), \ f \in \mathbb{C}^G\}$.

In the following, we address the question whether for some given Abelian group G and (k, l) chosen with $l \geq \theta(G, k) \geq \frac{|G|}{d_1 d_2}(d_1 + d_2 - k)$ there exists $f \in \mathbb{C}^G$ with $\|f\|_0 = k$ and $\|\widehat{f}\|_0 = l$. This question has been considered earlier in [FKLM05] where the set $\{(\|f\|_0, \|\widehat{f}\|_0), \ f \in G\}$ has been described for $G = \mathbb{Z}_6$ and $G = \mathbb{Z}_8$.

First, we state an affirmative positive result for cyclic groups. It follows from Example 5.6 in [Smi90] and the proof of Proposition 4.5 in [Kut03].

PROPOSITION 3.6. *Let $G = \mathbb{Z}_n$, $n \in \mathbb{N}$. If $0 < k, l \leq |G|$ satisfy $l + k \geq |G| + 1$, then there exists a function $f \in \mathbb{C}^G$ with $\|f\|_0 = k$ and $\|\widehat{f}\|_0 = l$.*

For product groups $G = H_1 \times H_2$, positive results on possible support pairs $(\|f\|_0, \|\widehat{f}\|_0)$ can be obtained on the basis of the support pairs for H_1 and H_2. These follow from the fact that for $f_1 \otimes f_2, f_1 \in \mathbb{C}^{H_1}, f_2 \in \mathbb{C}^{H_2}$ we have $\|f_1 \otimes f_2\|_0 = \|f_1\|_0 \cdot \|f_2\|_0$ and $\|\widehat{f_1 \otimes f_2}\|_0 = \|\widehat{f_1} \otimes \widehat{f_2}\|_0 = \|\widehat{f_1}\|_0 \cdot \|\widehat{f_2}\|_0$ [Grö06].

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Figure 1. Color coding used in Figures 2–5 to describe membership to subsets of $\mathbb{N}^2$ or $\mathbb{N}^3$. Y-pr indicates a proof is known that the corresponding value is in the set considered. Y-nu implies that there is numerical evidence that the value is in the set and Y-co indicates that we conjecture that the value is in the set. N-pr indicates a proof is known that the corresponding value is not in the set, and N-nu and N-co are defined accordingly. The color adjacent to ? implies that no judgement is made here.

A negative result for the groups $G = \mathbb{Z}_{2p}$ for, $p \geq 5$ prime, stating that there exists no $f \in \mathbb{C}^{\mathbb{Z}_{2p}}$ with $\|f\|_0 = 3$ and $\|\widehat{f}\|_0 = p - 1$ is included in [KPR07].

The numerical results collected in Figure 2 are based on Lemma 3.4 and they show that the set of all possible pairs $(\|f\|_0, \|\widehat{f}\|_0)$ is not easily described in general. The computations needed to obtain Figure 2 are quite involved. For example, the computations showing that there is no vector on $\mathbb{Z}_{16}$ with five nonzero entries and whose Fourier transform has nine nonzero entries include the calculation of the singular values of $\binom{16}{5}\binom{16}{7} = 49969920$ five by seven matrices.

4. UNCERTAINTY PRINCIPLES FOR SHORT–TIME FOURIER TRANSFORMS ON FINITE ABELIAN GROUPS

We now turn to discuss minimum support conditions on time–frequency representations of elements in $\mathbb{C}^G$, in particular, for the short–time Fourier transform $V_g f \in \mathbb{C}^{G \times \widehat{G}}$ of a function $f \in \mathbb{C}^G$ with respect to a window $g \in \mathbb{C}^G$. For background on uncertainty principles in joint time–frequency representations see [Grö03, HL05]

But first, we consider the simplest joint time–frequency representation of f which is given by the tensor product $f \otimes \widehat{f}$. Similarly, in electrical engineering the so-called Rihaczek distribution $R : G \times \widehat{G} \longrightarrow \mathbb{C}$ given by $Rf(x, \omega) = f(x)\overline{\widehat{f}(\omega)}\,\overline{\langle \omega, x \rangle}$ is considered. Theorem 3.1 implies that $\|Rf\|_0 = \|f \otimes \widehat{f}\|_0 = \|f\|_0 \|\widehat{f}\|_0 \geq |G|$. Figure 3 lists all possible pairs $(\|f\|_0, \|Rf\|_0)$ for $f \in \mathbb{C}^{\mathbb{Z}_4}$ and $f \in \mathbb{C}^{\mathbb{Z}_2^2}$.

The following result resembles Theorem 3.1. It is given for functions on the real line as so-called weak uncertainty principle in [Grö03].

PROPOSITION 4.1. $\|V_g f\|_0 \geq |G|$ for $f, g \in \mathbb{C}^G \backslash \{0\}$ with equality for $f = g = \delta$.

Proof. Clearly $\|V_\delta \delta\|_0 = |G|$. For $f, g \in \mathbb{C}^G \backslash \{0\}$, the result follows from

$$|G|\, \|f\|_2^2\, \|g\|_2^2 = \|V_g f\|_2^2 \leq \|V_g f\|_0\, \|V_g f\|_\infty^2 \leq \|V_g f\|_0\, \|f\|_2^2\, \|g\|_2^2. \qquad \square$$

Next, we derive lower bounds on $\|V_g f\|_0$ which depend on both, $\|f\|_0$ and $\|g\|_0$.

PROPOSITION 4.2. *For $f, g \in \mathbb{C}^G \backslash \{0\}$, we have for θ defined in (4)*

