INTEGRAL OPERATORS ON THE OSHIMA COMPACTIFICATION OF A RIEMANNIAN SYMMETRIC SPACE OF NON-COMPACT TYPE. REGULARIZED TRACES AND CHARACTERS

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ABSTRACT. Consider a Riemannian symmetric space $\mathbb{X} = G/K$ of non-compact type, where G denotes a connected, real, semi-simple Lie group with finite center, and K a maximal compact subgroup of G. Let $\widetilde{\mathbb{X}}$ be its Oshima compactification, and $(\pi, \mathrm{C}(\widetilde{\mathbb{X}}))$ the regular representation of G on $\widetilde{\mathbb{X}}$. In this paper, a regularized trace for the convolution operators $\pi(f)$ is defined, yielding a distribution on G which can be interpreted as global character of π. In case that f has compact support in a certain set of transversal elements, this distribution is a locally integrable function, and given by a fixed point formula analogous to the formula for the global character of an induced representation of G.

CONTENTS

1. INTRODUCTION

Let G be a connected, real, semi-simple Lie group with finite center, K a maximal compact subgroup, and G/K the corresponding Riemannian symmetric space which is assumed to be of non-compact type. In this paper, a distribution character for the regular representation of G on the Oshima compactification of G/K is introduced, and a corresponding character formula is proved. The paper is a continuation of [8], to which we shall refer in the following as Part I.

In his early work on infinite dimensional representations of semi-simple Lie groups, Harish–Chandra [6] realized that the correct generalization of the character of a finite-dimensional representation was a distribution on the group given by the trace of a convolution operator on representation space. This distribution character is given by a locally integrable function which is analytic on the set of regular elements, and satisfies character formulas analogous to the finite dimensional

case. Later, Atiyah and Bott [3] gave a similar description of the character of a parabolically induced representation in their work on Lefschetz fixed point formulae for elliptic complexes. More precisely, let H be a closed co-compact subgroup of G, and ϱ a representation of H on a finite dimensional vector space V. If $T(g) = (\iota_* \varrho)(g)$ is the representation of G induced by ϱ in the space of sections over G/H with values in the homogeneous vector bundle $G \times_H V$, then its distribution character is given by the distribution

$$\Theta_T : \mathrm{C}_\mathrm{c}^\infty(G) \ni f \longmapsto \mathrm{Tr}\ T(f), \qquad T(f) = \int_G f(g)T(g)d_G(g),$$

where d_G denotes a Haar measure on G. The point to be noted is that $T(f)$ is a smooth operator, and since G/H is compact, does have a well-defined trace. On the other hand, assume that $g \in G$ acts on G/H only with simple fixed points. In this case, a transversal trace $\mathrm{Tr}^\flat T(g)$ of $T(g)$ can be defined within the framework of pseudodifferential operators, which is given by a sum over fixed points of g. Atiyah and Bott then showed that, on an open set $G_T \subset G$,

$$\Theta_T(f) = \int_{G_T} f(g)\,\mathrm{Tr}^\flat T(g)d_G(g), \qquad f \in \mathrm{C}_\mathrm{c}^\infty(G_T).$$

This means that, on G_T, the character Θ_T of the induced representation T is represented by the locally integrable function $\mathrm{Tr}^\flat T(g)$, and its computation reduced to the evaluation of a sum over fixed points. When G is a p-adic reductive group defined over a non-Archimdean local field of characteristic zero, a similar analysis of the character of a parabolically induced representation was carried out in [5].

In this paper, we consider the regular representation π of G on the Oshima compactification $\widetilde{\mathbb{X}}$ of a Riemannian symmetric space $\mathbb{X} = G/K$ of non–compact type given by

$$\pi(g)\varphi(\tilde{x}) = \varphi(g^{-1} \cdot \tilde{x}), \qquad \varphi \in \mathrm{C}^\infty(\widetilde{\mathbb{X}}).$$

