INTEGRAL OPERATORS ON THE OSHIMA COMPACTIFICATION OF A RIEMANNIAN SYMMETRIC SPACE OF NON-COMPACT TYPE. MICROLOCAL ANALYSIS AND KERNEL ASYMPTOTICS

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ABSTRACT. Let $\mathbb{X} \simeq G/K$ be a Riemannian symmetric space of non-compact type, $\widetilde{\mathbb{X}}$ its Oshima compactification, and $(\pi, \mathrm{C}(\widetilde{\mathbb{X}}))$ the regular representation of G on $\widetilde{\mathbb{X}}$. We study integral operators on $\widetilde{\mathbb{X}}$ of the form $\pi(f)$, where f is a rapidly falling function on G, and characterize them within the framework of pseudodifferential operators, describing the singular nature of their kernels. In particular, we consider the holomorphic semigroup generated by a strongly elliptic operator associated to the representation π, as well as its resolvent, and describe the asymptotic behavior of the corresponding semigroup and resolvent kernels.

CONTENTS

1. INTRODUCTION

Let $\mathbb{X}$ be a Riemannian symmetric space of non-compact type. Then $\mathbb{X}$ is isomorphic to G/K, where G is a connected real semisimple Lie group, and K a maximal compact subgroup. Consider further the Oshima compactification [8] $\widetilde{\mathbb{X}}$ of $\mathbb{X}$, a simply connected closed real-analytic manifold on which G acts analytically. The orbital decomposition of $\widetilde{\mathbb{X}}$ is of normal crossing type, and the open orbits are isomorphic to G/K, the number of them being equal to 2^l, where l denotes the rank of G/K. In this paper, we shall study the invariant integral operators

$$\pi(f) = \int_G f(g)\pi(g)d_G(g), \tag{1}$$

where π is the regular representation of G on the Banach space $\mathrm{C}(\widetilde{\mathbb{X}})$ of continuous functions on $\widetilde{\mathbb{X}}$, f a smooth, rapidly decreasing function on G, and d_G a Haar measure on G. These operators play an important role in representation theory, and our interest will be directed towards the

1991 *Mathematics Subject Classification.* 22E46, 53C35, 32J05, 58J40, 58J37, 58J35, 47A10.
Key words and phrases. Riemannian symmetric spaces of non-compact type, Oshima compactification, totally characteristic pseudodifferential operators, elliptic operators on Lie groups, semigroup and resolvent kernels.
The authors wish to thank Toshio Oshima for valuable conversations on the subject. We are also grateful to Jean-Philippe Anker for helpful remarks on the classical heat kernel on Riemannian symmetric spaces. This work was financed by the DFG-grant RA 1370/2-1.

elucidation of their microlocal structure within the theory of pseudodifferential operators. Since the underlying group action on $\widetilde{\mathbb{X}}$ is not transitive, the operators $\pi(f)$ are not smooth, and the orbit structure of $\widetilde{\mathbb{X}}$ is reflected in the singular behavior of their Schwartz kernels. As it turns out, the operators in question can be characterized as pseudodifferential operators belonging to a particular class which was first introduced in [7] in connection with boundary problems. In fact, if $\widetilde{\mathbb{X}}_\Delta$ denotes a component in $\widetilde{\mathbb{X}}$ isomorphic to G/K, we prove that the restrictions

$$\pi(f)_{|\overline{\widetilde{\mathbb{X}}_\Delta}} : \mathrm{C}^\infty_\mathrm{c}(\overline{\widetilde{\mathbb{X}}_\Delta}) \longrightarrow \mathrm{C}^\infty(\overline{\widetilde{\mathbb{X}}_\Delta})$$

of the operators $\pi(f)$ to the manifold with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$ are totally characteristic pseudodifferential operators of class $\mathrm{L}^{-\infty}_b$. A similar description of invariant integral operators on prehomogeneous vector spaces was obtained by the second author in [9]. We then consider the holomorphic semigroup generated by a strongly elliptic operator Ω associated to the regular representation $(\pi, \mathrm{C}(\widetilde{\mathbb{X}}))$ of G, as well as its resolvent. Since both the holomorphic semigroup and the resolvent can be characterized as operators of the form (1), they can be studied with the previous methods, and relying on the theory of elliptic operators on Lie groups [10] we obtain a description of the asymptotic behavior of the semigroup and resolvent kernels on $\widetilde{\mathbb{X}}_\Delta \simeq \mathbb{X}$ at infinity. In the particular case of the Laplace-Beltrami operator on $\mathbb{X}$, these questions have been intensively studied before. While for the classical heat kernel on $\mathbb{X}$ precise upper and lower bounds were previously obtained in [1] using spherical analysis, a detailed description of the analytic properties of the resolvent of the Laplace-Beltrami operator on $\mathbb{X}$ was given in [5], [6].

The paper is organized as follows. In Section 2 we briefly recall those parts of the structure theory of real semisimple Lie groups that are relevant to our purposes. We then describe the G-action on the homogeneous spaces $G/P_\Theta(K)$, where $P_\Theta(K)$ is a closed subgroup of G associated naturally to a subset Θ of the set of simple roots, and the corresponding fundamental vector fields. This leads to the definition of the Oshima compactification $\widetilde{\mathbb{X}}$ of the symmetric space $\mathbb{X} \simeq G/K$, together with a description of the orbital decomposition of $\widetilde{\mathbb{X}}$. Since this decomposition is of normal crossing type, it is well-suited for our analytic purposes. A thorough and unified description of the various compactifications of a symmetric space is given in [2]. Section 3 contains a summary with some of the basic facts in the theory pseudodifferential operators needed in the sequel. In particular, the class of totally characteristic pseudodifferential operators on a manifold with corners is introduced. Section 4 is the central part of this paper. By analyzing the orbit structure of the G-action on $\widetilde{\mathbb{X}}$, we are able to elucidate the microlocal structure of the convolution operators $\pi(f)$, and characterize them as totally characteristic pseudodifferential operators on the manifold with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$. This leads to a description of the asymptotic behavior of their Schwartz kernels on $\widetilde{\mathbb{X}}_\Delta \simeq \mathbb{X}$ at infinity. In Section 5, we consider the holomorphic semigroup S_τ generated by the closure $\overline{\Omega}$ of a strongly elliptic differential operator Ω associated to the representation π. Since $S_\tau = \pi(K_\tau)$, where $K_\tau(g)$ is a smooth and rapidly decreasing function on G, we can apply our previous results to describe the Schwartz kernel of S_τ. The Schwartz kernel of the resolvent $(\lambda \mathbf{1} + \overline{\Omega})^{-\alpha}$, where $\alpha > 0$, and $\operatorname{Re} \lambda$ is sufficiently large, can be treated similarly, but is more subtle due to the singularity of the corresponding group kernel $R_{\alpha,\lambda}(g)$ at the identity.

2. The Oshima compactification of a Riemannian symmetric space

Let G be a connected real semisimple Lie group with finite centre and Lie algebra $\mathfrak{g}$, and denote by $\langle X, Y\rangle = \mathrm{tr}\,(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$ the *Cartan-Killing form* on $\mathfrak{g}$. Let θ be the Cartan involution of $\mathfrak{g}$, and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

the Cartan decomposition of $\mathfrak{g}$ into the eigenspaces of θ, corresponding to the eigenvalues $+1$ and -1 , respectively, and put $\langle X, Y\rangle_\theta := -\langle X, \theta Y\rangle$. Note that the Cartan decomposition is orthogonal with respect to $\langle,\rangle_\theta$. Consider further a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Then $\mathrm{ad}\,(\mathfrak{a})$ is a commuting family of self-adjoint operators on $\mathfrak{g}$. Indeed, for $X, Y, Z \in \mathfrak{g}$ one computes

$$\begin{aligned}\langle \mathrm{ad}\, X(Z), Y\rangle_\theta &= -\langle [X, Z], \theta Y\rangle = -\langle Z, [\theta Y, X]\rangle = -\langle Z, \theta[Y, \theta X]\rangle = \langle Z, [Y, \theta X]\rangle_\theta \\ &= \langle Z, -[\theta X, Y]\rangle_\theta = \langle Z, -\mathrm{ad}\, \theta X(Y)\rangle_\theta.\end{aligned}$$

Therefore $-\mathrm{ad}\,\theta X$ is the adjoint of $\mathrm{ad}\, X$ with respect to $\langle,\rangle_\theta$. So, if we take $X \in \mathfrak{p}$, the -1 eigenspace of θ, $\mathrm{ad}\, X$ is self-adjoint with respect to $\langle,\rangle_\theta$. The dimension l of $\mathfrak{a}$ is called the *real rank of* G and *the rank of the symmetric space* G/K. Next, one defines for each $\alpha \in \mathfrak{a}^*$, the dual of $\mathfrak{a}$, the simultaneous eigenspaces $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$ of $\mathrm{ad}\,(\mathfrak{a})$. A functional $0 \neq \alpha \in \mathfrak{a}^*$ is called a *(restricted) root* of $(\mathfrak{g}, \mathfrak{a})$ if $\mathfrak{g}^\alpha \neq \{0\}$, and setting $\Sigma = \{\alpha \in \mathfrak{a}^* : \alpha \neq 0, \mathfrak{g}^\alpha \neq \{0\}\}$, we obtain the decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}^\alpha,$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Note that this decomposition is orthogonal with respect to $\langle\cdot,\cdot\rangle_\theta$. With respect to an ordering of $\mathfrak{a}^*$, let $\Sigma^+ = \{\alpha \in \Sigma : \alpha > 0\}$ denote the *set of positive roots*, and $\Delta = \{\alpha_1, \dots, \alpha_l\}$ the *set of simple roots*. Let $\varrho = \frac{1}{2}\Sigma_{\alpha\in\Sigma^+}\alpha$, and put $m(\alpha) = \dim\, \mathfrak{g}^\alpha$ which is, in general, greater than 1. Define $\mathfrak{n}^+ = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha$, $\mathfrak{n}^- = \theta(\mathfrak{n}^+)$, and write K, A, N^+ and N^- for the analytic subgroups of G corresponding to $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}^+$, and $\mathfrak{n}^-$, respectively. The *Iwasawa decomposition* of G is then given by

$$G = KAN^\pm.$$

Next, let $M = \{k \in K : \mathrm{Ad}\,(k)H = H \text{ for all } H \in \mathfrak{a}\}$ be the centralizer of $\mathfrak{a}$ in K and $M^* = \{k \in K : \mathrm{Ad}\,(k)\mathfrak{a} \subset \mathfrak{a}\}$ the normalizer of $\mathfrak{a}$ in K. The quotient $W = M^*/M$ is the *Weyl group* corresponding to $(\mathfrak{g}, \mathfrak{a})$, and acts on $\mathfrak{a}$ as a group of linear transformations via the adjoint action. Alternatively, W can be characterized as follows. For each $\alpha_i \in \Delta$, define a reflection in $\mathfrak{a}^*$ with respect to the Cartan-Killing form $\langle\cdot,\cdot\rangle$ by

$$w_{\alpha_i} : \lambda \mapsto \lambda - 2\alpha_i\langle\lambda, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle,$$

where $\langle\lambda, \alpha\rangle = \langle H_\lambda, H_\alpha\rangle$. Here H_λ is the unique element in $\mathfrak{a}$ corresponding to a given $\lambda \in \mathfrak{a}^*$, and determined by the non-degeneracy of the Cartan-Killing form. One can then identify the Weyl group W with the group generated by the reflections $\{w_{\alpha_i} : \alpha_i \in \Delta\}$. For a subset Θ of Δ, let now W_Θ denote the subgroup of W generated by reflections corresponding to elements in Θ, and define

$$P_\Theta = \bigcup_{w\in W_\Theta} P m_w P,$$

where m_w denotes a representative of w in M^*, and $P = MAN^+$ is a minimal parabolic subgroup. It is then a classical result in the theory of parabolic subgroups [12] that, as Θ ranges over the subsets of Δ, one obtains all the parabolic subgroups of G containing P. In particular, if $\Theta = \emptyset$, $P_\Theta = P$. Let us now introduce for $\Theta \subset \Delta$ the subalgebras

$$\mathfrak{a}_\Theta = \{H \in \mathfrak{a} : \alpha(H) = 0 \text{ for all } \alpha \in \Theta\}, \qquad \mathfrak{a}(\Theta) = \{H \in \mathfrak{a} : \langle H, X\rangle_\theta = 0 \text{ for all } X \in \mathfrak{a}_\Theta\}.$$

Note that, when restricted to the $+1$ or the -1 eigenspace of θ, the orthogonal complement of a subspace with respect to $\langle\cdot,\cdot\rangle$ is the same as its orthogonal complement with respect to $\langle\cdot,\cdot\rangle_\theta$. We further define

$$\mathfrak{n}_\Theta^+ = \sum_{\alpha\in\Sigma^+\setminus\langle\Theta\rangle^+} \mathfrak{g}^\alpha, \qquad \mathfrak{n}_\Theta^- = \theta(\mathfrak{n}_\Theta^+),$$

$$\mathfrak{n}^+(\Theta) = \sum_{\alpha\in\langle\Theta\rangle^+} \mathfrak{g}^\alpha, \qquad \mathfrak{n}^-(\Theta) = \theta(\mathfrak{n}^+(\Theta)),$$

$$\mathfrak{m}_\Theta = \mathfrak{m} + \mathfrak{n}^+(\Theta) + \mathfrak{n}^-(\Theta) + \mathfrak{a}(\Theta), \qquad \mathfrak{m}_\Theta(K) = \mathfrak{m}_\Theta \cap \mathfrak{k},$$

where $\langle\Theta\rangle^+ = \Sigma^+ \cap \sum_{\alpha_i\in\Theta} \mathbb{R}\alpha_i$, and denote by $A_\Theta, A(\Theta)$, $N_\Theta^\pm, N^\pm(\Theta), M_{\Theta,0}$, and $M_\Theta(K)_0$ the corresponding connected analytic subgroups of G, obtaining the decompositions $A = A_\Theta A(\Theta)$ and $N^\pm = N_\Theta^\pm N(\Theta)^\pm$, the second being a semi-direct product. Let next $M_\Theta = MM_{\Theta,0}$, $M_\Theta(K) = MM_\Theta(K)_0$. One has the *Iwasawa decompositions*

$$M_\Theta = M_\Theta(K)A(\Theta)N^\pm(\Theta),$$

and the *Langlands decompositions*

$$P_\Theta = M_\Theta A_\Theta N_\Theta^+ = M_\Theta(K)AN^+.$$

In particular, $P_\Delta = M_\Delta = G$, since $m_\Delta = \mathfrak{m}\oplus\mathfrak{a}\oplus\bigoplus_{\alpha\in\Sigma}\mathfrak{g}^\alpha$, and $\mathfrak{a}_\Delta, \mathfrak{n}_\Delta^+$ are trivial. One then defines

$$P_\Theta(K) = M_\Theta(K)A_\Theta N_\Theta^+.$$

$P_\Theta(K)$ is a closed subgroup, and G is a union of the open and dense submanifold $N^-A(\Theta)P_\Theta(K) = N_\Theta^- P_\Theta$, and submanifolds of lower dimension, see [8], Lemma 1. For $\Delta = \{\alpha_1,\dots,\alpha_l\}$, let next $\{H_1,\dots,H_l\}$ be the basis of $\mathfrak{a}$, dual to Δ, i.e. $\alpha_i(H_j) = \delta_{ij}$. Fix a basis $\{X_{\lambda,i} : 1 \le i \le m(\lambda)\}$ of $\mathfrak{g}^\lambda$ for each $\lambda\in\Sigma^+$. Clearly,

$$[H, -\theta X_{\lambda,i}] = -\theta[\theta H, X_{\lambda,i}] = -\lambda(H)(-\theta X_{\lambda,i}), \qquad H\in\mathfrak{a},$$

so that setting $X_{-\lambda,i} = -\theta(X_{\lambda,i})$ one obtains a basis $\{X_{-\lambda,i} : 1\le i\le m(\lambda)\}$ of $\mathfrak{g}^{-\lambda}\subset\mathfrak{n}^-$. One now has the following lemma due to Oshima.