$$\|V_g f\|_0 \geq \max\{\ \theta(G, \|g\|_0)\, \theta(G, \|\widehat{f}\|_0)\,,\ \theta(G, \|f\|_0)\, \theta(G, \|\widehat{g}\|_0)\ \}. \tag{6}$$
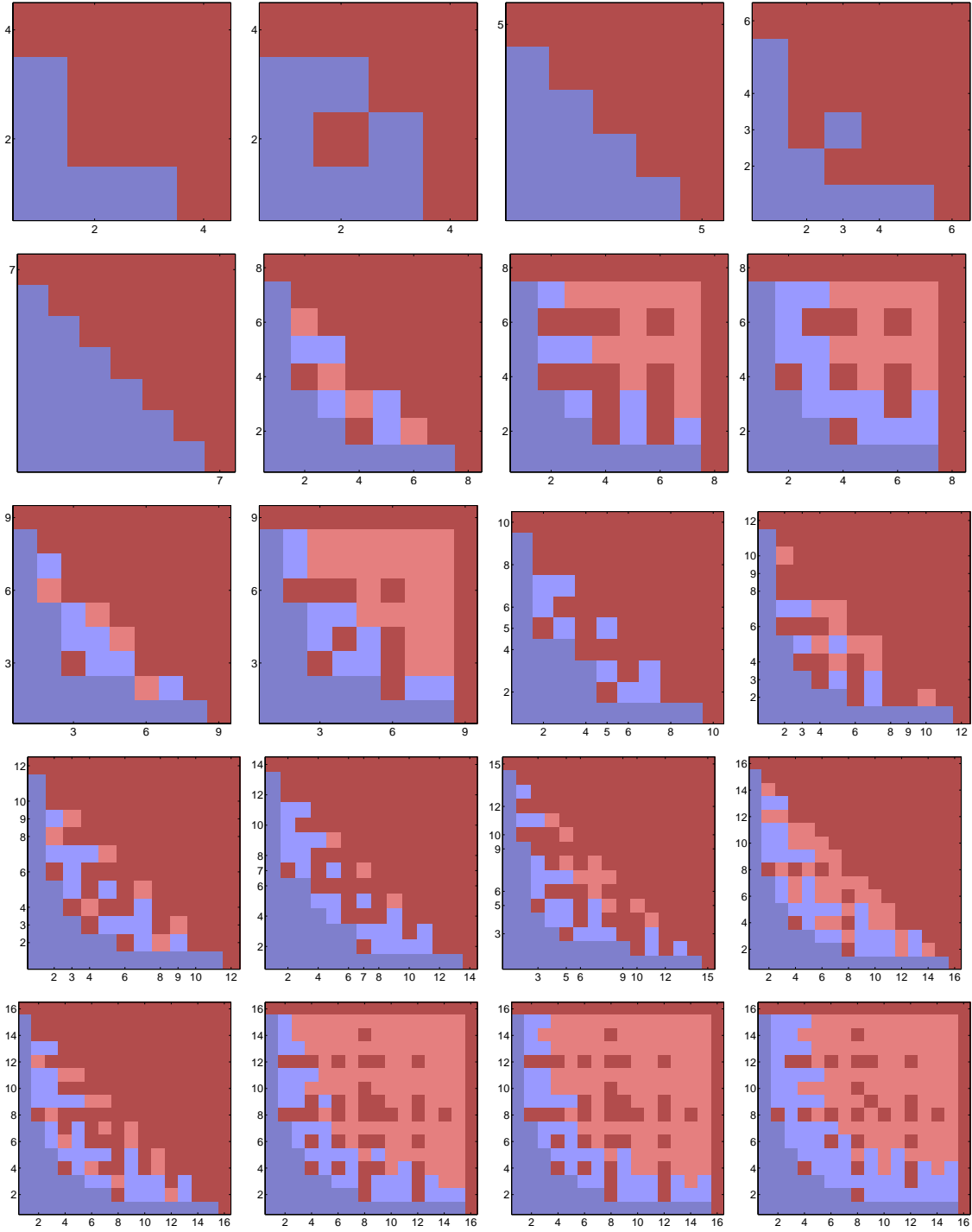
Figure 2. The set $\left\{(\|f\|_0, \|\widehat{f}\|_0),\ f \in \mathbb{C}^G \backslash \{0\}\right\}$ for all Abelian groups of order less than or equal to 16 with exception of the groups of prime order $\mathbb{Z}_{11}$ and $\mathbb{Z}_{13}$. The groups (row wise from left to right) are $\mathbb{Z}_4$, $\mathbb{Z}_2^2$, $\mathbb{Z}_5$, $\mathbb{Z}_6$; $\mathbb{Z}_7$, $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2^3$; $\mathbb{Z}_9$, $\mathbb{Z}_3^2$, $\mathbb{Z}_{10}$, $\mathbb{Z}_{12}$; $\mathbb{Z}_2 \times \mathbb{Z}_6$, $\mathbb{Z}_{14}$, $\mathbb{Z}_{15}$, $\mathbb{Z}_{16}$; $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_4^2$, $\mathbb{Z}_2^2 \times \mathbb{Z}_4$, $\mathbb{Z}_2^4$. The color code used is given in Figure 1, and justified by the results in Section 3.
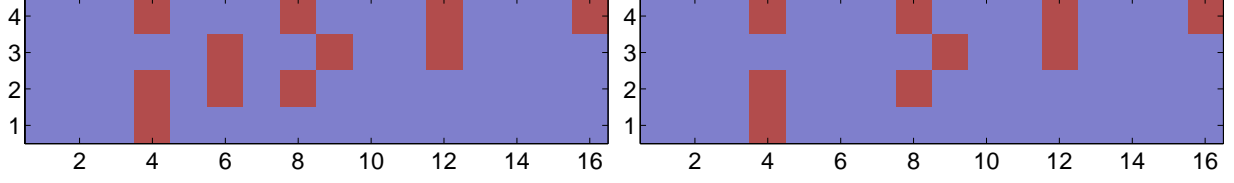
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Figure 3. For groups $G = \mathbb{Z}_4$ and $\mathbb{Z}_2^2$ all possible pairs $(\|f\|_0, \|Rf\|_0)$ are colored red, those pairs that are not achieved by some $f \in \mathbb{C}^G$ are colored blue in accordance with the color code given in Figure 1.

Proof. We shall prove $\|V_g f\|_0 \geq \theta(G, \|f\|_0)\theta(\widehat{G}, \|\widehat{g}\|_0)$. Then (6) follows from $\|V_g f\|_0 = \|V_{\widehat{g}}\widehat{f}\|_0$ and $\theta(G, k) = \theta(\widehat{G}, k)$ for any k, or, alternatively from $\|V_g f\|_0 = \|V_f g\|_0$. To see (6), observe first that the so-called symplectic Fourier transformation $\mathcal{F}_s = R \circ \mathcal{F}_{\widehat{G}}^{-1} \circ \mathcal{F}_G$, that is, the composition of a Fourier transformation $\mathcal{F}_G$ on G, an inverse Fourier transformation $\mathcal{F}_{\widehat{G}}^{-1}$ on $\widehat{G}$, and the axis transformation $R : F \mapsto F \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ obeys the same uncertainty principle as the Fourier transformation on the group $G \times \widehat{G}$. For $f, g \in \mathbb{C}^G$, we calculate

$$
\begin{aligned}
\mathcal{F}_s V_g f(r, \rho) &= \sum_{x \in G}\sum_{\xi \in \widehat{G}} V_g f(x, \xi)\overline{\langle \rho, x \rangle}\langle \xi, r \rangle = \sum_{x \in G}\sum_{\xi \in \widehat{G}}\sum_{t \in G} f(t)\overline{g(t-x)}\,\overline{\langle \xi, t \rangle}\,\overline{\langle \rho, x \rangle}\langle \xi, r \rangle \\
&= \sum_{x \in G}\sum_{t \in G} f(t)\overline{g(t-x)}\,\overline{\langle \rho, x \rangle}\sum_{\xi \in \widehat{G}}\langle \xi, r - t \rangle = |G|\sum_{x \in G} f(r)\overline{g(r-x)}\,\overline{\langle \rho, x \rangle} \\
&= |G|\overline{\langle \rho, r \rangle}f(r)\overline{\widehat{g}(\rho)}
\end{aligned}
$$

and note that $\operatorname{supp}\mathcal{F}_s V_g f = \operatorname{supp} f \times \operatorname{supp}\widehat{g}$. A simple tensor argument implies that $\|V_g f\|_0 = \|\mathcal{F}_s^{-1}(\mathcal{F}_s V_g f)\|_0 \geq \theta(G, \|f\|_0)\theta(\widehat{G}, \|\widehat{g}\|_0)$. (See Proposition 3.9 in [KPR07] for details.) $\qquad\square$

For $f, g \in \mathbb{C}^{\mathbb{Z}_p}\backslash\{0\}$, p prime, Proposition 4.2 gives the lower bound

$$
\|V_g f\|_0 \geq \max\{\,(p+1-\|g\|_0)(p+1-\|\widehat{f}\|_0)\,,\,(p+1-\|f\|_0)(p+1-\|\widehat{g}\|_0)\,\}
$$

which is improved below.