Since the G-action on $\widetilde{\mathbb{X}}$ is not transitive, the corresponding convolution operators $\pi(f)$, $f \in \mathrm{C}_\mathrm{c}^\infty(G)$, are not smooth, and therefore do not have a well-defined trace. Nevertheless, it was shown in Part I that they can be characterized as totally characteristic pseudodifferential operators of order $-\infty$. Using this fact, we are able to define a regularized trace $\mathrm{Tr}_{reg}\,\pi(f)$ for the operators $\pi(f)$, and in this way obtain a map

$$\Theta_\pi : \mathrm{C}_\mathrm{c}^\infty(G) \ni f \mapsto \mathrm{Tr}_{reg}(f) \in \mathbb{C},$$

which is shown to be a distribution on G. This distribution is defined to be the character of the representation π. We then show that, on a certain open set $G(\widetilde{\mathbb{X}})$ of transversal elements,

$$\mathrm{Tr}_{reg}\,\pi(f) = \int_{G(\widetilde{\mathbb{X}})} f(g)\,\mathrm{Tr}^\flat\,\pi(g)d_G(g), \qquad f \in \mathrm{C}_\mathrm{c}^\infty(G(\widetilde{\mathbb{X}})),$$

where, with the notation $\Phi_g(\tilde{x}) = g \cdot \tilde{x}$,

$$\mathrm{Tr}^\flat\,\pi(g) = \sum_{\tilde{x} \in \mathrm{Fix}(g)} \frac{1}{|\det\left(1 - d\Phi_{g^{-1}}(\tilde{x})\right)|},$$

the sum being over the (simple) fixed points of $g \in G(\widetilde{\mathbb{X}})$ on $\widetilde{\mathbb{X}}$. Thus, on the open set $G(\widetilde{\mathbb{X}})$, Θ_π is represented by the locally integrable function $\mathrm{Tr}^\flat\,\pi(g)$, which is given by a formula similar to the character of a parabolically induced representation. It is likely that similar distribution characters could be introduced for G-manifolds with a dense union of open orbits, or for spherical varieties, and that corresponding character formulae could be proved.

This paper is structured as follows. In Section 2, the main results of Part I that will be needed in the sequel are recalled. The regularized trace for the convolution operators $\pi(f)$ is defined

in Section 3, while the transversal trace of a pseudodifferential operator is introduced in Section 4, followed by a discussion of the global character of an induced representation. After studying G-actions on homogeneous spaces in Section 5, we prove that the distribution Θ_π is regular on the set of transversal elements $G(\widetilde{\mathbb{X}})$, and given by the locally integrable function $\mathrm{Tr}^\flat \pi(g)$. This is done in Section 6. In the last section, the Oshima compactification of $\mathbb{X} = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ is described in detail.

2. Preliminaries

In this section we shall briefly recall the main results of Part I relevant to our purposes. Let G be a connected, real, semi-simple Lie group with finite centre and Lie algebra $\mathfrak{g}$, and denote by $\langle X, Y \rangle = \mathrm{Tr}\,(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$ the *Cartan-Killing form* on $\mathfrak{g}$. Let θ be a Cartan involution on $\mathfrak{g}$, and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

the corresponding Cartan decomposition. Put $\langle X, Y \rangle_\theta := -\langle X, \theta Y \rangle$. Consider further a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Then $\mathrm{ad}\,(\mathfrak{a})$ constitutes a commuting family of self-adjoint operators on $\mathfrak{g}$ relative to $\langle,\rangle_\theta$, and one defines for each $\alpha \in \mathfrak{a}^*$ the simultaneous eigenspaces $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$. Let $\Sigma = \{\alpha \in \mathfrak{a}^* : \alpha \neq 0, \mathfrak{g}^\alpha \neq \{0\}\}$ be the set of roots of $(\mathfrak{g}, \mathfrak{a})$, $\Sigma^+ = \{\alpha \in \Sigma : \alpha > 0\}$ a *set of positive roots*, and $\Delta = \{\alpha_1, \dots, \alpha_l\}$ the *set of simple roots*. Define $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$, $\mathfrak{n}^- = \theta(\mathfrak{n}^+)$, and write K, A, N^+ and N^- for the analytic subgroups of G corresponding to $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}^+$, and $\mathfrak{n}^-$, respectively. Let M and M^* be the centralizer and the normalizer of $\mathfrak{a}$ in K, respectively. Consider then the Oshima compactification $\widetilde{\mathbb{X}}$ of the Riemannian symmetric space $\mathbb{X} = G/K$ which is assumed to be of non-compact type. It is a simply connected, compact, real-analytic manifold without boundary carrying a real-analytic G-action. The corresponding orbital decomposition of $\widetilde{\mathbb{X}}$ is of the form