Lemma 1. *Fix an element $g\in G$, and identify $N^-\times A(\Theta)$ with an open dense submanifold of the homogeneous space $G/P_\Theta(K)$ by the map $(n,a)\mapsto gnaP_\Theta(K)$. For $Y\in\mathfrak{g}$, let $Y_{|G/P_\Theta(K)}$ be the fundamental vector field corresponding to the action of the one-parameter group $\exp(sY), s\in\mathbb{R}$, on $G/P_\Theta(K)$. Then, at any point $p = (n,a)\in N^-\times A(\Theta)$, we have*

$$(Y_{|G/P_\Theta(K)})_p = \sum_{\lambda\in\Sigma^+}\sum_{i=1}^{m(\lambda)} c_{-\lambda,i}(g,n)(X_{-\lambda,i})_p + \sum_{\lambda\in\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)e^{-2\lambda(\log a)}(X_{-\lambda,i})_p$$
$$+ \sum_{\alpha_i\in\Theta} c_i(g,n)(H_i)_p$$

with the identification $T_nN^-\bigoplus T_a(A(\Theta)) \simeq T_p(N^-\times A(\Theta)) \simeq T_{gnaP_\Theta(K)}G/P_\Theta(K)$. The coefficient functions $c_{\lambda,i}(g,n), c_{-\lambda,i}(g,n), c_i(g,n)$ are real-analytic, and are determined by the equation

$$\text{(2)} \qquad \operatorname{Ad}^{-1}(gn)Y = \sum_{\lambda\in\Sigma^+}\sum_{i=1}^{m(\lambda)} (c_{\lambda,i}(g,n)X_{\lambda,i} + c_{-\lambda,i}(g,n)X_{-\lambda,i}) + \sum_{i=1}^{l} c_i(g,n)H_i \mod \mathfrak{m}.$$

Proof. Due to its importance, and for the convenience of the reader, we shall give a detailed proof of the lemma, following the original proof given in [8], Lemma 3. Let $s \in \mathbb{R}$, and assume that $|s|$ is small. According to the direct sum decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{a} \oplus \mathfrak{n}^+ \oplus \mathfrak{m}$ one has for an arbitrary $Y \in \mathfrak{g}$

$$(gn)^{-1} \exp(sY) gn = \exp N_1^-(s) \exp A_1(s) \exp N_1^+(s) \exp M_1(s), \tag{3}$$

where $N_1^-(s) \in \mathfrak{n}^-$, $A_1(s) \in \mathfrak{a}$, $N_1^+(s) \in \mathfrak{n}^+$, and $M_1(s) \in \mathfrak{m}$. The action of $\exp(sY)$ on the homogeneous space $G/P_\Theta(K)$ is therefore given by

$$\begin{aligned} \exp(sY) gna P_\Theta(K) &= gn \exp N_1^-(s) \exp A_1(s) \exp N_1^+(s) \exp M_1(s) a P_\Theta(K) \\ &= gn \exp N_1^-(s) \exp A_1(s) \exp N_1^+(s) a \exp M_1(s) P_\Theta(K) \\ &= gn \exp N_1^-(s) \exp A_1(s) \exp N_1^+(s) a P_\Theta(K), \end{aligned}$$

since M is the centralizer of A in K, and $\exp M_1(s) \in M M_\Theta(K)_0 \subset P_\Theta(K)$. The Lie algebra of $P_\Theta(K)$ is $\mathfrak{m}_\Theta(K) \oplus \mathfrak{a}_\Theta \oplus \mathfrak{n}_\Theta^+$, which we shall henceforth denote by $\mathfrak{p}_\Theta(K)$. Using the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{a}(\Theta) \oplus \mathfrak{p}_\Theta(K)$ we see that

$$a^{-1} \exp N_1^+(s) a = \exp N_2^-(s) \exp A_2(s) \exp P_2(s), \tag{4}$$

where $N_2^-(s) \in \mathfrak{n}^-$, $A_2(s) \in \mathfrak{a}(\Theta)$, and $P_2(s) \in \mathfrak{p}_\Theta(K)$. From this we obtain that

$$\begin{aligned} &gn \exp N_1^-(s) \exp A_1(s) \exp N_1^+(s) a P_\Theta(K) \\ &\qquad = gn \left(\exp N_1^-(s) \exp A_1(s) a \exp N_2^-(s)\right) \exp A_2(s) \exp P_2(s) P_\Theta(K) \\ &\qquad = gn \left(\exp N_1^-(s) \exp A_1(s) a \exp N_2^-(s) a^{-1}\right) a \exp A_2(s) P_\Theta(K). \end{aligned}$$

Noting that $[\mathfrak{a}, \mathfrak{n}^-] \subset \mathfrak{n}^-$ one deduces the equality $\exp N_1^-(s) \exp A_1(s) a \exp N_2^-(s) a^{-1} \exp A_1(s)^{-1} = \exp N_3^-(s) \in N^-$, and consequently

$$\exp N_1^-(s) \exp A_1(s) a \exp N_2^-(s) a^{-1} = \exp N_3^-(s) \exp A_1(s), \tag{5}$$

which in turn yields

$$\begin{aligned} gn \exp N_1^-(s) \exp A_1(s) \exp N_1^+(s) a P_\Theta(K) &= gn \exp N_3^-(s) \exp A_1(s) a \exp A_2(s) P_\Theta(K) \\ &= gn \exp N_3^-(s) a \exp(A_1(s) + A_2(s)) P_\Theta(K). \end{aligned}$$

The action of $\mathfrak{g}$ on $G/P_\Theta(K)$ can therefore be characterized as

$$\exp(sY) gna P_\Theta(K) = gn \exp N_3^-(s) a \exp(A_1(s) + A_2(s)) P_\Theta(K). \tag{6}$$

Set $dN_i^-(s)/ds|_{s=0} = N_i^-$, $dN_1^+(s)/ds|_{s=0} = N_1^+$, $dA_i(s)/ds|_{s=0} = A_i$, and $dP_2(s)/ds|_{s=0} = P_2$, where $i = 1, 2$, or 3. By differentiating equations (3)-(5) at $s = 0$ one computes

$$\operatorname{Ad}^{-1}(gn) Y = N_1^- + A_1 + N_1^+ \quad \mod \mathfrak{m}, \tag{7}$$

$$\operatorname{Ad}^{-1}(a) N_1^+ = N_2^- + A_2 + P_2, \tag{8}$$

$$N_1^- + \operatorname{Ad}(a) N_2^- = N_3^-. \tag{9}$$

In what follows, we express $N_1^\pm \in \mathfrak{n}^\pm$ in terms of the basis of $\mathfrak{n}^\pm$, and A_1 in terms of the one of $\mathfrak{a}$, as

$$\begin{aligned} N_1^\pm &= \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} c_{\pm\lambda, i}(g, n) X_{\pm\lambda, i}, \\ A_1 &= \sum_{i=1}^{l} c_i(g, n) H_i = \sum_{\alpha_i \in \Theta} c_i(g, n) H_i \quad \mod \mathfrak{a}_\Theta. \end{aligned}$$

For a fixed $X_{\lambda,i}$ one has $[H, X_{\lambda,i}] = \lambda(H)X_{\lambda,i}$ for all $H \in \mathfrak{a}$. Setting $H = -\log a$, $a \in A$, we get $\mathrm{ad}\,(-\log a)X_{\lambda,i} = -\lambda(\log a)X_{\lambda,i}$. By exponentiating we obtain $e^{\mathrm{ad}\,(-\log a)}X_{\lambda,i} = e^{-\lambda(\log a)}X_{\lambda,i}$, which together with the relation $e^{\mathrm{ad}\,(-\log a)} = \mathrm{Ad}\,(\exp(-\log a))$ yields

$$\mathrm{Ad}^{\,-1}(a)X_{\lambda,i} = e^{-\lambda(\log a)}X_{\lambda,i}.$$

Analogously, one has $[H, X_{-\lambda,i}] = \theta[\theta H, -X_{\lambda,i}] = -\lambda(H)X_{-\lambda,i}$ for all $H \in \mathfrak{a}$, so that

$$\mathrm{Ad}^{\,-1}(a)X_{-\lambda,i} = e^{\lambda(\log a)}X_{-\lambda,i}. \tag{10}$$

We therefore arrive at

$$\begin{aligned}\mathrm{Ad}^{\,-1}(a)X_{\lambda,i} &= e^{-\lambda(\log a)}(X_{\lambda,i} - X_{-\lambda,i}) + e^{-\lambda(\log a)}X_{-\lambda,i}\\ &= e^{-\lambda(\log a)}(X_{\lambda,i} - X_{-\lambda,i}) + e^{-2\lambda(\log a)}\mathrm{Ad}^{\,-1}(a)X_{-\lambda,i}.\end{aligned}$$

Now, since $\theta(X_{\lambda,i} - X_{-\lambda,i}) = \theta(X_{\lambda,i}) - \theta(X_{-\lambda,i}) = -X_{-\lambda,i} - (-X_{\lambda,i}) = X_{\lambda,i} - X_{-\lambda,i}$, we see that $X_{\lambda,i} - X_{-\lambda,i} \in \mathfrak{k}$. Consequently, if λ is in $\langle\Theta\rangle^+$, one deduces that $X_{\lambda,i} - X_{-\lambda,i} \in (\mathfrak{m} + \mathfrak{n}^+(\Theta) + \mathfrak{n}^-(\Theta) + \mathfrak{a}(\Theta)) \cap \mathfrak{k} = \mathfrak{m}_\Theta(K)$. On the other hand, if λ is in $\Sigma^+ - \langle\Theta\rangle^+$, then $\mathrm{Ad}^{\,-1}(a)X_{\lambda,i} = e^{-\lambda(\log a)}X_{\lambda,i}$ belongs to $\mathfrak{n}^+_\Theta$. Collecting everything we obtain

$$\begin{aligned}\mathrm{Ad}^{\,-1}(a)N_1^+ &= \sum_{\lambda\in\Sigma^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)\mathrm{Ad}^{\,-1}(a)X_{\lambda,i}\\
&= \sum_{\lambda\in\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)\mathrm{Ad}^{\,-1}(a)X_{\lambda,i} + \sum_{\lambda\in\Sigma^+-\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)\mathrm{Ad}^{\,-1}(a)X_{\lambda,i}\\
&= \sum_{\lambda\in\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)\left(e^{-2\lambda(\log a)}\mathrm{Ad}^{\,-1}(a)X_{-\lambda,i} + e^{-\lambda(\log a)}(X_{\lambda,i} - X_{-\lambda,i})\right)\\
&+ \sum_{\lambda\in\Sigma^+-\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)e^{-\lambda(\log a)}X_{\lambda,i}\\
&= \sum_{\lambda\in\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)e^{-2\lambda(\log a)}\mathrm{Ad}^{\,-1}(a)X_{-\lambda,i}\\
&+ \sum_{\lambda\in\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)e^{-\lambda(\log a)}(X_{\lambda,i} - X_{-\lambda,i}) + \sum_{\lambda\in\Sigma^+-\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)e^{-\lambda(\log a)}X_{\lambda,i}.\end{aligned}$$

Comparing this with the expression (8) we had obtained earlier for $\mathrm{Ad}^{\,-1}(a)N_1^+$, we obtain that

$$A_2 = 0, \tag{11}$$

and $N_2^- = \sum_{\lambda\in\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)e^{-2\lambda(\log a)}\mathrm{Ad}^{\,-1}(a)X_{-\lambda,i}$, since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^-$, and $\mathfrak{p}_\Theta(K) \cap \mathfrak{a}(\Theta) = \{0\}$. Therefore

$$\begin{aligned}N_3^- &= N_1^- + \mathrm{Ad}\,(a)N_2^-\\
&= \sum_{\lambda\in\Sigma^+}\sum_{i=1}^{m(\lambda)} c_{-\lambda,i}(g,n)X_{-\lambda,i} + \sum_{\lambda\in\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)e^{-2\lambda(\log a)}X_{-\lambda,i}, \qquad (12)\\
A_1 + A_2 &= \sum_{\alpha_i\in\Theta} c_i(g,n)H_i \mod \mathfrak{a}_\Theta.\end{aligned}$$

As $N^- \times A(\Theta)$ can be identified with an open dense submanifold of the homogeneous space $G/P_\Theta(K)$, we have the isomorphisms $T_{gnaP_\Theta(K)}G/P_\Theta(K) \simeq T_p(N^- \times A(\Theta)) \simeq T_n N^- \bigoplus T_a(A(\Theta))$, where $p = (n,a) \in N^- \times A(\Theta)$. Therefore, by equation (6) and the expressions for N_3^- and $A_1 + A_2$, we finally deduce that the fundamental vector field $Y_{|G/P_\Theta(K)}$ at a point p corresponding to the action of $\exp(sY)$ on $G/P_\Theta(K)$ is given by

$$(Y_{|G/P_\Theta(K)})_p = \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} c_{-\lambda,i}(g,n)(X_{-\lambda,i})_p + \sum_{\lambda \in \langle\Theta\rangle^+} \sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n) e^{-2\lambda \log a}(X_{-\lambda,i})_p + \sum_{\alpha_i \in \Theta} c_i(g,n)(H_i)_p,$$

where $Y \in \mathfrak{g}$, and the coefficients are given by (2). □

Let us next state the following

Lemma 2. *Let $Y \in \mathfrak{n}^- \oplus \mathfrak{a}$ be given by $Y = \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} c_{-\lambda,i} X_{-\lambda,i} + \sum_{j=1}^{l} c_j H_j$, and introduce the notation $t^\lambda = t_1^{\lambda(H_1)} \cdots t_l^{\lambda(H_l)}$. Then, via the identification of $N^- \times \mathbb{R}_+^l$ with $N^- A$ by $(n,t) \mapsto n \cdot exp(-\sum_{j=1}^{l} H_j \log t_j)$, the left invariant vector field on the Lie group $N^- A$ corresponding to Y is expressed as*

$$\tilde{Y}_{|N^- \times \mathbb{R}_+^l} = \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} c_{-\lambda,i} t^\lambda X_{-\lambda,i} - \sum_{j=1}^{l} c_j t_j \frac{\partial}{\partial t_j},$$

and can analytically be extended to a vector field on $N^- \times \mathbb{R}^l$.

Proof. The lemma is stated in Oshima, [8], Lemma 8, but for greater clarity, we include a proof of it here. Let $X_{-\lambda,i}$ be a fixed basis element of $\mathfrak{n}^-$. The corresponding left-invariant vector field on the Lie group $N^- A$ at the point na is given by

$$\frac{d}{ds} f(na \exp(sX_{-\lambda,i}))_{|s=0} = \frac{d}{ds} f(n(a \exp(sX_{-\lambda,i})a^{-1})a)_{|s=0} = \frac{d}{ds} f(n\, \mathrm{e}^{sAd(a)X_{-\lambda,i}}\, a)_{|s=0},$$

where f is a smooth function on $N^- A$. Regarded as a left invariant vector field on $N^- \times \mathbb{R}_+^l$, it is therefore given by

$$\tilde{X}_{-\lambda,i|N^- \times \mathbb{R}_+^l} = \mathrm{Ad}\,(a) X_{-\lambda,i} = e^{-\lambda(\log a)} X_{-\lambda,i} = t^\lambda X_{-\lambda,i},$$

compare (10). Similarly, for a basis element H_i of $\mathfrak{a}$ the corresponding left invariant vector field on $N^- A$ reads

$$\frac{d}{ds} f(na \exp(sH_i))_{|s=0} = \frac{d}{ds} f(n \exp(-\sum_{j=1}^{l} \log t_j H_j) \exp(sH_i))_{|s=0}$$

$$= \frac{d}{ds} f\Big(n \exp(-\sum_{j=1}^{l} \log t_j H_j + sH_i)\Big)_{|s=0} = \frac{d}{ds} f\Big(n \exp(-\sum_{j \neq i} \log t_j H_j - \log(t_i e^{-s}) H_i)\Big)_{|s=0},$$

and with the identification $N^- A \simeq N^- \times \mathbb{R}_+^l$ we obtain

$$\tilde{H}_{i|N^- \times \mathbb{R}_+^l} = -t_i \frac{\partial}{\partial t_i}.$$

As there are no negative powers of t, $\tilde{Y}_{N^- \times \mathbb{R}_+^l}$ can be extended analytically to $N^- \times \mathbb{R}^l$, and the lemma follows. □

