SINGULAR EQUIVARIANT ASYMPTOTICS AND THE MOMENT MAP II

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Contents

1. Introduction

This is the second of a series of papers dealing with the asymptotic behavior of certain integrals occuring in the description of the spectrum of an invariant elliptic operator on a compact Riemannian manifold M carrying the action of a compact, connected Lie group of isometries G [4, 15, 5], and in the study of its equivariant cohomology via the moment map $\mathbb{J}: T^*M \to \mathfrak{g}^*$, where T^*M and $\mathfrak{g}$ denote the cotangent bundle of M and the Lie algebra of G, respectively [8, 1, 19, 2]. The mentioned integrals are essentially of the form

$$I(\mu) = \int_{T^*M\times\mathfrak{g}} e^{i\mathbb{J}(\eta)(X)/\mu} a(\eta, X)\, d\eta\, dX, \qquad \mu \to 0^+,$$

where $a \in \mathrm{C^\infty_c}(T^*M\times\mathfrak{g})$ is an amplitude, $d\eta$ a density on T^*M, and dX, up to a constant factor, the Lebesgue measure in $\mathfrak{g}$. While asymptotics for $I(\mu)$ have been obtained for free group actions, one meets with serious difficulties when singular orbits are present. The reason is that, when trying to examine these integrals via the generalized stationary phase theorem in the case of general effective actions, the critical set of the phase function $\mathbb{J}(\eta)(X)$ is no longer a smooth manifold, so that, a priori, the principle of the stationary phase can not be applied in this case. Nevertheless, in what follows, we shall show how to circumvent this obstacle by partially resolving the singularities of the critical set of $\mathbb{J}(\eta)(X)$, and in this way obtain asymptotics for $I(\mu)$ with remainder estimates in the case of singular group actions. Similar asymptotics were already obtained in [16] for orthogonal actions in Euclidean space, and the present paper globalizes those results, while applications will be treated in a forthcoming paper.

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2. Compact group actions and the moment map

Let M be a closed, connected Riemannian manifold, and G a compact, connected Lie group with Lie algebra $\mathfrak{g}$ acting on M by isometries. Consider the cotangent bundle $\pi : T^*M \to M$, as well as the tangent bundle $\tau : T(T^*M) \to T^*M$, and define on T^*M the Liouville form

$$\Theta(\mathfrak{X}) = \tau(\mathfrak{X})[\pi_*(\mathfrak{X})], \qquad \mathfrak{X} \in T(T^*M).$$

We regard T^*M as a symplectic manifold with symplectic form

$$\omega = d\Theta,$$

and define for every $X \in \mathfrak{g}$ the function

$$J_X : T^*M \longrightarrow \mathbb{R}, \quad \eta \mapsto \Theta(\tilde{X})(\eta),$$

where $\tilde{X}$ denotes the fundamental vector field on T^*M, respectively M, generated by an element X of $\mathfrak{g}$. Note that $\Theta(\tilde{X})(\eta) = \eta(\tilde{X}_{\pi(\eta)})$. Indeed, put $\gamma(s) = \mathrm{e}^{-sX} \cdot \eta$, $s \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, so that $\gamma(0) = \eta$, $\dot{\gamma}(0) = \tilde{X}_\eta$. Since $\pi(\mathrm{e}^{-sX} \cdot \eta) = \mathrm{e}^{-sX} \cdot \pi(\eta)$, one computes

$$\pi_*(\tilde{X}_\eta) = \frac{d}{ds}\pi \circ \gamma(s)_{|s=0} = \frac{d}{ds}\mathrm{e}^{-sX} \cdot \pi(\eta)_{|s=0} = \tilde{X}_{\pi(\eta)}.$$

Therefore

$$\Theta(\tilde{X})(\eta) = \tau(\tilde{X}_\eta)(\pi_*(\tilde{X}_\eta)) = \eta(\tilde{X}_{\pi(\eta)}),$$

as asserted. The function J_X is linear in X, and due to the invariance of the Liouville form

$$\mathcal{L}_{\tilde{X}}\Theta = dJ_X + \iota_{\tilde{X}}\omega = 0, \qquad \forall X \in \mathfrak{g},$$

where $\mathcal{L}_{\mathfrak{X}}$ denotes the Lie derivative. This means that G acts on T^*M in a Hamiltonian way. The corresponding symplectic moment map is then given by

$$\mathbb{J} : T^*M \to \mathfrak{g}^*, \quad \mathbb{J}(\eta)(X) = J_X(\eta).$$

We are interested in the asymptotic behavior of integrals of the form

$$I(\mu) = \int_{T^*M}\int_{\mathfrak{g}} e^{i\psi(\eta,X)/\mu} a(\eta, X)\, dX\, d\eta, \qquad \mu \to 0^+, \tag{1}$$

where $a \in \mathrm{C}^\infty_{\mathrm{c}}(T^*M \times \mathfrak{g})$ is an amplitude, $d\eta$ a density on T^*M, and dX, up to a constant factor, the Lebesgue measure in $\mathfrak{g}$, while

$$\psi(\eta, X) = \mathbb{J}(\eta)(X).$$

We would like to study these integrals by means of the generalized stationary phase theorem, and for this we have to consider the critical set of the phase function $\psi(\eta, X)$. Let $\{X_1, \dots, X_d\}$ be a basis of $\mathfrak{g}$, and write $X = \sum s_i X_i$. Due to the linear dependence of J_X in X,

$$\partial_{s_i} \psi(\eta, X) = J_{X_i}(\eta),$$

and because of the non-degeneracy of ω,

$$J_{X,*} = 0 \quad\Longleftrightarrow\quad dJ_X = -\iota_{\tilde{X}}\omega = 0 \quad\Longleftrightarrow\quad \tilde{X} = 0.$$

Thus we see that

$$\mathrm{Crit}(\psi) = \{(\eta, X) \in T^*M \times \mathfrak{g} : \psi_*(\eta, X) = 0\} = \left\{(\eta, X) \in \Omega \times \mathfrak{g} : \tilde{X}_\eta = 0\right\},$$

where

$$\Omega = \mathbb{J}^{-1}(0)$$

represents the zero level of the moment map. Note that

$$\eta \in \Omega \quad\Longleftrightarrow\quad \eta_m \in \mathrm{Ann}(T_m(G \cdot m)) \quad \forall m \in M, \tag{2}$$

where $\mathrm{Ann}\,(V_m) \subset T^*_m M$ denotes the annihilator of a vector subspace $V_m \subset T_m M$. Now, the major difficulty in applying the generalized stationary phase theorem in our setting stems from the fact that, due to the singular orbit structure of the underlying group action, the zero level Ω of the moment map, and, consequently, the considered critical set $\mathrm{Crit}(\psi)$, are in general singular varieties. In fact, if the G-action on T^*M is not free, the considered moment map is no longer a submersion, so that Ω and the symplectic quotient Ω/G are no longer smooth. Nevertheless, it can be shown that these spaces have Whitney stratifications into smooth submanifolds, see Lerman-Sjamaar [17], and Ortega-Ratiu [14], Theorems 8.3.1 and 8.3.2, which correspond to the stratifications of T^*M, and M by orbit types, see Duistermaat-Kolk [9]. In particular, if (H_L) denotes the principal isotropy type of the G-action in M, Ω has a principal stratum given by

$$\mathrm{Reg}\,\Omega = \{\eta \in \Omega : G_\eta \sim H_L\}\,, \tag{3}$$

where G_η denotes the isotropy group of $\eta \in T^*M$. To see this, let $\eta \in \Omega$, and $m = \pi(\eta)$ be such that $G_m \sim H_L$. In view of (2) one computes for $g \in G_m$, and $\mathfrak{X} = \mathfrak{X}_T + \mathfrak{X}_N \in T_m M = T_m(G \cdot m) \oplus N_m(G \cdot m)$

$$g \cdot \eta_m(\mathfrak{X}) = \eta_m\big((L_{g^{-1}})_{*,m}(\mathfrak{X}_N)\big) = \eta_m(\mathfrak{X}),$$

since G_m acts trivially on $N_m(G \cdot m)$, see Bredon [3], pages 308 and 181. But $G_\eta \subset G_{\pi(\eta)}$ for arbitrary η, so that we conclude

$$\eta \in \Omega, \quad G_{\pi(\eta)} \sim H_L \quad \Rightarrow \quad G_\eta = G_{\pi(\eta)}, \tag{4}$$

and the assertion follows. Note that the stratum $\mathrm{Reg}\,\Omega$ is an open and dense subset of Ω, and a smooth submanifold in T^*M of codimension equal to the dimension κ of a principal G-orbit in M. Since the Lie algebra of G_η is given by $\mathfrak{g}_\eta = \{X \in \mathfrak{g} : \tilde{X}_\eta = 0\}$, it is clear that the smooth part of $\mathrm{Crit}(\psi)$ corresponds to

$$\mathrm{Reg}\,\mathrm{Crit}(\psi) = \{(\eta, X) \in \mathrm{Reg}\,\Omega \times \mathfrak{g} : X \in \mathfrak{g}_\eta\}\,, \tag{5}$$

and constitutes a submanifold of codimension 2κ. To obtain an asymptotic description of $I(\mu)$, we shall partially resolve the singularities of $\mathrm{Crit}(\psi)$, for which we will need a suitable G-invariant covering of M. In its construction, we shall follow Kawakubo [11], Theorem 4.20. Thus, let $(H_1), \dots (H_L)$ denote the isotropy types of M, and arrange them in such a way that

$$H_j \text{ is conjugate to a subgroup of } H_i \quad \Rightarrow \quad i \leq j.$$

Let $H \subset G$ be a closed subgroup, and $M(H)$ the union of all orbits of type G/H. Then M has a stratification into orbit types according to

$$M = M(H_1) \cup \dots \cup M(H_L).$$

By the principal orbit theorem, the set $M(H_L)$ is open and dense in M, while $M(H_1)$ is a closed, G-invariant submanifold. Denote by ν_1 the normal G-vector bundle of $M(H_1)$, and by $f_1 : \nu_1 \to M$ a G-invariant tubular neighbourhood of $M(H_1)$ in M. Take a G-invariant metric on ν_1, and put

$$D_t(\nu_1) = \{v \in \nu_1 : \|v\| \leq t\}\,, \qquad t > 0.$$

We then define the compact, G-invariant submanifold with boundary

$$M_2 = M - f_1(\overset{\circ}{D}_{1/2}(\nu_1)),$$

on which the isotropy type (H_1) no longer occurs, and endow it with a G-invariant Riemannian metric with product form in a G-invariant collar neighborhood of ∂M_2 in M_2. Consider now the union $M_2(H_2)$ of orbits in M_2 of type G/H_2, a compact G-invariant submanifold of M_2 with boundary, and let $f_2 : \nu_2 \to M_2$ be a G-invariant tubular neighbourhood of $M_2(H_2)$ in M_2, which exists due to the particular form of the metric on M_2. Taking a G-invariant metric on ν_2, we define

$$M_3 = M_2 - f_2(\overset{\circ}{D}_{1/2}(\nu_2)),$$

which constitutes a compact G-invariant submanifold with corners and isotropy types $(H_3), \dots (H_L)$. Continuing this way, one finally obtains for M the decomposition

$$M = f_1(D_{1/2}(\nu_1)) \cup \dots \cup f_L(D_{1/2}(\nu_L)),$$

where we identified $f_L(D_{1/2}(\nu_L))$ with M_L, which leads to the covering

$$M = f_1(\overset{\circ}{D}_1(\nu_1)) \cup \dots \cup f_L(\overset{\circ}{D}_1(\nu_L)), \qquad f_L(\overset{\circ}{D}_1(\nu_L)) = \overset{\circ}{M}_L .$$

3. The desingularization process

Let us now start resolving the singularities of the critical set $\mathrm{Crit}(\psi)$. For this, we will have to set up an iterative desingularization process along the strata of the underlying G-action, where each step in our iteration will consist of a decomposition, a monoidal transformation, and a reduction. For simplicity, we shall assume that at each iteration step the set of maximally singular orbits is connected. Otherwise each of the connected components, which might even have different dimensions, has to be treated separately.

First decomposition. As in the previous section, let $f_k : \nu_k \to M_k$ be an invariant tubular neighborhood of $M_k(H_k)$ in

$$M_k = M - \bigcup_{i=1}^{k-1} f_i(\overset{\circ}{D}_{1/2}(\nu_i)),$$

a manifold with corners on which G acts with the isotropy types $(H_k), (H_{k+1}), \dots, (H_L)$, and put $W_k = f_k(\overset{\circ}{D}_1(\nu_k))$. Introduce a partion of unity $\{\chi_k\}_{k=1,\dots,L}$ subordinate to the covering $\{W_k\}$, and define

$$I_k(\mu) = \int_{T^*W_k} \int_{\mathfrak{g}} e^{i\psi(\eta,X)/\mu}(a\chi_k)(\eta, X)\, dX\, d\eta,$$

so that $I(\mu) = I_1(\mu) + \dots + I_L(\mu)$. As will be explained in Lemma 3, the critical set of ψ is clean on the support of $a\chi_L$, so that we can apply directly the stationary phase theorem to compute the integral $I_L(\mu)$. But if $k \in \{1, \dots, L-1\}$, the sets

$$\Omega_k = \Omega \cap T^*W_k,$$
$$\mathrm{Crit}_k(\psi) = \Big\{(\eta, X) \in \Omega_k \times \mathfrak{g} : \tilde{X}_\eta = 0\Big\}$$

are no longer smooth manifolds, so that the stationary phase theorem can not a priori be applied in this situation. Instead, we shall resolve the singularities of $\mathrm{Crit}_k(\psi)$, and after this apply the principle of the stationary phase in a suitable resolution space. For this, introduce for each $x^{(k)} \in M_k(H_k)$ the decomposition

$$\mathfrak{g} = \mathfrak{g}_{x^{(k)}} \oplus \mathfrak{g}^\perp_{x^{(k)}},$$

where $\mathfrak{g}_{x^{(k)}}$ denotes the Lie algebra of the stabilizer $G_{x^{(k)}}$ of $x^{(k)}$, and $\mathfrak{g}^\perp_{x^{(k)}}$ its orthogonal complement with respect to the scalar product $\mathrm{tr}(^tAB)$ in $\mathfrak{g}$. Let further $A_1(x^{(k)}), \dots, A_{d^{(k)}}(x^{(k)})$ be an orthonormal basis of $\mathfrak{g}^\perp_{x^{(k)}}$, and $B_1(x^{(k)}), \dots, B_{e^{(k)}}(x^{(k)})$ an orthonormal basis of $\mathfrak{g}_{x^{(k)}}$. Consider the isotropy algebra bundle over $M_k(H_k)$

$$\mathfrak{iso}\, M_k(H_k) \to M_k(H_k),$$

as well as the canonical projection

$$\pi_k : W_k \to M_k(H_k), \qquad m = f_k(x^{(k)}, v^{(k)}) \mapsto x^{(k)}, \qquad x^{(k)} \in M_k(H_k),\ v^{(k)} \in (\nu_k)_{x^{(k)}},$$

where $f_k(x^{(k)}, v^{(k)}) = (\exp_{x^{(k)}} \circ \gamma^{(k)})(v^{(k)})$, and $\gamma^{(k)}$ is an equivariant diffeomorphism from $(\nu_k)_{x^{(k)}}$ onto its image, see Bredon [3], pages 306-307. We consider then the induced bundle

$$\pi_k^* \mathfrak{iso}\, M_k(H_k) = \left\{ (f_k(x^{(k)}, v^{(k)}), X) \in W_k \times \mathfrak{g} : X \in \mathfrak{g}_{x^{(k)}} \right\},$$

and denote by

$$\Pi_k : W_k \times \mathfrak{g} \to \pi_k^* \mathfrak{iso}\, M_k(H_k)$$

the canonical projection which is obtained by considering geodesic normal coordinates around $\pi_k^* \, \mathfrak{iso} M_k(H_k)$, and identifying $W_k \times \mathfrak{g}$ with a neighborhood of the zero section in the normal bundle $N \pi_k^* \, \mathfrak{iso}\, M_k(H_k)$. Note that the fiber of the normal bundle to $\pi^* \mathfrak{iso}\, M_k(H_k)$ at a point $(f_k(x^{(k)}, v^{(k)}), X)$ can be identified with $\mathfrak{g}^\perp_{x^{(k)}}$. Integrating along the fibers of the normal bundle to $\pi_k^* \, \mathfrak{iso} M_k(H_k)$ we therefore obtain for $I_k(\mu)$ the expression

$$I_k(\mu) = \int_{\pi_k^* \, \mathfrak{iso} M_k(H_k)} \left[\int_{\Pi_k^{-1}(m, B^{(k)}) \times T_m^* W_k} e^{i\psi/\mu} a \chi_k \, \Phi_k \; d(T_m^* W_k)(\eta)\, dA^{(k)} \right] dB^{(k)}\, dm$$

$$= \int_{M_k(H_k)} \left[\int_{\mathfrak{g} \times \pi_k^{-1}(x^{(k)}) \times T^*_{\exp_{x^{(k)}} v^{(k)}} W_k} e^{i\psi/\mu} a \chi_k \, \Phi_k \; d(T^*_{\exp_{x^{(k)}} v^{(k)}} W_k)(\eta)\, dA^{(k)}\, dB^{(k)}\, dv^{(k)} \right] dx^{(k)},$$

where

$$\gamma^{(k)}\big(\overset{\circ}{D}_1 \, (\nu_k)_{x^{(k)}} \big) \times \mathfrak{g}^\perp_{x^{(k)}} \times \mathfrak{g}_{x^{(k)}} \ni (v^{(k)}, A^{(k)}, B^{(k)}) \mapsto (\exp_{x^{(k)}} v^{(k)}, A^{(k)} + B^{(k)}) = (m, X)$$

are coordinates on $\pi_k^{-1}(x^{(k)}) \times \mathfrak{g}$, while dm, $dx^{(k)}$, $dA^{(k)}, dB^{(k)}, dv^{(k)}$, and $d(T_m^* W_k)(\eta)$ are suitable measures on W_k, $M_k(H_k)$, $\mathfrak{g}^\perp_{x^{(k)}}$, $\mathfrak{g}_{x^{(k)}}$, $\gamma^{(k)}(\overset{\circ}{D}_1 \, (\nu_k)_{x^{(k)}})$, and $T_m^* W_k$, respectively, such that

$$dX \, d\eta \equiv \Phi_k \, d(T^*_{\exp_{x^{(k)}} v^{(k)}} W_k)(\eta) dA^{(k)}\, dB^{(k)}\, dv^{(k)}\, dx^{(k)},$$

where Φ_k is a Jacobian.