PROPOSITION 4.3. *Let* $G = \mathbb{Z}_p$, *p prime. For $f, g \in \mathbb{C}^G\backslash\{0\}$,*

$$
\|V_g f\|_0 \geq \begin{cases} |G|(|G|+1) - \|f\|_0\|g\|_0 & \text{if } \|f\|_0 + \|g\|_0 > |G|; \\ |G|(|G|+1) - (|G|+1-\|f\|_0)(|G|+1-\|g\|_0) & \text{if } \|f\|_0 + \|g\|_0 \leq |G|. \end{cases}
$$

Proof. Note that for all $x \in G$, $V_g f(x, \cdot) = \langle f, \pi(x, \cdot)g \rangle$ represents the Fourier transform of a vector of the form $f \cdot T_x \bar{g}$, that is,

$$
V_g f(x, \xi) = \langle f, \pi(x, \xi)g \rangle = \sum_{y \in G} f(y)\overline{g(y-x)}\langle \xi, x \rangle = \widehat{f \cdot T_x \bar{g}}(\xi), \quad x \in G, \xi \in \widehat{G}.
$$

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As long as $f \cdot T_x \bar{g} \neq 0$, Theorem 3.2 applies and so $\|f \cdot T_x \bar{g}\|_0 + \|\widehat{f \cdot T_x \bar{g}}\|_0 \geq |G| + 1$. For $K := \{x : f \cdot T_x \bar{g} \neq 0\}$ we get

$$\|V_g f\|_0 = \sum_{x \in K} \|\widehat{f \cdot T_x \bar{g}}\|_0 \geq |K|(|G| + 1) - \sum_{x \in G} \|f \cdot T_x \bar{g}\|_0 = |K|(|G| + 1) - \|f\|_0 \|g\|_0,$$

where $\sum_{x \in G} \|f \cdot T_x \bar{g}\|_0 = \|f\|_0 \|g\|_0$ follows from a simple counting argument.

We shall now estimate $|K|$ using the Cauchy-Davenport inequality, which states that for nonempty subsets A and B of $G = \mathbb{Z}_p$, p prime, $|A+B| \geq \min(|A|+|B|-1, |G|)$, where $A+B = \{a+b : a \in A, b \in B\}$ [Kár05]. Now $K = \{x : f \cdot T_x \bar{g} \neq 0\} = \{x : (\text{supp}\,\bar{g}) + x) \cap \text{supp}\,f \neq \varnothing\} = \text{supp}\,f - \text{supp}\,\bar{g}$. We set $A = \text{supp}\,f$, $B = \text{supp}\,\bar{g}$, and obtain $|K| = |\text{supp}\,f - \text{supp}\,\bar{g}| \geq \min(\|f\|_0 + \|g\|_0 - 1, |G|)$.

If $\|f\|_0 + \|g\|_0 \geq |G| + 1$, then $|K| = |G|$ and, hence, $\|V_g f\|_0 \geq |G|(|G| + 1) - \|f\|_0 \|g\|_0$. If $\|f\|_0 + \|g\|_0 \leq |G|$, then $|K| \geq \|f\|_0 + \|g\|_0 - 1$ and so

$$\|V_g f\|_0 \geq (\|f\|_0 + \|g\|_0 - 1)(|G| + 1) - \|f\|_0 \|g\|_0 = |G|(|G| + 1) - (|G| + 1 - \|f\|_0)(|G| + 1 - \|g\|_0).$$

$\square$

To establish support size constraints for short–time Fourier transformations for a given group G analytically is quite tedious since it requires to check all combinations of $\|f\|_0$ and $\|g\|_0$. For the case $G = \mathbb{Z}_3$, however, we have assembled all possible and impossible combinations in Figure 4. A derivation of the entries can be found in Appendix 6.4 in [KPR07].

In the following, we shall fix the window g and vary only the analyzed function f. First we provide a short–time Fourier transform version of Theorem 3.2.

THEOREM 4.4. *Let $G = \mathbb{Z}_p$, p prime. For almost every $g \in \mathbb{C}^G$, we have*

$$\|f\|_0 + \|V_g f\|_0 \geq |G|^2 + 1 \tag{7}$$

for all $f \in \mathbb{C}^G \setminus \{0\}$. Moreover, for $1 \leq k \leq |G|$ and $1 \leq l \leq |G|^2$ with $k + l \geq |G|^2 + 1$ there exists f with $\|f\|_0 = k$ and $\|V_g f\|_0 = l$.
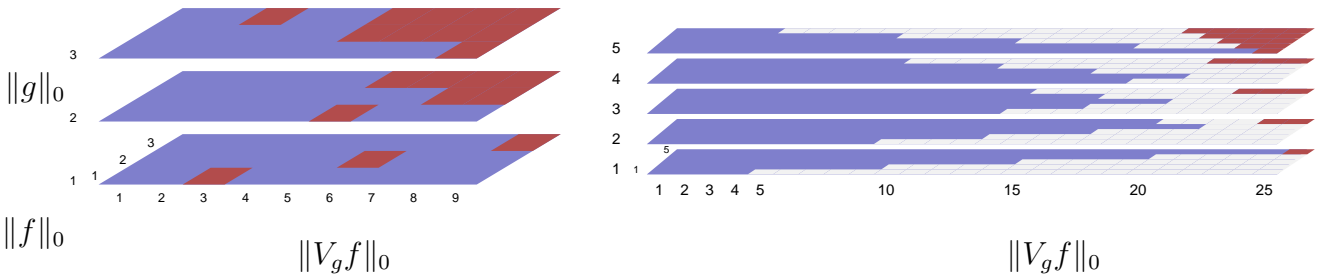

Figure 4. The set $\{(\|f\|_0, \|g\|_0, \|V_g f\|_0),\ f, g \in \mathbb{C}^G \setminus \{0\}\}$ for $G = \mathbb{Z}_3, \mathbb{Z}_5$. The color coding is given in Figure 1 and justified by Proposition 4.3 and Theorem 4.4.

We picture this result for $G = \mathbb{Z}_5$ and $G = \mathbb{Z}_7$ in Figure 5. Note that Theorem 4.4 follows from Proposition 3.3 together with Theorem 4 from [LPW05] which we state as

THEOREM 4.5. *For almost every $g \in \mathbb{C}^{\mathbb{Z}_p}$, p prime, we have that every minor of $A_{\mathbb{Z}_p,g}$ is nonzero.*

Outline of a proof of Theorem 4.5. It suffices to show that each square submatrix $(A_{\mathbb{Z}_p,g})_{A,B}$ has determinant nonzero for almost every g.

To this end, choose $A \subseteq \mathbb{Z}_p$ and $B \subseteq \mathbb{Z}_p \times \widehat{\mathbb{Z}_p}$ with $|A| = |B|$ and set $P_{A,B}(z) = \det(A_{\mathbb{Z}_p,z})_{A,B}$, $z = (z_0, z_1, \ldots, z_{p-1})$. To show that $P_{A,B} \neq 0$, we shall locate a term in the polynomial in standard form which has a nonzero coefficient. To construct this term, we determine first the maximal possible exponent of z_0 in one of the terms of P that are not trivially zero. Next, we determine the maximal exponent that z_1 can have in a monomial where the maximal exponent of z_0 is attained and so on.

Using generalized Vandermonde determinants, it can then be shown that the coefficient of this "maximal" term within $P_{A,B}$ can be expressed as a product of different minors of the discrete Fourier matrix $W_{\mathbb{Z}_p}$. For p prime, all these minors are nonzero, so the polynomial P has a nonzero coefficient for this "maximal term", hence is not identically 0, and nonzero almost everywhere. We have $P = \prod\limits_{A,B:\,|B|=|A|} P_{A,B} \not\equiv 0$, which implies that for $g \notin Z_P = \{z : P(z) = 0\}$, every minor of $A_{\mathbb{Z}_p,g}$ is nonzero. Since $P \not\equiv 0$, Z_P has Lebesgue measure 0. $\qquad\square$

Clearly, this proof of Theorem 4.5 is also based on Chebotarev's theorem on roots of unity. Also, Chebotarev's theorem on roots of unity and, therefore, Theorem 3.2 can be obtained as a corollary to Theorem 4.5 as shown in Appendix 6.5 of [KPR07].