Similarly, by the identification $G/K \simeq N^- \times A \simeq N^- \times \mathbb{R}^l_+$ via the mappings $(n,t) \mapsto n \cdot exp(-\sum_{i=1}^l H_i \log t_i) \cdot a \mapsto gnaK$ one sees that the action on G/K of the fundamental vector field corresponding to $\exp(sY)$, $Y \in \mathfrak{g}$, is given by

$$Y_{|N^- \times \mathbb{R}^l_+} = \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} (c_{\lambda,i}(g,n)t^{2\lambda} + c_{-\lambda,i}(g,n))X_{-\lambda,i} - \sum_{i=1}^{l} c_i(g,n)t_i \frac{\partial}{\partial t_i}, \tag{13}$$

where the coefficients are given by (2). Again, the vector field (13) can be extended analytically to $N^- \times \mathbb{R}^l$, but in contrast to the left invariant vector field $\tilde{Y}_{N^- \times \mathbb{R}^l}$, $Y_{N^- \times \mathbb{R}^l}$ does not necessarily vanish if $t_1 = \dots t_l = 0$. We come now to the description of the Oshima compactification of the Riemannian symmetric space G/K. For this, let $\hat{\mathbb{X}}$ be the product manifold $G \times N^- \times \mathbb{R}^l$. Take $\hat{x} = (g,n,t) \in \hat{\mathbb{X}}$, where $g \in G$, $n \in N^-$, $t = (t_1, \dots, t_l) \in \mathbb{R}^l$, and define an action of G on $\hat{\mathbb{X}}$ by $g' \cdot (g,n,t) := (g'g,n,t)$, $g' \in G$. For $s \in \mathbb{R}$, let

$$\operatorname{sgn} s = \begin{cases} s/|s|, & s \neq 0, \\ 0, & s = 0, \end{cases}$$

and put $\operatorname{sgn} \hat{x} = (\operatorname{sgn} t_1, \dots, \operatorname{sgn} t_l) \in \{-1,0,1\}^l$. We then define the subsets $\Theta_{\hat{x}} = \{\alpha_i \in \Delta : t_i \neq 0\}$. Similarly, let $a(\hat{x}) = \exp(-\sum_{t_i \neq 0} H_i \log |t_i|) \in A(\Theta_{\hat{x}})$. On $\hat{\mathbb{X}}$, define now an equivalence relation by setting

$$\hat{x} = (g,n,t) \sim \hat{x}' = (g',n,'t') \quad \Longleftrightarrow \quad \begin{cases} a) \operatorname{sgn} \hat{x} = \operatorname{sgn} \hat{x}', \\ b) \, g\, n\, a(\hat{x})\, P_{\Theta_{\hat{x}}}(K) = g'\, n'\, a(\hat{x}')\, P_{\Theta_{\hat{x}'}}(K). \end{cases}$$

Note that the condition $\operatorname{sgn} \hat{x} = \operatorname{sgn} \hat{x}'$ implies that $\hat{x}, \hat{x}'$ determine the same subset $\Theta_{\hat{x}}$ of Δ, and consequently the same group $P_{\Theta_{\hat{x}}}(K)$, as well as the same homogeneous space $G/P_{\Theta_{\hat{x}}}(K)$, so that condition $b)$ makes sense. It says that $gna(\hat{x})$, $g'n'a(\hat{x}')$ are in the same $P_{\Theta_{\hat{x}}}(K)$ orbit on G, corresponding to the right action by $P_{\Theta_{\hat{x}}}(K)$ on G. We now define

$$\widetilde{\mathbb{X}} := \hat{\mathbb{X}}/\sim,$$

endowing it with the quotient topology, and denote by $\pi : \hat{\mathbb{X}} \to \widetilde{\mathbb{X}}$ the canonical projection. The action of G on $\hat{\mathbb{X}}$ is compatible with the equivalence relation $\sim$, yielding a G-action $g' \cdot \pi(g,n,t) := \pi(g'g,n,t)$ on $\widetilde{\mathbb{X}}$. For each $g \in G$, one can show that the maps

$$\varphi_g : N^- \times \mathbb{R}^l \to \widetilde{U}_g : (n,t) \mapsto \pi(g,n,t), \qquad \widetilde{U}_g = \pi(\{g\} \times N^- \times \mathbb{R}^l), \tag{14}$$

are bijections. One has then the following

Theorem 1. (1) *$\widetilde{\mathbb{X}}$ is a simply connected, compact, real-analytic manifold without boundary.*

(2) *$\widetilde{\mathbb{X}} = \cup_{w \in W} \widetilde{U}_{m_w} = \cup_{g \in G} \widetilde{U}_g$. For $g \in G$, $\widetilde{U}_g$ is an open submanifold of $\widetilde{\mathbb{X}}$ topologized in such a way that the coordinate map φ_g defined above is a real-analytic diffeomorphism. Furthermore, $\widetilde{\mathbb{X}} \backslash \widetilde{U}_g$ is the union of a finite number of submanifolds of $\widetilde{\mathbb{X}}$ whose codimensions in $\widetilde{\mathbb{X}}$ are not lower than 2.*

(3) *The action of G on $\widetilde{\mathbb{X}}$ is real-analytic. For a point $\hat{x} \in \hat{\mathbb{X}}$, the G-orbit of $\pi(\hat{x})$ is isomorphic to the homogeneous space $G/P_{\Theta_{\hat{x}}}(K)$, and for $\hat{x}, \hat{x}' \in \hat{\mathbb{X}}$ the G-orbits of $\pi(\hat{x})$ and $\pi(\hat{x}')$ coincide if and only if $\operatorname{sgn} \hat{x} = \operatorname{sgn} \hat{x}'$. Hence the orbital decomposition of $\widetilde{\mathbb{X}}$ with respect to the action of G is of the form*

$$\widetilde{\mathbb{X}} \simeq \bigsqcup_{\Theta \subset \Delta} 2^{\#\Theta} (G/P_\Theta(K)) \quad \textit{(disjoint union)}, \tag{15}$$

where $\#\Theta$ is the number of elements of Θ and $2^{\#\Theta}(G/P_\Theta(K))$ is the disjoint union of $2^{\#\Theta}$ copies of $G/P_\Theta(K)$.

Proof. See Oshima, [8], Theorem 5. □

Next, for $\hat{x} = (g, n, t)$ define the set $B_{\hat{x}} = \{(t'_1 \dots t'_l) \in \mathbb{R}^l : \operatorname{sgn} t_i = \operatorname{sgn} t'_i, 1 \leq i \leq l\}$. By analytic continuation, one can restrict the vector field (13) to $N^- \times B_{\hat{x}}$, and with the identifications $G/P_{\Theta_{\hat{x}}}(K) \simeq N^- \times A(\Theta_{\hat{x}}) \simeq N^- \times B_{\hat{x}}$ via the maps

$$gnaP_{\Theta_{\hat{x}}} \leftarrow (n, a) \mapsto (n, \operatorname{sgn} t_1 e^{-\alpha_1(\log a)}, \dots, \operatorname{sgn} t_l e^{-\alpha_l(\log a)}),$$

one actually sees that this restriction coincides with the vector field in Lemma 1. The action of the fundamental vector field on $\widetilde{\mathbb{X}}$ corresponding to $\exp sY, Y \in \mathfrak{g}$, is therefore given by the extension of (13) to $N^- \times \mathbb{R}^l$. Note that for a simply connected nilpotent Lie group N with Lie algebra $\mathfrak{n}$, the exponential $\exp : \mathfrak{n} \to N$ is a diffeomorphism. So, in our setting, we can identify N^- with $\mathbb{R}^k$. Thus, for every point in $\widetilde{\mathbb{X}}$, there exists a local coordinate system $(n_1, \dots, n_k, t_1, \dots, t_l)$ in a neighbourhood of that point such that two points $(n_1, \dots, n_k, t_1, \dots, t_l)$ and $(n'_1, \dots, n'_k, t'_1, \dots, t'_l)$ belong to the same G-orbit if, and only if, $\operatorname{sgn} t_j = \operatorname{sgn} t'_j$, for $j = 1, \dots, l$. This means that the orbital decomposition of $\widetilde{\mathbb{X}}$ is of *normal crossing type*. In what follows, we shall identify the open G-orbit $\pi(\{\hat{x} = (e, n, t) \in \hat{\mathbb{X}} : \operatorname{sgn} \hat{x} = (1, \dots, 1)\})$ with the Riemannian symmetric space G/K, and the orbit $\pi(\{\hat{x} \in \hat{\mathbb{X}} : \operatorname{sgn} \hat{x} = (0, \dots, 0)\}$ of lowest dimension with its Martin boundary G/P.

3. Review of pseudodifferential operators

Generalities. This section is devoted to an exposition of some basic facts about pseudodifferential operators needed to formulate our main results in the sequel. For a detailed introduction to the field, the reader is referred to [3] and [11]. Consider first an open set U in $\mathbb{R}^n$, and let $x_1, \dots, x_n$ be the standard coordinates. For any real number l, we denote by $\mathrm{S}^l(U \times \mathbb{R}^n)$ the class of all functions $a(x, \xi) \in \mathrm{C}^\infty(U \times \mathbb{R}^n)$ such that, for any multi-indices α, β, and any compact set $\mathcal{K} \subset U$, there exist constants $C_{\alpha,\beta,\mathcal{K}}$ for which

$$|(\partial_\xi^\alpha \, \partial_x^\beta \, a)(x, \xi)| \leq C_{\alpha,\beta,\mathcal{K}} \, \langle \xi \rangle^{l - |\alpha|}, \qquad x \in \mathcal{K}, \quad \xi \in \mathbb{R}^n, \tag{16}$$

where $\langle \xi \rangle$ stands for $(1 + |\xi|^2)^{1/2}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We further put $\mathrm{S}^{-\infty}(U \times \mathbb{R}^n) = \bigcap_{l \in \mathbb{R}} \mathrm{S}^l(U \times \mathbb{R}^n)$. Note that, in general, the constants $C_{\alpha,\beta,K}$ also depend on $a(x, \xi)$. For any such $a(x, \xi)$ one then defines the continuous linear operator

$$A : \mathrm{C}^\infty_{\mathrm{c}}(U) \longrightarrow \mathrm{C}^\infty(U)$$

by the formula

$$Au(x) = \int e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) \bar{d}\xi, \tag{17}$$

where $\hat{u}$ denotes the Fourier transform of u, and $\bar{d}\xi = (2\pi)^{-n} \, d\xi$. [1] An operator A of this form is called a *pseudodifferential operator of order* l, and we denote the class of all such operators for which $a(x, \xi) \in \mathrm{S}^l(U \times \mathbb{R}^n)$ by $\mathrm{L}^l(U)$. The set $\mathrm{L}^{-\infty}(U) = \bigcap_{l \in \mathbb{R}} \mathrm{L}^l(U)$ consists of all operators with smooth kernel. They are called *smooth operators*. By inserting in (17) the definition of $\hat{u}$, we obtain for Au the expression

$$Au(x) = \int \int e^{i(x-y) \cdot \xi} a(x, \xi) u(y) \, dy \, \bar{d}\xi, \tag{18}$$

[1] Here and in what follows we use the convention that, if not specified otherwise, integration is to be performed over Euclidean space.

which has a suitable regularization as an oscillatory integral. The Schwartz kernel of A is a distribution $K_A \in \mathcal{D}'(U \times U)$ which is given the oscillatory integral

$$K_A(x,y) = \int e^{i(x-y)\cdot\xi} a(x,\xi)\, đ\xi. \tag{19}$$

It is a smooth function off the diagonal in $U \times U$. Consider next a n-dimensional paracompact C^∞ manifold $\mathbf{X}$, and let $\{(\kappa_\gamma, \widetilde{U}^\gamma)\}$ be an atlas for $\mathbf{X}$. Then a linear operator

$$A : \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X}) \longrightarrow \mathrm{C}^\infty(\mathbf{X}) \tag{20}$$

is called a *pseudodifferential operator on* $\mathbf{X}$ *of order* l if for each chart diffeomorphism $\kappa_\gamma : \widetilde{U}^\gamma \to U^\gamma = \kappa_\gamma(\widetilde{U}^\gamma)$, the operator $A^\gamma u = [A_{|\widetilde{U}^\gamma}(u \circ \kappa_\gamma)] \circ \kappa_\gamma^{-1}$ given by the diagram

$$\begin{array}{ccc} \mathrm{C}^\infty_{\mathrm{c}}(\widetilde{U}^\gamma) & \xrightarrow{A_{|\widetilde{U}^\gamma}} & \mathrm{C}^\infty(\widetilde{U}^\gamma) \\ {\scriptstyle \kappa_\gamma^*}\uparrow & & \uparrow{\scriptstyle \kappa_\gamma^*} \\ \mathrm{C}^\infty_{\mathrm{c}}(U^\gamma) & \xrightarrow{A^\gamma} & \mathrm{C}^\infty(U^\gamma) \end{array}$$

is a pseudodifferential operator on U^γ of order l, and its kernel K_A is smooth off the diagonal. In this case we write $A \in \mathrm{L}^l(\mathbf{X})$. Note that, since the $\widetilde{U}^\gamma$ are not necessarily connected, we can choose them in such a way that $\mathbf{X} \times \mathbf{X}$ is covered by the open sets $\widetilde{U}^\gamma \times \widetilde{U}^\gamma$. Therefore the condition that K_A is smooth off the diagonal can be dropped. Now, in general, if $\mathbf{X}$ and $\mathbf{Y}$ are two smooth manifolds, and

$$A : \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X}) \longrightarrow \mathrm{C}^\infty(\mathbf{Y}) \subset \mathcal{D}'(\mathbf{Y})$$

is a continuous linear operator, where $\mathcal{D}'(\mathbf{Y}) = (\mathrm{C}^\infty_{\mathrm{c}}(\mathbf{Y}, \Omega))'$ and $\Omega = |\Lambda^n(\mathbf{Y})|$ is the density bundle on $\mathbf{Y}$, its Schwartz kernel is given by the distribution section $K_A \in \mathcal{D}'(\mathbf{Y} \times \mathbf{X}, \mathbf{1} \boxtimes \Omega_{\mathbf{X}})$, where $\mathcal{D}'(\mathbf{Y} \times \mathbf{X}, 1 \boxtimes \Omega_{\mathbf{X}}) = (\mathrm{C}^\infty_{\mathrm{c}}(\mathbf{Y} \times \mathbf{X}, (\mathbf{1} \boxtimes \Omega_{\mathbf{X}})^* \otimes \Omega_{\mathbf{Y}\times\mathbf{X}}))'$. Observe that $\mathrm{C}^\infty_{\mathrm{c}}(\mathbf{Y}, \Omega_{\mathbf{Y}}) \otimes \mathrm{C}^\infty(\mathbf{X}) \simeq \mathrm{C}^\infty(\mathbf{Y} \times \mathbf{X}, (\mathbf{1} \boxtimes \Omega_{\mathbf{X}})^* \otimes \Omega_{\mathbf{Y}\times\mathbf{X}})$. In case that $\mathbf{X} = \mathbf{Y}$ and $A \in \mathrm{L}^l(\mathbf{X})$, A is given locally by the operators A^γ, which can be written in the form

$$A^\gamma u(x) = \int\int e^{i(x-y)\cdot\xi} a^\gamma(x,\xi) u(y)\, dy đ\xi,$$

where $u \in \mathrm{C}^\infty_{\mathrm{c}}(U^\gamma)$, $x \in U^\gamma$, and $a^\gamma(x,\xi) \in \mathrm{S}^l(U^\gamma, \mathbb{R}^n)$. The kernel of A is then determined by the kernels $K_{A^\gamma} \in \mathcal{D}'(U^\gamma \times U^\gamma)$. For $l < -\dim \mathbf{X}$, they are continuous, and given by absolutely convergent integrals. In this case, their restrictions to the respective diagonals in $U^\gamma \times U^\gamma$ define continuous functions

$$k^\gamma(m) = K_{A^\gamma}(\kappa_\gamma(m), \kappa_\gamma(m)), \qquad m \in \widetilde{U}^\gamma,$$

which, for $m \in \widetilde{U}^{\gamma_1} \cap \widetilde{U}^{\gamma_2}$, satisfy the relations $k^{\gamma_2}(m) = |\det(\kappa_{\gamma_1} \circ \kappa_{\gamma_2}^{-1})'| \circ \kappa_{\gamma_2}(m) k^{\gamma_1}(m)$, and therefore define a density $k \in C(\mathbf{X}, \Omega)$ on $\Delta_{\mathbf{X}} \times \mathbf{X} \simeq \mathbf{X}$. If $\mathbf{X}$ is compact, this density can be integrated, yielding the trace of the operator A,

$$\operatorname{tr} A = \int_{\mathbf{X}} k = \sum_\gamma \int_{U^\gamma} (\alpha_\gamma \circ \kappa_\gamma^{-1})(x)\, K_{A^\gamma}(x,x)\, dx, \tag{21}$$

where $\{\alpha_\gamma\}$ denotes a partition of unity subordinated to the atlas $\{(\kappa_\gamma, \widetilde{U}^\gamma)\}$, and dx is Lebesgue measure in $\mathbb{R}^n$.