First monoidal transformation. Let now $k \in \{1, \dots, L-1\}$ be fixed. For the further analysis of the integral $I_k(\mu)$, we shall sucessively resolve the singularities of $\mathrm{Crit}_k(\psi)$, until we are in position to apply the principle of the stationary phase in a suitable resolution space. To begin with, we perform a monoidal transformation

$$\zeta_k : B_{Z_k}(W_k \times \mathfrak{g}) \longrightarrow W_k \times \mathfrak{g}$$

in $W_k \times \mathfrak{g}$ with center $Z_k = \mathfrak{iso}\, M_k(H_k)$. For this, let us write $A^{(k)}(x^{(k)}, \alpha^{(k)}) = \sum \alpha_i^{(k)} A_i^{(k)}(x^{(k)})$, $B^{(k)}(x^{(k)}, \beta^{(k)}) = \sum \beta_i^{(k)} B_i^{(k)}(x^{(k)})$, and

$$v^{(k)} = \sum_{i=1}^{c^{(k)}} q_i^{(k)} v_i^{(k)}(x^{(k)}) \in \gamma^{(k)}\big(\overset{\circ}{D}_1 \, (\nu_k)_{x^{(k)}} \big),$$

where $\{v_1^{(k)}(x^{(k)}), \dots, v_{c^{(k)}}^{(k)}(x^{(k)})\}$ denotes an orthonormal frame in ν_k. With respect to these coordinates we have $Z_k = \{\alpha^{(k)} = 0,\, q^{(k)} = 0\}$, where $q^{(k)} = (q_1^{(k)}, \dots, q_{c^{(k)}}^{(k)})$, so that

$$B_{Z_k}(W_k \times \mathfrak{g}) = \left\{ (m, X, [t]) \in W_k \times \mathfrak{g} \times \mathbb{RP}^{c^{(k)} + d^{(k)} - 1} : q_i^{(k)} t_j = q_j^{(k)} t_i,\, \alpha_i^{(k)} t_{c^{(k)}+j} = \alpha_j^{(k)} t_{c^{(k)}+i} \right\},$$

$$\zeta_k : (m, X, [t]) \longmapsto (m, X).$$

Let us now cover $B_{Z_k}(W_k \times \mathfrak{g})$ with the charts $\{(\varphi_\varrho, U_\varrho)\}$, $U_\varrho = B_{Z_k}(W_k \times \mathfrak{g}) \cap (W_k \times \mathfrak{g} \times V_\varrho)$, where $V_\varrho = \Big\{[t] \in \mathbb{RP}^{c^{(k)}+d^{(k)}-1} : t_\varrho \neq 0\Big\}$. We obtain for ζ_k in each of the $q^{(k)}$-charts $\{U_\varrho\}_{1 \leq \varrho \leq c^{(k)}}$ the expressions

$$^{\varrho}\zeta_k = \zeta_k \circ \varphi_\varrho : (x^{(k)}, \tau_k, {}^{\varrho}\tilde{v}^{(k)}, A^{(k)}, B^{(k)}) \mapsto (\exp_{x^{(k)}} \tau_k\, {}^{\varrho}\tilde{v}^{(k)}, \tau_k A^{(k)} + B^{(k)}) \equiv (m, X),$$

where $\tau_k \in (-1, 1)$,

$${}^{\varrho}\tilde{v}^{(k)}(x^{(k)}, q^{(k)}) = \gamma^{(k)}\Big(\big(v_\varrho^{(k)}(x^{(k)}) + \sum_{i \neq \varrho}^{c^{(k)}} q_i^{(k)} v_i^{(k)}(x^{(k)})\big)\Big/\sqrt{1 + \sum_{i\neq\varrho}(q_i^{(k)})^2}\Big) \in \gamma^{(k)}({}^{\varrho}S_k^+)_{x^{(k)}},$$

and

$${}^{\varrho}S_k^+ = \Big\{v \in \nu_k : v = \sum s_i v_i, s_\varrho > 0, \|v\| = 1\Big\}.$$

Note that for each $1 \leq \varrho \leq c^{(k)}$,

$$W_k \simeq f_k({}^{\varrho}S_k^+ \times (-1, 1))$$

up to a set of measure zero. Now, for given $m \in M$, let $Z_m \subset T_m M$ be a neighborhood of zero such that $\exp_m : Z_m \longrightarrow M$ is a diffeomorphism onto its image. Then

$$(\exp_m)_{*,v} : T_v Z_m \longrightarrow T_{\exp_m v} M, \quad v \in Z_m,$$

and $g \cdot \exp_m v = L_g(\exp_m v) = \exp_{L_g(m)}(L_g)_{*,m}(v)$. As a consequence, since $B^{(k)} \in \mathfrak{g}_{x^{(k)}}$, we obtain

$$\begin{aligned}\widetilde{B^{(k)}}_{\exp_{x^{(k)}} \tau_k\, {}^{\varrho}\tilde{v}^{(k)}} &= \frac{d}{dt} \exp_{x^{(k)}} \big(L_{\mathrm{e}^{-tB^{(k)}}}\big)_{*,x^{(k)}} (\tau_k\, {}^{\varrho}\tilde{v}^{(k)})_{|t=0} = (\exp_{x^{(k)}})_{*,\tau_k\, {}^{\varrho}\tilde{v}^{(k)}} \big(\lambda(B^{(k)})(\tau_k\, {}^{\varrho}\tilde{v}^{(k)})\big) \\ &= \tau_k (\exp_{x^{(k)}})_{*,\tau_k\, {}^{\varrho}\tilde{v}^{(k)}} \big(\lambda(B^{(k)})({}^{\varrho}\tilde{v}^{(k)})\big),\end{aligned}$$

where we denoted by

$$\lambda : \mathfrak{g}_{x^{(k)}} \longrightarrow \mathfrak{gl}(\nu_{k,x^{(k)}}), \quad B^{(k)} \mapsto \frac{d}{dt}(L_{\mathrm{e}^{-tB^{(k)}}})_{*,x^{(k)}|t=0}$$

the linear representation of $\mathfrak{g}_{x^{(k)}}$ in $\nu_{k,x^{(k)}}$, and made the canonical identification $T_v(\nu_{k,x^{(k)}}) \equiv \nu_{k,x^{(k)}}$ for any $v \in (\nu_k)_{x^{(k)}}$. With $\pi(\eta) = m$ we therefore obtain for the phase function the factorization

$$\begin{aligned}\psi(\eta, X) = \eta(\tilde{X}_{\pi(\eta)}) &= \eta\big((\widetilde{\tau_k A^{(k)} + B^{(k)}})_{\exp_{x^{(k)}} \tau_k\, {}^{\varrho}\tilde{v}^{(k)}}\big) \\ &= \tau_k\Big[\eta\big(\widetilde{A^{(k)}}_{\exp_{x^{(k)}} \tau_k\, {}^{\varrho}\tilde{v}^{(k)}}\big) + \eta\big((\exp_{x^{(k)}})_{*,\tau_k\, {}^{\varrho}\tilde{v}^{(k)}}[\lambda(B^{(k)})^{\varrho}\tilde{v}^{(k)}]\big)\Big].\end{aligned}$$

Similar considerations hold for ζ_k in the $\alpha^{(k)}$-charts $\{U_\varrho\}_{c^{(k)}+1 \leq \varrho \leq c^{(k)}+d^{(k)}}$, so that we get

$$\psi \circ (\mathrm{id}_{fiber} \otimes \zeta_k) = {}^{(k)}\tilde{\psi}^{tot} = \tau_k \cdot {}^{(k)}\tilde{\psi}^{wk},$$

${}^{(k)}\tilde{\psi}^{tot}$ and ${}^{(k)}\tilde{\psi}^{wk}$ being the *total* and *weak transform* of the phase function ψ, respectively.[1] Introducing a partition $\{u_\varrho\}$ of unity subordinated to the covering $\{U_\varrho\}$ now yields

$$I_k(\mu) = \sum_{\varrho=1}^{c^{(k)}} {}^{\varrho}I_k(\mu) + \sum_{\varrho=c^{(k)}+1}^{d^{(k)}} {}^{\varrho}\tilde{I}_k(\mu),$$

[1] For an explanation of this notation, see section 6.

where the integrals ${}^{\varrho}I_k(\mu)$ and ${}^{\varrho}\tilde I_k(\mu)$ are given by the expressions

$$\int_{M_k(H_k)}\Bigg[\int_{(\mathrm{id}_{fiber}\otimes{}^{\varrho}\zeta)_k^{-1}(\mathfrak{g}\times\pi_k^{-1}(x^{(k)})\times T^*_{\exp_{x^{(k)}}v^{(k)}}W_k)}(u_\varrho\circ\varphi_\varrho)\,(\mathrm{id}_{fiber}\otimes{}^{\varrho}\zeta_k)^*(e^{i\psi/\mu}a\chi_k$$
$$\Phi_k\,d(T^*_{\exp_{x^{(k)}}v^{(k)}}W_k)(\eta)\,dA^{(k)}\,dB^{(k)}\,dv^{(k)})\Bigg]\,dx^{(k)}.$$

As we shall see in section 8, the weak transform ${}^{(k)}\tilde\psi^{wk}$ has no critical points in the $\alpha^{(k)}$-charts, which implies that the integrals ${}^{\varrho}\tilde I_k(\mu)$ contribute to $I(\mu)$ only with higher order terms. In what follows, we shall therefore restrict ourselves to the situation where $a_k\circ(\mathrm{id}_{fiber}\otimes\zeta_k)$ has compact support in one of the $q^{(k)}$-charts. Thus we can assume $I_k(\mu)$ to be given by

$$\int_{M_k(H_k)}\Big[\int_{\zeta_k^{-1}(\mathfrak{g}\times\pi_k^{-1}(x^{(k)}))\times T^*_{\exp_{x^{(k)}}\tau_k\tilde v^{(k)}}W_k} e^{i\frac{\tau_k}{\mu}\,{}^{(k)}\tilde\psi^{wk}}(a\chi_k\circ(\mathrm{id}_{fiber}\otimes\zeta_k))\,\tilde\Phi_k$$
$$d(T^*_{\exp_{x^{(k)}}\tau_k\tilde v^{(k)}}W_k)(\eta)\,dA^{(k)}\,dB^{(k)}\,d\tilde v^{(k)}\,d\tau_k\Big]\,dx^{(k)}$$
$$=\int_{M_k(H_k)\times(-1,1)}\Big[\int_{\gamma^{(k)}((S_k^+)_{x^{(k)}})\times\mathfrak{g}_{x^{(k)}}\times\mathfrak{g}^{\perp}_{x^{(k)}}\times T^*_{\exp_{x^{(k)}}\tau_k\tilde v^{(k)}}W_k} e^{i\frac{\tau_k}{\mu}\,{}^{(k)}\tilde\psi^{wk}}(a\chi_k\circ(\mathrm{id}_{fiber}\otimes\zeta_k))\,\tilde\Phi_k$$
$$d(T^*_{\exp_{x^{(k)}}\tau_k\tilde v^{(k)}}W_k)(\eta)\,dA^{(k)}\,dB^{(k)}\,d\tilde v^{(k)}\Big]\,d\tau_k\;dx^{(k)},$$

where we skipped the index ϱ, in particular identifying ζ_k with ${}^{\varrho}\zeta_k$, and took into account that

$$\zeta_k^{-1}(\mathfrak{g}\times\pi_k^{-1}(x^{(k)}))=\{x^{(k)}\}\times(-1,1)\times\gamma^{(k)}((S_k^+)_{x^{(k)}})\times\mathfrak{g}_{x^{(k)}}\times\mathfrak{g}^{\perp}_{x^{(k)}}.$$

Here $d\tilde v^{(k)}$ is a suitable measure on the set $\gamma^{(k)}((S_k^+)_{x^{(k)}})$ such that

$$dX\,d\eta\equiv\tilde\Phi_k\;d(T^*_{\exp_{x^{(k)}}\tau_k\tilde v^{(k)}}W_k)(\eta)\,dA^{(k)}\,dB^{(k)}\,d\tilde v^{(k)}\,d\tau_k\,dx^{(k)}.$$

Furthermore, a computation shows that

$$\tilde\Phi_k=|\tau_k|^{c^{(k)}+d^{(k)}-1}\,\Phi_k\circ\zeta_k.$$

First reduction. Let us now assume that there exists a $m\in W_k$ with orbit type G/H_j, and let $x^{(k)}\in M_k(H_k), v^{(k)}\in(\nu_k)_{x^{(k)}}$ be such that $m=f_k(x^{(k)},v^{(k)})$. Since we can assume that m lies in a slice at $x^{(k)}$ around the G-orbit of $x^{(k)}$, we have $G_m\subset G_{x^{(k)}}$, see Kawakubo [11], pages 184-185, and Bredon [3], page 86. Hence, $H_j\simeq G_m$ must be conjugate to a subgroup of $H_k\simeq G_{x^{(k)}}$. Now, G acts on M_k with the isotropy types $(H_k),(H_{k+1}),\dots,(H_L)$. The isotropy types occuring in W_k are therefore those for which the corresponding isotropy groups $H_k,H_{k+1},\dots,H_L$ are conjugate to a subgroup of H_k, and we shall denote them by

$$(H_k)=(H_{i_1}),(H_{i_2}),\dots,(H_L).$$

Now, for every $x^{(k)}\in M_k(H_k)$, $(\nu_k)_{x^{(k)}}$ is an orthogonal $G_{x^{(k)}}$-space; therefore $G_{x^{(k)}}$ acts on $(S_k)_{x^{(k)}}$ with isotropy types $(H_{i_2}),\dots,(H_L)$, cp. Donnelly [7], pp. 34. Furthermore, by the invariant tubular neighborhood theorem, one has the isomorphism

$$W_k/G\simeq(\nu_k)_{x^{(k)}}/G_{x^{(k)}},$$

so that G acts on $S_k=\{v\in\nu_k:\|v\|=1\}$ with isotropy types $(H_{i_2}),\dots,(H_L)$ as well. As will turn out, if G acted on S_k only with type (H_L), the critical set of ${}^{(k)}\tilde\psi^{wk}$ would be clean in the sense of Bott, and we could proceed to apply the stationary phase theorem to compute $I_k(\mu)$. But in general this will not be the case, and we are forced to continue with the iteration.

Second decomposition. Let now $x^{(k)} \in M_k(H_k)$ be fixed. Since $\gamma^{(k)}: \nu_k \to \nu_k$ is an equivariant diffeomorphism onto its image, $\gamma^{(k)}((S_k)_{x^{(k)}})$ is a compact $G_{x^{(k)}}$-manifold, and we consider the covering

$$\gamma^{(k)}((S_k)_{x^{(k)}}) = W_{ki_2} \cup \cdots \cup W_{kL}, \qquad W_{ki_j} = f_{ki_j}(\overset{\circ}{D}_1\,(\nu_{ki_j})), \quad W_{kL} = \mathrm{Int}(\gamma^{(k)}((S_k)_{x^{(k)}})_L),$$

where $f_{ki_j}: \nu_{ki_j} \to \gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}$ is an invariant tubular neighborhood of $\gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})$ in

$$\gamma^{(k)}((S_k)_{x^{(k)}})_{i_j} = \gamma^{(k)}((S_k)_{x^{(k)}}) - \bigcup_{r=2}^{j-1} f_{ki_r}(\overset{\circ}{D}_{1/2}\,(\nu_{ki_r})), \qquad j \geq 2,$$

and $f_{ki_j}(x^{(i_j)}, v^{(i_j)}) = (\exp_{x^{(i_j)}} \circ \gamma^{(i_j)})(v^{(i_j)})$, $x^{(i_j)} \in \gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})$, $v^{(i_j)} \in (\nu_{ki_j})_{x^{(i_j)}}$, $\gamma^{(i_j)}: \nu_{ki_j} \to \nu_{ki_j}$ being an equivariant diffeomorphism onto its image. Let further $\{\chi_{ki_j}\}$ denote a partition of the unity subordinated to the covering $\{W_{ki_j}\}$, and define

$$I_{ki_j}(\mu) = \int_{M_k(H_k)\times(-1,1)} \Big[\int_{\gamma^{(k)}((S_k^+)_{x^{(k)}})\times \mathfrak{g}_{x^{(k)}} \times \mathfrak{g}^\perp_{x^{(k)}} \times T^*_{\exp_{x^{(k)}} \tau_k \tilde v^{(k)}} W_k} e^{i\frac{\tau_k}{\mu}\,{}^{(k)}\tilde\psi^{wk}} (a\chi_k \circ (\mathrm{id}_{\,fiber} \otimes \zeta_k))$$
$$\chi_{ki_j}\, \tilde\Phi_k \; d(T^*_{\exp_{x^{(k)}} \tau_k \tilde v^{(k)}} W_k)(\eta)\, dA^{(k)}\, dB^{(k)}\, d\tilde v^{(k)} \Big]\, d\tau_k \; dx^{(k)},$$

so that $I_k(\mu) = I_{ki_2}(\mu) + \cdots + I_{kL}(\mu)$. It is important to note that the partition functions χ_{ki_j} depend smoothly on $x^{(k)}$ as a consequence of the tubular neighborhood theorem, by which in particular $\gamma^{(k)}(S_k)/G \simeq \gamma^{(k)}((S_k)_{x^{(k)}})/G_{x^{(k)}}$, and the smooth dependence in $x^{(k)}$ of the induced Riemannian metric on $\gamma^{(k)}((S_k)_{x^{(k)}})$, and the metrics on the normal bundles ν_{ki_j}. Since $G_{x^{(k)}}$ acts on W_{kL} only with type (H_L), the iteration process for $I_{kL}(\mu)$ ends here. For the remaining integrals $I_{ki_j}(\mu)$ with $k < i_j < L$, let us denote by

$$\mathfrak{iso}\, \gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j}) \to \gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})$$

the isotropy algebra bundle over $\gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})$, and by $\pi_{ki_j}: W_{ki_j} \to \gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})$ the canonical projection. For $x^{(i_j)} \in \gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})$, consider the decomposition

$$\mathfrak{g} = \mathfrak{g}_{x^{(k)}} \oplus \mathfrak{g}^\perp_{x^{(k)}} = (\mathfrak{g}_{x^{(i_j)}} \oplus \mathfrak{g}^\perp_{x^{(i_j)}}) \oplus \mathfrak{g}^\perp_{x^{(k)}}.$$

Let further $A_1^{(i_j)}, \dots, A^{(i_j)}_{d^{(i_j)}}$ be an orthonormal frame in $\mathfrak{g}^\perp_{x^{(i_j)}}$, as well as $B_1^{(i_j)}, \dots, B^{(i_j)}_{e^{(i_j)}}$ be an orthonormal frame in $\mathfrak{g}_{x^{(i_j)}}$, and $v_1^{(ki_j)}, \dots, v^{(ki_j)}_{c^{(i_j)}}$ an orthonormal frame in $(\nu_{ki_j})_{x^{(i_j)}}$. Integrating along the fibers in a neighborhood of $\pi^*_{ki_j}\mathfrak{iso}\,\gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j}) \subset W_{ki_j} \times \mathfrak{g}_{x^{(k)}}$ then yields for $I_{ki_j}(\mu)$ the expression

$$\int_{M_k(H_k)\times(-1,1)} \Big[\int_{\gamma^{(k)}((S_k^+)_{x^{(k)}})_{i_j}(H_{i_j})} \Big[\int_{\pi^{-1}_{ki_j}(x^{(i_j)})\times \mathfrak{g}_{x^{(k)}} \times \mathfrak{g}^\perp_{x^{(k)}} \times T^*_{\exp_{x^{(k)}} \tau_k \exp_{x^{(i_j)}} v^{(i_j)}} W_k} e^{i\frac{\tau_k}{\mu}\,{}^{(k)}\tilde\psi^{wk}} \times$$
$$(a\chi_k \circ (\mathrm{id}_{\,fiber} \otimes \zeta_k))\chi_{ki_j}\, \Phi_{ki_j} \; d(T^*_{\exp_{x^{(k)}} \tau_k \exp_{x^{(i_j)}} v^{(i_j)}}(W_k)(\eta)\, dA^{(k)}\, dA^{(i_j)}\, dB^{(i_j)}\, dv^{(i_j)} \Big] dx^{(i_j)} \Big] d\tau_k dx^{(k)},$$

where Φ_{ki_j} is a Jacobian, and

$$\gamma^{(i_j)}(\,\overset{\circ}{D}_1\,(\nu_{ki_j})_{x^{(i_j)}}) \times \mathfrak{g}^\perp_{x^{(i_j)}} \times \mathfrak{g}_{x^{(i_j)}} \ni (v^{(i_j)}, A^{(i_j)}, B^{(i_j)}) \mapsto (\exp_{x^{(i_j)}} v^{(i_j)}, A^{(i_j)} + B^{(i_j)}) = (\tilde v^{(k)}, B^{(k)})$$

are coordinates on $\pi^{-1}_{ki_j}(x^{(i_j)}) \times \mathfrak{g}_{x^{(k)}}$, while $dx^{(i_j)}$, and $dA^{(i_j)}, dB^{(i_j)}, dv^{(i_j)}$ are suitable measures in the spaces $\gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})$, and $\mathfrak{g}^\perp_{x^{(i_j)}}$, $\mathfrak{g}_{x^{(i_j)}}$, $\overset{\circ}{D}_1\,(\nu_{ki_j})_{x^{(i_j)}}$, respectively, such that we have the equality $\tilde\Phi_k\, dB^{(k)}\, d\tilde v^{(k)} \equiv \Phi_{ki_j}\, dA^{(i_j)}\, dB^{(i_j)}\, dv^{(i_j)}\, dx^{(i_j)}$.