It is easy to see that if $g \in \mathbb{C}^{\mathbb{Z}_p}$ satisfies (7) then $\|g\|_0 = \|\widehat{g}\|_0 = p$, that is, $g(x) \neq 0$ for all $x \in \mathbb{Z}_p$ and $\widehat{g}(\xi) \neq 0$ for all $\xi \in \widehat{\mathbb{Z}_p}$ [LPW05]. In addition, we have

PROPOSITION 4.6. *There exists a unimodular $g \in \mathbb{C}^{\mathbb{Z}_p}$, p prime, that is, a g with $|g(x)| = 1$ for all $x \in \mathbb{Z}_p$, satisfying the conclusions of Theorem 4.4.*

Proof. Theorem 4.5 implies that all minors of $A_{\mathbb{Z}_p,g}$ are nonzero polynomials in the polynomial ring $\mathbb{C}[z_0, ..., z_{n-1}]$. Let P be the product of all these minor polynomials, which, by assumption, is nonzero. We have to show that $P(g) \neq 0$ for some $g \in \mathbb{C}^{\mathbb{Z}_p}$ with $|g(x)| = 1$ for all $x \in G$.

This follows since the only polynomial P with $P(g) = 0$ whenever $|g(x)| = 1$ for all $x \in G$ is trivial, $P \equiv 0$, which we show below using induction over the number of variables n.

The case $n = 1$ follows since any nonzero polynomial in one variable has only finitely many zeros; only $P \equiv 0$ vanishes for all $z \in S^1 = \{z : |z| = 1\}$. Next, we consider a polynomial P of n variables which we regard as a polynomial in z_{n-1} with coefficients in the polynomial ring $\mathbb{C}[z_0, ..., z_{n-2}]$, that is,

$$P(z_{n-1}) = Q_m(z_0, ..., z_{n-2})z_{n-1}^m + Q_{m-1}(z_0, ..., z_{n-2})z_{n-1}^{m-1} + \cdots + Q_0(z_0, ..., z_{n-2})$$

For any fixed $(c_0, \ldots, c_{n-2}) \in (S^1)^{n-1}$ we have

$$Q_m(c_0, ..., c_{n-2})z_{n-1}^m + Q_{m-1}(c_0, ..., c_{n-2})z_{n-1}^{m-1} + \cdots + Q_0(c_0, ..., c_{n-2}) = 0$$

for all $z_{n-1} \in S^1$, hence, all its coefficients $Q_k(c_0, ..., c_{n-2})$, $k = 0, \ldots, m$ vanish. In other words, we have that $Q_k \in \mathbb{C}[z_0, ..., z_{n-2}]$, $k = 0, \ldots, m$ vanish on $(S^1)^{n-1}$, which, by induction hypothesis, implies that all $Q_k \equiv 0$ and therefore $P \equiv 0$. $\qquad\square$

Proposition 3.3 and Proposition 4.7 below combine to show that the condition "$G = \mathbb{Z}_p$ with p prime" is necessary for the existence of $g \in \mathbb{C}^G$ satisfying (7).

PROPOSITION 4.7. *If $|G|$ is not prime, then $A_{G,g}$ has at least one zero minor for all $g \in \mathbb{C}^G$.*

Proof. Let $|G| = k \cdot m$, $k, m \neq 1$. We consider only $G = \mathbb{Z}_{km}$, the general case follows since the Fourier matrix W_G for any non-cyclic G is a Kronecker product of Fourier matrices of cyclic groups.

For a primitive $|G|$-th root of unity ω, we have $(\omega^k)^m = \omega^{|G|} = 1$, so the discrete Fourier matrix W_G has a 1 in its (k, m)-entry. Now the matrix given by the first $|G|$ columns of $A_{G,g}$ results from W_G by multiplying the i-th row by c_i. So the minor given by the columns 0 and k and the rows 0 and m of A is $\det \begin{pmatrix} c_0 & c_0 \\ c_m & c_m \end{pmatrix} = 0$. Hence $A_{G,g}$ has a zero minor. $\qquad\square$

Recall Proposition 4.1, namely, the fact that for any G the estimates $|G| \leq \|V_g f\|_0 \leq |G|^2$, $g, f \in \mathbb{C}^G$ are sharp. In other words, for all G and $0 < k \leq |G|$ we have

$$\min_{g \in \mathbb{C}^G \setminus \{0\}} \min \left\{ \|V_g f\|_0 : \ f \in \mathbb{C}^G \text{ and } 0 < \|f\|_0 \leq k \right\} = |G| \,,$$

and

$$\max_{g \in \mathbb{C}^G \setminus \{0\}} \max \left\{ \|V_g f\|_0 : \ f \in \mathbb{C}^G \text{ and } 0 < \|f\|_0 \leq k \right\} = |G|^2 \,.$$

Certainly, $\|V_g f\|_0 = |G|$ is a rare event. In fact, it is reasonable to assume that $\|V_g f\|_0 = |G|^2$ for almost every pair (f, g). We shall now address the question whether for an appropriately chosen window g, we can achieve a lower bound $\|V_g f\|_0 \geq l$ for some $|G| < l \leq |G|^2$ and all $f \in \mathbb{C}^G$.

To this end, we define for $1 \leq k \leq |G|$

$$\phi(G, k) := \max_{g \in \mathbb{C}^G \setminus \{0\}} \min \left\{ \|V_g f\|_0 : \ f \in \mathbb{C}^G \text{ and } 0 < \|f\|_0 \leq k \right\}. \tag{8}$$

Using this notation, Theorem 4.4 indicates that $\phi(\mathbb{Z}_p, k) = p^2 + 1 - k$ for p prime. In fact, we have

PROPOSITION 4.8. *For almost every $g \in \mathbb{C}^G$, $\min_{0 < \|f\|_0 \leq k} \|V_g f\|_0 = \phi(G, k)$ for all $k \leq |G|$.*

Proof. In the following, we set $Q_{A,B}(z) = \det(A_{G,z})^*_{A,B}(A_{G,z})_{A,B}$, $z = (z_0, z_1, \ldots, z_{|G|-1})$, for $A \subseteq G$ and $B \subseteq G \times \widehat{G}$. $Q_{A,B}$ is a homogeneous polynomial in $z_0, z_1, \ldots, z_{|G|-1}$ of degree $2|A|$. We use the following result, whose proof can be found in [KPR07];

LEMMA 4.9. *The vector $g \in \mathbb{C}^G$ satisfies $\min_{0 < \|f\|_0 \leq k} \|V_g f\|_0 \geq l$ if and only if $Q_{A,B}(g) \neq 0$ for all $A \subseteq G$ with $|A| = k$ and all $B \subseteq G \times \widehat{G}$ with $|B| = |G|^2 - l + 1$.*

Lemma 4.9 and $\min_{0 < \|f\|_0 = k} \|V_{g_k} f\|_0 \geq \phi(G, k)$, $k \leq |G|$, for some $g_k \in \mathbb{C}^G \setminus \{0\}$ imply that $Q_{A,B} \not\equiv 0$ for all pairs $A \subseteq G$ and $B \subseteq G \times \widehat{G}$ with $|B| = |G|^2 - \phi(G, |A|) + 1$. Hence, $Q = \prod_{A,B: |B| = \phi(G, |A|) + 1} Q_{A,B} \not\equiv 0$. This implies that $Q(g) \neq 0$ for almost every $g \in \mathbb{C}^G$ and therefore, for almost every $g \in \mathbb{C}^G$ we have $\min_{0 < \|f\|_0 \leq k} \|V_g f\|_0 \geq \phi(G, k)$ for all $k \leq |G|$, from which the desired equality follows. $\qquad\square$