Totally characteristic pseudodifferential operators. We introduce now a special class of pseudodifferential operators associated in a natural way to a C^∞ manifold $\mathbf{X}$ with boundary $\partial\mathbf{X}$. Our main reference will be [7] in this case. Let $\mathrm{C}^\infty(\mathbf{X})$ be the space of functions on $\mathbf{X}$ which are

C^∞ up to the boundary, and $\dot{\mathrm{C}}^\infty(\mathbf{X})$ the subspace of functions vanishing to all orders on $\partial\,\mathbf{X}$. The standard spaces of distributions over $\mathbf{X}$ are

$$\mathcal{D}'(\mathbf{X}) = (\dot{\mathrm{C}}^\infty_{\mathrm{c}}(\mathbf{X},\Omega))', \qquad \dot{\mathcal{D}}(\mathbf{X})' = (\mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X},\Omega))',$$

the first being the space of *extendible distributions*, whereas the second is the space of *distributions supported by* $\mathbf{X}$. Consider now the translated partial Fourier transform of a symbol $a(x,\xi) \in \mathrm{S}^l(\mathbb{R}^n \times \mathbb{R}^n)$,

$$Ma(x,\xi';t) = \int e^{i(1-t)\xi_1} a(x,\xi_1,\xi')d\xi_1,$$

where we wrote $\xi = (\xi_1,\xi')$. $Ma(x,\xi';t)$ is C^∞ away from $t = 1$, and one says that $a(x,\xi)$ is *lacunary* if it satisfies the condition

$$Ma(x,\xi';t) = 0 \qquad \text{for } t < 0. \tag{22}$$

The subspace of lacunary symbols will be denoted by $\mathrm{S}^l_{la}(\mathbb{R}^n \times \mathbb{R}^n)$. Let $Z = \overline{\mathbb{R}^+} \times \mathbb{R}^{n-1}$ be the standard manifold with boundary with the natural coordinates $x = (x_1,x')$. In order to define on Z operators of the form (18), where now $a(x,\xi) = \widetilde{a}(x_1,x',x_1\xi_1,\xi')$ is a more general amplitude and $\widetilde{a}(x,\xi)$ is lacunary, one rewrites the formal adjoint of A by making a singular coordinate change. Thus, for $u \in \mathrm{C}^\infty_{\mathrm{c}}(Z)$, one considers

$$A^*u(y) = \int\int e^{i(y-x)\cdot\xi}\overline{a}(x,\xi)u(x)\ dx đ\xi.$$

By putting $\lambda = x_1\xi_1$, $s = x_1/y_1$, this can be rewritten as

$$A^*u(y) = (2\pi)^{-n}\int\int\int\int e^{i(1/s-1,y'-x')\cdot(\lambda,\xi')}\overline{\widetilde{a}}(y_1 s,x',\lambda,\xi')u(y_1 s,x')d\lambda\frac{ds}{s}dx'd\xi'. \tag{23}$$

According to [7], Propositions 3.6 and 3.9, for every $\widetilde{a} \in \mathrm{S}^{-\infty}_{la}(Z \times \mathbb{R}^n)$, the successive integrals in (23) converge absolutely and uniformly, thus defining a continuous bilinear form

$$\mathrm{S}^{-\infty}_{la}(Z \times \mathbb{R}^n) \times \mathrm{C}^\infty_{\mathrm{c}}(Z) \longrightarrow \mathrm{C}^\infty(Z),$$

which extends to a separately continuous form

$$\mathrm{S}^{\infty}_{la}(Z \times \mathbb{R}^n) \times \mathrm{C}^\infty_{\mathrm{c}}(Z) \longrightarrow \mathrm{C}^\infty(Z).$$

If $\widetilde{a} \in \mathrm{S}^\infty_{la}(Z \times \mathbb{R}^n)$ and $a(x,\xi) = \widetilde{a}(x_1,x',x_1\xi_1,\xi')$, one then defines the operator

$$A : \dot{\mathcal{E}}'(Z) \longrightarrow \dot{\mathcal{D}}'(Z), \tag{24}$$

written formally as (18), as the adjoint of A^*. In this way, the oscillatory integral (18) is identified with a separately continuous bilinear mapping

$$\mathrm{S}^\infty_{la}(Z \times \mathbb{R}^n) \times \dot{\mathcal{E}}'(Z) \longrightarrow \dot{\mathcal{D}}'(Z).$$

The space $\mathrm{L}^l_b(Z)$ of *totally characteristic pseudodifferential operators on* Z *of order* l consists of those continuous linear maps (24) such that for any $u,v \in \mathrm{C}^\infty_{\mathrm{c}}(Z)$, vAu is of the form (18) with $a(x,\xi) = \widetilde{a}(x_1,x',x_1\xi_1,\xi')$ and $\widetilde{a}(x,\xi) \in \mathrm{S}^l_{la}(Z \times \mathbb{R}^n)$. Similarly, a continuous linear map (20) on a smooth manifold $\mathbf{X}$ with boundary $\partial\,\mathbf{X}$ is said to be an element of the space $\mathrm{L}^l_b(\mathbf{X})$ of *totally characteristic pseudodifferential operators on* $\mathbf{X}$ *of order* l, if for a given atlas $(\kappa_\gamma,\widetilde{U}^\gamma)$ the operators $A^\gamma u = [A_{|\widetilde{U}^\gamma}(u \circ \kappa_\gamma)] \circ \kappa_\gamma^{-1}$ are elements of $\mathrm{L}^l_b(Z)$, where the $\widetilde{U}^\gamma$ are coordinate patches isomorphic to subsets in Z.

In an analogous way, it is possible to introduce the concept of a totally characteristic pseudodifferential operator on a manifold with corners. As the standard manifold with corners, consider

$$\mathbb{R}^{n,k} = [0,\infty)^k \times \mathbb{R}^{n-k}, \qquad 0 \le k \le n,$$

with coordinates $x = (x_1, \dots, x_k, x')$. A *totally characteristic pseudodifferential operator on* $\mathbb{R}^{n,k}$ *of order* l is locally given by an oscillatory integral (18) with $a(x,\xi) = \widetilde{a}(x, x_1\xi_1, \dots, x_k\xi_k, \xi')$, where now $\widetilde{a}(x,\xi)$ is a symbol of order l that satisfies the lacunary condition for each of the coordinates $x_1, \dots, x_k$, i.e.

$$\int e^{i(1-t)\xi_j} a(x,\xi)\, d\xi_j = 0 \qquad \text{for } t < 0 \text{ and } 1 \leq j \leq k.$$

In this case we write $\widetilde{a}(x,\xi) \in \mathrm{S}^l_{la}(\mathbb{R}^{n,k} \times \mathbb{R}^n)$. A continuous linear map (20) on a smooth manifold $\mathbf{X}$ with corners is then said to be an element of the space $\mathrm{L}^l_b(\mathbf{X})$ of *totally characteristic pseudodifferential operators on* $\mathbf{X}$ *of order* l, if for a given atlas $(\kappa_\gamma, \widetilde{U}^\gamma)$ the operators $A^\gamma u = [A_{|\widetilde{U}^\gamma}(u \circ \kappa_\gamma)] \circ \kappa_\gamma^{-1}$ are totally characteristic pseudodifferential operator on $\mathbb{R}^{n,k}$ of order l, where the $\widetilde{U}^\gamma$ are coordinate patches isomorphic to subsets in $\mathbb{R}^{n,k}$. For an extensive treatment, we refer the reader to [4].

4. Invariant integral operators

Let $\widetilde{\mathbb{X}}$ be the Oshima compactification of a Riemannian symmetric space $\mathbb{X} \simeq G/K$ of noncompact type. As was already explained, G acts analytically on $\widetilde{\mathbb{X}}$, and the orbital decomposition is of normal crossing type. Consider the Banach space $\mathrm{C}(\widetilde{\mathbb{X}})$ of continuous, complex valued functions on $\widetilde{\mathbb{X}}$, equipped with the supremum norm, and let $(\pi, \mathrm{C}(\widetilde{\mathbb{X}}))$ be the corresponding continuous regular representation of G given by

$$\pi(g)\varphi(\tilde{x}) = \varphi(g^{-1} \cdot \tilde{x}), \qquad \varphi \in \mathrm{C}(\widetilde{\mathbb{X}}).$$

The representation of the universal enveloping algebra $\mathfrak{U}$ of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$ on the space of differentiable vectors $\mathrm{C}(\widetilde{\mathbb{X}})_\infty$ will be denoted by $d\pi$. We will also consider the regular representation of G on $\mathrm{C}^\infty(\widetilde{\mathbb{X}})$ which, equipped with the topology of uniform convergence on compact subsets, becomes a Fréchet space. This representation will be denoted by π as well. Let $(L, \mathrm{C}^\infty(G))$ be the left regular representation of G. With respect to the left-invariant metric on G given by $\langle , \rangle_\theta$, we define $d(g,h)$ as the distance between two points $g, h \in G$, and set $|g| = d(g,e)$, where e is the identity element of G. A function f on G is *at most of exponential growth*, if there exists a $\kappa > 0$ such that $|f(g)| \leq Ce^{\kappa|g|}$ for some constant $C > 0$. As before, denote a Haar measure on G by d_G. Consider next the space $\mathcal{S}(G)$ of rapidly decreasing functions on G introduced in [9].

Definition 1. *The space of rapidly decreasing functions on* G*, denoted by* $\mathcal{S}(G)$*, is given by all functions* $f \in \mathrm{C}^\infty(G)$ *satisfying the following conditions:*

i) *For every* $\kappa \geq 0$*, and* $X \in \mathfrak{U}$*, there exists a constant* C *such that*

$$|dL(X)f(g)| \leq Ce^{-\kappa|g|};$$

ii) *for every* $\kappa \geq 0$*, and* $X \in \mathfrak{U}$*, one has* $dL(X)f \in \mathrm{L}^1(G, e^{\kappa|g|}d_G)$.

For later purposes, let us recall the following integration formulas.

Proposition 1. *Let* $f_1 \in \mathcal{S}(G)$*, and assume that* $f_2 \in \mathrm{C}^\infty(G)$*, together with all its derivatives, is at most of exponential growth. Let* $X_1, \dots, X_d$ *be a basis of* $\mathfrak{g}$*, and for* $X^\gamma = X_{i_1}^{\gamma_1} \dots X_{i_r}^{\gamma_r}$ *write* $X^{\tilde{\gamma}} = X_{i_r}^{\gamma_r} \dots X_{i_1}^{\gamma_1}$*, where* γ *is an arbitrary multi-index. Then*

$$\int_G f_1(g) dL(X^\gamma) f_2(g) d_G(g) = (-1)^{|\gamma|} \int_G dL(X^{\tilde{\gamma}}) f_1(g) f_2(g) d_G(g).$$

Proof. See [9], Proposition 1. ∎

Next, we associate to every $f \in \mathcal{S}(G)$ and $\varphi \in \mathrm{C}(\widetilde{\mathbb{X}})$ the element $\int_G f(g)\pi(g)\varphi \, d_G(g) \in \mathrm{C}(\widetilde{\mathbb{X}})$. It is defined as a Bochner integral, and the continuous linear operator on $\mathrm{C}(\widetilde{\mathbb{X}})$ obtained this way is denoted by (1). Its restriction to $\mathrm{C}^\infty(\widetilde{\mathbb{X}})$ induces a continuous linear operator

$$\pi(f) : \mathrm{C}^\infty(\widetilde{\mathbb{X}}) \longrightarrow \mathrm{C}^\infty(\widetilde{\mathbb{X}}) \subset \mathcal{D}'(\widetilde{\mathbb{X}}),$$

with Schwartz kernel given by the distribution section $\mathcal{K}_f \in \mathcal{D}'(\widetilde{\mathbb{X}} \times \widetilde{\mathbb{X}}, \mathbf{1} \boxtimes \Omega_{\widetilde{\mathbb{X}}})$. The properties of the Schwartz kernel $\mathcal{K}_f$ will depend on the analytic properties of f, as well as the orbit structure of the underlying G-action, and our main effort will be directed towards the elucidation of the structure of $\mathcal{K}_f$. For this, let us consider the orbital decomposition (15) of $\widetilde{\mathbb{X}}$, and remark that the restriction of $\pi(f)\varphi$ to any of the connected components isomorphic to $G/P_\Theta(K)$ depends only on the restriction of $\varphi \in \mathrm{C}(\widetilde{\mathbb{X}})$ to that component, so that we obtain the continuous linear operators

$$\pi(f)_{|\widetilde{\mathbb{X}}_\Theta} : \mathrm{C}^\infty_{\mathrm{c}}(\widetilde{\mathbb{X}}_\Theta) \longrightarrow \mathrm{C}^\infty(\widetilde{\mathbb{X}}_\Theta),$$

where $\widetilde{\mathbb{X}}_\Theta$ denotes a component in $\widetilde{\mathbb{X}}$ isomorphic to $G/P_\Theta(K)$. Let us now assume that $\Theta = \Delta$, so that $P_\Theta(K) = K$. Since G acts transitively on $\widetilde{\mathbb{X}}_\Delta$ one deduces that $\pi(f)_{|\widetilde{\mathbb{X}}_\Delta} \in \mathrm{L}^{-\infty}(\widetilde{\mathbb{X}}_\Delta)$, c.p. [9], Section 4. The main goal of this section is to prove that the restrictions of the operators $\pi(f)$ to the manifolds with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$ are totally characteristic pseudodifferential operators of class $\mathrm{L}_b^{-\infty}$.

Let $\left\{(\widetilde{U}_{m_w}, \varphi_{m_w}^{-1})\right\}_{w \in W}$ be the finite atlas on the Oshima compactification $\widetilde{\mathbb{X}}$ defined earlier. For each $\tilde{x} \in \widetilde{\mathbb{X}}$, let $\widetilde{W}_{\tilde{x}}$ be an open neighborhood of $\tilde{x}$ contained in some $\widetilde{U}_{m_w}$ such that $\left\{h \in G : h\widetilde{W}_{\tilde{x}} \subset \widetilde{U}_{m_w}\right\}$ acts transitively on the G-orbits of $\widetilde{W}_{\tilde{x}}$, c.p. [9], Section 6. We obtain a finite atlas $\left\{(\widetilde{W}_\gamma, \varphi_{m_{w_\gamma}}^{-1})\right\}_{\gamma \in I}$ of $\widetilde{\mathbb{X}}$ satisfying the following properties:

- i) For each $\widetilde{W}_\gamma$, there exist open sets $V_\gamma \subset V_\gamma^1 \subset G$, stable under inverse, that act transitively on the G-orbits of $\widetilde{W}_\gamma$;
- ii) For all $\gamma \in I$ one has $V_\gamma^1 \cdot \widetilde{W}_\gamma \subset \widetilde{U}_{m_{w_\gamma}}$ for some $m_{w_\gamma} \in M^*$.

To simplify notation, we shall write φ_γ instead of $\varphi_{m_{w_\gamma}}$. Consider now the localization of the operators $\pi(f)$ with respect to the finite atlas $\left\{(\widetilde{W}_\gamma, \varphi_\gamma^{-1})\right\}_{\gamma \in I}$ given by

$$A_f^\gamma u = [\pi(f)_{|\widetilde{W}_\gamma}(u \circ \varphi_\gamma^{-1})] \circ \varphi_\gamma, \qquad u \in \mathrm{C}^\infty_{\mathrm{c}}(W_\gamma),\, W_\gamma = \varphi_\gamma^{-1}(\widetilde{W}_\gamma),$$

see Section 3. Writing $\varphi_\gamma^g = \varphi_\gamma^{-1} \circ g^{-1} \circ \varphi_\gamma$ and $x = (n, t) \in W_\gamma$ we obtain

$$A_f^\gamma u(x) = \int_G f(g)\pi(g)(u \circ \varphi_\gamma^{-1})(\varphi_\gamma(x))dg = \int_G f(g)(u \circ \varphi_\gamma^g)(x)dg.$$

Since we can restrict the domain of integration to V_γ, the latter integral can be rewritten as

$$A_f^\gamma u(x) = \int_G c_\gamma(g)f(g)(u \circ \varphi_\gamma^g)(x)dg,$$

where c_γ is a smooth bounded function on G with support in V_γ^1 such that $c_\gamma \equiv 1$ on V_γ. Define next

$$\hat{f}_\gamma(x, \xi) = \int_G e^{i\varphi_\gamma^g(x)\cdot\xi} c_\gamma(g)f(g)dg, \qquad a_f^\gamma(x, \xi) = e^{-ix\cdot\xi}\hat{f}_\gamma(x, \xi). \tag{25}$$

Differentiating under the integral we see that $\hat{f}_\gamma(x, \xi), a_f^\gamma(x, \xi) \in \mathrm{C}^\infty(W_\gamma \times \mathbb{R}^{k+l})$. Let us next state the following

Lemma 3. *For any $\tilde{x} = \varphi_\gamma(n,t) \in \widetilde{W}_\gamma$ and $g \in V_\gamma^1$ we have the power series expansion*

$$t_j(g \cdot \tilde{x}) = \sum_{\substack{\alpha,\beta \\ \beta_j \neq 0}} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}), \qquad j = 1, \dots, l, \tag{26}$$

where the coefficients $c^j_{\alpha,\beta}(g)$ depend real-analytically on g, and α, β are multi-indices.