Second monoidal transformation. Let us fix an l such that $k < l < L$, and consider in the $q^{(k)}$-chart $(-1,1) \times \gamma^{(k)}(S_k^+) \times \mathfrak{g}$ a monoidal transformation

$$\zeta_{kl} : B_{Z_{kl}}((-1,1) \times \gamma^{(k)}(S_k^+) \times \mathfrak{g}) \longrightarrow (-1,1) \times \gamma^{(k)}(S_k^+) \times \mathfrak{g}$$

with center

$$Z_{kl} = (-1,1) \times \mathfrak{iso}\, \Gamma^+_{k,l}(H_l), \qquad \Gamma^+_{k,l} = \bigcup_{x^{(k)} \in M_k(H_k)} \gamma^{(k)}((S_k^+)_{x^{(k)}})_l.$$

Writing $A^{(l)}(x^{(k)}, x^{(l)}, \alpha^{(l)}) = \sum \alpha_i^{(l)} A_i^{(l)}(x^{(k)}, x^{(l)})$, $B^{(l)}(x^{(k)}, x^{(l)}, \beta^{(l)}) = \sum \beta_i^{(l)} B_i^{(l)}(x^{(l)})$, and

$$v^{(l)}(x^{(k)}, x^{(l)}, q^{(l)}) = \sum_{i=1}^{c^{(l)}} q_i^{(l)} v_i^{(kl)}(x^{(k)}, x^{(l)}),$$

one has $Z_{kl} = \left\{\alpha^{(k)} = 0,\, \alpha^{(l)} = 0,\, q^{(l)} = 0\right\}$, which in particular shows that Z_{kl} is a manifold. If we now cover $B_{Z_{kl}}((-1,1) \times \gamma^{(k)}(S_k^+) \times \mathfrak{g})$ with the standard charts, we shall see again in section 8 that modulo higher order terms we can assume that $((a\chi_k \circ (\mathrm{id}_{\,fiber} \otimes \zeta_k))\chi_{kl}) \circ \zeta_{kl}$ has compact support in one of the $q^{(l)}$-charts. Therefore it suffices to examine ζ_{kl} in one of these charts, in which it reads

$$\zeta_{kl} : (x^{(k)}, \tau_k, x^{(l)}, \tau_l, \tilde v^{(l)}, A^{(k)}, A^{(l)}, B^{(l)}) \mapsto$$
$$\mapsto (x^{(k)}, \tau_k, \exp_{x^{(l)}} \tau_l \tilde v^{(l)}, \tau_l A^{(k)}, \tau_l A^{(l)} + B^{(l)}) \equiv (x^{(k)}, \tau_k, \tilde v^{(k)}, A^{(k)}, B^{(k)}),$$

where

$$\tilde v^{(l)}(x^{(k)}, x^{(l)}, q^{(l)}) = \gamma^{(l)} \left(\Big(v_\varrho^{(kl)} + \sum_{i \neq \varrho}^{c^{(l)}} q_i^{(l)} v_i^{(kl)}\Big) \Big/ \sqrt{1 + \sum_{i\neq\varrho} (q_i^{(l)})^2} \right) \in \gamma^{(l)}\big((S^+_{kl})_{x^{(l)}}\big)$$

for some ϱ. Note that Z_{kl} has normal crossings with the exceptional divisor $E_k = \zeta_k^{-1}(Z_k) = \{\tau_k = 0\}$, and that

$$W_{kl} \simeq f_{kl}(S^+_{kl} \times (-1,1))$$

up to a set of measure zero, where S_{kl} denotes the sphere subbundle in ν_{kl}, and we set $S^+_{kl} = \Big\{v \in S_{kl} : v = \sum v_i v_i^{(kl)},\, v_\varrho > 0\Big\}$. Consequently, the phase function factorizes according to

$$\psi \circ (\mathrm{id}_{\,fiber} \otimes (\zeta_k \circ \zeta_{kl})) = {}^{(kl)}\tilde\psi^{tot} = \tau_k\, \tau_l \cdot {}^{(kl)}\tilde\psi^{wk},$$

which in the given charts reads

$$\begin{aligned}
\psi(\eta, X) = \tau_k & \Big[\eta\Big(\widetilde{\tau_l A^{(k)}}_{\exp_{x^{(k)}} \tau_k \exp_{x^{(l)}} \tau_l \tilde v^{(l)}}\Big) \\
& + \eta\Big((\exp_{x^{(k)}})_{*, \tau_k \exp_{x^{(l)}} \tau_l \tilde v^{(l)}} [\lambda(\tau_l A^{(l)} + B^{(l)}) \exp_{x^{(l)}} \tau_l \tilde v^{(l)}]\Big)\Big] \\
= \tau_k \tau_l & \Big[\eta\Big(\widetilde{A^{(k)}}_{\exp_{x^{(k)}} \tau_k \exp_{x^{(l)}} \tau_l \tilde v^{(l)}}\Big) + \eta\Big((\exp_{x^{(k)}})_{*, \tau_k \exp_{x^{(l)}} \tau_l \tilde v^{(l)}} [\lambda(A^{(l)}) \exp_{x^{(l)}} \tau_l \tilde v^{(l)}]\Big) \\
& + \eta\Big((\exp_{x^{(k)}})_{*, \tau_k \exp_{x^{(l)}} \tau_l \tilde v^{(l)}} \big[(\exp_{x^{(l)}})_{*, \tau_l \tilde v^{(l)}} [(\lambda(B^{(l)}) \tilde v^{(l)}]\big]\Big)\Big]
\end{aligned}$$

where we took into account that

$$\lambda(B^{(l)}) \exp_{x^{(l)}} \tau_l \tilde v^{(l)} = \frac{d}{dt} \exp_{x^{(l)}} \big(L_{\mathrm{e}^{-tB^{(l)}}}\big)_{*, x^{(k)}} \tau_l \tilde v^{(l)}_{|t=0} = (\exp_{x^{(l)}})_{*, \tau_l \tilde v^{(l)}} \big(\lambda(B^{(l)}) \tau_l \tilde v^{(l)}\big).$$

Since

$$\begin{aligned}&\zeta_{kl}^{-1}(\{x^{(k)}\}\times\{\tau_k\}\times\pi_{kl}^{-1}(x^{(l)})\times\mathfrak{g}_{x^{(k)}}\times\mathfrak{g}_{x^{(k)}}^{\perp})\\ &=\{x^{(k)}\}\times\{\tau_k\}\times\{x^{(l)}\}\times(-1,1)\times\gamma^{(l)}\big((S_{kl}^+)_{x^{(l)}}\big)\times\mathfrak{g}_{x^{(l)}}\times\mathfrak{g}_{x^{(l)}}^{\perp}\times\mathfrak{g}_{x^{(k)}}^{\perp},\end{aligned}$$

we obtain for $I_{kl}(\mu)$ the expression

$$\begin{aligned}&\int_{M_k(H_k)\times(-1,1)}\Big[\int_{\gamma^{(k)}((S_k^+)_{x^{(k)}})_l(H_l)}\Big[\int_{\zeta_{kl}^{-1}(\{x^{(k)}\}\times\{\tau_k\}\times\pi_{kl}^{-1}(x^{(l)})\times\mathfrak{g}_{x^{(k)}}\times\mathfrak{g}_{x^{(k)}}^{\perp})\times T^*_{m^{(kl)}}W_k} e^{i\frac{\tau_k\tau_l}{\mu}\,{}^{(kl)}\tilde{\psi}^{wk}}\\ &\times((a_k\circ(\mathrm{id}_{fiber}\otimes\zeta_k))\chi_{kl})\circ\zeta_{kl}\,\tilde{\Phi}_{kl}\;d(T^*_{m^{(kl)}}W_k)(\eta)\,dA^{(k)}\,dA^{(l)}\,dB^{(l)}\,d\tilde{v}^{(l)}\,d\tau_l\Big]dx^{(l)}\Big]\,d\tau_k\,dx^{(k)}\\ &=\int_{M_k(H_k)\times(-1,1)}\Big[\int_{\gamma^{(k)}((S_k^+)_{x^{(k)}})_l(H_l)\times(-1,1)}\Big[\int_{\gamma^{(l)}((S_{kl}^+)_{x^{(l)}})\times\mathfrak{g}_{x^{(l)}}\times\mathfrak{g}_{x^{(l)}}^{\perp}\times\mathfrak{g}_{x^{(k)}}^{\perp})\times T^*_{m^{(kl)}}W_k} e^{i\frac{\tau_k\tau_l}{\mu}\,{}^{(kl)}\tilde{\psi}^{wk}}\\ &\times((a\chi_k\circ(\mathrm{id}_{fiber}\otimes\zeta_k))\chi_{kl})\circ\zeta_{kl}\,\tilde{\Phi}_{kl}\;d(T^*_{m^{(kl)}}W_k)(\eta)\,dA^{(k)}\,dA^{(l)}\,dB^{(l)}\,d\tilde{v}^{(l)}\Big]d\tau_l\,dx^{(l)}\Big]\,d\tau_k\,dx^{(k)},\end{aligned}$$

where $m^{(kl)}=\exp_{x^{(k)}}\tau_k\exp_{x^{(l)}}\tau_l\tilde{v}^{(l)}$, and $d\tilde{v}^{(l)}$ is a suitable measure in $\gamma^{(l)}((S_{kl}^+)_{x^{(l)}})$ such that we have the equality

$$dX\,d\eta\equiv\tilde{\Phi}_{kl}\;d(T^*_{m^{(kl)}}W_k)(\eta)\,dA^{(k)}\,dA^{(l)}\,dB^{(l)}\,d\tilde{v}^{(l)}\,d\tau_l\,dx^{(l)}\,d\tau_k\,dx^{(k)}.$$

Furthermore, $\tilde{\Phi}_{kl}=|\tau_l|^{c^{(l)}+d^{(k)}+d^{(l)}-1}\Phi_{kl}\circ\zeta_{kl}$.

Second reduction. Now, the group $G_{x^{(k)}}$ acts on $\gamma^{(l)}((S_k)_{x^{(k)}})_l$ with the isotropy types $(H_l)=(H_{i_j}),(H_{i_{j+1}}),\dots,(H_L)$. By the same arguments given in the first reduction, the isotropy types occuring in W_{kl} constitute a subset of these types, and we shall denote them by

$$(H_l)=(H_{i_{r_1}}),(H_{i_{r_2}}),\dots,(H_L).$$

Consequently, $G_{x^{(k)}}$ acts on S_{kl} with the isotropy types $(H_{i_{r_2}}),\dots,(H_L)$. Again, if G acted on S_{kl} only with type (H_L), we shall see in the next section that the critical set of ${}^{(kl)}\tilde{\psi}^{wk}$ would be clean. However, in general this will not be the case, and we have to continue with the iteration.

N-th decomposition. Once one arrives at a sphere bundle $S_{klmn\dots}$ on which G acts only with the isotropy type (H_L), the end of the iteration will be reached. More precisely, let $N\geq 3$, $(H_{i_1}),\dots,(H_{i_{N+1}})=(H_L)$ be a branch of the isotropy tree of the G-action on M, and f_{i_1}, $f_{i_1i_2}$, S_{i_1}, $S_{i_1i_2}$, as well as $x^{(i_1)}\in M_{i_1}(H_{i_1})$, $\quad x^{(i_2)}\in\gamma^{(i_1)}\big((S_{i_1}^+)_{x^{(i_1)}}\big)_{i_2}(H_{i_2})$ be defined as in the first two iteration steps. Let now $N\geq j\geq 3$, and assume that $f_{i_1\dots i_{j-1}}$, $S_{i_1\dots i_{j-1}}$,... have already been defined. Let $\gamma^{(i_{j-1})}((S_{i_1\dots i_{j-1}})_{x^{(i_{j-1})}})_{i_j}$ be the submanifold with corners of $\gamma^{(i_{j-1})}((S_{i_1\dots i_{j-1}})_{x^{(i_{j-1})}})$ from which all the isotropy types less than (H_{i_j}) have been removed. Consider the invariant tubular neighborhood $f_{i_1\dots i_j}=\exp\circ\gamma^{(i_j)}:\nu_{i_1\dots i_j}\to\gamma^{(i_{j-1})}((S_{i_1\dots i_{j-1}})_{x^{(i_{j-1})}})_{i_j}$ of the set of maximal singular orbits $\gamma^{(i_{j-1})}((S_{i_1\dots i_{j-1}})_{x^{(i_{j-1})}})_{i_j}(H_{i_j})$, and define $S_{i_1\dots i_j}$ as the sphere subbundle in $\nu_{i_1\dots i_j}$. For $x^{(i_j)}\in\gamma^{(i_{j-1})}((S^+_{i_1\dots i_{j-1}})_{x^{(i_{j-1})}})_{i_j}(H_{i_j})$ we then consider the decomposition

$$\mathfrak{g}_{x^{(i_{j-1})}}=\mathfrak{g}_{x^{(i_j)}}\oplus\mathfrak{g}_{x^{(i_j)}}^{\perp},$$

and set $d^{(i_j)}=\dim\mathfrak{g}^{\perp}_{p^{(i_j)}}$, $e^{(i_j)}=\dim\mathfrak{g}_{p^{(i_j)}}$. After N iterations, one arrives at the decomposition

$$\mathfrak{g}=\mathfrak{g}_{x^{(i_1)}}\oplus\mathfrak{g}_{x^{(i_1)}}^{\perp}=\big(\mathfrak{g}_{x^{(i_2)}}\oplus\mathfrak{g}_{x^{(i_2)}}^{\perp}\big)\oplus\mathfrak{g}_{x^{(i_1)}}^{\perp}=\dots=\mathfrak{g}_{x^{(i_N)}}\oplus\mathfrak{g}_{x^{(i_N)}}^{\perp}\oplus\dots\oplus\mathfrak{g}_{x^{(i_1)}}^{\perp},$$

and we denote by $\{A_r^{(i_j)}(x^{(i_1)},\dots,x^{(i_j)})\}$ a basis of $\mathfrak{g}^\perp_{x^{(i_j)}}$, and by $\{B_r^{(i_N)}(x^{(i_1)},\dots,x^{(i_N)})\}$ a basis of $\mathfrak{g}_{x^{(i_N)}}$. Let further

$$A^{(i_j)}=\sum_{r=1}^{d^{(i_j)}}\alpha_r^{(i_j)}A_r^{(i_j)}(x^{(i_1)},\dots,x^{(i_j)}),\qquad B^{(i_N)}=\sum_{r=1}^{e^{(i_N)}}\beta_r^{(i_N)}B_r^{(i_N)}(x^{(i_1)},\dots,x^{(i_N)}),$$

and put

$$\tilde v^{(i_N)}(x^{(i_j)},\theta^{(i_N)})=\gamma^{(i_N)}\left(\left(v_\varrho^{(i_1\dots i_N)}(x^{(i_j)})+\sum_{r\neq\varrho}^{c^{(i_N)}}q_r^{(i_N)}v_r^{(i_1\dots i_N)}(x^{(i_j)})\right)\Big/\sqrt{1+\sum_{r\neq\varrho}(q_r^{(i_N)})^2}\right)$$

for some ϱ, where $\left\{v_r^{(i_1\dots i_N)}(x^{(i_1)},\dots x^{(i_N)})\right\}$ is an orthonormal frame in $(\nu_{i_1\dots i_N})_{x^{(i_N)}}$. Finally, we shall use the notations

$$m^{(i_j\dots i_N)}=\exp_{x^{(i_j)}}[\tau_{i_j}\exp_{x^{(i_{j+1})}}[\tau_{i_{j+1}}\exp_{x^{(i_{j+2})}}[\dots[\tau_{i_{N-2}}\exp_{x^{(i_{N-1})}}[\tau_{i_{N-1}}\exp_{x^{(i_N)}}[\tau_{i_N}\tilde v^{(i_N)}]]]\dots]]],$$
$$X^{(i_j\dots i_N)}=\tau_{i_j}\cdots\tau_{i_N}A^{(i_j)}+\tau_{i_{j+1}}\cdots\tau_{i_N}A^{(i_{j+1})}+\dots+\tau_{i_{N-1}}\tau_{i_N}A^{(i_{N-1})}+\tau_{i_N}A^{(i_N)}+B^{(i_N)},$$

where $j=1,\dots,N$. Consider now for every fixed $x^{(i_{N-1})}\in\gamma^{(i_{N-2})}((S_{i_1\dots i_{N-2}})_{x^{(i_{N-2})}})_{i_{N-1}}(H_{i_{N-1}})$ the decomposition of the closed $G_{x^{(i_{N-1})}}$-manifold $\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})$ given by

$$\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})=W_{i_1\dots i_N}\cup W_{i_1\dots i_{N-1}L},$$
$$W_{i_1\dots i_N}=f_{i_1\dots i_N}(\overset{\circ}{D}_1(\nu_{i_1\dots i_N})),\quad W_{i_1\dots i_{N-1}L}=\mathrm{Int}(\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_L),$$

where $f_{i_1\dots i_N}:\nu_{i_1\dots i_N}\to\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}$ is an invariant tubular neighborhood of the closed invariant submanifold $\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}(H_{i_N})$ in $\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}=\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})$, and

$$\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_L=\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})-f_{i_1\dots i_N}(\overset{\circ}{D}_{1/2}(\nu_{i_1\dots i_N})).$$

Let $\{\chi_{i_1\dots i_N},\chi_{i_1\dots i_{N-1}L}\}$ denote a partition of unity subordinated to the covering by the open sets $\{W_{i_1\dots i_N},W_{i_1\dots i_{N-1}L}\}$, and decompose $I_{i_1\dots i_{N-1}}(\mu)$ accordingly, so that

$$I_{i_1\dots i_{N-1}}(\mu)=I_{i_1\dots i_N}(\mu)+I_{i_1\dots i_{N-1}L}(\mu).$$

N-th monoidal transformation. In the chart $(-1,1)^{N-1}\times\gamma^{(i_{N-1})}(S^+_{i_1\dots i_{N-1}})\times\mathfrak{g}$ consider the monoidal transformation

$$\zeta_{i_1\dots i_N}:B_{Z_{i_1\dots i_N}}((-1,1)^{N-1}\times\gamma^{(i_{N-1})}(S^+_{i_1\dots i_{N-1}})\times\mathfrak{g})\longrightarrow(-1,1)^{N-1}\times\gamma^{(i_{N-1})}(S^+_{i_1\dots i_{N-1}})\times\mathfrak{g}$$

with center

$$Z_{i_1\dots i_N}=(-1,1)^{N-1}\times\mathfrak{iso}\,\Gamma^+_{i_1\dots i_{N-1},i_N}(H_{i_N}),$$
$$\Gamma_{i_1\dots i_{N-1},i_N}=\bigcup_{x^{(i_{N-1})}}\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}=\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})).$$

For an arbitrary element $A^{(i_j)}\in\mathfrak{g}^\perp_{i_j}$ one computes

$$\begin{aligned}(\tilde{A}^{i_j)})_{m^{(i_1\dots i_N)}}&=\frac{d}{dt}\,\mathrm{e}^{-tA^{(i_j)}}\cdot m^{(i_1\dots i_N)}_{|t=0}=\frac{d}{dt}\exp_{x^{(i_1)}}\left[(\mathrm{e}^{-tA^{(i_j)}})_{*,x^{(i_1)}}[\tau_{i_1}m^{(i_2\dots i_N)}]\right]_{|t=0}\\&=(\exp_{x^{(i_1)}})_{*,\tau_{i_1}m^{(i_2\dots i_N)}}[\lambda(A^{(i_j)})\tau_{i_1}m^{(i_2\dots i_N)}],\end{aligned}$$

successively obtaining

$$
\begin{aligned}
(\tilde{A}^{i_j})_{m^{(i_1\dots i_N)}} &= \frac{d}{dt}\exp_{x^{(i_1)}}\big[\tau_{i_1}\exp_{x^{(i_2)}}[\dots[\tau_{i_{j-1}}(\,\mathrm{e}^{-tA^{(i_j)}}\,)_{*,x^{(i_1)}}m^{(i_j\dots i_N)}]\dots]\big]_{|t=0}\\
&= (\exp_{x^{(i_1)}})_{*,\tau_{i_1}m^{(i_2\dots i_N)}}\big[\tau_{i_1}(\exp_{x^{(i_2)}})_{*,\tau_{i_2}m^{(i_3\dots i_N)}}[\dots[\tau_{i_{j-1}}\lambda(A^{(i_j)})m^{(i_j\dots i_N)}]\dots]\big].
\end{aligned}
$$