$$(1) \qquad \widetilde{\mathbb{X}} \simeq \bigsqcup_{\Theta \subset \Delta} 2^{\#\Theta}(G/P_\Theta(K)) \quad \text{(disjoint union)},$$

the union being over subsets of Δ, where $\#\Theta$ is the number of elements of Θ, and $2^{\#\Theta}(G/P_\Theta(K))$ denotes the disjoint union of $2^{\#\Theta}$ copies of the homogeneous space $G/P_\Theta(K)$, where $P_\Theta(K)$ is a certain closed subgroup of G associated to Θ. In particular, for $\Theta = \Delta$, one has $P_\Delta(K) = K$, while for $\Theta = \emptyset$, $P_\emptyset(K) = P$. In what follows, denote by $\widetilde{\mathbb{X}}_\Theta$ a component in $\widetilde{\mathbb{X}}$ isomorphic to $G/P_\Theta(K)$. The orbital decomposition is of normal crossing type, meaning that for every point in $\widetilde{\mathbb{X}}$ there exists a local coordinate system $(n_1, \dots, n_k, t_1, \dots, t_l)$ in a neighbourhood of that point such that two points with coordinates $(n_1, \dots, n_k, t_1, \dots, t_l)$ and $(n'_1, \dots, n'_k, t'_1, \dots, t'_l)$ belong to the same G-orbit if, and only if, $\mathrm{sgn}\, t_j = \mathrm{sgn}\, t'_j$ for all $j = 1, \dots, l$. For a detailed description of $\widetilde{\mathbb{X}}$, the reader is referred to Part I. On $\widetilde{\mathbb{X}}$, there is a natural representation of G given by

$$\pi(g)\varphi(\tilde{x}) = \varphi(g^{-1} \cdot \tilde{x}), \qquad \varphi \in \mathrm{C}(\widetilde{\mathbb{X}}),$$

where $\mathrm{C}(\widetilde{\mathbb{X}})$ denotes the Banach space of continuous, complex-valued functions on $\widetilde{\mathbb{X}}$. Let d_G be a Haar measure on G, and $\mathcal{S}(G)$ the space of rapidly decreasing functions on G introduced in Part I, Definition 1. Let further Ω be the density bundle on $\widetilde{\mathbb{X}}$, and consider for every $f \in \mathcal{S}(G)$ the continuous linear operator

$$\pi(f) : \mathrm{C}^\infty(\widetilde{\mathbb{X}}) \longrightarrow \mathrm{C}^\infty(\widetilde{\mathbb{X}}) \subset \mathcal{D}'(\widetilde{\mathbb{X}}),$$

with Schwartz kernel given by the distributional section $\mathcal{K}_f \in \mathcal{D}'(\widetilde{\mathbb{X}} \times \widetilde{\mathbb{X}}, \mathbf{1} \boxtimes \Omega)$. Let

$$\left\{ (\widetilde{U}_{m_w}, \varphi_{m_w}^{-1}) \right\}_{w \in W}$$

be the finite atlas of $\widetilde{\mathbb{X}}$ constructed in Part I, where $W = M^*/M$ denotes the Weyl group of $(\mathfrak{g}, \mathfrak{a})$, and $m_w \in M^*$, a representative of $w \in W$. The coordinates on each of the charts of this atlas are then precisely of the form $(n, t) = (n_1, \ldots, n_k, t_1, \ldots, t_l)$ described above. For each point $\tilde{x} \in \widetilde{\mathbb{X}}$, choose open neighborhoods $\widetilde{W}_{\tilde{x}} \subset \widetilde{W}'_{\tilde{x}}$ of $\tilde{x}$ contained in a chart $\widetilde{U}_{m_w(\tilde{x})}$. Since $\widetilde{\mathbb{X}}$ is compact, we can find a finite subcover of the cover $\left\{ \widetilde{W}_{\tilde{x}} \right\}_{\tilde{x} \in \widetilde{\mathbb{X}}}$, and in this way obtain a finite atlas $\left\{ \widetilde{W}_\gamma, \varphi_\gamma^{-1} \right\}_{\gamma \in I}$ of $\widetilde{\mathbb{X}}$, where for simplicity we wrote $\varphi_\gamma = \varphi_{m_w(\tilde{x})}$. Further, let $\{\alpha_\gamma\}_{\gamma \in I}$ be a partition of unity subordinate to the above atlas, and let $\{\bar{\alpha}_\gamma\}_{\gamma \in I}$ be another set of functions satisfying $\bar{\alpha}_\gamma \in \mathrm{C}_\mathrm{c}^\infty(\widetilde{W}'_\gamma)$ and $\bar{\alpha}_{\gamma|\widetilde{W}_\gamma} \equiv 1$. Consider now the localization of $\pi(f)$ with respect to the atlas above given by