Proof. By Theorem 1, a G-orbit in $\widetilde{\mathbb{X}}$ is locally determined by the signature of any of its elements. In particular, for $\tilde{x} \in \widetilde{W}_\gamma$, $g \in V^1_\gamma$ we have $\operatorname{sgn} t_j(g \cdot \tilde{x}) = \operatorname{sgn} t_j(\tilde{x})$ for all $j = 1, \dots, l$. Hence, $t_j(g \cdot \tilde{x}) = 0$ if and only if $t_j(\tilde{x}) = 0$. Now, due to the analyticity of the coordinates $(\varphi_\gamma, \widetilde{W}_\gamma)$, there is a power series expansion

$$t_j(g \cdot \tilde{x}) = \sum_{\alpha,\beta} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}), \qquad \tilde{x} \in \widetilde{W}_\gamma,\ g \in V^1_\gamma,$$

for every $j = 1, \dots, l$, which can be rewritten as

$$t_j(g \cdot \tilde{x}) = \sum_{\substack{\alpha,\beta \\ \beta_j \neq 0}} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}) + \sum_{\substack{\alpha,\beta \\ \beta_j = 0}} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}). \tag{27}$$

Suppose $t_j(\tilde{x}) = 0$. Then the first summand of the last equation must vanish, as in each term of the summation a non-zero power of $t_j(\tilde{x})$ occurs. Also, $t_j(g \cdot \tilde{x}) = 0$. Therefore (27) implies that the second summand must vanish, too. But the latter is independent of t_j. So we conclude

$$\sum_{\substack{\alpha,\beta \\ \beta_j = 0}} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}) \equiv 0$$

for all $\tilde{x} \in \widetilde{W}_\gamma$, $g \in V^1_\gamma$, and the assertion follows. □

From Lemma 3 we deduce that

$$t_j(g \cdot \tilde{x}) = t_j^{q_j}(\tilde{x}) \chi_j(g, \tilde{x}), \qquad \tilde{x} \in \widetilde{W}_\gamma,\ g \in V^1_\gamma, \tag{28}$$

where $\chi_j(g, \tilde{x})$ is a function that is real-analytic in g and in $\tilde{x}$, and q_j is the lowest power of t_j that occurs in the expansion (26), so that

$$\chi_j(g, \tilde{x}) \neq 0 \qquad \forall\, \tilde{x} \in \widetilde{W}_\gamma,\ g \in V^1_\gamma. \tag{29}$$

Indeed, $\chi_j(g, \tilde{x})$ can only vanish if $t_j(\tilde{x}) = 0$. But if this were the case, q_j would not be the lowest power, and we obtain (28). Furthermore, since $t_j(g \cdot \tilde{x}) = t_j(\tilde{x})$ for $g = e$, one has $q_1 = \dots = q_l$. Thus, for $\tilde{x} = \varphi_\gamma(x) \in \widetilde{W}_\gamma$, $x = (n,t)$, $g \in V^1_\gamma$, we have

$$\varphi^g_\gamma(x) = (n_1(g \cdot \tilde{x}), \dots, n_k(g \cdot \tilde{x}), t_1(\tilde{x}) \chi_1(g, \tilde{x}), \dots, t_l(\tilde{x}) \chi_l(g, \tilde{x})).$$

Note that similar formulas hold for $\tilde{x} \in \widetilde{U}_{m_w}$ and g sufficiently close to the identity. The following lemma describes the G-action on $\widetilde{\mathbb{X}}$ as far as the t-coordinates are concerned.

Lemma 4. *Let $X_{-\lambda,i}$ and H_j the basis elements for $\mathfrak{n}^-$ and $\mathfrak{a}$ introduced in Section 2, $w \in W$, and $\tilde{x} \in \widetilde{U}_{m_w}$. Then, for small $s \in \mathbb{R}$,*

$$\chi_j(\mathrm{e}^{sH_i}, \tilde{x}) = e^{-c_{ij}(m_w)s},$$

where the $c_{ij}(m_w)$ represent the matrix coefficients of the adjoint representation of M^* on $\mathfrak{a}$, and are given by $\mathrm{Ad}\,(m_w^{-1}) H_i = \sum_{j=1}^l c_{ij}(m_w) H_j$. Furthermore, when $\tilde{x} = \pi(e, n, t)$,

$$\chi_j(\mathrm{e}^{sX_{-\lambda,i}}, \tilde{x}) \equiv 1.$$

Proof. Let $Y \in \mathfrak{g}$. As we saw in the proof of Lemma 1, the action of the one-parameter group $\exp(sY)$ on the homogeneous space $G/P_\Theta(K)$ is given by equation (6), where $N_3^-(s) \in \mathfrak{n}^-, A_1(s) \in \mathfrak{a}, A_2(s) \in \mathfrak{a}(\Theta)$. Denote the derivatives of $N_3^-(s)$, $A_1(s)$, and $A_2(s)$ at $s = 0$ by N_3^-, A_1, and A_2 respectively. The analyticity of the G-action implies that $N_3^-(s), A_1(s), A_2(s)$ are real-analytic functions in s. Furthermore, from (6) it is clear that $N_3^-(0) = 0$, $A_1(0) + A_2(0) = 0$, so that for small s we have

$$A_1(s) + A_2(s) = (A_1 + A_2)\, s + \frac{1}{2}\frac{d^2}{ds^2}(A_1(s) + A_2(s))|_{s=0}\, s^2 + \dots$$

$$N_3^-(s) = N_3^-\, s + \frac{1}{2}\frac{d^2}{ds^2}N_3^-(s)|_{s=0}\, s^2 + \dots .$$

Next, fix $m_w \in M^*$ and let $\Theta = \Delta$. The action of the one-parameter group corresponding to H_i at $\tilde{x} = \pi(m_w, n, t) \in \widetilde{U}_{m_w} \cap \widetilde{\mathbb{X}}_\Delta$ is given by

$$\exp(sH_i)m_w naK = m_w \left(m_w^{-1} \exp(sH_i)m_w\right) naK = m_w \exp(s\mathrm{Ad}\,(m_w^{-1})H_i)naK.$$

As m_w lies in M^*, $\exp(s\mathrm{Ad}\,(m_w^{-1})H_i)$ lies in A. Since A normalizes N^-, we conclude that $\exp(s\mathrm{Ad}\,(m_w^{-1})H_i)n \exp(-s\mathrm{Ad}\,(m_w^{-1})H_i)$ belongs to N^-. Writing

$$n^{-1} \exp(s\mathrm{Ad}\,(m_w^{-1})H_i)n \exp(-s\mathrm{Ad}\,(m_w^{-1})H_i) = \exp N_3^-(s)$$

we get

$$\exp(sH_i)m_w naK = m_w n \exp N_3^-(s) a \exp(s\mathrm{Ad}\,(m_w^{-1})H_i)K.$$

In the notation of (6) we therefore obtain $A_1(s)+A_2(s) = s\mathrm{Ad}\,(m_w^{-1})H_i$, and by writing $\mathrm{Ad}\,(m_w^{-1})H_i = \sum_{j=1}^l c_{ij}(m_w)H_j$ we arrive at

$$a \exp(A_1(s) + A_2(s)) = \exp\Big(\sum_{j=1}^{l}(c_{ij}(m_w)s - \log t_j)H_j\Big).$$

In terms of the coordinates this shows that $t_j(\exp(sH_i) \cdot \tilde{x}) = t_j(\tilde{x})e^{-c_{ij}(m_w)s}$ for $\tilde{x} \in \widetilde{U}_{m_w} \cap \widetilde{\mathbb{X}}_\Delta$, and by analyticity we obtain that $\chi_j(\,\mathrm{e}^{sH_i}\,, \tilde{x}) = e^{-c_{ij}(m_w)s}$ for arbitrary $\tilde{x} \in \widetilde{U}_{m_w}$. On the other hand, let $Y = X_{-\lambda,i}$, and $\tilde{x} = \varphi_e(n, t) \in \widetilde{U}_e \cap \widetilde{\mathbb{X}}_\Delta$. Then the action corresponding to $X_{-\lambda,i}$ at $\tilde{x}$ is given by

$$\exp(sX_{-\lambda,i})naK = n \exp N_3^-(s)aK,$$

where we wrote $\exp N_3^-(s) = s\mathrm{Ad}\,(n^{-1}) \exp X_{-\lambda,i}$. In terms of the coordinates this implies that $t_j(\exp(sX_{-\lambda,i}) \cdot \tilde{x}) = t_j(\tilde{x})$ showing that $\chi_j(\,\mathrm{e}^{sX_{-\lambda,i}}\,, \tilde{x}) \equiv 1$ for $\tilde{x} \in \widetilde{U}_e \cap \widetilde{\mathbb{X}}_\Delta$, and, by analyticity, for general $\tilde{x} \in \widetilde{U}_e$, finishing the proof of the lemma. □

Let now $x = (n, t) \in W_\gamma$, and define the matrix

$$T_x = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_l \end{pmatrix}, \tag{30}$$

so that for $\tilde{x} = \varphi_\gamma(x) \in \widetilde{W}_\gamma,\ g \in V_\gamma^1$,

$$(\mathbf{1}_k \otimes T_x^{-1})(\varphi_\gamma^g(x)) = (x_1(g \cdot \tilde{x}), \dots, x_k(g \cdot \tilde{x}), \chi_1(g, \tilde{x}), \dots, \chi_l(g, \tilde{x})),$$

and set

$$\psi_{\xi,x}^\gamma(g) = e^{i(\mathbf{1}_k \otimes T_x^{-1})(\varphi_\gamma^g(x))\cdot\xi},$$

where $\xi = (\xi_1, \dots, \xi_{k+l}) \in \mathbb{R}^{k+l}$. Also, introduce the auxiliary symbol

$$\tilde{a}^\gamma_f(x, \xi) = a^\gamma_f(x, (\mathbf{1}_k \otimes T_x^{-1})\xi) = e^{-i(x_1,\dots,x_k,1,\dots,1).\xi} \int_G \psi^\gamma_{\xi,x}(g) c_\gamma(g) f(g) dg. \tag{31}$$

Clearly, $\tilde{a}^\gamma_f(x, \xi) \in \mathrm{C}^\infty(W_\gamma \times \mathbb{R}^{k+l})$. Our next goal is to show that $\tilde{a}^\gamma_f(x, \xi)$ is a lacunary symbol. To do so, we shall need the following

Proposition 2. *Let $(L, \mathrm{C}^\infty(G))$ be the left regular representation of G. Let $X_{-\lambda,i}, H_j$ be the basis elements of $\mathfrak{n}^-$ and $\mathfrak{a}$ introduced in Section 2, and $(\widetilde{W}_\gamma, \varphi_\gamma)$ an arbitrary chart. With $x = (n, t) \in W_\gamma$, $\tilde{x} = \varphi_\gamma(x) \in \widetilde{W}_\gamma$, $g \in V^1_\gamma$ one has*

$$\begin{pmatrix} dL(X_{-\lambda,1})\psi^\gamma_{\xi,x}(g) \\ \vdots \\ dL(H_l)\psi^\gamma_{\xi,x}(g) \end{pmatrix} = i\psi^\gamma_{\xi,x}(g)\Gamma(x, g)\xi, \tag{32}$$

with

$$\Gamma(x, g) = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{pmatrix} = \left(\begin{array}{c|c} dL(X_{-\lambda,i})n_{j,\tilde{x}}(g) & dL(X_{-\lambda,i})\chi_j(g, \tilde{x}) \\ \hline dL(H_i)n_{j,\tilde{x}}(g) & dL(H_i)\chi_j(g, \tilde{x}) \end{array}\right) \tag{33}$$

belonging to $\mathrm{GL}(l + k, \mathbb{R})$, where $n_{j,\tilde{x}}(g) = n_j(g \cdot \tilde{x})$.

Proof. Fix a chart $(\widetilde{W}_\gamma, \varphi_\gamma)$, and let x, $\tilde{x}$, g be as above. For $X \in \mathfrak{g}$, one computes that

$$\begin{aligned} dL(X)\psi^\gamma_{\xi,x}(g) = \frac{d}{ds} e^{i(\mathbf{1}_k \otimes T_t^{-1})\varphi_\gamma^{e^{-sX}\, g}(x)\cdot\xi}|_{s=0} = i\psi^\gamma_{\xi,x}(g)\Big[\sum_{i=1}^k \xi_i dL(X) n_{i,\tilde{x}}(g) \\ + \sum_{j=k+1}^{l+k} \xi_j dL(X)\chi_j(g, \tilde{x})\Big], \end{aligned}$$

showing the first equality. To see the invertibility of the matrix $\Gamma(x, g)$, note that for small s

$$\chi_j(\mathrm{e}^{-sX}\, g, \tilde{x}) = \chi_j(g, \tilde{x})\chi_j(\mathrm{e}^{-sX}\,, g \cdot \tilde{x}).$$

Lemma 4 then yields

$$dL(H_i)\chi_j(g, \tilde{x}) = \chi_j(g, \tilde{x})\frac{d}{ds}\Big(e^{c_{ij}(m_{w_\gamma})s}\Big)_{|s=0} = \chi_j(g, \tilde{x})c_{ij}(m_{w_\gamma}).$$

This means that Γ_4 is the product of the matrix $\big(c_{ij}(m_{w_\gamma})\big)_{i,j}$ with the diagonal matrix whose j-th diagonal entry is $\chi_j(g, \tilde{x})$. Since $\big(c_{ij}(m_{w_\gamma})\big)_{i,j}$ is just the matrix representation of $\mathrm{Ad}\,(m^{-1}_{w_\gamma})$ relative to the basis $\{H_1, \dots, H_l\}$ of $\mathfrak{a}$, it is invertible. On the other hand, $\chi_j(g, \tilde{x})$ is non-zero for all $j \in \{1, \dots, l\}$ and arbitrary g and $\tilde{x}$. Therefore Γ_4, being the product of two invertible matrices, is invertible. Next, let us show that the matrix Γ_1 is non-singular. Its $(ij)^{th}$ entry reads

$$dL(X_{-\lambda,i})n_{j,\tilde{x}}(g) = \frac{d}{ds} n_{j,\tilde{x}}(\mathrm{e}^{-sX_{-\lambda,i}} \cdot g)_{|s=0} = (-X_{-\lambda,i|\widetilde{\mathbb{X}}})_{g\cdot\tilde{x}}(n_j).$$

For $\Theta \subset \Delta$, $q \in \mathbb{R}^l$, we define the k-dimensional submanifolds

$$\mathfrak{L}_\Theta(q) = \{\tilde{x} = \varphi_\gamma(n, q) \in \widetilde{W}_\gamma : q_i \neq 0 \Leftrightarrow \alpha_i \in \Theta\},$$

and consider the decomposition $T_{g\cdot\tilde{x}}\widetilde{\mathbb{X}}_\Theta = T_{g\cdot\tilde{x}}\mathfrak{L}_\Theta(q) \oplus N_{g\cdot\tilde{x}}\mathfrak{L}_\Theta(q)$ of $T_{g\cdot\tilde{x}}\widetilde{\mathbb{X}}_\Theta$ into the tangent and normal space to $\mathfrak{L}_\Theta(q)$ at the point $g \cdot \tilde{x} \in \widetilde{\mathbb{X}}_\Theta$. Since $\widetilde{\mathbb{X}}_\Theta$ is a G-orbit, the group G acts transitively on it. Now, as g varies over G in Lemma 1, one deduces that $N^- \times A(\Theta)$ acts locally transitively on

$\widetilde{\mathbb{X}}_\Theta$. In addition, by the definition of $\mathfrak{L}_\Theta(q)$, $N_{g\cdot\tilde{x}}\mathfrak{L}_\Theta(q)$ is spanned by the vector fields $\{-t_i\frac{\partial}{\partial t_i}\}_{\alpha_i\in\Theta}$. Consequently, $T_{g\cdot\tilde{x}}\mathfrak{L}_\Theta(q)$ must be equal to the span of the vector fields $\{X_{-\lambda,i|\widetilde{\mathbb{X}}}\}$, which means that N^- acts locally transitively on $\mathfrak{L}_\Theta(q)$ for arbitrary Θ. Since the latter is parametrized by the coordinates $(n_1,\dots,n_k)$, one concludes that the matrix $((X_{-\lambda,i|\widetilde{\mathbb{X}}})_{g\cdot\tilde{x}}(n_j))_{ij}$ has full rank. Thus, Γ_1 is non-singular. On the other hand, if $\tilde{x}=\pi(e,n,t)\in\tilde{U}_e$, Lemma 4 implies

$$dL(X_{-\lambda,i})\chi_j(g,\tilde{x})=\chi_j(g,\tilde{x})\frac{d}{ds}\Big(\chi_j(e^{-sX_{-\lambda,i}},g\cdot\tilde{x})\Big)_{|s=0}=0,$$

showing that Γ_2 is identically zero, while Γ_4 is a non-singular diagonal matrix in this case. Geometrically, this amounts to the fact that the fundamental vector field corresponding to H_j is transversal to the hypersurface defined by $t_j=q\in\mathbb{R}\setminus\{0\}$, while the vector fields corresponding to the Lie algebra elements $X_{-\lambda,i},H_i$, $i\neq j$, are tangential. We therefore conclude that $\Gamma(x,g)$ is non-singular if $\tilde{x}\in\widetilde{U}_e$. But since the different copies $\widetilde{\mathbb{X}}_{\Theta_{(e,n,t)}}$ of $G/P_{\Theta_{(e,n,t)}}(K)\simeq N^-\times B_{(e,n,t)}\subset N^-\times\mathbb{R}^l\simeq\widetilde{U}_e$ in $\widetilde{\mathbb{X}}$ are isomorphic to each other, the same must hold if $\tilde{x}$ lies in one of the remaining charts $\widetilde{U}_{m_{w_\gamma}}$, and the assertion of the lemma follows.