As a consequence, the phase function factorizes according to

$$
{}^{(i_1\dots i_N)}\tilde{\psi}^{tot} = \mathbb{J}(\eta_{m^{(i_1\dots i_N)}})(X^{(i_1\dots i_N)}) = \tau_{i_1}\cdots\tau_{i_N}\;{}^{(i_1\dots i_N)}\tilde{\psi}^{wk},
$$

where $\eta_{m^{(i_1\dots i_N)}} \in \pi^{-1}(m^{(i_1\dots i_N)})$, and

$$
\begin{aligned}
{}^{(i_1\dots i_N)}\tilde{\psi}^{wk} = \eta_{m^{(i_1\dots i_N)}}\Big(\widetilde{A^{(i_1)}}_{m^{(i_1\dots i_N)}}\Big) + \sum_{j=2}^{N}\eta_{m^{(i_1\dots i_N)}}\Big((\exp_{x^{(i_1)}})_{*,\tau_{i_1}m^{(i_2\dots i_N)}}\\
\big[(\exp_{x^{(i_2)}})_{*,\tau_{i_2}m^{(i_3\dots i_N)}}\big[\dots(\exp_{x^{(i_{j-1})}})_{*,\tau_{i_{j-1}}m^{(i_j\dots i_N)}}[\lambda(A^{(i_j)})m^{(i_j\dots i_N)}]\dots\big]\big]\Big)\\
+\eta_{m^{(i_1\dots i_N)}}\Big((\exp_{x^{(i_1)}})_{*,\tau_{i_1}m^{(i_2\dots i_N)}}\big[(\exp_{x^{(i_2)}})_{*,\tau_{i_2}m^{(i_3\dots i_N)}}\big[\dots\\
(\exp_{x^{(i_N)}})_{*,\tau_{i_N}\tilde{v}^{(i_N)}}[\lambda(B^{(i_N)})\tilde{v}^{(i_N)}]\dots\big]\big]\Big)
\end{aligned}
$$

in the given charts. With $S_{i_1\dots i_N}$ equal to the sphere bundle over $\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}(H_{i_N})$, one finally obtains for the integral $I_{i_1\dots i_N}(\mu)$ the expression

$$
\begin{aligned}
&I_{i_1\dots i_N}(\mu)\\
&= \int_{M_{i_1}(H_{i_1})\times(-1,1)}\Big[\int_{\gamma^{(i_1)}((S^+_{i_1})_{x^{(i_1)}})_{i_2}(H_{i_2})\times(-1,1)}\cdots\Big[\int_{\gamma^{(i_{N-1})}((S^+_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}(H_{i_N})\times(-1,1)}\\
&\Big[\int_{\gamma^{(i_N)}((S^+_{i_1\dots i_N})_{x^{(i_N)}})\times\mathfrak{g}_{x^{(i_N)}}\times\mathfrak{g}^\perp_{x^{(i_N)}}\times\dots\times\mathfrak{g}^\perp_{x^{(i_1)}}\times T^*_{m^{(i_1\dots i_N)}}W_{i_1}} e^{i\frac{\tau_1\dots\tau_N}{\mu}\,{}^{(i_1\dots i_N)}\tilde{\psi}^{wk}}\,a_{i_1\dots i_N}\,\tilde{\Phi}_{i_1\dots i_N}\\
&\quad d(T^*_{m^{(i_1\dots i_N)}}W_{i_1})(\eta)\,dA^{(i_1)}\dots dA^{(i_N)}\,dB^{(i_N)}\,d\tilde{v}^{(i_N)}\Big]\,d\tau_{i_N}\,dx^{(i_N)}\dots\Big]\,d\tau_{i_2}\,dx^{(i_2)}\Big]\,d\tau_{i_1}\,dx^{(i_1)}.
\end{aligned}
\tag{6}
$$

Here

$$
a_{i_1\dots i_N} = [a\,\chi_{i_1}\circ(\mathrm{id}_{fiber}\otimes\zeta_{i_1}\circ\zeta_{i_1i_2}\circ\dots\circ\zeta_{i_1\dots i_N})]\,[\chi_{i_1i_2}\circ\zeta_{i_1i_2}\circ\dots\circ\zeta_{i_1\dots i_N}]\dots[\chi_{i_1\dots i_N}\circ\zeta_{i_1\dots i_N}]
$$

is supposed to have compact support in one of the $\theta^{(i_N)}$-charts, and

$$
\tilde{\Phi}_{i_1\dots i_N} = \prod_{j=1}^{N}|\tau_{i_j}|^{c^{(i_j)}+\sum_{r=1}^{j}d^{(i_r)}-1}\Phi_{i_1\dots i_N},
$$

where $\Phi_{i_1\dots i_N}$ is a smooth function which does not depend on the variables τ_{i_j}.

N-th reduction. By assumption, G acts on $S_{i_1\dots i_N}$ only with type (H_L), and the iteration process ends here.

4. Phase analysis of the weak transform. The first fundamental theorem

We are now in position to state the first fundamental theorem in the derivation of equivariant spectral asymptotics. For this end, let us define certain geometric distributions $E^{(i_j)}$ and $F^{(i_N)}$

on M associated to the iteration of N steps along the branch $((H_{i_1}),\dots,(H_{i_{N+1}})=(H_L))$ of the isotropy tree of the G-action on M by setting

$$
\begin{aligned}
&E^{(i_1)}_{m^{(i_1\dots i_N)}} = \mathrm{Span}\{\tilde Y_{m^{(i_1\dots i_N)}} : Y \in \mathfrak{g}^\perp_{x^{(i_1)}}\},\\
&E^{(i_j)}_{m^{(i_1\dots i_N)}} = (\exp_{x^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}} \dots (\exp_{x^{(i_{j-1})}})_{*,\tau_{i_{j-1}} m^{(i_j\dots i_N)}} [\lambda(\mathfrak{g}^\perp_{x^{(i_j)}}) m^{(i_j\dots i_N)}],\\
&F^{(i_N)}_{m^{(i_1\dots i_N)}} = (\exp_{x^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}} \dots (\exp_{x^{(i_N)}})_{*,\tau_{i_N} \tilde v^{(i_N)}} [\lambda(\mathfrak{g}_{x^{(i_N)}}) \tilde v^{(i_N)}],
\end{aligned}
\tag{7}
$$

where $2 \le j \le N$, the notation being as in the previous section. By construction, for $\tau_{i_j} \neq 0$, $1 \le j \le N$, the G-orbit through $m^{(i_1\dots i_N)}$ is of principal type G/H_L, which amounts to the fact that G acts on $S_{i_1\dots i_N}$ only with the isotropy type (H_L). Let $\eta_{m^{(i_1\dots i_N)}} \in \pi^{-1}(m^{(i_1\dots i_N)})$. We then have the following

Theorem 1. *Consider the factorization*

$$\mathbb{J}(\eta_{m^{(i_1\dots i_N)}})(X^{(i_1\dots i_N)}) = {}^{(i_1\dots i_N)}\tilde\psi^{tot} = \tau_{i_1}\cdots\tau_{i_N}\, {}^{(i_1\dots i_N)}\tilde\psi^{wk,\,pre}$$

of the phase function ψ after N iteration steps, where ${}^{(i_1\dots i_N)}\tilde\psi^{wk,pre}$ is given by

$$
\begin{aligned}
&\eta_{m^{(i_1\dots i_N)}}\Big(\widetilde{A^{(i_1)}}_{m^{(i_1\dots i_N)}}\Big) + \sum_{j=2}^N \eta_{m^{(i_1\dots i_N)}}\Big((\exp_{x^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}}\big[(\exp_{x^{(i_2)}})_{*,\tau_{i_2} m^{(i_3\dots i_N)}}\big[\dots\\
&(\exp_{x^{(i_{j-1})}})_{*,\tau_{i_{j-1}} m^{(i_j\dots i_N)}}[\lambda(A^{(i_j)}) m^{(i_j\dots i_N)}]\dots\big]\big]\Big) + \eta_{m^{(i_1\dots i_N)}}\Big((\exp_{m^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}}\\
&\big[(\exp_{x^{(i_2)}})_{*,\tau_{i_2} m^{(i_3\dots i_N)}}\big[\dots(\exp_{x^{(i_N)}})_{*,\tau_{i_N}\tilde v^{(i_N)}}[\lambda(B^{(i_N)})\tilde v^{(i_N)}]\dots\big]\big]\Big),
\end{aligned}
$$

Let further

$${}^{(i_1\dots i_N)}\tilde\psi^{wk}$$

denote the pullback of ${}^{(i_1\dots i_N)}\tilde\psi^{wk,\,pre}$ along the substitution $\tau = \delta_{i_1\dots i_N}(\sigma)$ given by the sequence of monoidal transformations

$$
\begin{aligned}
\delta_{i_1\dots i_N} : (\sigma_{i_1},\dots\sigma_{i_N}) &\mapsto \sigma_{i_1}(1,\sigma_{i_2},\dots,\sigma_{i_N}) = (\sigma'_{i_1},\dots,\sigma'_{i_N}) \mapsto \sigma'_{i_2}(\sigma'_{i_1},1,\dots,\sigma'_{i_N}) = (\sigma''_{i_1},\dots,\sigma''_{i_N})\\
&\mapsto \sigma''_{i_3}(\sigma''_{i_1},\sigma''_{i_2},1,\dots,\sigma''_{i_N}) = \dots \mapsto \dots = (\tau_{i_1},\dots,\tau_{i_N}).
\end{aligned}
$$

Then the critical set $\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde\psi^{wk})$ *of* ${}^{(i_1\dots i_N)}\tilde\psi^{wk}$ *is given by all points*

$$(\sigma_{i_1},\dots,\sigma_{i_N},x^{(i_1)},\dots,x^{(i_N)},\tilde v^{(i_N)},A^{(i_1)},\dots,A^{(i_N)},B^{(i_N)},\eta_{m^{(i_1\dots i_N)}})$$

satisfying the conditions

(I) $A^{(i_j)} = 0$ *for all* $j = 1,\dots,N$, *and* $\lambda(B^{(i_N)})\tilde v^{(i_N)} = 0$;

(II) $\eta_{m^{(i_1\dots i_N)}} \in \mathrm{Ann}\big(E^{(i_j)}_{m^{(i_1\dots i_N)}}\big)$ *for all* $j = 1,\dots,N$;

(III) $\eta_{m^{(i_1\dots i_N)}} \in \mathrm{Ann}\big(F^{(i_N)}_{m^{(i_1\dots i_N)}}\big)$.

Furthermore, $\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde\psi^{wk})$ *is a* C^∞*-submanifold of codimension* 2κ*, where* $\kappa = \dim G/H_L$ *is the dimension of a principal orbit.*

Proof. To begin with, let $\sigma_{i_1}\cdots\sigma_{i_N} \neq 0$. In this case, the sequence of monoidal transformations $\zeta = \zeta_{i_1}\circ\zeta_{i_1 i_2}\circ\dots\circ\zeta_{i_1\dots i_N}\circ\delta_{i_1\dots i_N}$ constitutes a diffeomorphism, so that

$$
\begin{aligned}
\mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{tot})_{\sigma_{i_1}\cdots\sigma_{i_N}\neq 0} = \{&(\sigma_{i_1},\dots,\sigma_{i_N},x^{(i_1)},\dots,x^{(i_N)},\tilde v^{(i_N)},A^{(i_1)},\dots,A^{(i_N)},B^{(i_N)},\eta_{m^{(i_1\dots i_N)}})\\
&\in \tilde{\mathcal{C}}^{tot}, \quad \sigma_{i_1}\cdots\sigma_{i_N}\neq 0\},
\end{aligned}
$$

where $\tilde{\mathcal{C}}^{tot} = ((\zeta \otimes \mathrm{id}_{fiber})^{-1}(\mathrm{Crit}(\psi))$ denotes the total transform of the critical set of ψ. Now,

$$(\eta_{m^{(i_1\dots i_N)}}, X^{(i_1\dots i_N)}) \in \mathrm{Crit}(\psi) \quad \Leftrightarrow \quad \eta_{m^{(i_1\dots i_N)}} \in \Omega, \quad \tilde{X}^{(i_1\dots i_N)}_{\eta_{m^{(i_1\dots i_N)}}} = 0.$$

Furthermore, $\tilde{X}_\eta = 0$ clearly implies $\tilde{X}_{\pi(\eta)} = \pi_*(\tilde{X}_\eta) = 0$. Since the point $m^{(i_1\dots i_N)}$ lies in a slice at $x^{(i_1)}$, the condition $\tilde{X}^{(i_1\dots i_N)}_{m^{(i_1\dots i_N)}} = 0$ means that the vector field $\tilde{X}^{(i_1\dots i_N)}$ must vanish at $x^{(i_1)}$ as well. But

$$\mathfrak{g}_m = \mathrm{Lie}(G_m) = \left\{ X \in \mathfrak{g} : \tilde{X}_m = 0 \right\},$$

so that $X^{(i_1\dots i_N)} \in \mathfrak{g}_{x^{(i_1)}}$. Next

$$\mathfrak{g}_{x^{(i_N)}} \subset \mathfrak{g}_{x^{(i_{N-1})}} \subset \dots \subset \mathfrak{g}_{x^{(i_1)}}$$

and $\mathfrak{g}^\perp_{x^{(i_{j+1})}} \subset \mathfrak{g}_{x^{(i_j)}}$ imply

$$\tilde{X}^{(i_1\dots i_N)}_{x^{(i_1)}} = \tau_{i_1} \dots \tau_{i_N} \sum \alpha_r^{(i_1)} (\tilde{A}_r^{(i_1)})_{x^{(i_1)}} = 0.$$

Thus we conclude $\alpha^{(i_1)} = 0$, which gives $X^{(i_2\dots i_N)} \in \mathfrak{g}_{m^{(i_1\dots i_N)}}$, and consequently $X^{(i_2\dots i_N)} \in \mathfrak{g}_{m^{(i_2\dots i_N)}}$. Repeating the above argument we actually obtain for $\sigma_{i_j} \neq 0$

$$\mathfrak{g}_{m^{(i_1\dots i_N)}} = \mathfrak{g}_{\tilde{v}^{(i_N)}}, \tag{8}$$

since $\mathfrak{g}_{\tilde{v}^{(i_N)}} \subset \mathfrak{g}_{x^{(i_N)}}$. Therefore the condition $\tilde{X}^{(i_1\dots i_N)}_{m^{(i_1\dots i_N)}} = 0$ is equivalent to (I) in the case that all σ_{i_j} are different from zero. Now, $\eta_{m^{(i_1\dots i_N)}} \in \Omega$ means that

$$\mathbb{J}(\eta_{m^{(i_1\dots i_N)}})(X) = \eta_{m^{(i_1\dots i_N)}}(\tilde{X}_{m^{(i_1\dots i_N)}}) = 0 \qquad \forall X \in \mathfrak{g},$$

which is equivalent to $\eta_{m^{(i_1\dots i_N)}} \in \mathrm{Ann}(T_{m^{(i_1\dots i_N)}}(G \cdot m^{(i_1\dots i_N)}))$. If $\sigma_{i_j} \neq 0$ for all $j = 1, \dots, N$, (II) and (III) imply that

$$\eta_{m^{(i_1\dots i_N)}}\Big((\exp_{x^{(i_1)}})_{*, \tau_{i_1} m^{(i_2\dots i_N)}} [\dots (\exp_{x^{(i_{j-1})}})_{*, \tau_{i_{N-1}} m^{(i_N)}} [\lambda(Z) m^{(i_N)}] \dots] \Big) = 0 \quad \forall Z \in \mathfrak{g}_{x^{(i_{N-1})}},$$

since $\mathfrak{g}_{x^{(i_{N-1})}} = \mathfrak{g}_{x^{(i_N)}} \oplus \mathfrak{g}^\perp_{x^{(i_N)}}$. By repeatedly using this argument, we conclude that for $\sigma_{i_j} \neq 0$

$$\text{(II), (III)} \quad \Leftrightarrow \quad \eta_{m^{(i_1\dots i_N)}} \in \mathrm{Ann}(T_{m^{(i_1\dots i_N)}}(G \cdot m^{(i_1\dots i_N)})). \tag{9}$$

Taking everything together therefore gives

$$\begin{aligned} &\mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{tot})_{\sigma_{i_1}\cdots\sigma_{i_N}\neq 0} \\ &\qquad = \{ (\sigma_{i_1}, \dots, \sigma_{i_N}, x^{(i_1)}, \dots, x^{(i_N)}, \tilde{v}^{(i_N)}, A^{(i_1)}, \dots, A^{(i_N)}, B^{(i_N)}, \eta_{m^{(i_1\dots i_N)}}) : \\ &\qquad \sigma_{i_1}\cdots\sigma_{i_N} \neq 0, \text{ (I)-(III) are fulfilled and } \tilde{B}^{(i_N),\mathrm{v}}_{\eta_{m^{(i_1\dots i_N)}}} = 0 \}. \end{aligned} \tag{10}$$

Here $\mathfrak{X}^{\mathrm{v}}_\eta$ denotes the vertical component of a vector field $\mathfrak{X} \in T(T^*M)$ with respect to the decomposition $T_\eta(T^*M) = T^{\mathrm{v}} \oplus T^{\mathrm{h}}$, T^{v} being the tangent space to the fiber $T^*_\eta M$ at zero, and T^{h} the tangent space to the zero section $M \subset T^*M$ at η. We now assert that

$$\mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{wk}) = \overline{\mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{tot})_{\sigma_{i_1}\cdots\sigma_{i_N}\neq 0}}.$$

To show this, let us write $\eta_{m^{(i_1\dots i_N)}} = \sum p_i \, dq_i$ with respect to some local coordinates $q_1, \dots, q_n$, and still assume that all σ_{i_j} are different from zero. Then all τ_{i_j} are different from zero, too, and $\partial_p \, {}^{(i_1\dots i_N)}\tilde{\psi}^{wk} = 0$ is equivalent to

$$\partial_p \, \mathbb{J}(\eta_{m^{(i_1\dots i_N)}})(X^{(i_1\dots i_N)}) = (\, dq_1(\tilde{X}^{(i_1\dots i_N)}_{m^{(i_1\dots i_N)}}), \dots, \, dq_n(\tilde{X}^{(i_1\dots i_N)}_{m^{(i_1\dots i_N)}})) = 0,$$

which gives us the condition $\tilde X^{(i_1\dots i_N)}_{m^{(i_1\dots i_N)}}=0$. By (8) we therefore obtain condition I) in the case that all σ_{i_j} are different from zero. Let now one of the σ_{i_j} be equal to zero, so that all τ_{i_j} are zero. With the identification $T_0(T_mM)\simeq T_mM$ one has

$$(\exp_m)_{*,0}:T_0(T_mM)\longrightarrow T_mM, \qquad (\exp_m)_{*,0}\simeq \mathrm{id}\,,$$

and similarly $(\exp_{x^{(i_j)}})_{*,0}\simeq \mathrm{id}$ for all $j=2,\dots,N$, so that

$$ {}^{(i_1\dots i_N)}\tilde\psi^{wk}=\sum p_i\,dq_i\Big(\widetilde{A^{(i_1)}}_{x^{(i_1)}}+\sum_{j=2}^N \lambda(A^{(i_j)})x^{(i_j)}+\lambda(B^{(i_N)})\tilde v^{(i_N)}\Big). \tag{11}$$

Therefore $\partial_p\ {}^{(i_1\dots i_N)}\tilde\psi^{wk}=0$ is equivalent to

$$\widetilde{A^{(i_1)}}_{x^{(i_1)}}+\sum_{j=2}^N \lambda(A^{(i_j)})x^{(i_j)}+\lambda(B^{(i_N)})\tilde v^{(i_N)}=0.$$