$$A_f^\gamma u = [\pi(f)_{|\widetilde{W}_\gamma}(u \circ \varphi_\gamma^{-1})] \circ \varphi_\gamma, \qquad u \in \mathrm{C}_\mathrm{c}^\infty(W_\gamma), \ W_\gamma = \varphi_\gamma^{-1}(\widetilde{W}_\gamma) \subset \mathbb{R}^{k+l}.$$

Writing $\varphi_\gamma^g = \varphi_\gamma^{-1} \circ g^{-1} \circ \varphi_\gamma$ and $x = (x_1, \ldots, x_{k+l}) = (n, t) \in W_\gamma$ we obtain

$$A_f^\gamma u(x) = \int_G f(g)[(u \circ \varphi_\gamma^{-1})\bar{\alpha}_\gamma](g^{-1} \cdot \varphi_\gamma(x)) \, d_G(g) = \int_G f(g) c_\gamma(x, g)(u \circ \varphi_\gamma^g)(x) d_G(g),$$

where we put $c_\gamma(x, g) = \bar{\alpha}_\gamma(g^{-1} \cdot \varphi_\gamma(x))$. Next, define the smooth functions

$$(2) \qquad a_f^\gamma(x, \xi) = \int_G e^{i(\varphi_\gamma^g(x)-x)\cdot\xi} c_\gamma(x, g) f(g) d_G(g),$$

and let T_x be the diagonal $(l \times l)$-matrix with entries $x_{k+1}, \ldots, x_{k+l}$. Introduce the auxiliary symbol

$$(3) \qquad \tilde{a}_f^\gamma(x, \xi) = a_f^\gamma(x, (\mathbf{1}_k \otimes T_x^{-1})\xi) = \int_G e^{i\Psi_\gamma(g,x)\cdot\xi} c_\gamma(x, g) f(g) d_G(g),$$

where we put

$$\Psi_\gamma(g, x) = [(\mathbf{1}_k \otimes T_x^{-1})(\varphi_\gamma^g(x) - x)] = (x_1(g\cdot\tilde{x}) - x_1(\tilde{x}), \ldots, x_k(g\cdot\tilde{x}) - x_k(\tilde{x}), \chi_1(g, \tilde{x}) - 1, \ldots, \chi_l(g, \tilde{x}) - 1),$$

the $\chi_j(g, \tilde{x})$ being analytic functions, see Part I, equation (28). One of the main results of Part I is the following

Theorem 1. *Let $f \in \mathcal{S}(G)$. The restrictions of the operators $\pi(f)$ to the manifolds with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$ are totally characteristic pseudodifferential operators of class $\mathrm{L}_b^{-\infty}$. More precisely, the operators $\pi(f)$ are locally of the form* [1]

$$(4) \qquad A_f^\gamma u(x) = \int e^{ix\cdot\xi} a_f^\gamma(x, \xi) \hat{u}(\xi) d\xi, \qquad u \in \mathrm{C}_\mathrm{c}^\infty(W_\gamma),$$

where $a_f^\gamma(x, \xi) = \tilde{a}_f^\gamma(x, \xi_1, \ldots, \xi_k, x_{k+1}\xi_{k+1}, \ldots, \xi_{k+l}x_{k+l})$, and $\tilde{a}_f^\gamma(x, \xi) \in \mathrm{S}_\mathrm{la}^{-\infty}(W_\gamma \times \mathbb{R}_\xi^{k+l})$ is a lacunary symbol given by (3). In particular, the kernel of the operator A_f^γ is determined by its restrictions to $W_\gamma^* \times W_\gamma^*$, where $W_\gamma^* = \{x \in W_\gamma : x_{k+1}\cdots x_{k+l} \neq 0\}$, and given by the oscillatory integral

$$(5) \qquad K_{A_f^\gamma}(x, y) = \int e^{i(x-y)\cdot\xi} a_f^\gamma(x, \xi) d\xi.$$

Proof. See Part I, Theorem 2. $\square$

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[1] Here and in what follows we shall adhere to the convention that, if not specified otherwise, integration is to be performed over whole Euclidean space $\mathbb{R}^n$, with n appropriate. In addition, we shall use the notation $d\xi = (2\pi)^{-n} d\xi$.