□

We can now state the main result of this paper. In what follows, $\{(\widetilde{W}_\gamma,\varphi_\gamma)\}_{\gamma\in I}$ will always denote the atlas of $\widetilde{\mathbb{X}}$ constructed above.

Theorem 2. *Let $\widetilde{\mathbb{X}}$ be the Oshima compactification of a Riemannian symmetric space $\mathbb{X}\simeq G/K$ of non-compact type, and $f\in\mathcal{S}(G)$ a rapidly decaying function on G. Let further $\Big\{(\widetilde{W}_\gamma,\varphi_\gamma^{-1})\Big\}_{\gamma\in I}$ be the atlas of $\widetilde{\mathbb{X}}$ construced above. Then the operators $\pi(f)$ are locally of the form*

$$A_f^\gamma u(x)=\int e^{ix\cdot\xi}a_f^\gamma(x,\xi)\hat{u}(\xi)d\xi,\qquad u\in \mathrm{C_c^\infty}(W_\gamma),\tag{34}$$

where $a_f^\gamma(x,\xi)=\tilde{a}_f^\gamma(x,\xi_1,\dots,\xi_k,x_{k+1}\xi_{k+1},\dots,\xi_{k+l}x_{k+l})$, and $\tilde{a}_f^\gamma(x,\xi)\in\mathrm{S_{la}^{-\infty}}(W_\gamma\times\mathbb{R}^{k+l}_\xi)$ is given by (31). *In particular, the kernel of the operator A_f^γ is determined by its restrictions to $W_\gamma^*\times W_\gamma^*$, where $W_\gamma^*=\{x=(n,t)\in W_\gamma: t_1\cdots t_l\neq 0\}$, and given by the oscillatory integral*

$$K_{A_f^\gamma}(x,y)=\int e^{i(x-y)\cdot\xi}a_f^\gamma(x,\xi)d\xi.\tag{35}$$

As a consequence, we obtain the following

Corollary 1. *Let $\widetilde{\mathbb{X}}_\Delta$ be an open G-orbit in $\widetilde{\mathbb{X}}$ isomorphic to G/K. Then the continuous linear operators*

$$\pi(f)_{|\overline{\widetilde{\mathbb{X}}_\Delta}}:\mathrm{C_c^\infty}(\overline{\widetilde{\mathbb{X}}_\Delta})\longrightarrow \mathrm{C^\infty}(\overline{\widetilde{\mathbb{X}}_\Delta}),$$

are totally characteristic pseudodifferential operators of class $\mathrm{L}_b^{-\infty}$ on the manifolds with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$.

□

Proof of Theorem 2. Our considerations will essentially follow the proof of Theorem 4 in [9]. Let $\Gamma(x,g)$ be the matrix defined in (33), and consider its extension as an endomorphism in $\mathbb{C}^1[\mathbb{R}^{k+l}_\xi]$ to the symmetric algebra $\mathrm{S}(\mathbb{C}^1[\mathbb{R}^{k+l}_\xi])\simeq\mathbb{C}[\mathbb{R}^{k+l}_\xi]$. Since for $x\in W_\gamma$, $g\in V^1_\gamma$, $\Gamma(x,g)$ is invertible, its extension to $\mathrm{S}^N(\mathbb{C}^1[\mathbb{R}^{k+l}_\xi])$ is also an automorphism for any $N\in\mathbb{N}$. Regarding the polynomials $\xi_1,\dots,\xi_{k+l}$ as a basis in $\mathbb{C}^1[\mathbb{R}^{k+l}_\xi]$, let us denote the image of the basis vector ξ_j under the endomorphism $\Gamma(x,g)$ by $\Gamma\xi_j$, so that by (32)

$$\Gamma\xi_j = -i\psi^\gamma_{-\xi,x}(g) dL(X_{-\lambda,j})\psi^\gamma_{\xi,x}(g), \qquad 1 \leq j \leq k,$$
$$\Gamma\xi_j = -i\psi^\gamma_{-\xi,x}(g) dL(H_j)\psi^\gamma_{\xi,x}(g), \qquad k+1 \leq j \leq k+l.$$

Every polynomial $\xi_{j_1} \otimes \cdots \otimes \xi_{j_N} \equiv \xi_{j_1} \dots \xi_{j_N}$ can then be written as a linear combination

$$\xi^\alpha = \sum_\beta \Lambda^\alpha_\beta(x,g)\Gamma\xi_{\beta_1}\cdots\Gamma\xi_{\beta_{|\alpha|}}, \tag{36}$$

where the $\Lambda^\alpha_\beta(x,g)$ are real-analytic functions on $W_\gamma \times V^1_\gamma$. We need now the following

Lemma 5. *For arbitrary indices $\beta_1, \dots, \beta_r$, one has*

$$\begin{aligned} i^r\psi^\gamma_{\xi,x}(g)\Gamma\xi_{\beta_1}\cdots\Gamma\xi_{\beta_r} &= dL(X_{\beta_1}\cdots X_{\beta_r})\psi^\gamma_{\xi,x}(g) \\ &+ \sum_{s=1}^{r-1}\sum_{\alpha_1,\dots,\alpha_s} d^{\beta_1,\dots,\beta_r}_{\alpha_1,\dots,\alpha_s}(x,g) dL(X_{\alpha_1}\cdots X_{\alpha_s})\psi^\gamma_{\xi,x}(g), \end{aligned} \tag{37}$$

where the coefficients $d^{\beta_1,\dots,\beta_r}_{\alpha_1,\dots,\alpha_s}(x,g) \in \mathrm{C}^\infty(\tilde W_\gamma \times \operatorname{supp} c_\gamma)$ are at most of exponential growth in g, and independent of ξ.

Proof. The lemma is proved by induction. For $r = 1$ one has $i\psi^\gamma_{\xi,x}(g)\Gamma\xi_p = dL(X_p)\psi^\gamma_{\xi,x}(g)$, where $1 \leq p \leq d$. Differentiating the latter equation with respect to X_j, and writing $\Gamma\xi_p = \sum_{s=1}^{k+l}\Gamma_{ps}(x,g)\,\xi_s$, we obtain with (36) the equality

$$-\psi^\gamma_{\xi,x}(g)\Gamma\xi_j\Gamma\xi_p = dL(X_jX_p)\psi^\gamma_{\xi,x}(g) - \sum_{s,r=1}^{k+l}(dL(X_j)\Gamma_{ps})(x,g)\Lambda^s_r(x,g)dL(X_r)\psi^\gamma_{\xi,x}(g).$$

Hence, the assertion of the lemma is correct for $r = 1, 2$. Now, assume that it holds for $r \leq N$. Setting $r = N$ in equation (37), and differentiating with respect to X_p, yields for the left hand side

$$i^{N+1}\psi^\gamma_{\xi,x}(g)\Gamma\xi_p\Gamma\xi_{\beta_1}\cdots\Gamma\xi_{\beta_N}$$
$$+i^N\psi^\gamma_{\xi,x}(g)\Big(\sum_{s,q=1}^{k+l}(dL(X_p)\Gamma_{\beta_1 s})(x,g)\Lambda^s_q(x,g)\Gamma\xi_q\Big)\Gamma\xi_{\beta_2}\cdots\Gamma\xi_{\beta_N} + \dots .$$

By assumption, we can apply (37) to the products $\Gamma\xi_q\Gamma\xi_{\beta_2}\cdots\Gamma\xi_{\beta_N}, \dots$ of at most N factors. Since the functions $n_{i,m}(g)$ and $\chi_j(g,m)$, and consequently the coefficients of $\Gamma(x,g)$, are at most of exponential growth in g, the assertion of the lemma follows. ☐

End of proof of Theorem 2. Let us next show that $\tilde a^\gamma_f(x,\xi) \in \mathrm{S}^{-\infty}(W_\gamma \times \mathbb{R}^{k+l}_\xi)$. As already noted, $\tilde a^\gamma_f(x,\xi) \in \mathrm{C}^\infty(W_\gamma \times \mathbb{R}^{k+l}_\xi)$. While differentiation with respect to ξ does not alter the growth properties of $\tilde a^\gamma_f(x,\xi)$, differentiation with respect to x yields additional powers in ξ. Now, as an immediate consequence of equations (36) and (37), one computes for arbitrary $N \in \mathbb{N}$

$$\psi^\gamma_{\xi,x}(g)(1+\xi^2)^N = \sum_{r=0}^{2N}\sum_{|\alpha|=r} b^N_\alpha(x,g)dL(X^\alpha)\psi^\gamma_{\xi,x}(g), \tag{38}$$

where the coefficients $b^N_\alpha(x,g) \in \mathrm{C}^\infty(W_\gamma \times V^1_\gamma)$ are at most of exponential growth in g. Now, $(\partial^\alpha_\xi\,\partial^\beta_x\,\tilde a^\gamma_f)(x,\xi)$ is a finite sum of terms of the form

$$\xi^\delta e^{-i(x_1,\dots x_k,1,\dots,1)\cdot\xi}\int_G f(g)d_{\delta\beta}(x,g)\psi^\gamma_{\xi,x}(g)c_\gamma(g)dg,$$

the functions $d_{\delta\beta}(x,g) \in \mathrm{C}^\infty(W_\gamma \times V^1_\gamma)$ being at most of exponential growth in g. Making use of equation (38), and integrating according to Proposition 1, we finally obtain for arbitrary α, β the estimate

$$|(\partial^\alpha_\xi \, \partial^\beta_x \, \tilde{a}^\gamma_f)(x,\xi)| \leq \frac{1}{(1+\xi^2)^N} C_{\alpha,\beta,\mathcal{K}} \qquad x \in \mathcal{K},$$

where $\mathcal{K}$ denotes an arbitrary compact set in W_γ, and $N \in \mathbb{N}$. This proves that $\tilde{a}^\gamma_f(x,\xi) \in \mathrm{S}^{-\infty}(W_\gamma \times \mathbb{R}^{k+l}_\xi)$. Since equation (34) is an immediate consequence of Fourier inversion formula, it remains to show that $\tilde{a}^\gamma_f(x,\xi)$ satisfies the lacunary condition (22) for each of the coordinates t_i. Now, it is clear that $a^\gamma_f \in \mathrm{S}^{-\infty}(W^*_\gamma \times \mathbb{R}^{k+l}_\xi)$, since G acts transitively on each $\widetilde{\mathbb{X}}_\Delta$. As a consequence, the Schwartz kernel of the restriction of the operator $A^\gamma_f : \mathrm{C}^\infty_{\mathrm{c}}(W_\gamma) \to \mathrm{C}^\infty(W_\gamma)$ to W^*_γ is given by the absolutely convergent integral

$$\int e^{i(x-y)\cdot\xi} a^\gamma_f(x,\xi) đ\xi \in \mathrm{C}^\infty(W^*_\gamma \times W^*_\gamma).$$

Next, let us write $W_\gamma = \bigcup_{\Theta\subset\Delta} W^\Theta_\gamma$, where $W^\Theta_\gamma = \{x = (n,t) : t_i \neq 0 \Leftrightarrow \alpha_i \in \Theta\}$. Since on W^Θ_γ the function $A^\gamma_f u$ depends only on the restriction of $u \in \mathrm{C}^\infty_{\mathrm{c}}(W_\gamma)$ to W^Θ_γ, one deduces that

$$\operatorname{supp} K_{A^\gamma_f} \subset \bigcup_{\Theta\subset\Delta} \overline{W^\Theta_\gamma} \times \overline{W^\Theta_\gamma}. \tag{39}$$

Therefore, each of the integrals

$$\int e^{i(x_j-y_j)\xi_j} \tilde{a}^\gamma_f(x, (\mathbf{1}_k \otimes T_x)\xi) \, d\xi_j, \qquad j = k+1, \dots, k+l,$$

which are smooth functions on $W^*_\gamma \times W^*_\gamma$, must vanish if x_j and y_j do not have the same sign. With the substitution $r_j = y_j/x_j - 1$, $\xi_j x_j = \xi'_j$ one finally arrives at the conditions

$$\int e^{-ir_j\xi_j} \tilde{a}^\gamma_f(x,\xi) \, d\xi_j = 0 \qquad \text{for } r_j < -1, \, x \in W^*_\gamma.$$

But since $\tilde{a}^\gamma_f$ is rapidly decreasing in ξ, the Lebesgue bounded convergence theorem implies that these conditions must also hold for $x \in W_\gamma$. Thus, the lacunarity of the symbol $\tilde{a}^\gamma_f$ follows. The fact that the kernel $K_{A^\gamma_f}$ must be determined by its restriction to $W^*_\gamma \times W^*_\gamma$, and hence by the oscillatory integral (35), is now a consequence of [7], Lemma 4.1, completing the proof of Theorem 2. ∎

As a consequence of Theorem 2, we can locally write the kernel of $\pi(f)$ in the form

$$\begin{aligned} K_{A^\gamma_f}(x,y) &= \int e^{i(x-y)\cdot\xi} a^\gamma_f(x,\xi) đ\xi = \int e^{i(x-y)\cdot(\mathbf{1}_k\otimes T_x^{-1})\xi} \tilde{a}^\gamma_f(x,\xi) |\det{(\mathbf{1}_k \otimes T_x^{-1})'(\xi)}| đ\xi \\ &= \frac{1}{|x_{k+1}\cdots x_{k+l}|} \tilde{A}^\gamma_f(x, x_1 - y_1, \dots, 1 - \frac{y_{k+1}}{x_{k+1}}, \dots), \qquad x_{k+1}\cdots x_{k+l} \neq 0, \end{aligned} \tag{40}$$

where $\tilde{A}^\gamma_f(x,y)$ denotes the inverse Fourier transform of $\tilde{a}^\gamma_f(x,\xi)$,

$$\tilde{A}^\gamma_f(x,y) = \int e^{iy\cdot\xi} \tilde{a}^\gamma_f(x,\xi) \, đ\xi. \tag{41}$$

Since for $x \in W^\gamma$ the amplitude $\tilde{a}^\gamma_f(x,\xi)$ is rapidly falling in ξ, it follows that $\tilde{A}^\gamma_f(x,y) \in \mathcal{S}(\mathbb{R}^n_y)$, the Fourier transform being an isomorphism on the Schwartz space. Therefore $K_{A^\gamma_f}(x,y)$ is rapidly decreasing as $|x_j| \to 0$ if $x_j \neq y_j$ and $k+1 \leq j \leq k+l$. Furthermore, by the lacunarity of $\tilde{a}^\gamma_f$, $K_{A^\gamma_f}(x,y)$ is also rapidly decaying as $|y_j| \to 0$ if $x_j \neq y_j$ and $k+1 \leq j \leq k+l$.

5. Holomorphic semigroup and resolvent kernels

In this section, we shall study the holomorphic semigroup generated by a strongly elliptic operator Ω associated to the regular representation $(\pi, \mathrm{C}(\widetilde{\mathbb{X}}))$ of G, as well as its resolvent. Both the holomorphic semigroup and the resolvent can be characterized as convolution operators of the type considered before, so that we can study them by the methods developed in the previous section. In particular, this will allow us to obtain a description of the asymptotic behavior of the semigroup and resolvent kernels on $\widetilde{\mathbb{X}}_\Delta \simeq \mathbb{X}$ at infinity.