Now, let $N_{x^{(i_1)}}(G\cdot x^{(i_1)})$ be the normal space in $T_{x^{(i_1)}}M$ to the orbit $G\cdot x^{(i_1)}$, on which $G_{x^{(i_1)}}$ acts, and define $N_{x^{(i_{j+1})}}(G_{x^{(i_j)}}\cdot x^{(i_{j+1})})$ successively as the normal space to the orbit $G_{x^{(i_j)}}\cdot x^{(i_{j+1})}$ in the $G_{x^{(i_j)}}$-space $N_{x^{(i_j)}}(G_{x^{(i_{j-1})}}\cdot x^{(i_j)})$, where we understand that $G_{x^{(i_0)}}=G$. By Bredon [3], page 308, these actions can be assumed to be orthogonal. Set

$$V^{(i_1\dots i_j)}=\bigcap_{r=1}^j N_{x^{(i_r)}}(G_{x^{(i_{r-1})}}\cdot x^{(i_r)})=N_{x^{(i_j)}}(G_{x^{(i_{j-1})}}\cdot x^{(i_j)}). \tag{12}$$

Since $x^{(i_j)}\in\gamma^{(i_{j-1})}((S^+_{i_1\dots i_{j-1}})_{x^{(i_{j-1})}})\subset V^{(i_1\dots i_{j-1})}$, we see that for every $j=2,\dots,N$

$$\lambda\Big(\sum_r \alpha_r^{(i_j)}A_r^{(i_j)}\Big)\,x^{(i_j)}\in T_{x^{(i_j)}}(G_{x^{(i_{j-1})}}\cdot x^{(i_j)})\subset V^{(i_1\dots i_{j-1})}.$$

In addition, $(\tilde A_r^{(i_1)})_{x^{(i_1)}}\in T_{x^{(i_1)}}(G\cdot x^{(i_1)})$, and $\lambda\Big(\sum_r\beta_r^{(i_N)}B_r^{(i_N)}\Big)\tilde v^{(i_N)}\in V^{(i_1\dots i_N)}$, so that taking everything together we obtain for arbitrary σ_{i_j}

$$\partial_p\ {}^{(i_1\dots i_N)}\tilde\psi^{wk}=0 \quad\iff\quad (\mathrm{I}).$$

In particular, one concludes that ${}^{(i_1\dots i_N)}\tilde\psi^{wk}$ must vanish on its critical set. Since

$$d({}^{(i_1\dots i_N)}\psi^{tot})=d(\tau_{i_1}\dots\tau_{i_N})\cdot{}^{(i_1\dots i_N)}\psi^{wk}+\tau_{i_1}\dots\tau_{i_N}\,d\,({}^{(i_1\dots i_N)}\psi^{wk}),$$

one sees that

$$\mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{wk})\subset\mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{tot}).$$

In turn, the vanishing of ψ on its critical set implies

$$\mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{wk})_{\sigma_{i_1}\dots\sigma_{i_N}\neq0}=\mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{tot})_{\sigma_{i_1}\dots\sigma_{i_N}\neq0}.$$

Therefore, by continuity,

$$\overline{\mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{tot})_{\sigma_{i_1}\dots\sigma_{i_N}\neq0}}\subset\mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{wk}). \tag{13}$$

In order to see the converse inclusion, let us consider next the α-derivatives. Clearly,

$$\partial_{\alpha^{(i_1)}}\ {}^{(i_1\dots i_N)}\tilde\psi^{wk}=0\quad\iff\quad \eta_{m^{(i_1\dots i_N)}}(\tilde Y_{m^{(i_1\dots i_N)}})=0\quad \forall\, Y\in\mathfrak g^\perp_{x^{(i_1)}}.$$

For the remaining derivatives one computes

$$\partial_{\alpha_r^{(i_j)}}\ {}^{(i_1\dots i_N)}\tilde\psi^{wk}$$
$$=\eta_{m^{(i_1\dots i_N)}}\Big((\exp_{x^{(i_1)}})_{*,\tau_{i_1}m^{(i_2\dots i_N)}}\big[\dots(\exp_{x^{(i_{j-1})}})_{*,\tau_{i_{j-1}}m^{(i_j\dots i_N)}}[\lambda(A_r^{(i_j)})m^{(i_j\dots i_N)}]\dots\big]\Big),$$

from which one deduces that for $j = 2, \dots, N$

$$\partial_{\alpha^{(i_j)}} \, {}^{(i_1\dots i_N)}\tilde{\psi}^{wk} = 0 \quad \iff \quad \forall\, Y \in \mathfrak{g}^{\perp}_{x^{(i_j)}}$$

$$\eta_{m^{(i_1\dots i_N)}}\Big((\exp_{x^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}}\big[\dots(\exp_{x^{(i_{j-1})}})_{*,\tau_{i_{j-1}} m^{(i_j\dots i_N)}}[\lambda(Y) m^{(i_j\dots i_N)}]\dots\big]\Big) = 0.$$

In a similar way, it is not difficult to see that

$$\partial_{\beta^{(i_j)}} \, {}^{(i_1\dots i_N)}\tilde{\psi}^{wk} = 0 \quad \iff \quad \forall\, Z \in \mathfrak{g}_{x^{(i_N)}}$$

$$\eta_{m^{(i_1\dots i_N)}}\Big((\exp_{x^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}}\big[\dots(\exp_{x^{(i_N)}})_{*,\tau_{i_N} \tilde{v}^{(i_N)}}[\lambda(Z)\tilde{v}^{(i_N)}]\dots\big]\Big) = 0.$$

by which the necessity of the conditions (I)–(III) is established. In order to see their sufficiency, let them be fulfilled, and assume again that $\sigma_{i_j} \neq 0$ for all $j = 1, \dots, N$. Then (9) implies that $\eta_{m^{(i_1\dots i_N)}} \in \mathrm{Ann}(T_{m^{(i_1\dots i_N)}}(G \cdot m^{(i_1\dots i_N)}))$. Now, if $\sigma_{i_j} \neq 0$, $G \cdot m^{(i_1\dots i_N)}$ is of principal type G/H_L in M, so that the isotropy group of $m^{(i_1\dots i_N)}$ must act trivially on $N_{m^{(i_1\dots i_N)}}(G \cdot m^{(i_1\dots i_N)})$, compare Bredon [3], page 181. If therefore $\mathfrak{X} = \mathfrak{X}_T + \mathfrak{X}_N$ denotes an arbitrary element in $T_{m^{(i_1\dots i_N)}}M = T_{m^{(i_1\dots i_N)}}(G \cdot m^{(i_1\dots i_N)})) \oplus N_{m^{(i_1\dots i_N)}}(G \cdot m^{(i_1\dots i_N)}))$, and $g \in G_{m^{(i_1\dots i_N)}}$, one computes

$$\begin{aligned} g \cdot \eta_{m^{(i_1\dots i_N)}}(\mathfrak{X}) &= [(L_{g^{-1}})^*_{gm^{(i_1\dots i_N)}}\eta_{m^{(i_1\dots i_N)}}](\mathfrak{X}) = \eta_{m^{(i_1\dots i_N)}}((L_{g^{-1}})_{*,m^{(i_1\dots i_N)}}(\mathfrak{X}_N)) \\ &= \eta_{m^{(i_1\dots i_N)}}(\mathfrak{X}_N) = \eta_{m^{(i_1\dots i_N)}}(\mathfrak{X}). \end{aligned}$$

In view of (8), and $\lambda(B^{(i_N)})\tilde{v}^{(i_N)} = 0$ we therefore get the condition $\tilde{B}^{(i_N),\mathrm{v}}_{\eta_{m^{(i_1\dots i_N)}}} = 0$. Let us now assume that one of the σ_{i_j} equals zero. Then

$$\text{(II), (III)} \quad \Leftrightarrow \quad \begin{cases} \eta_{x^{(i_1)}} \in \mathrm{Ann}(T_{x^{(i_j)}}(G_{x^{(i_{j-1})}} \cdot x^{(i_j)})) & \forall\, j = 1, \dots, N, \\ \eta_{x^{(i_1)}} \in \mathrm{Ann}(T_{\tilde{v}^{(i_N)}}(G_{x^{(i_N)}} \cdot \tilde{v}^{(i_N)})). \end{cases} \tag{14}$$

Lemma 1. *The orbit of the point $\tilde{v}^{(i_N)}$ in the $G_{x^{(i_N)}}$-space $V^{(i_1\dots i_N)}$ is of principal type.*

Proof of the lemma. By assumption, for $\sigma_{i_j} \neq 0$, $1 \leq j \leq N$, the G-orbit of $m^{(i_1\dots i_N)}$ is of principal type G/H_L in M. The theory of compact group actions then implies that this is equivalent to the fact that $m^{(i_2\dots i_N)} \in V^{(i_1)}$ is of principal type in the $G_{x^{(i_1)}}$-space $V^{(i_1)}$, see Bredon [3], page 181, which in turn is equivalent to the fact that $m^{(i_3\dots i_N)} \in V^{(i_1 i_2)}$ is of principal type in the $G_{x^{(i_2)}}$-space $V^{(i_1 i_2)}$, and so forth. Thus, $m^{(i_j\dots i_N)} \in V^{(i_1\dots i_{j-1})}$ must be of principal type in the $G_{x^{(i_{j-1})}}$-space $V^{(i_1\dots i_{j-1})}$ for all $j = 1, \dots N$, and the assertion follows. □

As a consequence of the previous lemma, the stabilizer of $\tilde{v}^{(i_N)}$ must act trivially on $N_{\tilde{v}^{(i_N)}}(G_{x^{(i_N)}} \cdot \tilde{v}^{(i_N)})$. If therefore $\mathfrak{X} = \mathfrak{X}_T + \mathfrak{X}_N$ denotes an arbitrary element in

$$\begin{aligned} T_{x^{(i_1)}}M &= T_{x^{(i_1)}}(G \cdot x^{(i_1)}) \oplus N_{x^{(i_1)}}(G \cdot x^{(i_1)}) \\ &= \bigoplus_{j=1}^{N} T_{x^{(i_j)}}(G_{x^{(i_{j-1})}} \cdot x^{(i_j)}) \oplus T_{\tilde{v}^{(i_N)}}(G_{x^{(i_N)}} \cdot \tilde{v}^{(i_N)}) \oplus N_{\tilde{v}^{(i_N)}}(G_{x^{(i_N)}} \cdot \tilde{v}^{(i_N)}), \end{aligned}$$

we obtain with (14)

$$\begin{aligned} g \cdot \eta_{x^{(i_1)}}(\mathfrak{X}) &= [(L_{g^{-1}})^*_{gx^{(i_1)}}\eta_{x^{(i_1)}}](\mathfrak{X}) = \eta_{x^{(i_1)}}((L_{g^{-1}})_{*,x^{(i_1)}}(\mathfrak{X}_N)) \\ &= \eta_{x^{(i_1)}}(\mathfrak{X}_N) = \eta_{x^{(i_1)}}(\mathfrak{X}), \qquad g \in G_{\tilde{v}^{(i_N)}}. \end{aligned}$$

Collecting everything together we have shown for arbitrary σ_{i_j} that

$$\partial_{p,\alpha^{(i_1)},\dots,\alpha^{(i_N)},\beta^{(i_N)}} \, {}^{(i_1\dots i_N)}\tilde{\psi}^{wk} = 0 \quad \iff \quad \text{(I), (II), (III)} \quad \Longrightarrow \quad \tilde{B}^{(i_N),\mathrm{v}}_{\eta_{m^{(i_1\dots i_N)}}} = 0. \tag{15}$$

By (10) and (13) we therefore conclude

$$\overline{\mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{tot})_{\sigma_{i_1}\dots\sigma_{i_N}\neq 0}} = \mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{wk}). \tag{16}$$

Thus we have computed the critical set of ${}^{(i_1\dots i_N)}\tilde{\psi}^{wk}$, and it remains to show that it is a C^∞-submanifold of codimension 2κ. For this end, let us note that if $\sigma_{i_j}=0$ for some j, then $E^{(i_1)}_{x^{(i_1)}} = T_{x^{(i_1)}}(G\cdot x^{(i_1)})$, and

$$E^{(i_j)}_{x^{(i_1)}} \equiv T_{x^{(i_j)}}(G_{x^{(i_{j-1})}}\cdot x^{(i_j)}) \subset V^{(i_1\dots i_{j-1})}, \qquad 2\leq j\leq N,$$

while $F^{(i_N)}_{x^{(i_1)}} \equiv T_{\tilde{v}^{(i_N)}}(G_{x^{(i_N)}}\cdot \tilde{v}^{(i_N)}) \subset V^{(i_1\dots i_N)}$. Therefore $E^{(i_j)}_{x^{(i_1)}} \cap V^{(i_1\dots i_j)} = \{0\}$, so that we obtain the direct sum of vector spaces

$$E^{(i_1)}_{x^{(i_1)}} \oplus E^{(i_2)}_{x^{(i_1)}} \oplus \dots \oplus E^{(i_N)}_{x^{(i_1)}} \oplus F^{(i_N)}_{x^{(i_1)}} \subset T_{x^{(i_1)}}M.$$

On the other hand, note that if $\sigma_{i_1}\cdots\sigma_{i_N}\neq 0$ one has

$$T_{m^{(i_1\dots i_N)}}(G\cdot m^{(i_1\dots i_N)}) = E^{(i_1)}_{m^{(i_1\dots i_N)}} \oplus \bigoplus_{j=2}^N \tau_{i_1}\dots\tau_{i_{j-1}} E^{(i_j)}_{m^{(i_1\dots i_N)}} \oplus \tau_{i_1}\dots\tau_{i_N} F^{(i_N)}_{m^{(i_1\dots i_N)}}$$

for dimensional reasons, so that we obtain the direct sum of geometric distributions $\sum_{j=1}^N E^{(i_j)} \oplus F^{(i_N)}$. Consequently, we arrive at the characterization

$$\begin{aligned} &\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}) \\ &= \Big\{ A^{(i_j)}=0, \quad \lambda(B^{(i_N)})\tilde{v}^{(i_N)}=0, \quad \eta_{m^{(i_1\dots i_N)}} \in \mathrm{Ann}\Big(\bigoplus_{j=1}^N E^{(i_j)}_{m^{(i_1\dots i_N)}} \oplus F^{(i_N)}_{m^{(i_1\dots i_N)}}\Big)\Big\}. \end{aligned} \tag{17}$$

Note that the condition $\tilde{B}^{(i_N),\mathrm{v}}_{\eta_{m^{(i_1\dots i_N)}}}=0$ is already implied by the others. Now, $\dim E^{(i_j)}_{m^{(i_1\dots i_N)}} = \dim G_{x^{(i_{j-1})}}\cdot x^{(i_j)}$. Since for $\sigma_{i_1}\cdots\sigma_{i_N}\neq 0$ the G-orbit of $m^{(i_1\dots i_N)}$ is of principal type G/H_L in M, one computes in this case

$$\begin{aligned} \kappa &= \dim G\cdot m^{(i_1\dots i_N)} = \dim T_{m^{(i_1\dots i_N)}}(G\cdot m^{(i_1\dots i_N)}) \\ &= \dim[E^{(i_1)}_{m^{(i_1\dots i_N)}} \oplus \bigoplus_{j=2}^N \tau_{i_1}\dots\tau_{i_{j-1}} E^{(i_j)}_{m^{(i_1\dots i_N)}} \oplus \tau_{i_1}\dots\tau_{i_N} F^{(i_N)}_{m^{(i_1\dots i_N)}}] \\ &= \sum_{j=1}^N \dim E^{(i_j)}_{m^{(i_1\dots i_N)}} + \dim F^{(i_N)}_{m^{(i_1\dots i_N)}}. \end{aligned}$$

But since the dimension of the spaces $E^{(i_j)}_{m^{(i_1\dots i_N)}}$ and $F^{(i_N)}_{m^{(i_1\dots i_N)}}$ does not depend on the variables σ_{i_j}, we obtain the equality

$$\kappa = \sum_{j=1}^N \dim E^{(i_j)}_{m^{(i_1\dots i_N)}} + \dim F^{(i_N)}_{m^{(i_1\dots i_N)}} \tag{18}$$

for arbitrary $m^{(i_1\dots i_N)}$. Note that, in contrast, the dimension of $T_{m^{(i_1\dots i_N)}}(G\cdot m^{(i_1\dots i_N)})$ collapses, as soon as one of the τ_{i_j} becomes zero. Since the annihilator of a subspace of T_mM is itself a linear subspace of T^*_mM, we arrive at a vector bundle with $(n-\kappa)$-dimensional fiber that is locally given by the trivialization

$$\Big(\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}, \mathrm{Ann}\big(\bigoplus_{j=1}^N E^{(i_j)}_{m^{(i_1\dots i_N)}} \oplus F^{(i_N)}_{m^{(i_1\dots i_N)}}\big)\Big) \mapsto (\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}).$$

Consequently, by equation (17) we see that $\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk})$ is equal to the fiber product of the mentioned vector bundle with the isotropy algebra bundle given by the local trivialization

$$(\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}, \mathfrak{g}_{\tilde{v}^{(i_N)}}) \mapsto (\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}).$$

Lastly, since by equation (8) we have $\mathfrak{g}_{\tilde{v}^{(i_N)}} = \mathfrak{g}_{m^{(i_1,\dots,i_N)}}$ in case that all σ_{i_j} are different from zero, we necessarily have $\dim \mathfrak{g}_{\tilde{v}^{(i_N)}} = d-\kappa$, which concludes the proof of the theorem. □

5. Phase analysis of the weak transform. The second fundamental theorem

In this section, we shall prove the second fundamental theorem in the derivation of equivariant spectral asymptotics for compact group actions. We begin with the following general observation. Let M be a n-dimensional Riemannian manifold, and C the critical set of a function $\psi \in \mathrm{C}^\infty(M)$, which is assumed to be a smooth submanifold in a chart $\mathcal{O} \subset M$. Let further

$$\alpha : (x,y) \mapsto p, \qquad \beta : (q_1, \dots, q_n) \mapsto m, \qquad m \in \mathcal{O},$$

be two systems of local coordinates on $\mathcal{O}$, such that $\alpha(x,y) \in C$ if and only if $y = 0$. One computes

$$\partial_{y_l}(\psi \circ \alpha)(x,y) = \sum_{i=1}^n \frac{\partial(\psi\circ\beta)}{\partial\, q_i}(\beta^{-1}\circ\alpha(x,y))\ \partial_{y_l}(\beta^{-1}\circ\alpha)_i(x,y),$$

as well as

$$\begin{aligned}\partial_{y_k}\,\partial_{y_l}(\psi\circ\alpha)(x,y) &= \sum_{i=1}^n \frac{\partial(\psi\circ\beta)}{\partial\, q_i}(\beta^{-1}\circ\alpha(x,y))\ \partial_{y_k}\,\partial_{y_l}(\beta^{-1}\circ\alpha)_i(x,y)\\ &\quad + \sum_{i,j=1}^n \frac{\partial^2(\psi\circ\beta)}{\partial\, q_i\,\partial\, q_j}(\beta^{-1}\circ\alpha(x,y))\,\partial_{y_k}(\beta^{-1}\circ\alpha)_j(x,y)\ \partial_{y_l}(\beta^{-1}\circ\alpha)_i(x,y).\end{aligned}$$

Since

$$\alpha_{*,(x,y)}(\partial_{y_k}) = \sum_{j=1}^n \partial_{y_k}(\beta^{-1}\circ\alpha)_j(x,y)\,\beta_{*,(\beta^{-1}\circ\alpha)(x,y)}(\partial_{q_j}),$$

this implies

$$\partial_{y_k}\,\partial_{y_l}(\psi\circ\alpha)(x,0) = \mathrm{Hess}\,\psi_{|\alpha(x,0)}(\alpha_{*,(x,0)}(\partial_{y_k}), \alpha_{*,(x,0)}(\partial_{y_l})), \tag{19}$$

by definition of the Hessian. Let us now write $x = (x', x'')$, and consider the restriction of ψ onto the C^∞-submanifold

$$M_{c'} = \{m \in \mathcal{O} : m = \alpha(c', x'', y)\}.$$

We write $\psi_{c'} = \psi_{|M_{c'}}$, and denote the critical set of $\psi_{c'}$ by $C_{c'}$, which contains $C \cap M_{c'}$ as a subset. Introducing on $M_{c'}$ the local coordinates

$$\alpha' : (x'', y) \mapsto \alpha(c', x'', y),$$

we obtain

$$\partial_{y_k}\,\partial_{y_l}(\psi_{c'}\circ\alpha')(x'',0) = \mathrm{Hess}\,\psi_{c'|\alpha(x'',0)}(\alpha'_{*,(x'',0)}(\partial_{y_k}), \alpha'_{*,(x'',0)}(\partial_{y_l})).$$

Let us now assume $C_{c'} = C \cap M_{c'}$, a transversal intersection. Then $C_{c'}$ is a submanifold of $M_{c'}$, and the normal space to $C_{c'}$ as a submanifold of $M_{c'}$ at a point $\alpha'(x'',0)$ is spanned by the vector fields $\alpha'_{*,(x'',0)}(\partial_{y_k})$. Since clearly

$$\partial_{y_k}\,\partial_{y_l}(\psi_{c'}\circ\alpha')(x'',0) = \partial_{y_k}\,\partial_{y_l}(\psi\circ\alpha)(x,0), \qquad x = (c', x''),$$

we thus have proven the following

Lemma 2. *Assume that $C_{c'} = C \cap M_{c'}$. Then the restriction*

$$\operatorname{Hess}\psi(\alpha(c', x'', 0))_{|N_{\alpha(c',x'',0)}C}$$

of the Hessian of ψ to the normal space $N_{\alpha(c',x'',0)}C$ defines a non-degenerate quadratic form if, and only if the restriction

$$\operatorname{Hess}\psi_{c'}(\alpha'(x'', 0))_{|N_{\alpha'(x'',0)}C_{c'}}$$

of the Hessian of $\psi_{c'}$ to the normal space $N_{\alpha'(x'',0)}C_{c'}$ defines a non-degenerate quadratic form.