3. Regularized traces

We shall now define a regularized trace for the convolution operators $\pi(f)$ introduced in the previous section. To begin with, note that, as a consequence of Theorem 1, we can write the kernel of $\pi(f)$ locally in the form

$$
K_{A_f^\gamma}(x,y) = \int e^{i(x-y)\cdot\xi} a_f^\gamma(x,\xi) d\xi = \int e^{i(x-y)\cdot(\mathbf{1}_k \otimes T_x^{-1})\xi} \tilde{a}_f^\gamma(x,\xi) |\det(\mathbf{1}_k \otimes T_x^{-1})'(\xi)| d\xi
$$

(6)

$$
= \frac{1}{|x_{k+1}\cdots x_{k+l}|} \tilde{A}_f^\gamma(x, x_1 - y_1, \ldots, 1 - \frac{y_{k+1}}{x_{k+1}}, \ldots), \qquad x_{k+1}\cdots x_{k+l} \neq 0,
$$

where $\tilde{A}_f^\gamma(x,y)$ denotes the inverse Fourier transform of the lacunary symbol $\tilde{a}_f^\gamma(x,\xi)$ given by

$$
(7) \qquad\qquad\qquad \tilde{A}_f^\gamma(x,y) = \int e^{iy\cdot\xi} \tilde{a}_f^\gamma(x,\xi)\, d\xi.
$$

Since for $x \in W_\gamma$, the amplitude $\tilde{a}_f^\gamma(x,\xi)$ is rapidly falling in ξ, it follows that $\tilde{A}_f^\gamma(x,y) \in \mathcal{S}(\mathbb{R}_y^n)$, the Fourier transform being an isomorphism on the Schwartz space. Therefore, $K_{A_f^\gamma}(x,y)$ is rapidly decreasing as $|x_j| \to 0$ for $x_j \neq y_j$ and $k+1 \leq j \leq k+l$. Furthermore, by the lacunarity of $\tilde{a}_f^\gamma$, $K_{A_f^\gamma}(x,y)$ is also rapidly decaying as $|y_j| \to 0$, $x_j \neq y_j$ and $k+1 \leq j \leq k+l$.

Consider now the partition of unity $\{\alpha_\gamma\}$ subordinate to the atlas $\{(\widetilde{W}_\gamma, \varphi_\gamma^{-1})\}$. By equation (6), the restriction of the kernel of A_f^γ to the diagonal is given by

$$
K_{A_f^\gamma}(x,x) = \frac{1}{|x_{k+1}\cdots x_{k+l}|} \tilde{A}_f^\gamma(x,0), \qquad x_{k+1}\cdots x_{k+l} \neq 0.
$$

These restrictions yield a family of smooth functions $k_f^\gamma(\tilde{x}) = K_{A_f^\gamma}(\varphi_\gamma^{-1}(\tilde{x}), \varphi_\gamma^{-1}(\tilde{x}))$ which define a density k_f on

$$
2^{\#l}(G/K) \subset \widetilde{\mathbb{X}}.
$$

Nevertheless, the functions $k_f^\gamma(\tilde{x})$ are not locally integrable on the entire compactification $\widetilde{\mathbb{X}}$, so that we cannot define a trace of $\pi(f)$ by integrating the density k_f over the diagonal $\Delta_{\widetilde{\mathbb{X}} \times \widetilde{\mathbb{X}}} \simeq \widetilde{\mathbb{X}}$. Instead, we have the following