Let us begin by recalling some basic facts about elliptic operators and parabolic evolution equations on Lie groups, our main reference being [10]. Let $\mathcal{G}$ be a Lie group, and π a continuous representation of $\mathcal{G}$ on a Banach space $\mathcal{B}$. Let further $X_1, \dots, X_d$ be a basis of the Lie algebra $\mathrm{Lie}(\mathcal{G})$ of $\mathcal{G}$, and

$$\Omega = \sum_{|\alpha| \leq q} c_\alpha \, d\pi(X^\alpha)$$

a *strongly elliptic differential operator of order* q associated with π, meaning that for all $\xi \in \mathbb{R}^d$ one has the inequality $\mathrm{Re}\,(-1)^{q/2} \sum_{|\alpha|=q} c_\alpha \xi^\alpha \geq \kappa |\xi|^q$ for some $\kappa > 0$. By the general theory of strongly continuous semigroups, its closure generates a strongly continuous holomorphic semigroup of bounded operators given by

$$S_\tau = \frac{1}{2\pi i} \int_\Gamma e^{\lambda \tau} (\lambda \mathbf{1} + \overline{\Omega})^{-1} d\lambda,$$

where Γ is a appropriate path in $\mathbb{C}$ coming from infinity and going to infinity such that $\lambda \notin \sigma(\overline{\Omega})$ for $\lambda \in \Gamma$. Here $|\arg \tau| < \alpha$ for an appropriate $\alpha \in (0, \pi/2]$, and the integral converges uniformly with respect to the operator norm. Furthermore, the subgroup S_τ can be characterized by a convolution semigroup of complex measures μ_τ on $\mathcal{G}$ according to

$$S_\tau = \int_{\mathcal{G}} \pi(g) d\mu_\tau(g),$$

π being measurable with respect to the measures μ_τ. The measures μ_τ are absolutely continuous with respect to Haar measure $d_{\mathcal{G}}$ on $\mathcal{G}$, and denoting by $K_\tau(g) \in L^1(\mathcal{G}, d_{\mathcal{G}})$ the corresponding Radon-Nikodym derivative, one has

$$S_\tau = \pi(K_\tau) = \int_{\mathcal{G}} K_\tau(g) \pi(g) d_{\mathcal{G}}(g).$$

The function $K_\tau(g) \in \mathrm{L}^1(\mathcal{G}, d_{\mathcal{G}})$ is analytic in τ and g, and universal for all Banach representations. It satisfies the parabolic differential equation

$$\frac{\partial K_\tau}{\partial \tau}(g) + \sum_{|\alpha| \leq q} c_\alpha \, dL(X^\alpha) K_\tau(g) = 0, \qquad \lim_{\tau \to 0} K_\tau(g) = \delta(g),$$

where $(L, \mathrm{C}^\infty(\mathcal{G}))$ denotes the left regular representation of $\mathcal{G}$. As a consequence, K_τ must be supported on the identity component $\mathcal{G}_0$ of $\mathcal{G}$. It is called the *Langlands kernel* of the holomorphic semigroup S_τ, and satisfies the following L^1- and L^∞-bounds.

Theorem 3. *For each $\kappa \geq 0$, there exist constants $a, b, c > 0$, and $\omega \geq 0$ such that*

$$\int_{\mathcal{G}_0} |dL(X^\alpha)\, \partial_\tau^\beta \, K_\tau(g)| e^{\kappa |g|} \, d_{\mathcal{G}_0}(g) \leq a b^{|\alpha|} c^\beta |\alpha|! \, \beta! (1 + \tau^{-\beta - |\alpha|/q}) e^{\omega \tau}, \tag{42}$$

for all $\tau > 0$, $\beta = 0, 1, 2, \dots$ and multi-indices α. Furthermore,

$$|dL(X^\alpha)\, \partial_\tau^\beta \, K_\tau(g)| \leq a b^{|\alpha|} c^\beta |\alpha|! \, \beta! (1 + \tau^{-\beta - (|\alpha| + d + 1)/q}) e^{\omega \tau} e^{-\kappa |g|}, \tag{43}$$

for all $g \in \mathcal{G}_0$, where $d = \dim \mathcal{G}_0$, and q denotes the order of Ω.

A detailed exposition of these facts can be found in [10], pages 30, 152, 166, and 167. Let now $\mathcal{G} = G$, and $(\pi, \mathcal{B})$ be the regular representation of G on $C(\widetilde{\mathbb{X}})$. Theorem 3 implies that the Langlands kernel K_τ belongs to the space $\mathcal{S}(G)$ of rapidly falling functions on G. As a consequence of the previous considerations we obtain

Theorem 4. *Let Ω be a strongly elliptic differential operator of order q associated with the regular representation $(\pi, C(\widetilde{\mathbb{X}}))$, and $S_\tau = \pi(K_\tau)$ the holomorphic semigroup of bounded operators generated by $\overline{\Omega}$. Then the operators S_τ are locally of the form (34) with f being replaced by K_τ, and totally characteristic pseudodifferential operators of class $\mathrm{L}_b^{-\infty}$ on the manifolds with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$. Furthermore, on $W_\gamma \times W_\gamma$, the kernel of S_τ is given by*

$$S_\tau^\gamma(x, y) = K_{A^\gamma_{K_\tau}}(x, y) = \int e^{i(x-y)\cdot\xi} a^\gamma_{K_\tau}(x, \xi) d\xi = \frac{1}{|x_{k+1} \cdots x_{k+l}|} \tilde{A}^\gamma_{K_\tau}(x, (\mathbf{1}_k \otimes T_x^{-1})(x-y)),$$

where $x_{k+1} \cdots x_{k+l} \neq 0$, and $\tilde{A}^\gamma_{K_\tau}(x, y)$ was defined in (41). In particular, $S_\tau^\gamma(x, y)$ is rapidly falling at infinity as $|x_j| \to 0$, or $|y_j| \to 0$, as long as $x_j \neq y_j$, where $k+1 \leq j \leq k+l$. In addition,

$$|\tilde{A}^\gamma_{K_\tau}(x, y)| \leq \begin{cases} c_1(1 + \tau^{-(l+k+1)/q}), & 0 < \tau \leq 1, \\ c_2 e^{\omega\tau}, & 1 < \tau, \end{cases} \tag{44}$$

uniformly on compact subsets of $W_\gamma \times W_\gamma$ for some constants $c_i > 0$.

Proof. The first assertions are immediate consequences of Theorem 2, and its corollary. In order to prove (44), note that for large $N \in \mathbb{N}$ one computes with (31), (38), and (41)

$$\begin{aligned} |\tilde{A}^\gamma_{K_\tau}(x, y)| &\leq \int_{\mathbb{R}^{k+l}} |\tilde{a}^\gamma_{K_\tau}(x, \xi)| \, d\!\!{}^-\xi = \int_{\mathbb{R}^{k+l}} \Big| \int_G \psi^\gamma_{\xi,x}(g) c_\gamma(g) K_\tau(g) d_G(g) \Big| d\!\!{}^-\xi \\ &= \int_{\mathbb{R}^{k+l}} (1 + |\xi|^2)^{-N} \Big| \int_G c_\gamma(g) K_\tau(g) \sum_{r=0}^{2N} \sum_{|\alpha|=r} b^N_\alpha(x, g) dL(X^\alpha) \psi^\gamma_{\xi,x}(g) d_G(g) \Big| d\!\!{}^-\xi. \end{aligned}$$

If we now apply Proposition 1, and take into account the estimate (42) we obtain

$$\begin{aligned} |\tilde{A}^\gamma_{K_\tau}(x, y)| &\leq \int (1 + |\xi|^2)^{-N} \Big| \int_G \psi^\gamma_{\xi,x}(g) \sum_{r=0}^{2N} \sum_{|\alpha|=r} dL(X^{\tilde{\alpha}}) [b^N_\alpha(x, g) c_\gamma(g) K_\tau(g)] d_G(g) \Big| d\!\!{}^-\xi \\ &\leq \begin{cases} c_1(1 + \tau^{-2N/q}), & 0 < \tau \leq 1, \\ c_2 e^{\omega\tau}, & 1 < \tau, \end{cases} \end{aligned}$$

for certain constants $c_i > 0$. Expressing $\xi_j^{k+l+1} \psi^\gamma_{\xi,x}(g)$ on $\{\xi \in \mathbb{R}^n : |\xi_i| \leq |\xi_j| \text{ for all } i\}$ as left derivatives of $\psi^\gamma_{\xi,x}(g)$ according to (36) and (37), and estimating the maximum norm on $\mathbb{R}^n$ by the usual norm, a similar argument shows that the last estimate is also valid for $N = (k + l + 1)/2$, compare (50). The proof is now complete. □

Let us now turn to the resolvent of the closure of the strongly elliptic operator Ω. By (42) one has the bound $\|S_\tau\| \leq c e^{\omega\tau}$ for some constants $c \geq 1, \omega \geq 0$. For $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \omega$, the resolvent of $\overline{\Omega}$ can then be expressed by means of the Laplace transform according to

$$(\lambda \mathbf{1} + \overline{\Omega})^{-1} = \Gamma(1)^{-1} \int_0^\infty e^{-\lambda\tau} S_\tau \, d\tau,$$

where Γ is the Γ-function. More generally, one can consider for arbitrary $\alpha > 0$ the integral transforms

$$(\lambda \mathbf{1} + \overline{\Omega})^{-\alpha} = \Gamma(\alpha)^{-1} \int_0^\infty e^{-\lambda\tau} \tau^{\alpha-1} S_\tau \, d\tau.$$

As it turns out, the functions

$$R_{\alpha,\lambda}(g) = \Gamma(\alpha)^{-1} \int_0^\infty e^{-\lambda\tau} \tau^{\alpha-1} K_\tau(g)\, d\tau$$

are in $\mathrm{L}^1(G, e^{\kappa|g|} d_G)$, where $\kappa \geq 0$ is such that $\|\pi(g)\| \leq c e^{\kappa|g|}$ for some $c \geq 1$. This implies that the resolvent of $\overline{\Omega}$ can be expressed as the convolution operator

$$(\lambda \mathbf{1} + \overline{\Omega})^{-\alpha} = \pi(R_{\alpha,\lambda}) = \int_G R_{\alpha,\lambda}(g) \pi(g)\, d_G(g).$$

The resolvent kernels $R_{\alpha,\lambda}$ decrease exponentially as $|g| \to \infty$, but they are singular at the identity if $d \geq q\alpha$. More precisely, one has the following

Theorem 5. *There exist constants $b, c, \lambda_0 > 0$, and $a_{\alpha,\lambda} > 0$, such that*

$$|dL(X^\delta) R_{\alpha,\lambda}(g)| \leq \begin{cases} a_{\alpha,\lambda} |g|^{-(d+|\delta|-q\alpha)} e^{-(b(\operatorname{Re}\lambda)^{1/q}-c)|g|}, & d > q\alpha, \\ a_{\alpha,\lambda}(1 + |\log|g||) e^{-(b(\operatorname{Re}\lambda)^{1/q}-c)|g|}, & d = q\alpha, \\ a_{\alpha,\lambda} e^{-(b(\operatorname{Re}\lambda)^{1/q}-c)|g|}, & d < q\alpha \end{cases}$$

for each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \lambda_0$.

A proof of these estimates is given in [10], pages 238 and 245. Our next aim is to understand the microlocal structure of the operators $\pi(R_{\alpha,\lambda})$ on the Oshima compactification $\widetilde{\mathbb{X}}$ of $\mathbb{X} \simeq G/K$. Consider again the atlas $\left\{(\widetilde{W}_\gamma, \varphi_\gamma^{-1})\right\}_{\gamma \in I}$ of $\widetilde{\mathbb{X}}$ introduced in Section 4, and the local operators

$$A^\gamma_{R_{\alpha,\lambda}} u = [\pi(R_{\alpha,\lambda})_{|\widetilde{W}_\gamma} (u \circ \varphi_\gamma^{-1})] \circ \varphi_\gamma, \tag{45}$$

where $u \in \mathrm{C}^\infty_\mathrm{c}(W_\gamma)$ and $W_\gamma = \varphi_\gamma^{-1}(\widetilde{W}_\gamma)$. By the Fourier inversion formula, $A^\gamma_{R_{\alpha,\lambda}}$ is given by the absolutely convergent integral

$$A^\gamma_{R_{\alpha,\lambda}} u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} a^\gamma_{R_{\alpha,\lambda}}(x,\xi) \hat{u}(\xi) đ\xi, \tag{46}$$

where

$$a^\gamma_{R_{\alpha,\lambda}}(x,\xi) = \int_G e^{i(\varphi^g_\gamma(x)-x)\cdot\xi} c_\gamma(g) R_{\alpha,\lambda}(g) d_G(g),$$
$$\tilde{a}^\gamma_{R_{\alpha,\lambda}}(x,\xi) = \int_G e^{i[(\mathbf{1}_k \otimes T_x^{-1})(\varphi^g_\gamma(x)-x)]\cdot\xi} c_\gamma(g) R_{\alpha,\lambda}(g) d_G(g)$$

are smooth functions on $W_\gamma \times \mathbb{R}^{k+l}$, since $R_{\alpha,\lambda} \in \mathrm{L}^1(G, e^{\kappa|g|} d_G)$, the notation being the same as in Section 4. Moreover, in view of the L^1-bound (42), the functions $e^{-\lambda\tau}\tau^{\alpha-1}\tilde{a}^\gamma_{K_\tau}(x,\xi)$ and $e^{-\lambda\tau}\tau^{\alpha-1} a^\gamma_{K_\tau}(x,\xi)$ are integrable in τ over $(0,\infty)$, and by Fubini we obtain the equalities

$$a^\gamma_{R_{\alpha,\lambda}}(x,\xi) = \Gamma(\alpha)^{-1} \int_0^\infty e^{-\lambda\tau}\tau^{\alpha-1} a^\gamma_{K_\tau}(x,\xi) d\tau,$$
$$\tilde{a}^\gamma_{R_{\alpha,\lambda}}(x,\xi) = \Gamma(\alpha)^{-1} \int_0^\infty e^{-\lambda\tau}\tau^{\alpha-1} \tilde{a}^\gamma_{K_\tau}(x,\xi) d\tau.$$

In what follows, we shall describe the microlocal structure of the resolvent $(\lambda\mathbf{1} + \overline{\Omega})^{-\alpha}$ on $\widetilde{\mathbb{X}}$, and in particular, its kernel.