□

Let us now state the second fundamental theorem, the notation being the same as in the previous sections.

Theorem 2. *Let*

$$^{(i_1\dots i_N)}\tilde{\psi}^{tot} = \tau_{i_1} \dots \tau_{i_N} \, ^{(i_1\dots i_N)}\tilde{\psi}^{wk,\, pre} = \tau_{i_1}(\sigma) \dots \tau_{i_N}(\sigma) \, ^{(i_1\dots i_N)}\tilde{\psi}^{wk}$$

denote the factorization of the phase function after N iteration steps along the isotropy branch $((H_{i_1}), \dots, (H_{i_{N+1}}) = (H_L))$. By construction, for $\tau_{i_j} \neq 0$, $1 \leq j \leq N$, the G-orbit through $m^{(i_1\dots i_N)}$ is of principal type G/H_L. Then, for each point of the critical manifold $\operatorname{Crit}(\,^{(i_1\dots i_N)}\tilde{\psi}^{wk})$, the restriction of

$$\operatorname{Hess}\,^{(i_1\dots i_N)}\tilde{\psi}^{wk}$$

to the normal space to $\operatorname{Crit}(\,^{(i_1\dots i_N)}\tilde{\psi}^{wk})$ at the given point defines a non-degenerate symmetric bilinear form.

For the proof of Theorem 2 we need the following

Lemma 3. *Let $(\eta, X) \in \operatorname{Crit}(\psi)$, and $\pi(\eta) \in M(H_L)$. Then $(\eta, X) \in \operatorname{Reg}\operatorname{Crit}(\psi)$. Furthermore, the restriction of the Hessian of ψ at the point (η, X) to the normal space $N_{(\eta,X)}\operatorname{Reg}\operatorname{Crit}(\psi)$ defines a non-degenerate quadratic form.*

Proof. The first assertion is clear from (3) - (5). To see the second, note that by (4)

$$\eta \in \Omega \cap T^*M(H_L), \tilde{X}_{\pi(\eta)} = 0 \quad \Longrightarrow \quad \tilde{X}_\eta = 0.$$

Let now $\{q_1, \dots, q_n\}$ be local coordinates on M, $m = m(q)$, and write $\eta_m = \sum p_i (dq_i)_m$, $X = \sum s_i X_i$, where $\{X_1, \dots, X_d\}$ denotes a basis of $\mathfrak{g}$. Then

$$\psi(\eta, X) = \sum p_i (dq_i)_m(\tilde{X}_m),$$

and

$$\partial_p \, \psi(\eta, X) = 0 \quad \Longleftrightarrow \quad \tilde{X}_m = 0, \qquad \partial_s \, \psi(\eta, X) = 0 \quad \Longleftrightarrow \quad \eta \in \Omega.$$

On $T^*M(H_L) \times \mathfrak{g}$ we therefore get

$$\partial_{p,s} \, \psi(\eta, X) = 0 \quad \Longrightarrow \quad \partial_q \, \psi(\eta, X) = 0.$$

Let $\psi_q(p, s)$ denote the phase function regarded as a function of the coordinates p, s alone, while q is regarded as a parameter. Lemma 2 then implies that on $T^*M(H_L) \times \mathfrak{g}$ the study of the transversal Hessian of ψ can be reduced to the study of the transversal Hessian of ψ_q. Now, with respect to the coordinates s, p, the Hessian of ψ_q is given by

$$\begin{pmatrix} 0 & (dq_i)_m((\tilde{X}_j)_m) \\ (dq_j)_m((\tilde{X}_i)_m) & 0 \end{pmatrix}.$$

A computation then shows that the kernel of the corresponding linear transformation is isomorphic to $T_{p,s}(\operatorname{Crit}\psi_q) = \big\{(\tilde{p}, \tilde{s}) : \sum \tilde{p}_j (dq_j)_{m(q)} \in \operatorname{Ann}(T_{m(q)}(G \cdot m(q))), \sum \tilde{s}_j X_j \in \mathfrak{g}_{m(q)}\big\}$. The lemma now follows with the following general observation. Let $\mathcal{B}$ be a symmetric bilinear form on an n-dimensional $\mathbb{K}$-vector space V, and $B=(B_{ij})_{i,j}$ the corresponding Gramsian matrix with respect to a basis $\{v_1,\dots,v_n\}$ of V such that

$$\mathcal{B}(u,w)=\sum_{i,j} u_i w_j B_{ij}, \qquad u=\sum u_i v_i, \quad w=\sum w_i v_i.$$

We denote the linear operator given by B with the same letter, and write

$$V=\ker B\oplus W.$$

Consider the restriction $\mathcal{B}_{|W\times W}$ of $\mathcal{B}$ to $W\times W$, and assume that $\mathcal{B}_{|W\times W}(u,w)=0$ for all $u\in W$, but $w\neq 0$. Since the Euclidean scalar product in V is non-degenerate, we necessarily must have $Bw=0$, and consequently $w\in\ker B\cap W=\{0\}$, which is a contradiction. Therefore $\mathcal{B}_{|W\times W}$ defines a non-degenerate symmetric bilinear form. □

Proof of second fundamental theorem. Let us begin by noting that for $\sigma_{i_1}\cdots\sigma_{i_N}\neq 0$, the sequence of monoidal transformations $\zeta=\zeta_{i_1}\circ\zeta_{i_1 i_2}\circ\cdots\circ\zeta_{i_1\dots i_N}\circ\delta_{i_1\dots i_N}$ constitutes a diffeomorphism, so that by the previous lemma the restriction of

$$\operatorname{Hess}^{(i_1\dots i_N)}\tilde{\psi}^{tot}(\sigma_{i_j},x^{(i_j)},\tilde{v}^{(i_N)},\alpha^{(i_j)},\beta^{(i_N)},p)$$

to the normal space of

$$\operatorname{Crit}({}^{(i_1\dots i_N)}\psi^{tot})_{\sigma_{i_1}\cdots\sigma_{i_N}\neq 0}$$

defines a non-degenerate quadratic form. Next, one computes

$$\left(\frac{\partial^2\ {}^{(i_1\dots i_N)}\tilde{\psi}^{tot}}{\partial\gamma_k\,\partial\gamma_l}\right)_{k,l}=\tau_{i_1}(\sigma)\cdots\tau_{i_N}(\sigma)\left(\frac{\partial^2\ {}^{(i_1\dots i_N)}\tilde{\psi}^{wk}}{\partial\gamma_k\,\partial\gamma_l}\right)_{k,l}$$
$$+\begin{pmatrix}\left(\frac{\partial^2(\tau_{i_1}(\sigma)\cdots\tau_{i_N}(\sigma))}{\partial\sigma_{i_r}\sigma_{i_s}}\right)_{r,s} & 0\\ 0 & 0\end{pmatrix}{}^{(i_1\dots i_N)}\tilde{\psi}^{wk}+R$$

where R represents a matrix whose entries contain first order derivatives of ${}^{(i_1\dots i_N)}\tilde{\psi}^{wk}$ as factors. But since

$$\operatorname{Crit}({}^{(i_1\dots i_N)}\psi^{tot})_{\sigma_{i_1}\cdots\sigma_{i_N}\neq 0}=\operatorname{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk})_{|\sigma_{i_1}\cdots\sigma_{i_N}\neq 0},$$

we conclude that the transversal Hessian of ${}^{(i_1\dots i_N)}\tilde{\psi}^{wk}$ does not degenerate along the manifold $\operatorname{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk})_{|\sigma_{i_1}\cdots\sigma_{i_N}\neq 0}$. Therefore, it remains to study the transversal Hessian of ${}^{(i_1\dots i_N)}\tilde{\psi}^{wk}$ in the case that any of the σ_{i_j} vanishes. Now, the proof of the first fundamental theorem, in particular (15), showed that

$$\partial_{p,\alpha^{(i_1)},\dots,\alpha^{(i_N)},\beta^{(i_N)}}\ {}^{(i_1\dots i_N)}\tilde{\psi}^{wk}=0\quad\Longrightarrow\quad\partial_{\sigma_{i_1},\dots\sigma_{i_N},x^{(i_1)},\dots,x^{(i_N)},\tilde{v}^{(i_N)}}\ {}^{(i_1\dots i_N)}\tilde{\psi}^{wk}=0.$$

If therefore

$${}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde{v}^{(i_N)}}(\alpha^{(i_j)},\beta^{(i_N)},p)$$

denotes the weak transform of the phase function ψ regarded as a function of the variables $(\alpha^{(i_1)},\dots,\alpha^{(i_N)},\beta^{(i_N)},p)$ alone, while the variables $(\sigma_{i_1},\dots,\sigma_{i_N},x^{(i_1)},\dots,x^{(i_N)},\tilde{v}^{(i_N)})$ are kept fixed,

$$\operatorname{Crit}\big({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde{v}^{(i_N)}}\big)=\operatorname{Crit}\big({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}\big)\cap\Big\{\sigma_{i_j},x^{(i_j)},\tilde{v}^{(i_N)}=\text{ constant}\Big\}.$$

Thus, the critical set of ${}^{(i_1\dots i_N)}\tilde\psi^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)}}$ is equal to the fiber over $(\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)})$ of the vector bundle

$$\Big((\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)}),\mathfrak{g}_{\tilde v^{(i_N)}}\times \mathrm{Ann}\big(\bigoplus_{j=1}^N E^{(i_j)}_{m^{(i_1\dots i_N)}}\oplus F^{(i_N)}_{m^{(i_1\dots i_N)}}\big)\Big)\mapsto(\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)}),$$

and in particular a smooth submanifold. Lemma 2 then implies that the study of the transversal Hessian of ${}^{(i_1\dots i_N)}\tilde\psi^{wk}$ can be reduced to the study of the transversal Hessian of ${}^{(i_1\dots i_N)}\tilde\psi^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)}}$. The crucial fact is now contained in the following

Proposition 1. *Assume that $\sigma_{i_1}\cdots\sigma_{i_N}=0$. Then*

$$\ker \mathrm{Hess}\,{}^{(i_1\dots i_N)}\tilde\psi^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)}}(0,\dots,0,\beta^{(i_N)},p)\simeq T_{(0,\dots,0,\beta^{(i_N)},p)}\mathrm{Crit}\big({}^{(i_1\dots i_N)}\tilde\psi^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)}}\big)$$

for all $(0,\dots,0,\beta^{(i_N)},p)\in\mathrm{Crit}\big({}^{(i_1\dots i_N)}\tilde\psi^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)}}\big)$, and arbitrary $x^{(i_j)}$, $\tilde v^{(i_j)}$.

Proof. With (11) one computes

$$\partial_{p_r}\,{}^{(i_1\dots i_N)}\tilde\psi^{wk}=dq_r\Big(\widetilde{A^{(i_1)}}_{x^{(i_1)}}+\sum_{j=2}^N\lambda(A^{(i_j)})x^{(i_j)}+\lambda(B^{(i_N)})\tilde v^{(i_N)}\Big).$$

The second derivatives therefore read

$$\begin{aligned}
\partial_{p_r}\partial_{p_s}\,{}^{(i_1\dots i_N)}\tilde\psi^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)}}&=0,\\
\partial_{\alpha_s^{(i_1)}}\partial_{p_r}\,{}^{(i_1\dots i_N)}\tilde\psi^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)}}&=dq_r((\tilde A_s^{(i_1)})_{x^{(i_1)}}),\\
\partial_{\alpha_s^{(i_j)}}\partial_{p_r}\,{}^{(i_1\dots i_N)}\tilde\psi^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)}}&=dq_r(\lambda(A_s^{(i_j)})x^{(i_j)}),\\
\partial_{\beta_s^{(i_N)}}\partial_{p_r}\,{}^{(i_1\dots i_N)}\tilde\psi^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)}}&=dq_r(\lambda(B_s^{(i_N)})\tilde v^{(i_N)}).
\end{aligned}$$

Next, one has

$$\partial_{\alpha_s^{(i_j)}}\,{}^{(i_1\dots i_N)}\tilde\psi^{wk}=\sum p_i dq_i(\lambda(A_s^{(i_j)})x^{(i_j)}),\qquad j=2,\dots,N,$$

and similar expressions for the $\alpha^{(i_1)}$-derivatives, so that for $\sigma_{i_1}\cdots\sigma_{i_j}=0$ all the second order derivatives involving $\alpha^{(i_j)}$ must vanish, except the ones that were already computed. Finally, the computation of the $\beta^{(i_N)}$-derivatives yields

$$\partial_{\beta_r^{(i_N)}}\partial_{\beta_s^{(i_N)}}\,{}^{(i_1\dots i_N)}\tilde\psi^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)}}=0.$$

Collecting everything we see that for $\sigma_{i_1}\cdots\sigma_{i_j}=0$, the Hessian of the function ${}^{(i_1\dots i_N)}\tilde\psi^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)}}$ with respect to the coordinates $\alpha^{(i_j)},\beta^{(i_j)},p$ is given on its critical set by the matrix

$$\begin{pmatrix}
0 & dq_r((\tilde A_s^{(i_1)})_{x^{(i_1)}}) & \dots & dq_r(\lambda(A_s^{(i_N)})x^{(i_j)}) & dq_r(\lambda(B_s^{(i_N)})\tilde v^{(i_N)})\\
dq_s((\tilde A_r^{(i_1)})_{x^{(i_1)}}) & 0 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
dq_s(\lambda(A_r^{(i_N)})x^{(i_j)}) & 0 & \dots & 0 & 0\\
dq_s(\lambda(B_r^{(i_N)})\tilde v^{(i_N)}) & 0 & \dots & 0 & 0
\end{pmatrix}.$$

Let us now compute the kernel of the linear transformation corresponding to this matrix. Cleary, the vector $(\tilde p,\tilde\alpha^{(i_1)},\dots,\tilde\alpha^{(i_N)},\tilde\beta^{(i_N)})$ lies in the kernel if and only if

(a) $\sum \tilde{\alpha}_r^{(i_1)} (\tilde{A}_r^{(i_1)})_{x^{(i_1)}} + \cdots + \sum \tilde{\alpha}_r^{(i_N)} \lambda(A_r^{(i_N)}) x^{(i_N)} + \sum \tilde{\beta}_r^{(i_N)} \lambda(B_r^{(i_N)}) \tilde{v}^{(i_N)} = 0$;
(b) $\sum \tilde{p}_s dq_s((\tilde{Y}^{(i_1)})_{x^{(i_1)}}) = 0$ for all $Y^{(i_1)} \in \mathfrak{g}^{\perp}_{x^{(i_1)}}$, $\sum \tilde{p}_s dq_s(\lambda(\mathfrak{g}^{\perp}_{x^{(i_j)}}) x^{(i_j)}) = 0$, $2 \leq j \leq N$;
(c) $\sum \tilde{p}_s dq_s(\lambda(\mathfrak{g}_{x^{(i_N)}}) \tilde{v}^{(i_N)}) = 0$.

Let $E^{(i_j)}$, $F^{(i_N)}$, and $V^{(i_1 \dots i_N)}$ be defined as in (7) and (12). Then

$$\sum \tilde{\alpha}_r^{(i_j)} (\tilde{A}_r^{(i_1)})_{x^{(i_1)}} + \cdots \sum \tilde{\alpha}_r^{(i_N)} \lambda(A_r^{(i_N)}) x^{(i_N)} + \sum \tilde{\beta}_r^{(i_N)} \lambda(B_r^{(i_N)}) \tilde{v}^{(i_N)} \in \bigoplus_{j=1}^N E^{(i_j)}_{x^{(i_1)}} \oplus F^{(i_N)}_{x^{(i_1)}},$$

so that for condition (a) to hold, it is necessary and sufficient that

$$\tilde{\alpha}^{(i_j)} = 0, \quad 1 \leq j \leq N, \qquad \sum \tilde{\beta}_r^{(i_N)} \lambda(B_r^{(i_N)}) \tilde{v}^{(i_N)} = 0.$$

Since $\mathfrak{g}^{\perp}_{x^{(i_j)}} \subset \mathfrak{g}_{x^{(i_{j-1})}}$, condition (b) is equivalent to $\sum \tilde{p}_s (dq_s)_{x^{(i_1)}} \in \mathrm{Ann}(E^{(i_j)}_{x^{(i_1)}})$ for al $j = 1, \dots, N$. Similarly, condition (c) is equivalent to $\sum \tilde{p}_s (dq_s)_{x^{(i_1)}} \in \mathrm{Ann}(F^{(i_N)}_{x^{(i_1)}})$. On the other hand, by (17),

$$T_{(0,\dots,0,\beta^{(i_N)},p)} \mathrm{Crit}\big({}^{(i_1 \dots i_N)}\tilde{\psi}^{wk}_{\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}} \big) = \Big\{ (\tilde{\alpha}^{(i_1)}, \dots, \tilde{\alpha}^{(i_N)}, \tilde{\beta}^{(i_N)}, \tilde{p}) : \tilde{\alpha}^{(i_j)} = 0,$$
$$\sum \tilde{\beta}_r^{(i_N)} \lambda(B_r^{(i_N)}) \in \mathfrak{g}_{\tilde{v}^{(i_N)}}, \sum \tilde{p}_s (dq_s)_{x^{(i_1)}} \in \mathrm{Ann}\Big(\bigoplus_{j=1}^N E^{(i_j)}_{x^{(i_1)}} \oplus F^{(i_N)} \Big) \Big\},$$

and the proposition follows. □

The previous proposition now implies that for $\sigma_{i_1} \cdots \sigma_{i_N} = 0$

$$\mathrm{Hess}\, {}^{(i_1 \dots i_N)}\tilde{\psi}^{wk}_{\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}} (0, \dots, 0, \beta^{(i_N)}, p)_{|N_{(0,\dots,0,\beta^{(i_N)},p)} \mathrm{Crit}\big({}^{(i_1 \dots i_N)}\tilde{\psi}^{wk}_{\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}} \big)}$$

defines a non-degenerate symmetric bilinear form for all points $(0, \dots, 0, \beta^{(i_N)}, p)$ lying in the critical set of ${}^{(i_1 \dots i_N)}\tilde{\psi}^{wk}_{\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}}$, and the second fundamental theorem follows with Lemma 2. □

We are now in position to give an asymptotic description of the integral $I(\mu)$. But before, it might be in place to say a few words about the desingularization process.

6. Resolution of singularities and the stationary phase theorem

Let M be a smooth variety, $\mathcal{O}_M$ the structure sheaf of rings of M, and $I \subset \mathcal{O}_M$ an ideal sheaf. The aim in the theory of resolution of singularities is to construct a birational morphism $\pi : \tilde{M} \to M$ such that $\tilde{M}$ is smooth, and the pulled back ideal sheaf $\pi^* I$ is locally principal. This is called the *principalization* of I, and implies resolution of singularities. That is, for every quasi-projective variety X, there is a smooth variety $\tilde{X}$, and a birational and projective morphism $\pi : \tilde{X} \to X$. Vice versa, resolution of singularities implies principalization.