Proposition 1. *Let $f \in \mathcal{S}(G)$, $s \in \mathbb{C}$, and define for $\operatorname{Re} s > 0$*

$$
\operatorname{Tr}_s \pi(f) = \sum_\gamma \int_{W_\gamma} (\alpha_\gamma \circ \varphi_\gamma)(x) |x_{k+1}\cdots x_{k+l}|^s \widetilde{A}_f^\gamma(x,0) dx
$$

$$
= \left\langle |x_{k+1}\cdots x_{k+l}|^s, \sum_\gamma (\alpha_\gamma \circ \varphi_\gamma) \widetilde{A}_f^\gamma(\cdot, 0) \right\rangle.
$$

Then $\operatorname{Tr}_s \pi(f)$ can be continued analytically to a meromorphic function in s with at most poles at $-1, -3, \ldots$. Furthermore, for $s \in \mathbb{C} - \{-1, -3, \ldots\}$,

(8) $$\Theta_\pi^s : C_c^\infty(G) \ni f \mapsto \operatorname{Tr}_s \pi(f) \in \mathbb{C}$$

defines a distribution density on G.

Proof. The fact that $\operatorname{Tr}_s \pi(f)$ can be continued meromorphically is a consequence of the analytic continuation of $|x_{k+1}\cdots x_{k+l}|^s$ as a distribution in $\mathbb{R}^{k+l}$, proved by Bernshtein-Gel'fand in [4], Lemma 2. One even has that

$$
\langle |x_{k+1}|^{s_1} \cdots |x_{k+l}|^{s_l}, u \rangle, \qquad u \in C_c^\infty(\mathbb{R}^{k+l}),
$$

can be continued meromorphically in the variables $s_1, \ldots, s_l$ to $\mathbb{C}^l$ with poles $s_i = -1, -3, \ldots$. To see that (8) is a distribution density, note that $\Theta_\pi^s : C_c^\infty(G) \to \mathbb{C}$ is certainly linear. Since

$|x_{k+1} \cdots x_{k+l}|^s$ is a distribution, for any open, relatively compact subset $\omega \subset \mathbb{R}^{k+l}$ there exist $C_\omega > 0$ and $B_\omega \in \mathbb{N}$ such that

$$(9) \qquad |\langle |x_{k+1} \cdots x_{k+l}|^s, u \rangle | \leq C_\omega \sum_{|\beta| \leq B_\omega} \sup |\partial^\beta u|, \qquad u \in \mathrm{C}_c^\infty(\omega).$$

Let now $\mathcal{O} \subset G$ be an arbitrary open, relatively compact subset, and $f \in \mathrm{C}_c^\infty(\mathcal{O})$. With equation (7) one has

$$(10) \qquad \mathrm{Tr}_s \pi(f) = \left\langle |x_{k+1} \cdots x_{k+l}|^s, \sum_\gamma (\alpha_\gamma \circ \varphi_\gamma) \int \tilde{a}_f^\gamma(\cdot, \xi) d\!\!\!/\xi \right\rangle.$$

By equation (38) of Part I, one computes for arbitrary $N \in \mathbb{N}$ that

$$e^{i\Psi_\gamma(g,x) \cdot \xi} = \frac{1}{(1 + |\xi|^2)^N} \sum_{r=0}^{2N} \sum_{|\alpha|=r} b_\alpha^N(x,g) dL(X^\alpha) \left[e^{i\Psi_\gamma(g,x) \cdot \xi} \right],$$

where the coefficients $b_\alpha^N(x,g)$ are smooth, and at most of exponential growth in g. With (3) and Proposition 1 of Part I we therefore obtain for $\tilde{a}_f^\gamma(x,\xi)$ the expression

$$\tilde{a}_f^\gamma(x,\xi) = \frac{1}{(1+|\xi|^2)^N} \int_G e^{i\Psi_\gamma(g,x) \cdot \xi} \sum_{r=0}^{2N} \sum_{|\alpha|=r} (-1)^r dL(X^{\tilde{\alpha}}) \left[b_\alpha^N(x,g) c_\gamma(x,g) f(g) \right] d_G(g).$$

Inserting this in (10), and taking N sufficiently large, we obtain with (9) that

$$| \mathrm{Tr}_s \pi(f)| \leq C_\mathcal{O} \sum_{|\beta| \leq B_\mathcal{O}} \sup |dL(X^\beta) f|$$

for suitable $C_\mathcal{O} > 0$ and $B_\mathcal{O} \in \mathbb{N}$. Since the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ can be identified with the algebra of invariant differential operators on G, the assertion now follows with [9], page 480. $\qquad \square$