To obtain bounds on $\phi(G, k)$ for groups of non-prime order, we shall follow the roadmap used in [Mes06] to show Theorem 3.5. The proof is inductive and the induction step is based on

PROPOSITION 4.10. *Let H be a subgroup of the finite Abelian group G. For $k \in \mathbb{N}$ there exist $s, t \in \mathbb{N}$ with $st \leq k$ such that*

$$\phi(G, k) \geq \phi(H, s)\phi(G/H, t) \tag{9}$$

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Proof. In the following, we express the short–time Fourier transformation for functions defined on G as two consecutive short–time Fourier transformations. We use the following notation: let $H = \{x_i\} = \{y_i\}$ be a subgroup of G and, abusing notation, let $\{x_j\} = \{y_j\}$ be a set of coset representatives of the quotient group G/H. We let $H^\perp = \{\xi_j \in \widehat{G} : \xi_j(H) = 1\}$ and $\{\xi_i\}$ is a set of coset representatives of $\widehat{G}/H^\perp$.

Set

$$\phi_H(G, k) = \max_{g_1 \in \mathbb{C}^H, \ g_2 \in \mathbb{C}^{G/H}} \min \left\{ \|V_{g_1 \otimes g_2} f\|_0 : \ f \in \mathbb{C}^G \text{ and } 0 < \|f\|_0 \le k \right\},$$

where $g_1 \otimes g_2(x_i + x_j) = g_1(x_i) g_2(x_j + H)$. Clearly $\phi(G, k) \ge \phi_H(G, k)$, so (9) follows from $\phi_H(G, k) \ge \phi(H, s)\phi(G/H, t)$, which we shall show below. First, note that a similar argument as is used in Proposition 4.8 gives that for almost every pair (g_1, g_2),

$$\phi_H(G, k) = \min_{0 < \|f\|_0 \le k} \|V_{g_1 \otimes g_2} f\|_0, \quad 1 \le k \le |G|.$$

Therefore, we can pick g_1 and g_2 so that for all possible k, s, t,

$$\phi_H(G, k) = \min_{0 < \|f\|_0 \le k} \|V_{g_1 \otimes g_2} f\|_0, \quad \phi(H, s) = \min_{0 < \|f_1\|_0 \le s} \|V_{g_1} f_1\|_0, \quad \phi(G/H, t) = \min_{0 < \|f_2\|_0 \le t} \|V_{g_2} f_2\|_0. \ (10)$$

We fix $x = x_i + x_j$ and $\xi = \xi_i + \xi_j$, and compute

$$V_{g_1 \otimes g_2} f(x, \xi) = \sum_{y_j} \sum_{y_i} f(y_i + y_j) \overline{g_1(y_i - x_i)} \, \overline{g_2(y_j - x_j + H)} \, \overline{\langle \xi_i, y_i \rangle_H} \, \overline{\langle \xi_i, y_j \rangle_G} \, \overline{\langle \xi_j, y_j + H \rangle_{G/H}}$$

$$= \sum_{y_j} \overline{g_2(y_j - x_j + H)} \, \overline{\langle \xi_i, y_j \rangle_G} \, \overline{\langle \xi_j, y_j + H \rangle_{G/H}} \sum_{y_i} f(y_i + y_j) \overline{g_1(y_i - x_i) \langle \xi_i, y_i \rangle_H}$$

where we used $\xi_j \in H^\perp$, that is, $\langle \xi_j, y_i \rangle_G = 1$. For

$$F_H(x_i, \xi_i, y_j) := \overline{\langle \xi_i, y_j \rangle_G} \sum_{y_i} f(y_i + y_j) \overline{g_1(y_i - x_i)} \, \overline{\langle \xi_i, y_i \rangle_H}$$

we have

$$F_H(x_i, \xi_i, y_j) = \overline{\langle \xi_i, y_j \rangle_G} V_{g_1} T_{-y_j} f(x_i, \xi_i)$$

and $V_g f(x, \xi) = \big(V_{g_2} F_H(x_i, \xi_i, \cdot) \big)(x_j + H, \xi_j)$.

We fix now f such that $\|f\|_0 \le k$. Let $t = |\{y_j : \operatorname{supp} f \cap (y_j + H) \ne \varnothing\}|$. If for some y_j, $\operatorname{supp} f \cap (y_j + H) = \varnothing$, then $F_H(\cdot, \cdot, y_j) \equiv 0$ too. Therefore, $\|F_H(x_i, \xi_i, \cdot)\|_0 \le t$ and using (10) we obtain $\|V_{g_2} F_H(x_i, \xi_i, \cdot, \cdot)\|_0 \ge \phi(G/H, t)$. Also, by distributing $\operatorname{supp} f$ over t cosets of H in G, there is a coset $y_{j_0} + H$ with $|\operatorname{supp} f \cap (y_{j_0} + H)| = s \le k/t$. Because $F_H(\cdot, \cdot, y_{j_0})$ is, up to a nonzero factor, the partial short–time Fourier transform of $T_{-y_{j_0}} f$ with window g_1 on that coset,

$$\|F_H(\cdot, \cdot, y_{j_0})\|_0 = \|V_{g_1} T_{-y_{j_0}} f\|_0 \ge \phi(H, s).$$

We have obtained that the set $\Lambda = \{(x_i, \xi_i) \in H \times \widehat{H} : F_H(x_i, \xi_i, y_{j_0}) \ne 0\}$ has at least $\phi(H, s)$ elements so

$$\|V_g f(x_i + x_j, \xi_i + \xi_j)\|_0 = \sum_{(x_i, \xi_i') \in H \times \widehat{H}} \|V_g f(x_i, \xi_i, \cdot, \cdot)\|_0 \ge \sum_{(x_i, \xi_i) \in \Lambda} \|V_{g_2} F_H(x_i, \xi_i, \cdot, \cdot)\|_0$$

$$\ge \phi(H, s)\phi(G/H, t).$$

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This inequality holds for all $V_g f$ with $0 < \|f\|_0 \le k$ and therefore, $\phi_H(G,k) \ge \phi(H,s)\phi(G/H,t)$. $\square$

THEOREM 4.11. *For any finite Abelian group G and $k \le |G|$, let d_1 be the largest divisor of $|G|$ which is less than or equal to k and let d_2 be the smallest divisor of $|G|$ which is larger than or equal to k. Then*

$$\phi(G,k) \ge \frac{|G|^2}{d_1 d_2}(d_1 + d_2 - k). \tag{11}$$

Proof. The function $v(n,k) = n\,u(n,k) = \frac{n^2}{d_1 d_2}(d_1 + d_2 - k)$, is submultiplicative since $u(n,k) = \frac{n}{d_1 d_2}(d_1 + d_2 - k)$ in [Mes06] is submultiplicative, that is, we have $v(a,b)v(c,d) \ge v(ac,bd)$. We proceed by induction on $|G| = n$. Suppose (11) holds for $|G| = 1, \ldots, n-1$. If n is prime, then Proposition 4.4 implies $v(n,k) = n(1 + n - k) < n^2 - k + 1 = \phi(\mathbb{Z}_p, k)$ for all k. Else, we choose a nontrivial divisor d of n, and let H be a subgroup of G of order d. By Proposition 4.10, there exist s,t with $1 \le s \le d$, $1 \le t \le \min\{\frac{k}{s}, \frac{n}{d}\}$ such that $\phi(G,k) \ge \phi(H,s)\phi(G/H,t)$. Therefore, $\phi(G,k) \ge v(d,s)v(\frac{n}{d},t) \ge v(n,st) \ge v(n,k)$. $\square$

For the case $G = \mathbb{Z}_{pq}$, we can improve (11) by finding the convex hull of all pairs $(|H|, |G/H|)$ for all subgroups H of G as in [Mes06].