Consider next the derivative $D(I)$ of I, which is the sheaf ideal that is generated by all derivatives of elements of I. Let further $Z \subset M$ be a smooth subvariety, and $\pi : B_Z M \to M$ the corresponding monoidal transformation with center Z and exceptional divisor $F \subset B_Z M$. Assume that (I, m) is a marked ideal sheaf with $m \leq \mathrm{ord}_Z I$. The total transform $\pi^* I$ vanishes along F with multiplicity $\mathrm{ord}_Z I$, and by removing the ideal sheaf $\mathcal{O}_{B_Z M}(-\mathrm{ord}_Z I \cdot F)$ from $\pi^* I$ we obtain the *birational, or* *weak transform* $\pi_*^{-1}I$ of I. Take local coordinates $(x_1,\dots,x_n)$ on M such that $Z=(x_1=\cdots=x_r=0)$. As a consequence,

$$y_1=\frac{x_1}{x_r},\dots,y_{r-1}=\frac{x_{r-1}}{x_r},y_r=x_r,\dots,y_n=x_n$$

define local coordinates on $B_Z M$, and for $(f,m)\in(I,m)$ one has

$$\pi_*^{-1}(f(x_1,\dots,x_n),m)=(y_r^{-m}f(y_1y_r,\dots y_{r-1}y_r,y_r,\dots,y_n),m).$$

By computing the first derivatives of $\pi_*^{-1}(f(x_1,\dots,x_n),m)$, one then sees that for any composition $\Pi:\tilde M\to M$ of blowing-ups of order greater or equal than m,

$$\Pi_*^{-1}(D(I,m))\subset D(\Pi_*^{-1}(I,m)),$$

see Kollár [12], Theorem 71.

Let us now come back to our situation, and consider on $T^*M\times\mathfrak{g}$ the ideal $I_\psi=(\psi)$ generated by the phase function $\psi=\mathbb{J}(\eta)(X)$, together with its vanishing set V_ψ. The derivative of I is given by $D(I_\psi)=I_{\mathcal{C}}$, where $I_{\mathcal{C}}$ denotes the vanishing ideal of the critical set $\mathcal{C}=\mathrm{Crit}(\psi)$, and by the implicit function theorem $\mathrm{Sing}\,V_\psi\subset V_\psi\cap\mathcal{C}=\mathcal{C}$. Let $((H_{i_1}),\cdots,(H_{i_{N+1}})=(H_L))$ be an arbitrary branch of isotropy types, and consider the corresponding sequence of monoidal transformations $(\zeta_{i_1}\circ\zeta_{i_1i_2}\circ\cdots\circ\zeta_{i_1\dots i_N})\otimes\mathrm{id}_{fiber}$. Compose it with the sequence of monoidal transformations $\delta_{i_1\dots i_N}$, and denote the resulting transformation by ζ. We then have the diagram

$$\begin{array}{ccccc}
\zeta^*(I_{\mathcal{C}}) & \supset & \zeta^*(I_\psi) & =\prod_{i=1}^N\tau_{i_j}(\sigma)\cdot\zeta_*^{-1}(I_\psi) & \ni\tau_{i_1}(\sigma)\cdots\tau_{i_N}(\sigma)\,{}^{(i_1\dots i_N)}\tilde\psi^{wk}\\
\uparrow & & \uparrow & & \\
I_{\mathcal{C}} & \supset & I_\psi & & \ni\psi
\end{array}$$

According to the previous considerations, we have the inclusion

$$\zeta_*^{-1}(I_{\mathcal{C}})\subset D(\zeta_*^{-1}(I_\psi)).$$

It is easy to see that $\zeta_*^{-1}(I_\psi)$ is not resolved, so that $\prod_{i=1}^N\tau_{i_j}(\sigma)\cdot\zeta_*^{-1}(I_\psi)$ is only a partial principalization. On the other hand, the first fundamental theorem implies that $D(\zeta_*^{-1}(I_\psi))$ is a resolved ideal,

$$\overline{\mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{tot})_{\sigma_{i_1}\dots\sigma_{i_N}\neq0}}=\mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{wk})$$

being a smooth manifold. Nevertheless, this again results only in a partial resolution $\tilde{\mathcal{C}}$ of $\mathcal{C}$, since the induced global birational transform $\tilde{\mathcal{C}}\to\mathcal{C}$ is in general not surjective. This is because of the transformation $\delta_{i_1\dots i_N}$, and the fact that the centers of our monoidal transformations were only chosen in $M\times\mathfrak{g}$, to keep the phase analysis of the weak transform of ψ as simple as possible. In turn, the singularities of $\mathcal{C}$ along the fibers of T^*M were not completely resolved.

As we shall see in the next section, the principalization of the ideal I_ψ

$$\zeta^*(I_\psi)=\tau_{i_1}\cdots\tau_{i_N}\zeta_*^{-1}(I_\psi),$$

and the fact that the weak transform ${}^{(i_1\dots i_N)}\tilde\psi^{wk}$ has a clean critical set, are essential for an application of the stationary phase principle in the context of singular equivariant asymptotics, which is we why had to consider resolutions of both $\mathcal{C}$ and V_ψ in $T^*M\times\mathfrak{g}$. By Hironaka's theorem on resolution of singularities, such resolutions always exist, and are equivalent to the principalization of the corresponding ideals. But in general, they would not be explicit enough [2] to allow an application of the stationary phase theorem. This is the reason why we were forced to construct an explicit, though partial, resolution ζ of $\mathcal{C}$ and V_ψ in $T^*M\times\mathfrak{g}$, using as centers isotropy algebra

[2] In particular, the so-called numerical data of ζ are not known a priori, which in our case are given in terms of the dimensions $c^{(i_j)}$ and $d^{(i_j)}$.

bundles over sets of maximal singular orbits. Partial desingularizations of the zero level set Ω of the moment map and the symplectic quotient Ω/G have been obtained e.g. by Meinrenken-Sjamaar [13] for compact symplectic manifolds with a Hamiltonian compact Lie group action by performing blowing-ups along minimal symplectic suborbifolds containing the strata of maximal depth in Ω.

7. Asymptotics for the integrals $I_{i_1\dots i_N}(\mu)$

In this section, we will give an asymptotic description of the integrals $I_{i_1\dots i_N}(\mu)$ defined in (6). Since the considered integrals are absolutely convergent integral, we can interchange the order of integration by Fubini, and write

$$I_{i_1\dots i_N}(\mu)=\int_{(-1,1)^N} J_{\tau_{i_1},\dots,\tau_{i_N}}\Big(\frac{\mu}{\tau_{i_1}\cdots\tau_{i_N}}\Big)\prod_{j=1}^N |\tau_{i_j}|^{c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-1}\,d\tau_{i_N}\dots d\tau_{i_1},$$

where we set

$$J_{\tau_{i_1},\dots,\tau_{i_N}}(\nu) = \int e^{i\,{}^{(i_1\dots i_N)}\tilde\psi^{wk,pre}/\nu}\, a_{i_1\dots i_N}\,\Phi_{i_1\dots i_N}\, d(T^*_{m^{(i_1\dots i_N)}}W_{i_j})(\eta)\bigwedge_j dA^{(i_j)}\,dB^{(i_N)}\,d\tilde v^{(i_N)}\bigwedge_l dx^{(i_l)},$$

and introduced the new parameter

$$\nu=\frac{\mu}{\tau_{i_1}\cdots\tau_{i_N}}.$$

Now, for an arbitrary $0<\varepsilon<T$ to be chosen later we define

$$I^1_{i_1\dots i_N}(\mu)=\int_{((-1,1)\setminus(-\varepsilon,\varepsilon))^N} J_{\tau_{i_1},\dots,\tau_{i_N}}\Big(\frac{\mu}{\tau_{i_1}\cdots\tau_{i_N}}\Big)\prod_{j=1}^N |\tau_{i_j}|^{c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-1}\,d\tau_{i_N}\dots d\tau_{i_1},$$

$$I^2_{i_1\dots i_N}(\mu)=\int_{(-\varepsilon,\varepsilon)^N} J_{\tau_{i_1},\dots,\tau_{i_N}}\Big(\frac{\mu}{\tau_{i_1}\cdots\tau_{i_N}}\Big)\prod_{j=1}^N |\tau_{i_j}|^{c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-1}\,d\tau_{i_N}\dots d\tau_{i_1}.$$

Lemma 4. *One has $c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-1\geq\kappa$ for arbitrary $j=1,\dots,N$.*

Proof. We first note that

$$c^{(i_j)}=\dim(\nu_{i_1\dots i_j})_{x^{(i_j)}}\geq \dim G_{x^{(i_j)}}\cdot m^{(i_{j+1}\dots i_N)}+1.$$

Indeed, $(\nu_{i_1\dots i_j})_{x^{(i_j)}}$ is an orthogonal $G_{x^{(i_j)}}$-space, so that the dimension of the $G_{x^{(i_j)}}$-orbit of $m^{(i_{j+1}\dots i_N)}\in\gamma^{(i_j)}((S^+_{i_1\dots i_j})_{x^{(i_j)}})$ can be at most $c^{(i_j)}-1$. Now, under the assumption $\sigma_{i_1}\cdots\sigma_{i_N}\neq 0$ one has

$$\begin{aligned}T_{m^{(i_{j+1}\dots i_N)}}(G_{x^{(i_j)}}\cdot m^{(i_{j+1}\dots i_N)})&\simeq T_{m^{(i_1\dots i_N)}}(G_{x^{(i_j)}}\cdot m^{(i_1\dots i_N)})\\&=E^{(i_{j+1})}_{m^{(i_1\dots i_N)}}\oplus\bigoplus_{k=j+2}^N\tau_{i_{j+1}}\dots\tau_{i_{k-1}}E^{(i_k)}_{m^{(i_1\dots i_N)}}\oplus\tau_{i_{j+1}}\dots\tau_{i_N}F^{(i_N)}_{m^{(i_1\dots i_N)}},\end{aligned}$$

where the distributions $E^{(i_j)}$, $F^{(i_N)}$ where defined in (7). On then computes

$$\begin{aligned}\dim G_{x^{(i_j)}}\cdot m^{(i_{j+1}\dots i_N)}&=\dim T_{m^{(i_{j+1}\dots i_N)}}(G_{x^{(i_j)}}\cdot m^{(i_{j+1}\dots i_N)})\\&=\sum_{l=j+1}^N\dim E^{(i_l)}_{m^{(i_1\dots i_N)}}+\dim F^{(i_N)}_{m^{(i_1\dots i_N)}},\end{aligned}$$

which implies

$$c^{(i_j)} \geq \sum_{l=j+1}^{N} \dim E^{(i_l)}_{m^{(i_1\dots i_N)}} + \dim F^{(i_N)}_{m^{(i_1\dots i_N)}} + 1.$$

Here we used the same arguments as in the proof of equation (18). On the other hand, one has

$$d^{(i_j)} = \dim \mathfrak{g}^{\perp}_{x^{(i_j)}} = \dim[\lambda(\mathfrak{g}^{\perp}_{x^{(i_j)}}) \cdot x^{(i_j)}] = \dim[\lambda(\mathfrak{g}^{\perp}_{x^{(i_j)}}) \cdot m^{(i_j\dots i_N)}] = \dim E^{(i_j)}_{m^{(i_1\dots i_N)}}.$$

The assertion now follows with (18). □

As a consequence of the lemma, we obtain for $I^2_{i_1\dots i_N}(\mu)$ the estimate

$$\begin{aligned} I^2_{i_1\dots i_N}(\mu) &\leq C \int_{(-\varepsilon,\varepsilon)^N} \prod_{j=1}^{N} |\tau_{i_j}|^{c^{(i_j)}+\sum_{r=1}^{j} d^{(i_r)}-1}\, d\tau_{i_N}\dots d\tau_{i_1} \\ &\leq C \int_{(-\varepsilon,\varepsilon)^N} \prod_{j=1}^{N} |\tau_{i_j}|^{\kappa}\, d\tau_{i_N}\dots d\tau_{i_1} = \frac{2C}{\kappa+1}\varepsilon^{N(\kappa+1)} \end{aligned} \tag{20}$$

for some $C > 0$. Let us now turn to the integral $I^1_{i_1\dots i_N}(\mu)$. After performing the change of variables $\delta_{i_1\dots i_N}$ one obtains

$$I^1_{i_1\dots i_N}(\mu) = \int\limits_{\varepsilon<|\tau_{i_j}(\sigma)|<1} J_{\sigma_{i_1},\dots,\sigma_{i_N}}\Big(\frac{\mu}{\tau_{i_1}(\sigma)\cdots\tau_{i_N}(\sigma)}\Big) \prod_{j=1}^{N} |\tau_{i_j}(\sigma)|^{c^{(i_j)}+\sum_{r=1}^{j} d^{(i_r)}-1} |\det D\delta_{i_1\dots i_N}(\sigma)|\, d\sigma,$$

where

$$J_{\sigma_{i_1},\dots,\sigma_{i_N}}(\nu) = \int e^{i\,{}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_{\sigma}/\nu}\, a_{i_1\dots i_N}\, \Phi_{i_1\dots i_N}\, d(T^{*}_{m^{(i_1\dots i_N)}} W_{i_j})(\eta) \bigwedge_j dA^{(i_j)}\, dB^{(i_N)}\, d\tilde{v}^{(i_N)} \bigwedge_l dx^{(i_l)}.$$

Here we denoted by ${}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_{\sigma}$ the weak transform of the phase function ψ as a function of the variables $x^{(i_j)}, \tilde{v}^{(i_N)}, \alpha^{(i_j)}, \beta^{(i_N)}, p$ alone, while the variables $\sigma = (\sigma_{i_1},\dots \sigma_{i_N})$ are regarded as parameters. The idea is now to make use of the principle of the stationary phase to give an asymptotic expansion of $J_{\sigma_{i_1},\dots,\sigma_{i_N}}(\nu)$.

Theorem 3 (Generalized stationary phase theorem for manifolds)**.** *Let M be a n-dimensional Riemannian manifold, $\psi \in \mathrm{C}^\infty(M)$ be a real valued phase function, $\mu > 0$, and set*

$$I(\mu) = \int_M e^{i\psi(m)/\mu} a(m)\, dm,$$

where $a(m)dm$ denotes a compactly supported C^∞-density on M. Let

$$\mathcal{C} = \big\{m \in M : \psi_* : T_m M \to T_{\psi(m)}\mathbb{R} \text{ is zero}\big\}$$

be the critical set of the phase function ψ, and assume that

1. *$\mathcal{C}$ is a smooth submanifold of M of dimension p in a neighborhood of the support of a;*
2. *for all $m \in \mathcal{C}$, the restriction $\psi''(m)_{|N_m\mathcal{C}}$ of the Hessian of ψ at the point m to the normal space $N_m\mathcal{C}$ is a non-degenerate quadratic form.*

Then, for all $N \in \mathbb{N}$, there exists a constant $C_{N,\psi} > 0$ such that

$$|I(\mu) - e^{i\psi_0/\mu}(2\pi\mu)^{\frac{n-p}{2}} \sum_{j=0}^{N-1} \mu^j Q_j(\psi; a)| \leq C_{N,\psi}\mu^N vol(\operatorname{supp} a \cap \mathcal{C}) \sup_{l\leq 2N} \big\|D^l a\big\|_{\infty,M},$$

where D^l is a differential operator on M of order l, and ψ_0 is the constant value of ψ on $\mathcal{C}$. Furthermore, for each j there exists a constant $\tilde C_{j,\psi} > 0$ such that

$$|Q_j(\psi; a)| \leq \tilde C_{j,\psi} \, vol(\operatorname{supp} a \cap \mathcal{C}) \sup_{l \leq 2j} \left\| D^l a \right\|_{\infty, \mathcal{C}},$$

and, in particular,

$$Q_0(\psi; a) = \int_{\mathcal{C}} \frac{a(m)}{|\det \psi''(m)_{|N_m \mathcal{C}}|^{1/2}} d\sigma_{\mathcal{C}}(m) e^{i\pi\sigma_{\psi''}},$$

where $\sigma_{\psi''}$ is the constant value of the signature of $\psi''(m)_{|N_m\mathcal{C}}$ for m in $\mathcal{C}$.

Proof. See for instance Hörmander, [10], Theorem 7.7.5, together with Combescure-Ralston-Robert [6], Theorem 3.3, as well as Varadarajan [18], pp. 199. □

Remark 1. An examination of the proof of the foregoing theorem shows that the constants $C_{N,\psi}$ are essentially bounded from above by

$$\sup_{m \in \mathcal{C} \cap \operatorname{supp} a} \left\| \Big(\psi''(m)_{|N_m \mathcal{C}} \Big)^{-1} \right\|.$$

Indeed, let $\alpha : (x, y) \to m \in \mathcal{O} \subset M$ be local normal coordinates such that $\alpha(x, y) \in \mathcal{C}$ if, and only if, $y = 0$. By (19), the transversal Hessian $\operatorname{Hess} \psi(m)_{|N_m\mathcal{C}}$ is given in these coordinates by the matrix

$$\Big(\partial_{y_k} \partial_{y_l} (\psi \circ \alpha)(x, 0) \Big)_{k,l}$$

where $m = \alpha(x, 0)$. If now the transversal Hessian of ψ is non-degenerate at the point $m = \alpha(x, 0)$, then $y = 0$ is a non-degenerate critical point of the function $y \mapsto (\psi \circ \alpha)(x, y)$, and therefore an isolated critical point by the lemma of Morse. As a consequence,

$$\frac{|y|}{|\, \partial_y (\psi \circ \alpha)(x, y)|} \leq 2 \left\| \Big(\partial_{y_k} \partial_{y_l} (\psi \circ \alpha)(x, 0) \Big)^{-1}_{k,l} \right\| \tag{21}$$

for y close to zero. The assertion now follows by applying Hörmander [10], Theorem 7.7.5, to the integral

$$\int_{\alpha^{-1}(\mathcal{O})} e^{i(\psi \circ \alpha)(x,y)/\mu} (a \circ \alpha)(x, y) \, dy \, dx$$

in the variable y, and with x as a parameter, since in our situation the constant C occuring in Hörmander [10], equation (7.7.12), is precisely bounded by (21), if we assume as we may that a is supported near $\mathcal{C}$. A similar observation holds with respect to the constants $\tilde C_{j,\psi}$.

We arrive now at the following

Theorem 4. *Let $\sigma = (\sigma_{i_1}, \dots, \sigma_{i_N})$ be a fixed set of parameters. Then, for every $\tilde N \in \mathbb{N}$ there exists a constant $C_{\tilde N, {}^{(i_1 \dots i_N)}\tilde\psi^{wk}_\sigma} > 0$ such that*

$$|J_{\sigma_{i_1}, \dots, \sigma_{i_N}}(\nu) - (2\pi|\nu|)^\kappa \sum_{j=0}^{\tilde N - 1} |\nu|^j Q_j({}^{(i_1 \dots i_N)}\tilde\psi^{wk}_\sigma; a_{i_1 \dots i_N} \Phi_{i_1 \dots i_N})| \leq C_{\tilde N, {}^{(i_1 \dots i_N)}\tilde\psi^{wk}_\sigma} |\nu|^{\tilde N},$$

with estimates for the coefficients Q_j, and an explicit expression for Q_0. Moreover, the constants $C_{\tilde N, {}^{(i_1 \dots i_N)}\tilde\psi^{wk}_\sigma}$ and the coefficients Q_j have uniform bounds in σ.

Proof. As a consequence of the fundamental theorems, and Lemma 2, together with the observations preceding Proposition 1, the phase function ${}^{(i_1 \dots i_N)}\tilde\psi^{wk}_\sigma$ has a clean critical set, meaning that

- the critical set $\operatorname{Crit}({}^{(i_1 \dots i_N)}\tilde\psi^{wk}_\sigma)$ is a C^∞-submanifold of codimension 2κ for arbitrary σ;

- the transversal Hessian

$$\mathrm{Hess}\,{}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_\sigma(x^{(i_j)},\tilde{v}^{(i_N)},\alpha^{(i_j)},\beta^{(i_N)},p)_{|N_{(x^{(i_j)},\tilde{v}^{(i_N)},\alpha^{(i_j)},\beta^{(i_N)},p)}\mathrm{Crit}\left({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_\sigma\right)}$$

  defines a non-degenerate symmetric bilinear form for arbitrary σ at every point of the critical set of ${}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_\sigma$.