Proposition 3. *Let Q be the largest integer such that $Q < q\alpha$. Then $\tilde{a}^{\gamma}_{R_{\alpha,\lambda}}(x,\xi) \in \mathrm{S}^{-Q}_{la}(W_\gamma \times \mathbb{R}^{k+l})$. That is, for any compactum $\mathcal{K} \subset W_\gamma$, and arbitrary multi-indices β, ε there exist constants $C_{\mathcal{K},\beta,\varepsilon} > 0$ such that*

$$|(\partial_x^\varepsilon \partial_\xi^\beta \tilde{a}^{\gamma}_{R_{\alpha,\lambda}})(x,\xi)| \leq C_{\mathcal{K},\beta,\varepsilon}(1+|\xi|^2)^{(-Q-|\beta|)/2}, \qquad x \in \mathcal{K},\ \xi \in \mathbb{R}^{k+l}, \tag{47}$$

and $\tilde{a}^{\gamma}_{R_{\alpha,\lambda}}$ satisfies the lacunary condition (22) *for each of the coordinates x_j, $k+1 \leq j \leq k+l$.*

Proof. For a fixed a chart chart $(\widetilde{W}_\gamma, \varphi_\gamma)$ of $\widetilde{\mathbb{X}}$ we write $x = (n,t) \in W_\gamma$, $\tilde{x} = \varphi_\gamma(x) \in \widetilde{W}_\gamma$ as usual. As a consequence of Proposition 2 and Lemma 5 one computes with (38) for arbitrary $N \in \mathbb{N}$

$$\begin{aligned}(\partial_\xi^{2\beta} \tilde{a}^{\gamma}_{R_{\alpha,\lambda}})(x,\xi) &= \int_G e^{i[(\mathbf{1}_k \otimes T_x^{-1})(\varphi^g_\gamma(x)-x)]\cdot\xi}[i(\mathbf{1}_k \otimes T_x^{-1})(\varphi^g_\gamma(x)-x)]^{2\beta} c_\gamma(g) R_{\alpha,\lambda}(g) d_G(g) \\ &= (1+|\xi|^2)^{-N} e^{-i(x_1,\dots,x_k,1,\dots,1)\cdot\xi} \sum_{r=0}^{2N} \sum_{|\delta|=r} \int_G b^N_\delta(x,g) dL(X^\delta) \psi^\gamma_{\xi,x}(g) \\ &\quad \cdot [i(\mathbf{1}_k \otimes T_x^{-1})(\varphi^g_\gamma(x)-x)]^{2\beta} c_\gamma(g) R_{\alpha,\lambda}(g) d_G(g).\end{aligned}$$

Now, $n_r(g\cdot\tilde{x}) \to n_r(\tilde{x})$ and $\chi_r(g,\tilde{x}) \to 1$ as $g \to e$, so that due to the analyticity of the G-action on $\widetilde{\mathbb{X}}$ one deduces

$$|(\mathbf{1}_k \otimes T_x^{-1})(\varphi^g_\gamma(x)-x)| = |(n_1(g\tilde{x})-n_1,\dots,\chi_1(g\tilde{x})-1,\dots)| = C_{\mathcal{K}}|g|, \qquad x \in \mathcal{K}. \tag{48}$$

Indeed, let

$$(\zeta_1,\dots,\zeta_d) \mapsto e^{\zeta_1 X_1 + \cdots + \zeta_d X_d} = g$$

be canonical coordinates of the first type near the identity $e \in G$. We then have the power expansions

$$\chi_r(g,\tilde{x}) - 1 = \sum_{\alpha,\beta,\gamma} c^r_{\alpha,\beta,\gamma} n^\alpha t^\beta \zeta^\gamma, \qquad n_r(g\cdot\tilde{x}) - n_r(\tilde{x}) = \sum_{\alpha,\beta,\gamma} d^r_{\alpha,\beta,\gamma} n^\alpha t^\beta \zeta^\gamma, \tag{49}$$

where the constant term vanishes, that is, $c^r_{\alpha,\beta,\gamma}$, $d^r_{\alpha,\beta,\gamma} = 0$ if $|\gamma| = 0$. Hence,

$$|n_r(g\cdot\tilde{x}) - n_r(\tilde{x})|,\ |\chi_r(g,\tilde{x}) - 1| \leq C_1|\zeta| \leq C_2|g|,$$

compare [10], pages 12-13, and we obtain (48). With Theorem 5, we therefore have the pointwise estimates

$$|[(\mathbf{1}_k \otimes T_x^{-1})(\varphi^g_\gamma(x)-x)]^{\beta'} dL(X^{\delta'}) R_{\alpha,\lambda}(g)| \leq C_{\mathcal{K},\alpha,\lambda}|g|^{-(d+|\delta'|-q\alpha-|\beta'|)} e^{-(b(\mathrm{Re}\,\lambda)^{1/q}-c)|g|}$$

for some constant $C_{\mathcal{K},\alpha,\lambda} > 0$ uniformly on $\mathcal{K} \times V^1_\gamma$. Now, let $2\tilde{Q}$ be the largest even number strictly smaller than $q\alpha$. Applying the same reasoning as in the proof of Proposition 1, one obtains for $N = \tilde{Q} + |\beta|$

$$\begin{aligned}(\partial_\xi^{2\beta} \tilde{a}^{\gamma}_{R_{\alpha,\lambda}})(x,\xi) &= (1+|\xi|^2)^{-\tilde{Q}-|\beta|} \sum_{r=0}^{2\tilde{Q}+2|\beta|} \sum_{|\delta|=r} (-1)^{|\delta|} \int_G e^{i[(\mathbf{1}_k \otimes T_x^{-1})(\varphi^g_\gamma(x)-x)]\cdot\xi} \\ &\quad \cdot dL(X^{\tilde{\delta}})\big[b^{\tilde{Q}+|\beta|}_\delta(x,g)[i(\mathbf{1}_k \otimes T_x^{-1})(\varphi^g_\gamma(x)-x)]^{2\beta} c_\gamma(g) R_{\alpha,\lambda}(g)\big] d_G(g),\end{aligned}$$

since all the occuring combinations $[(\mathbf{1}_k \otimes T_x^{-1})(\varphi^g_\gamma(x)-x)]^{\beta'} dL(X^{\delta'}) R_{\alpha,\lambda}(g)$ on the right hand side are such that $q\alpha + |\beta'| - |\delta'| > 0$, implying that the corresponding integrals over G converge. Equality then follows by the left-invariance of $d_G(g)$, and Lebesgue's Theorem on Dominated Convergence. To show the estimate (47) in general for $\varepsilon = 0$, let $x \in \mathcal{K}$, and $\xi \in \mathbb{R}^{k+l}$ be such that $|\xi| \geq 1$, and $|\xi|_{\max} = \max\{|\xi_r| : 1 \leq r \leq k+l\} = |\xi_j|$. Using (36) and (37) we can express

$\xi_j^{Q+|\beta|}\psi^\gamma_{\xi,x}(g)$ as left derivatives of $\psi^\gamma_{\xi,x}(g)$, and repeating the previous argument we obtain the estimate

$$\begin{aligned} |(\partial_\xi^\beta\, \tilde a^\gamma_{R_{\alpha,\lambda}})(x,\xi)| = |\xi_j|^{-Q-|\beta|}\Big| \sum_{r=0}^{Q+|\beta|}\sum_{|\delta|=r}\int_G b^j_\delta(x,g) dL(X^\delta)\psi^\gamma_{\xi,x}(g) \\ \cdot [i(\mathbf{1}_k\otimes T_x^{-1})(\varphi^g_\gamma(x)-x)]^\beta c_\gamma(g) R_{\alpha,\lambda}(g) d_G(g)\Big| \leq \tilde C_{\mathcal{K},\beta}\frac{1}{|\xi|^{Q+|\beta|}_{\max}} \leq C_{\mathcal{K},\beta}\frac{1}{|\xi|^{Q+|\beta|}}, \end{aligned} \tag{50}$$

where the coefficients $b^j_\delta(x,g)$ are at most of exponential growth in g. But since $\tilde a^\gamma_{R_{\alpha,\lambda}}(x,\xi)\in \mathrm{C}^\infty(W_\gamma\times\mathbb{R}^{k+l})$, we obtain (47) for $\varepsilon=0$. Let us now turn to the x-derivatives. We have to show that the powers in ξ that arise when differentiating $(\partial_\xi^\beta\, \tilde a^\gamma_{R_{\alpha,\lambda}})(x,\xi)$ with respect to x can be compensated by an argument similar to the previous considerations. Now, (49) clearly implies

$$\partial_x^\varepsilon\,(\chi_r(g,\tilde x)-1) = O(|g|), \qquad \partial_x^\varepsilon\,(n_r(g\cdot\tilde x)-n_r(\tilde x)) = O(|g|).$$

Thus, each time we differentiate the exponential $e^{i[(\mathbf{1}_k\otimes T_x^{-1})(\varphi^g_\gamma(x)-x)]\cdot\xi}$ with respect to x, the result is of order $O(|\xi||g|)$. Therefore, expressing the ocurring powers $\xi^{\varepsilon'}\psi^\gamma_{\xi,x}(g)$ as left derivatives of $\psi^\gamma_{\xi,x}(g)$, we can repeat the preceding argument to absorb the powers in ξ, and (47) follows. Note next that the previous argument also implies $a^\gamma_{R_{\alpha,\lambda}}(x,\xi)\in \mathrm{S}^{-Q}(W^*_\gamma\times\mathbb{R}^{k+l}_\xi)$, where $W^*_\gamma=\{x=(n,t)\in W_\gamma: t_1\cdots t_l\neq 0\}$, the G-action being transitive on each $\widetilde{\mathbb{X}}_\Delta$. The Schwartz kernel $K_{A^\gamma_{R_{\alpha,\lambda}}}$ of the restriction of the operator (45) to W^*_γ is therefore given by the oscillatory integral

$$\int e^{i(x-y)\cdot\xi} a^\gamma_{R_{\alpha,\lambda}}(x,\xi) đ\xi \in \mathcal{D}'(W^*_\gamma\times W^*_\gamma),$$

which is C^∞ off the diagonal. As in (39) we have $\operatorname{supp} K_{A^\gamma_{R_{\alpha,\lambda}}}\subset\bigcup_{\Theta\subset\Delta}\overline{W^\Theta_\gamma}\times\overline{W^\Theta_\gamma}$, so that each of the integrals

$$\int e^{i(x_j-y_j)\xi_j}\tilde a^\gamma_{R_{\alpha,\lambda}}(x,(\mathbf{1}_k\otimes T_x)\xi)\,d\xi_j, \qquad j=k+1,\dots,k+l,$$

must vanish if x_j and y_j do not have the same sign. Hence,

$$\int e^{-ir_j\xi_j}\tilde a^\gamma_{R_{\alpha,\lambda}}(x,\xi)\,d\xi_j = 0 \qquad \text{for } r_j<-1,\ x\in W^*_\gamma.$$

Since $\tilde a^\gamma_{R_{\alpha,\lambda}}(x,\xi)\in \mathrm{S}^{-Q}(W_\gamma\times\mathbb{R}^{k+l}_\xi)$, these integrals are absolutely convergent for $r_j\neq 0$. Lebesgue's Theorem on Bounded Convergence theorem then implies that these conditions must also hold for $x\in W_\gamma$. The proof of the proposition is now complete. □

Remark 1. One would actually expect that $\tilde a^\gamma_{R_{\alpha,\lambda}}(x,\xi)\in \mathrm{S}^{-q\alpha}_{la}(W_\gamma\times\mathbb{R}^{k+l})$, being the local symbol of the resolvent $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}$. Nevertheless, the general estimates of Theorem 5 for the resolvent kernels $R_{\alpha,\lambda}$, which correctly reflect the singular behavior at the identity, are not sufficient to show this, and more information about them is required. Indeed, $dL(X^\beta)R_{\alpha,\lambda}\in L_1(G,d_G(g))$ only holds if $0<q\alpha-|\beta|$.

We are now able to describe the microlocal structure of the resolvent $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}$.

Theorem 6. *Let Ω be a strongly elliptic differential operator of order q associated with the representation $(\pi, C(\widetilde{\mathbb{X}}))$ of G. Let $\omega\geq 0$ be given by Theorem 3, and $\lambda\in\mathbb{C}$ be such that $\operatorname{Re}\lambda>\omega$. Let further $\alpha>0$, and denote by Q the largest integer such that $Q<q\alpha$. Then $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}=\pi(R_{\alpha,\lambda})$ is locally of the form (46), where $a^\gamma_{R_{\alpha,\lambda}}(x,\xi)=\tilde a^\gamma_{R_{\alpha,\lambda}}(x,(\mathbf{1}_k\otimes T_x)\xi)$, and $\tilde a^\gamma_{R_{\alpha,\lambda}}(x,\xi)\in \mathrm{S}^{-Q}_{la}(W_\gamma\times\mathbb{R}^{k+l})$.*

In particular, $(\lambda \mathbf{1} + \overline{\Omega})^{-\alpha}$ *is a totally characteristic pseudodifferential operators of class* L_b^{-Q} *on the manifolds with corners* $\overline{\widetilde{\mathbb{X}}_\Delta}$. *Furthermore, its kernel is locally given by the oscillatory integral*

$$R^\gamma_{\alpha,\lambda}(x,y) = \int e^{i(x-y)\xi} a^\gamma_{R_{\alpha,\lambda}}(x,\xi) d\xi = \frac{1}{|x_{k+1}\cdots x_{k+l}|} \int e^{i(\mathbf{1}_k \otimes T_x^{-1})(x-y)\cdot\xi} a^\gamma_{R_{\alpha,\lambda}}(x,\xi) d\xi,$$

where $x_{k+1}\cdots x_{k+l} \neq 0$, $x, y \in W_\gamma$. $R^\gamma_{\alpha,\lambda}(x,y)$ *is smooth off the diagonal, and rapidly falling at infinity as* $|x_j| \to 0$, *or* $|y_j| \to 0$, *as long as* $x_j \neq y_j$, *where* $k+1 \leq j \leq k+l$.

Proof. The assertions of the theorem are direct consequences of our previous considerations, except for the behavior of $R^\gamma_{\alpha,\lambda}(x,y)$ at infinity. Let $k+1 \leq j \leq k+l$. While the behavior as $|y_j| \to 0$ is a direct consequence of the lacunarity of $\tilde{a}^\gamma_{R_{\alpha,\gamma}}$, the behavior as $|x_j| \to 0$ is a direct consequence of the fact that, as oscillatory integrals,

$$\int e^{i(x-y)\cdot\xi} a^\gamma_{R_{\alpha,\lambda}}(x,\xi) đ\xi = \frac{1}{|x-y|^{2N}} \int e^{i(x-y)\cdot\xi} (\partial^2_{\xi_1} + \cdots + \partial^2_{\xi_{k+l}})^N a^\gamma_{R_{\alpha,\lambda}}(x,\xi) đ\xi,$$

where $x \neq y$, and N is arbitrarily large. □

Remark 2. The singular behavior of $R_{\alpha,\lambda}(g)$ at the identity corresponds to the fact that, as a pseudodifferential operator of class L_b^{-Q}, $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}$ has a kernel which is singular at the diagonal.

To conclude, let us say some words about the classical heat kernel on a Riemannian symmetric space of non-compact type. Consider thus the regular representation $(\sigma, \mathrm{C}(\widetilde{\mathbb{X}}))$ of the solvable Lie group $S = AN^- \simeq \mathbb{X} \simeq G/K$ on the Oshima compactification $\widetilde{\mathbb{X}}$ of $\mathbb{X}$, and associate to every $f \in \mathcal{S}(S)$ the corresponding convolution operator

$$\int_S f(g)\sigma(g)\, d_S(g).$$

Its restriction to $\mathrm{C}^\infty(\widetilde{\mathbb{X}})$ induces again a continuous linear operator

$$\sigma(f) : \mathrm{C}^\infty(\widetilde{\mathbb{X}}) \longrightarrow \mathrm{C}^\infty(\widetilde{\mathbb{X}}) \subset \mathcal{D}'(\widetilde{\mathbb{X}}),$$

and an examination of the arguments in Section 4 shows that an analogous analysis applies to the operators $\sigma(f)$. In particular, Theorem 2 holds for them, too. Let ϱ be the half sum of all positiv roots, and

$$C = \sum_j H_j^2 - \sum_j Z_j^2 - \sum_j [X_j\theta(X_j) + \theta(X_j)X_j] \equiv \sum_j H_j^2 - 2\varrho + 2\sum X_j^2 \mod \mathfrak{U}(\mathfrak{g})\mathfrak{k}$$

be the Casimir operator in $\mathfrak{U}(\mathfrak{g})$, where $\{H_j\}$, $\{Z_j\}$, and $\{X_j\}$ are orthonormal basis of $\mathfrak{a}$, $\mathfrak{m}$, and $\mathfrak{n}^-$, respectively, and put $C' = \sum_j H_j^2 - 2\varrho + 2\sum X_j^2$. Though $-d\pi(C')$ is not a strongly elliptic operator in the sense defined above, $\Omega = -d\sigma(C')$ certainly is. Consequently, if $K'_\tau(g) \in \mathcal{S}(S)$ denotes the corresponding Langlands kernel, Theorems 4 and 6 yield descriptions of the Schwartz kernels of $\sigma(K'_\tau)$ and $(\lambda\mathbf{1} + \overline{\Omega})^{-\alpha}$ on $\widetilde{\mathbb{X}}$. On the other hand, denote by Δ the Laplace-Beltrami operator on $\mathbb{X}$. Then

$$\Delta\varphi(gK) = \varphi(g : C) = \varphi(g : C'), \qquad \varphi \in \mathrm{C}^\infty(\mathbb{X}),$$

and the associated heat kernel $h_\tau(g)$ on $\mathbb{X}$ coincides with the heat kernel on S associated to C'. But the latter is essentially given by the Langlands kernel $K'_\tau(g)$, being the solution of the parabolic equation

$$\frac{\partial K'_\tau}{\partial \tau}(g) - dL(C')K'_\tau(g) = 0, \qquad \lim_{\tau\to 0} K'_\tau(g) = \delta(g)$$

on S. In this particular case, optimal upper and lower bounds for h_τ and the Bessel-Green-Riesz kernels were given in [1] using spherical analysis under certain restrictions coming from the lack of control in the Trombi-Varadarajan expansion for spherical functions along the walls. Our asymptotics for the kernels of $\sigma(K'_\tau)$ and $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}$ on $\widetilde{\mathbb{X}}_\Delta \simeq \mathbb{X}$ are free of restrictions, and in concordance with those of [1], though, of course, less explicit. A detailed description of the resolvent of Δ on $\mathbb{X}$ was given in [5], [6].

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