PROPOSITION 4.12. *Let $G = \mathbb{Z}_{pq}$ with $q < p$ and p, q prime. Then*

$$\phi(G,k) \ge \begin{cases} p^2(q^2 - k + 1) & \text{if } k < q; \\ (p^2 - \frac{k}{q} + 1)(q^2 - q + 1) & \text{else.} \end{cases} \tag{12}$$

Proof. Proposition 4.10 implies that there exists s,t such that $st \le k$ and $\phi(G,k) \ge \phi(H,s)\phi(G/H,t)$. For $G = \mathbb{Z}_{pq}$ and $|H| = p$, we have $\phi(H,s) = p^2 - s + 1$ and $\phi(G/H,t) = q^2 - t + 1$. As $st \le k$, we can find $\bar{t} \in \mathbb{R}$ such that $q \ge \bar{t} \ge t$ and $p \ge \frac{k}{\bar{t}} \ge s$. Hence,

$$\phi(G,k) \;\ge\; (p^2 - s + 1)(q^2 - t + 1) \ge (p^2 - \tfrac{k}{\bar{t}} + 1)(q^2 - \bar{t} + 1)\,.$$

So $\phi(G,k)$ must exceed the minimum of $M(u) = (p^2 - \frac{k}{u} + 1)(q^2 - u + 1)$, where u ranges from $\frac{k}{p}$ to q since we assume $\frac{k}{u} \le p$ and $u \le q$. We have $M'(u) = -(p^2 + 1) + \frac{k(q^2+1)}{u^2} = 0$ if and only if $u = \pm\sqrt{k\frac{q^2+1}{p^2+1}}$. As $M(u) \to -\infty$ for $u \to 0^+$ and $u \to \infty$, the only positive extremum is a maximum and the minimum is attained at a boundary point. A simple calculation gives that $M(q) \le M\left(\frac{k}{p}\right)$.

For $k < q$, the condition $1 \le s$, $1 \le t$, implies that t ranges only from 1 to k. The same arguments as used above show again that the minimum is attained at a boundary point and that $M(1) \ge M(k)$. $\square$

At $k = q$, the two lower bounds in (12) coincide and lead to what a geometric argument shows to be the optimal value that can be obtained using $g = g_1 \otimes g_2$. So the two straight lines meeting in $(q, p^2 - q + 1)$ define a convex hull similar to that given in Theorem 3.5. However, as expected, the computational results are far better than those given in (11), since tensor windows cannot be used to find optimal bounds for $\phi(G,k)$. See Table 1 for an illustration of (12) for $G = \mathbb{Z}_6$.

For $|G|$ prime, Theorem 4.4 characterizes all pairs $(\|f\|_0, \|V_g f\|_0)$, $f \in \mathbb{C}^G$, which are achieved for almost every window function $g \in \mathbb{C}^G$. However, for general Abelian groups, it is quite difficult to

$\|f\|_0$	1	2	3	4	5	6
Theorem 4.11	36	18	12	10	8	6
Proposition 4.12	36	26	25	23	22	20
Numerical results	36	33	32	32	32	31

Table 1. Lower bounds for $\|V_g f\|_0$ given by Theorem 4.11, Proposition 4.12, and by numerical experiments for $G = \mathbb{Z}_6$ and randomly chosen $g \in \mathbb{C}^{\mathbb{Z}_6}$.

establish lower bounds for $\|V_g f\|_0$. Further, our limited numerical results for cyclic groups indicate a close correspondence between the achieved pairs $(\|f\|_0, \|\widehat{f}\|_0)$ and the achieved pairs $(\|f\|_0, \|V_g f\|_0)$ for a given window g. Consequently, we pose

QUESTION 4.13. *For every cyclic group G and almost every $g \in \mathbb{C}^G$, is it true that*

$$\{(\|f\|_0, \|V_g f\|_0),\ f \in \mathbb{C}^G \backslash \{0\}\} = \{(\|f\|_0,\ \|\widehat{f}\|_0 + |G|^2 - |G|),\ f \in \mathbb{C}^G \backslash \{0\}\} \ ?$$

The basis for this question is illustrated in Figure 5 by considering the cyclic groups $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, and $\mathbb{Z}_8$. The statement does not hold for non-cyclic groups, for example, in the diagram for $\mathbb{Z}_2^2$ in Figure 5 the existence of 4×4 zero minors in $A_{G,g}$ in (3), that is, the minor given by columns 1,3,13,14, leads to the possible pair $(4, 12)$.
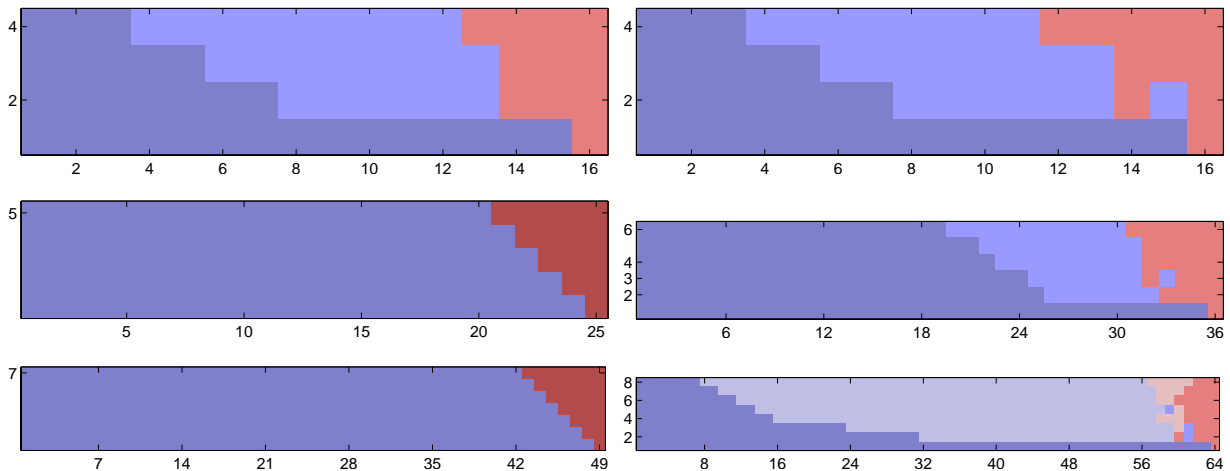

Figure 5. The set $\{(\|f\|_0, \|V_g f\|_0),\ f \in \mathbb{C}^G \backslash \{0\}\}$ for appropriately chosen $g \in \mathbb{C}^G \backslash \{0\}$ for $G = \mathbb{Z}_4$, $\mathbb{Z}_2^2$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, $\mathbb{Z}_7$ and $\mathbb{Z}_8$. The color coding from Figure 1 is applied in accordance with numerical experiments based on Lemma 3.4.

5. APPLICATIONS

We shall now turn to applications of the results stated in Section 4 to communications engineering and, in the subsequent section, to the problem of recovering sparse signals from incomplete data.