Thus, the necessary conditions for applying the principle of the stationary phase to the integral $J_{\sigma_{i_1},\dots,\sigma_{i_N}}(\nu)$ are fulfilled, and we obtain the desired asymptotic expansion by Theorem 3. To see the existence of the uniform bounds, note that by Remark 1 we have

$$C_{\tilde{N},{}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_\sigma} \leq C'_{\tilde{N}} \sup_{x^{(i_j)},\tilde{v}^{(i_N)},\alpha^{(i_j)},\beta^{(i_N)},p} \left\| \left(\mathrm{Hess}\,{}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_{\sigma\,|N\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_\sigma)}\right)^{-1}\right\|.$$

But since by Lemma 2 the transversal Hessian

$$\mathrm{Hess}\,{}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_{\sigma\,|N_{(x^{(i_j)},\tilde{v}^{(i_N)},\alpha^{(i_j)},\beta^{(i_N)},p)}\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_\sigma)}$$

is given by

$$\mathrm{Hess}\,{}^{(i_1\dots i_N)}\tilde{\psi}^{wk}{}_{|N_{(\sigma_{i_j},x^{(i_j)},\tilde{v}^{(i_N)},\alpha^{(i_j)},\beta^{(i_N)},p)}\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk})},$$

we finally obtain the estimate

$$C_{\tilde{N},{}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_\sigma} \leq C'_{\tilde{N}} \sup_{\sigma_{i_j},x^{(i_j)},\tilde{v}^{(i_N)},\alpha^{(i_j)},\beta^{(i_N)},p} \left\| \left(\mathrm{Hess}\,{}^{(i_1\dots i_N)}\tilde{\psi}^{wk}{}_{|N\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk})}\right)^{-1}\right\| \leq C_{\tilde{N},i_1\dots i_N}$$

by a constant independent of σ. Similarly, one can show the existence of bounds of the form

$$|Q_j({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_\sigma; a_{i_1\dots i_N}\Phi_{i_1\dots i_N})| \leq \tilde{C}_{j,i_1\dots i_N},$$

with constants $\tilde{C}_{j,i_1\dots i_N}$ independent of σ. □

Remark 2. Before going on, let us remark that for the computation of the integrals $I^1_{i_1\dots i_N}(\mu)$ it is only necessary to have an asymptotic expansion for the integrals $J_{\sigma_{i_1},\dots,\sigma_{i_N}}(\nu)$ in the case that $\sigma_{i_1}\cdots\sigma_{i_N} \neq 0$, which can also be obtained without the fundamental theorems using only the factorization of the phase function ψ given by the resolution process, together with Lemma 3. Nevertheless, the main consequence to be drawn from the fundamental theorems is that the constants $C_{\tilde{N},{}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_\sigma}$ and the coefficients Q_j in Theorem 4 have uniform bounds in σ.

As a consequence of Theorem 4, we obtain for arbitrary $\tilde{N}\in\mathbb{N}$

$$\begin{aligned}
&|J_{\sigma_{i_1},\dots,\sigma_{i_N}}(\nu) - (2\pi|\nu|)^\kappa Q_0({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_\sigma; a_{i_1\dots i_N}\Phi_{i_1\dots i_N})| \\
&\qquad\leq \left| J_{\sigma_{i_1},\dots,\sigma_{i_N}}(\nu) - (2\pi|\nu|)^\kappa \sum_{l=0}^{\tilde{N}-1} |\nu|^l Q_l({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_\sigma; a_{i_1\dots i_N}\Phi_{i_1\dots i_N})\right| \\
&\qquad+ (2\pi|\nu|)^\kappa \sum_{l=1}^{\tilde{N}-1} |\nu|^l |Q_l({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_\sigma; a_{i_1\dots i_N}\Phi_{i_1\dots i_N})| \leq c_1|\nu|^{\tilde{N}} + c_2|\nu|^\kappa \sum_{l=1}^{\tilde{N}-1}|\nu|^l
\end{aligned}$$

with constants $c_i > 0$ independent of both σ and ν. From this we deduce

$$\begin{aligned}
&\Big| I^1_{i_1 \dots i_N}(\mu) - (2\pi\mu)^\kappa \int_{\varepsilon < |\tau_{i_j}(\sigma)| < 1} Q_0 \prod_{j=1}^N |\tau_{i_j}(\sigma)|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1 - \kappa} |\det D\delta_{i_1 \dots i_N}(\sigma)| \, d\sigma \Big| \\
&\qquad \leq c_3 \mu^{\tilde N} \int_{\varepsilon < |\tau_{i_j}(\sigma)| < 1} \prod_{j=1}^N |\tau_{i_j}(\sigma)|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1 - \tilde N} |\det D\delta_{i_1 \dots i_N}(\sigma)| \, d\sigma \\
&\qquad + c_4 \mu^\kappa \sum_{l=1}^{\tilde N - 1} \mu^l \int_{\varepsilon < |\tau_{i_j}(\sigma)| < 1} \prod_{j=1}^N |\tau_{i_j}(\sigma)|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1 - \kappa - l} |\det D\delta_{i_1 \dots i_N}(\sigma)| \, d\sigma \\
&\qquad \leq c_5 \mu^{\tilde N} \max \Big\{ 1, \prod_{j=1}^N \varepsilon^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - \tilde N} \Big\} + c_6 \sum_{l=1}^{\tilde N - 1} \mu^{\kappa + l} \max \Big\{ 1, \prod_{j=1}^N \varepsilon^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - \kappa - l} \Big\}.
\end{aligned}$$

We now set

$$\varepsilon = \mu^{1/N}.$$

Taking into account Lemma 4, one infers that the right hand side of the last inequality can be estimated by

$$c_5 \max \Big\{ \mu^{\tilde N}, \mu^{\kappa+1} \Big\} + c_6 \sum_{l=1}^{\tilde N - 1} \max \big\{ \mu^{\kappa + l}, \mu^{\kappa+1} \big\},$$

so that for sufficiently large $\tilde N \in \mathbb{N}$ we finally obtain an asymptotic expansion for $I_{i_1 \dots i_N}(\mu)$ by taking into account (20), and the fact that

$$(2\pi\mu)^\kappa \int_{0 < |\tau_{i_j}| < \mu^{1/N}} Q_0 \prod_{j=1}^N |\tau_{i_j}|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1} \, d\tau_{i_N} \dots d\tau_{i_1} = O(\mu^{\kappa+1}).$$

Theorem 5. *Let the assumptions of the first fundamental theorem be fulfilled. Then*

$$I_{i_1 \dots i_N}(\mu) = (2\pi\mu)^\kappa L_{i_1 \dots i_N} + O(\mu^{\kappa+1}),$$

where the leading coefficient $L_{i_1 \dots i_N}$ is given by

$$L_{i_1 \dots i_N} = \int_{\mathrm{Crit}(^{(i_1 \dots i_N)}\tilde\psi^{wk})} \frac{a_{i_1 \dots i_N} \Phi_{i_1 \dots i_N} \, d\mathrm{Crit}(^{(i_1 \dots i_N)}\tilde\psi^{wk})}{|\mathrm{Hess}(^{(i_1 \dots i_N)}\tilde\psi^{wk})_{N\mathrm{Crit}(^{(i_1 \dots i_N)}\tilde\psi^{wk})}|^{1/2}}, \tag{22}$$

where $d\mathrm{Crit}(^{(i_1 \dots i_N)}\tilde\psi^{wk})$ denotes the induced Riemannian measure.

□

8. Statement of the main result

Let us now return to our departing point, that is, the asymptotic behavior of the integral $I(\mu)$ introduced in (1). For this, we still have to examine the contributions to $I(\mu)$ coming from integrals of the form

$$\tilde I_{i_1\dots i_\Theta}(\mu) =$$

$$\int_{M_{i_1}(H_{i_1})\times(-1,1)} \Big[\int_{\gamma^{(i_1)}((S^+_{i_1})_{x^{(i_1)}})_{i_2}(H_{i_2})\times(-1,1)} \dots \Big[\int_{\gamma^{(i_{\Theta-1})}((S^+_{i_1\dots i_{\Theta-1}})_{x^{(i_{\Theta-1})}})_{i_\Theta}(H_{i_\Theta})\times(-1,1)}$$

$$\Big[\int_{\gamma^{(i_\Theta)}((S^+_{i_1\dots i_\Theta})_{x^{(i_\Theta)}})\times\mathfrak{g}_{x^{(i_\Theta)}}\times\mathfrak{g}^\perp_{x^{(i_\Theta)}}\times\dots\times\mathfrak{g}^\perp_{x^{(i_1)}}\times T^*_{m^{(i_1\dots i_\Theta)}}W_{i_1}} e^{i\frac{\tau_1\dots\tau_\Theta}{\mu}\,{}^{(i_1\dots i_\Theta)}\tilde\psi^{wk}} a_{i_1\dots i_\Theta}\,\tilde\Phi_{i_1\dots i_\Theta}$$

$$d(T^*_{m^{(i_1\dots i_\Theta)}}W_{i_1})(\eta)\,dA^{(i_1)}\dots dA^{(i_\Theta)}\,dB^{(i_\Theta)}\,d\tilde v^{(i_\Theta)}\Big]\,d\tau_{i_\Theta}\,dx^{(i_\Theta)}\dots\Big]\,d\tau_{i_2}\,dx^{(i_2)}\Big]\,d\tau_{i_1}\,dx^{(i_1)},$$

where $((H_{i_1}),\dots,(H_{i_\Theta}))$ is an arbitrary isotropy branch, and

$$a_{i_1\dots i_\Theta} = [a\,\chi_{i_1}\circ(\mathrm{id}_{fiber}\otimes\zeta_{i_1}\circ\zeta_{i_1i_2}\circ\dots\circ\zeta_{i_1\dots i_\Theta})]\,[\chi_{i_1i_2}\circ\zeta_{i_1i_2}\circ\dots\circ\zeta_{i_1\dots i_\Theta}]\dots[\chi_{i_1\dots i_\Theta}\circ\zeta_{i_1\dots i_\Theta}]$$

is supposed to have compact support in one of the $\alpha^{(i_\Theta)}$-charts, and

$$\tilde\Phi_{i_1\dots i_\Theta} = \prod_{j=1}^{\Theta} |\tau_{i_j}|^{c^{(i_j)}+\sum_r^j d^{(i_r)}-1}\,\Phi_{i_1\dots i_\Theta},$$

$\Phi_{i_1\dots i_\Theta}$ being a smooth function which does not depend on the variables τ_{i_j}. Now, a computation of the ξ-derivatives of ${}^{(i_1\dots i_\Theta)}\tilde\psi^{wk}$ in any of the $\alpha^{(i_\Theta)}$-charts shows that ${}^{(i_1\dots i_\Theta)}\tilde\psi^{wk}$ has no critical points there. By the non-stationary phase theorem, see Hörmander [10], Theorem 7.7.1, one then computes for arbitrary $\tilde N\in\mathbb{N}$

$$|\tilde I_{i_1\dots i_\Theta}(\mu)| \le c_7\mu^{\tilde N}\int_{\varepsilon<|\tau_{i_j}|<1}\prod_{j=1}^{\Theta}|\tau_{i_j}|^{c^{(i_j)}+\sum_r^j d^{(i_r)}-1-\tilde N}d\tau + c_8\varepsilon^{\Theta(\kappa+1)} \le c_9\max\left\{\mu^{\tilde N},\mu^{\kappa+1}\right\},$$

where we took $\varepsilon=\mu^{1/\Theta}$. Choosing $\tilde N$ large enough, we conclude that

$$|\tilde I_{i_1\dots i_\Theta}(\mu)| = O(\mu^{\kappa+1}).$$

As a consequence of this we see that, up to terms of order $O(\mu^{\kappa+1})$, $I(\mu)$ can be written as a sum

$$\begin{aligned} I(\mu) &= \sum_{k<L} I_k(\mu) + I_L(\mu) = \sum_{k<l<L} I_{kl}(\mu) + \sum_{k<L} I_{kL}(\mu) + I_L(\mu) \\ &= \sum_N \sum_{i_1<\dots<i_N<i_{N+1}=L} I_{i_1\dots i_N}(\mu) + \sum_\Theta \sum_{i_1<\dots<i_\Theta<i_{\Theta+1}\neq L} I_{i_1\dots i_\Theta L}(\mu), \end{aligned}$$

where the first term in the last line is a sum to be taken over all the indices $i_1,\dots,i_N$ corresponding to all possible isotropy branches of the form $(H_{i_1},\dots,(H_{i_N}),(H_{i_{N+1}})=(H_L))$ of varying length N, while the second term is a sum over all indices $i_1,\dots,i_\Theta$ corresponding to branches $(H_{i_1},\dots,(H_{i_\Theta}),(H_{i_{\Theta+1}})\neq(H_L))$ of arbitrary length Θ. The asymptotic behavior of the integrals $I_{i_1\dots i_N}(\mu)$ has been determined in the previous section, and it is not difficult to see that the integrals $I_{i_1\dots i_\Theta L}$ have analogous asymptotic descriptions. We are now ready to state and prove the main result of this paper.

Theorem 6. *Let M be a connected, closed Riemannian manifold, and G a compact, connected Lie group G with Lie algebra $\mathfrak{g}$ acting isometrically and effectively on M. Consider the oscillatory integral*

$$I(\mu) = \int_{T^*M}\int_{\mathfrak{g}} e^{i\psi(\eta,X)/\mu}a(\eta,X)\,dX\,d\eta, \qquad \mu>0,$$

where the phase function

$$\psi(\eta,X) = \mathbb{J}(\eta)(X)$$

is given by the moment map $\mathbb{J}: T^*M \to \mathfrak{g}^*$ corresponding to the Hamiltonian action on T^*M, and $d\eta$ is a density on T^*M, while dX is, up to a constant factor, the Lebesgue measure on $\mathfrak{g}$, and $a \in \mathrm{C}_{\mathrm{c}}^{\infty}(T^*M \times \mathfrak{g})$. Then $I(\mu)$ has the asymptotic expansion

$$I(\mu) = (2\pi\mu)^{\kappa} L_0 + O(\mu^{\kappa+1}), \qquad \mu \to 0^+.$$

Here κ is the dimension of an orbit of principal type in M, and the leading coefficient is given by

$$L_0 = \int_{\operatorname{Reg}\mathcal{C}} \frac{a(\eta, X)}{|\operatorname{Hess}\psi(\eta, X)_{N_{(\eta,X)}\operatorname{Reg}\mathcal{C}}|^{1/2}} \, d(\operatorname{Reg}\mathcal{C})(\eta, X), \tag{23}$$

where $\operatorname{Reg}\mathcal{C}$ denotes the regular part of the critical set $\mathcal{C} = \operatorname{Crit}(\psi)$ of ψ, and $d(\operatorname{Reg}\mathcal{C})$ the induced Riemannian measure. In particular, the integral over $\operatorname{Reg}\mathcal{C}$ exists.

Remark 3. Note that equation (23) in particular means that the obtained asymptotic expansion for $I(\mu)$ is independent of the explicit partial resolution we used.

Proof. By the considerations at the beginning of this section, one has

$$I(\mu) = (2\pi\mu)^{\kappa} L_0 + O(\mu^{\kappa+1}), \qquad \mu \to 0^+,$$

where L_0 is given as a sum of integrals of the form (22). It therefore remains to show the equality (23). For this, let us remark that since M is compact, T^*M is a paracompact manifold, admitting a Riemannian metric, so that T^*M is a metric space with metric $|\cdot|$. We introduce now certain cut-off functions for the singular part $\operatorname{Sing}\Omega$ of Ω. Let K be a compact subset in T^*M, $\varepsilon > 0$, and consider the set

$$(\operatorname{Sing}\Omega \cap K)_{\varepsilon} = \{\eta \in T^*M : |\eta - \eta'| < \varepsilon \text{ for some } \eta' \in \operatorname{Sing}\Omega\}.$$

By using a partition of unity, one can show the existence of a test function $u_\varepsilon \in \mathrm{C}_{\mathrm{c}}^{\infty}((\operatorname{Sing}\Omega \cap K)_{3\varepsilon})$ satisfying $u_\varepsilon = 1$ on $(\operatorname{Sing}\Omega \cap K)_{\varepsilon}$, see Hörmander [10], Theorem 1.4.1. We then have the following

Lemma 5. *Let $a \in \mathrm{C}_{\mathrm{c}}^{\infty}(T^*M \times \mathfrak{g})$, K be a compact subset in T^*M such that $\operatorname{supp}_{\eta} a \subset K$, and u_ε as above. Then the limit*

$$\lim_{\varepsilon \to 0} \int_{\operatorname{Reg}\mathcal{C}} \frac{[a(1-u_\varepsilon)](\eta, X)}{|\det \psi''(\eta, X)_{|N_{(\eta,X)}\operatorname{Reg}\mathcal{C}}|^{1/2}} d(\operatorname{Reg}\mathcal{C})(\eta, X) \tag{24}$$

exists and is equal to L_0, where $d(\operatorname{Reg}\mathcal{C})$ is the induced Riemannian measure on $\operatorname{Reg}\mathcal{C}$.

Proof. We define

$$I_\varepsilon(\mu) = \int_{T^*M} \int_{\mathfrak{g}} e^{\frac{i}{\mu}\psi(\eta, X)} [a(1-u_\varepsilon)](\eta, X) \, dX \, d\xi \, dx.$$

Since $(\eta, X) \in \operatorname{Sing}\mathcal{C}$ implies $\eta \in \operatorname{Sing}\Omega$, a direct application of the generalized stationary phase theorem for fixed $\varepsilon > 0$ gives

$$|I_\varepsilon(\mu) - (2\pi\mu)^{\kappa} L_0(\varepsilon)| \leq C_\varepsilon \mu^{\kappa+1}, \tag{25}$$

where $C_\varepsilon > 0$ is a constant depending only on ε, and

$$L_0(\varepsilon) = \int_{\operatorname{Reg}\mathcal{C}} \frac{[a(1-u_\varepsilon)](\eta, X)}{|\det \psi''(\eta, X)_{|N_{(\eta,X)}\operatorname{Reg}\mathcal{C}}|^{1/2}} d(\operatorname{Reg}\mathcal{C})(\eta, X).$$

On the other hand, applying our previous considerations to $I_\varepsilon(\mu)$ instead of $I(\mu)$, we obtain again an asymptotic expansion of the form (25) for $I_\varepsilon(\mu)$, where now the first coefficient is given by a sum of integrals of the form (22) with a replaced by $a(1-u_\varepsilon)$. Since the first term in the asymptotic expansion (25) is uniquely determined, the two expressions for $L_0(\varepsilon)$ must be identical. The statement of the lemma now follows by the Lebesgue theorem on bounded convergence. $\square$

Note that existence of the limit in (24) has been established by partially resolving the singularities of the critical set $\mathcal{C}$, the corresponding limit being given by L_0. Let now $a^+ \in \mathrm{C}^\infty_{\mathrm{c}}(T^*M \times \mathfrak{g}), \mathbb{R}^+)$. Since one can assume that $|u_\varepsilon| \leq 1$, the lemma of Fatou implies that

$$\int_{\mathrm{Reg}\,\mathcal{C}} \lim_{\varepsilon \to 0} \frac{[a^+(1-u_\varepsilon)](\eta, X)}{|\det\, \psi''(\eta, X)_{|N_{(\eta,X)}\mathrm{Reg}\,\mathcal{C}}|^{1/2}} d(\mathrm{Reg}\,\mathcal{C})(\eta, X)$$

is mayorized by the limit (24), with a replaced by a^+. Lemma 5 then implies that

$$\int_{\mathrm{Reg}\,\mathcal{C}} \frac{a^+(\eta, X)}{|\det\, \psi''(\eta, X)_{|N_{(\eta,X)}\mathrm{Reg}\,\mathcal{C}}|^{1/2}} d(\mathrm{Reg}\,\mathcal{C})(\eta, X) < \infty.$$

Choosing now a^+ to be equal 1 on a neighborhood of the support of a, and applying the theorem of Lebesgue on bounded convergence to the limit (24), we obtain equation (23). □

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