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5.1. Gabor frames, erasure channels, and the identification of operators

In generic communication systems, information is transmitted in the form of the entries of a vector $f \in \mathbb{C}^G$ over a channel in such a way that recovery of the information at the receiver is robust to errors introduced by the channel. Here we will focus on two inherent problems. First, we shall discuss transmission over a channel with erasure, that is, some of the vector entries may be lost during transmission. Second, we discuss the so-called identification problem for another class of operators, namely, of linear time–varying operators which play a central role in wireless and mobile communications. Clearly, knowledge of the operator at hand would help to counteract disturbances that were caused during transmission.

But first, we give some preliminaries on frames in finite dimensional vector spaces will be used in this section. For details on frames and, in particular, Gabor frames we refer to the excellent expositions [Chr03, Grö01, KC06]. The geometry of finite frames is discussed in [BF03].

DEFINITION 5.1. *Let G be a finite Abelian group and let K be a finite or countably infinite index set. A family of functions $\{\varphi_k\} \subset \mathbb{C}^G$ with*

$$A\|f\|_2^2 \leq \sum_k |\langle f, \varphi_k \rangle|^2 \leq B\|f\|_2^2, \quad f \in \mathbb{C}^G,$$

for positive A and B is called a frame for $\mathbb{C}^G$. A frame is called tight if we can choose $A = B$. If we can choose $A = B = 1$, then the frame is called Parseval tight frame. If $\|\varphi_k\| = C > 0$ for all k, then the frame $\{\varphi_k\}$ is called equal norm frame and if in addition $C = 1$, then we have a unit norm frame.

In the following, we shall refer to a Gabor system which forms a frame as *Gabor frame*. A direct consequence of (1) is

PROPOSITION 5.2. *For any $g \in \mathbb{C}^G \backslash \{0\}$, the Gabor system $\{\pi(\lambda)g\}_{\lambda \in G \times \widehat{G}}$ is an equal norm tight Gabor frame for $\mathbb{C}^G$ with frame bound $A = B = |G| \|g\|_2^2$.*

The usefulness of frames stems largely from the existence of a reconstruction formula resembling (1).

PROPOSITION 5.3. *Let $\{\varphi_k\}$ be a frame for $\mathbb{C}^G$. Then there exists a so-called dual frame $\{\widetilde{\varphi}_k\}$, with*

$$f = \sum_k \langle f, \varphi_k \rangle \widetilde{\varphi}_k = \sum_k \langle f, \widetilde{\varphi}_k \rangle \varphi_k, \quad f \in \mathbb{C}^G. \tag{13}$$

Note that Parseval frames are self dual, that is, we can choose $\widetilde{\varphi}_k = \varphi_k$ for all k.

Now, we are in position to briefly discuss the recovery of information from a vector that suffered erasures [CK03, PK05, GK01, SH03]. In data transmission, rather then sending the information given as independent entries of a vector $f \in \mathbb{C}^G$ in raw form, that is, sending vector entries one-by-one, information is being coded prior to transmission. For example, we can choose a frame $\{\varphi_k\}_{k \in K}$ for $\mathbb{C}^G$ and send the coefficients $\langle f, \varphi_k \rangle$, $k \in K$. If none of the transmitted coefficients are lost, the receiver can use a dual frame $\{\widetilde{\varphi}_k\}$ of $\{\varphi_k\}$ and recover f using (13). But even if some coefficients are lost and only $\langle f, \varphi_k \rangle$ is received for $k \in K' \subset K$, the information can still be recovered if (and only if) $\{\varphi_k\}_{k \in K'}$ remains a frame. This necessitates that $|K'| \geq |G| = \dim \mathbb{C}^G$.

DEFINITION 5.4. *A frame $\mathcal{F} = \{\varphi_k\}_{k \in K}$ in $\mathbb{C}^G$ is maximally robust to erasures if the removal of any $l \leq |K| - |G|$ vectors from $\mathcal{F}$ leaves a frame.*

Similarly, we give

DEFINITION 5.5. *A set of m vectors in $\mathbb{C}^G$ is in general position, if any collection of at most $|G|$ of these vectors are linearly independent.*

Next, we introduce some vocabulary and notation regarding the previously mentioned operator identification problem.

DEFINITION 5.6. *A linear space of operators $\mathcal{H}$ mapping $\mathbb{C}^A$ to $\mathbb{C}^B$ is called identifiable with identifier $g \in \mathbb{C}^A$ if the linear map $\varphi_g : \mathcal{H} \longrightarrow \mathbb{C}^B$, $H \mapsto Hg$ is injective, that is, if $Hg \neq 0$ for all $H \in \mathcal{H}\backslash\{0\}$.*

Time–variant communication channels, for example, multi-path channels in wireless telephony, are often modeled through a combination of translation operators (time–shift, delay) and modulation operators (frequency shifts that are caused by the Doppler effect). Therefore, identification of $\mathcal{H}_\Lambda = \{\sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda),\ c_\lambda \in \mathbb{C}\}$ for $\Lambda \subseteq G \times \widehat{G}$ is quite relevant (see [PRT07] and references therein).

The following theorem is a straightforward generalization to general finite Abelian groups of Theorem 2 and Theorem 3 for cyclic groups in [LPW05]. The proofs of Theorem 2 and Theorem 3 in [LPW05] carry over to this setting.

THEOREM 5.7. *For $g \in \mathbb{C}^G\backslash\{0\}$, the following are equivalent:*

1. *Every minor of $A_{G,g}$ of order $|G|$ is nonzero.*

2. *The vectors from the Gabor system $\{\pi(\lambda)g\}_{\lambda \in G \times \widehat{G}}$ are in general position.*

3. *The Gabor system $\{\pi(\lambda)g\}_{\lambda \in G \times \widehat{G}}$ is an equal norm tight frame which is maximally robust to erasures.*

4. *For all $f \in \mathbb{C}^G\backslash\{0\}$ we have $\|V_g f\|_0 \geq |G|^2 - |G| + 1$.*

5. *For all $f \in \mathbb{C}^G$, $V_g f(\lambda)$, and, therefore, f, is completely determined by its values on any set Λ with $|\Lambda| = |G|$.*

6. *$\mathcal{H}_\Lambda$ is identifiable by g if and only if $|\Lambda| \leq |G|$*

For $|G|$ prime, Theorem 4.4 ensures the existence of $g \in \mathbb{C}^G$ which satisfies parts *1-6* in Theorem 5.7, and Proposition 4.6 allows us to choose g to be unimodular. A positive answer to Question 4.13 would also confirm the existence of $g \in \mathbb{C}^{\mathbb{Z}_n}$, $n \in \mathbb{N}$, satisfying Theorem 5.7 part *4*, and therefore Theorem 5.7 parts *1-6* for cyclic groups.

REMARK 5.8. To our knowledge, the only known *equal norm tight frames that are maximally robust to erasures* are so-called harmonic frames (see Conclusions in [CK03]). Harmonic frames for $\mathbb{C}^n$ with $m \geq n$ elements are obtained by deleting uniformly $m - n$ components of the characters of $\mathbb{Z}_m$ [CK03]. Similarly, Theorem 4.5 together with Proposition 4.6 provides us with equal norm tight frames with p^2 elements in $\mathbb{C}^n$ for $n \leq p$. Namely, we can choose a $g \in (S^1)^p$ and remove $p - n$ components of the equal norm tight frame $\{\pi(\lambda)g\}_{\lambda \in G \times \widehat{G}}$ in order to obtain an equal norm tight frame in $\mathbb{C}^n$ which is maximally robust to erasure. Note that this frame is not a Gabor frame proper. Reducing the number of vectors in the frame to $m \leq p^2$ vectors leaves an equal norm frame which is maximally robust to erasure but which might not be tight. This holds for harmonic frames, too.

With the restriction to frames with p^2 elements, p prime, we have shown the existence of Gabor frames which share the usefulness of harmonic frames when it comes to transmission of information through erasure channels.

5.2. Signals with sparse representations

In Section 5.1 we discussed the recovery of signals or operators from $|G|$ known complex numbers. Here, we will use the functions ϕ and θ which were defined in Section 3 and Section 4 to refine some of these findings. That is, we show that a function/signal which can be represented as a linear combination of a small number of pure frequencies or of a small number of time–frequency shifts of a fixed function g can be recovered from fewer than $|G|$ of its values. Our brief discussion is based on the most basic ideas and results from the theory of sparse signal recovery [Don06, Rau07, CRT06].

There exist a number of entry points to the theory of sparse signal recovery. Here, we shall consider dictionaries $\mathcal{D} = \{g_0, g_1, \ldots, g_{N-1}\}$ of N vectors in $\mathbb{C}^n$, or equivalently, in $\mathbb{C}^G$. For $k \leq n = |G|$ we shall examine the sets

$$\Sigma_k^{\mathcal{D}} = \{f \in \mathbb{C}^n : \ f = M_{\mathcal{D}}\, c = \sum_r c_r g_r, \text{ with } \|c\|_0 \leq k\}.$$

The central question is: *how many values of $f \in \Sigma_k^{\mathcal{D}}$ need to be known (or stored), in order that $c \in \mathbb{C}^N$ with $f = \sum_r c_r g_r$ and $\|c\|_0 \leq k$, and therefore f, is uniquely determined by the known data?*

To this end, we set

$$\psi(\mathcal{D}, k) = \min\left\{\|f\|_0 : \ f \in \Sigma_k^{\mathcal{D}}\right\},$$

and observe the following well known result.

PROPOSITION 5.9. *Any $f \in \Sigma_k^{\mathcal{D}}$ is fully determined by any choice of $n - \psi(\mathcal{D}, 2k) + 1$ values of f.*

Note that unlike in Theorem 5.7, we do not assume knowledge of the set $\operatorname{supp} c$ for c with $M_{\mathcal{D}}c = f$, $\|f\|_0$ in Proposition 5.9 and in the following.

Proof. Assume that for some $B \subset \mathbb{C}^n$ with $|B| = n - \psi(\mathcal{D}, 2k) + 1$, two coefficient vectors $c_1, c_2 \in \mathbb{C}^N$ exist that satisfy $M_{\mathcal{D}}c_1|_B = f|_B = M_{\mathcal{D}}c_2|_B$ and $\|c_1\|_0, \|c_2\|_0 \leq k$. Then $\|c_2 - c_1\|_0 \leq 2k$ with $\|M_{\mathcal{D}}(c_2 - c_1)\|_0 \leq n - |B| = n - (n - \psi(\mathcal{D}, 2k) + 1) = \psi(\mathcal{D}, 2k) - 1$, a contradiction. $\square$

A classical dictionary for $\mathbb{C}^G$ is $\mathcal{D}_G = \{\xi\}_{\xi \in \widehat{G}}$, where G is a finite Abelian group. Then

$$\psi(\mathcal{D}_G, k) = \min\left\{\|f\|_0 : \ f \in \Sigma_k^{\mathcal{D}}\right\} = \min\left\{\|\widehat{f}\|_0 : \ \|f\|_0 \leq k\right\} = \theta(G, k).$$

This equality together with Proposition 5.9 demonstrates the relevance of the results cited in Section 3 for the recovery of signals with limited spectral content. For example, Theorem 3.5 shows that for any finite Abelian group of order 16 we have $\theta(G, 6) \geq 3$. In fact, our computations that are illustrated in Figure 2 show that $\theta(G, 6) = 4$ for $|G| = 16$, and, hence, any $f \in \Sigma_3^{\mathcal{D}_G} = \{f : \|\widehat{f}\|_0 \leq 3\}$ can be recovered from any choice of $|G| - \theta(G, 2 \cdot 3) + 1 = 16 - 4 + 1 = 13$ values of f. For $f \in \Sigma_3^{\mathcal{D}_{\mathbb{Z}_{17}}}$ on the other side, Theorem 3.2 implies that f is already fully determined by $|\mathbb{Z}_{17}| - \theta(\mathbb{Z}_{17}, 2 \cdot 3) + 1 = 17 - (17 - 6 + 1) + 1 = 6$ of its values.

The results in Section 4 which involve the function ϕ are relevant to determine vectors which have sparse representations in the dictionary $\mathcal{D}_{A_{G,g}}$ which consists of the columns of $A_{G,g}$. In fact, we have $F \in \Sigma_k^{\mathcal{D}_{A_{G,g}}}$ if and only if $F = V_g f$ for some $f \in \mathbb{C}^G$ with $\|f\|_0 \leq k$ and, therefore,

$$\psi(\mathcal{D}_{A_{G,g}}, k) = \min \left\{ \|V_g f\|_0 : \|f\|_0 \leq k \right\} = \phi(G, k).$$

For $|G|$ prime for example, this leads to the following short–time Fourier transform version of Theorem 1.1 in [CRT06].

THEOREM 5.10. *Let $g \in \mathbb{C}^{\mathbb{Z}_p}$, p prime, satisfy the conclusion of Theorem 4.4. Then any $f \in \mathbb{C}^{\mathbb{Z}_p}$ with $\|f\|_0 \leq \frac{1}{2}|\Lambda|$, $\Lambda \subset \mathbb{Z}_p \times \widehat{\mathbb{Z}_p}$ is uniquely determined by Λ and $V_g f|_\Lambda$.*

In terms of sparse representations, the Gabor frame dictionary $\{\pi(\lambda)g\}_{\lambda \in G \times \widehat{G}}$ of time–frequency shifts of a prototype vector g, that is, the dictionary consisting of the rows of $A_{G,g}$, appears to be more interesting. Rudimentary numerical experiments give some indication that for any cyclic group G, and almost every $g \in \mathbb{C}^G$, we have for $k \leq |G|$

$$\psi(\{\pi(\lambda)g\}_{\lambda \in G \times \widehat{G}}, k) = \theta(G, k).$$

Note that this does not hold for all Abelian groups of finite order. For example, for any $g \in \mathbb{C}^{\mathbb{Z}_2 \times \mathbb{Z}_2}$ we have $\psi(\{\pi(\lambda)g\}_{\lambda \in (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (\mathbb{Z}_2 \times \mathbb{Z}_2)}, 4) = 0$ while $\theta(\mathbb{Z}_2 \times \mathbb{Z}_2, 4) = 1$.

For $|G|$ prime, Theorem 4.5 implies that $\psi(\{\pi(\lambda)g\}_{\lambda \in G \times \widehat{G}}, k) = p - k + 1 = \theta(G, k)$, and analogously to Theorem 5.10, we obtain

THEOREM 5.11. *Let $g \in \mathbb{C}^{\mathbb{Z}_p}$, p prime, satisfy the conclusion of Theorem 4.4. Then any $f \in \mathbb{C}^{\mathbb{Z}_p}$ with $f = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g$, $\Lambda \subset \mathbb{Z}_p \times \widehat{\mathbb{Z}_p}$ is uniquely determined by B and $f|_B$ whenever $|B| \geq 2|\Lambda|$.* Note that similar to before, the recovery of f from $2|\Lambda|$ samples of f in Theorem 5.11 does not require knowledge of Λ.

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