Remark 1. Using Hironaka's theorem on resolution of singularities, Bernshtein-Gel'fand [4] and Atiyah [1] even proved the following general result. Let M be a real analytic manifold and f a non-zero, real analytic function on M. Then $|f|^s$, which is locally integrable for $\mathrm{Re}\, s > 0$, extends analytically to a distribution on M which is a meromorphic function of s in the whole complex plane. The poles are located at the negative rational numbers, and their order does not exceed the dimension of M. From this one deduces that if $f : M \to \mathbb{C}$ is a non-zero analytic function, then there exists a distribution S on M such that $fS = 1$. This is the Hörmander-Lojasiewicz theorem on the division of distributions, and implies the existence of temperate fundamental solutions for constant-coefficient differential operators.

Consider next the Laurent expansion of $\Theta_\pi^s(f)$ at $s = -1$. For this, let $u \in \mathrm{C}_c^\infty(\mathbb{R}^{k+l})$ be a test function, and consider the expansion

$$\langle |x_{k+1} \cdots x_{k+l}|^s, u \rangle = \sum_{j=-q}^\infty S_j(u)(s+1)^j,$$

where $S_k \in \mathcal{D}'(\mathbb{R}^{k+l})$. Since $|x_{k+1} \cdots x_{k+l}|^{s+1}$ has no pole at $s = -1$, we necessarily must have

$$|x_{k+1} \cdots x_{k+l}| \cdot S_j = 0 \quad \text{for } j < 0, \qquad |x_{k+1} \cdots x_{k+l}| \cdot S_0 = 1$$

as distributions. Therefore $S_0 \in \mathcal{D}'(\mathbb{R}^{k+l})$ represents a distributional inverse of $|x_{k+1} \cdots x_{k+l}|$. By repeating the reasoning of the proof of Proposition 1 we arrive at the following

Proposition 2. *For $f \in \mathcal{S}(G)$, let the regularized trace of the operator $\pi(f)$ be defined by*

$$\mathrm{Tr}_{reg}\, \pi(f) = \left\langle S_0, \sum_{\gamma} (\alpha_{\gamma} \circ \varphi_{\gamma}) \tilde{A}_f^{\gamma}(\cdot, 0) \right\rangle.$$

Then $\Theta_{\pi} : \mathrm{C}_c^{\infty}(G) \ni f \mapsto \mathrm{Tr}_{reg}\, \pi(f) \in \mathbb{C}$ constitutes a distribution density on G, which is called the character of the representation π.

$\square$

Remark 2. An alternative definition of $\mathrm{Tr}_{reg}\, \pi(f)$ could be given within the calculus of b-pseudodifferential operators developed by Melrose. For a detailed description, the reader is referred to [7], Section 6.

In what follows, we shall identify distributions with distribution densities on G via the Haar measure d_G. Our next aim is to understand the distributions Θ_{π}^s and Θ_{π} in terms of the G-action on $\widetilde{\mathbb{X}}$. We shall actually show that on a certain open set of transversal elements, they are represented by locally integrable functions given in terms of fixed points. Similar expressions where derived by Atiyah and Bott for the global character of an induced representation of G. Their work is based on the concept of transversal trace of a pseudodifferential operator, and will be explained in the next section.

4. Transversal trace and characters of induced representations

In [2], Atiyah and Bott extended the classical Lefschetz fixed point theorem to geometric endomorphisms on elliptic complexes. Their work relies on the concept of transversal trace of a smooth operator, and its extension by continuity to pseudodifferential operators. The Lefschetz theorem then follows by showing that the Lefschetz number of a geometric endomorphism is given by an alternating sum of transversal traces, and extending an analogous alternating sum formula for smooth endomorphisms. To explain the notion of transversal trace of a pseudodifferential operator, let us introduce the following

Definition 1. *Let M be a smooth manifold. A fixed point x_0 of a smooth map $f : M \to M$ is said to be* simple *if* $\det(\mathbf{1} - df_{x_0}) \neq 0$, *where df_{x_0} denotes the differential of f at x_0. The map f is called* transversal *if it has only simple fixed points.*

Note that the non-vanishing condition on the determinant is equivalent to the requirement that the graph of f intersects the diagonal transversally at $(x_0, x_0) \in M \times M$, and hence the terminology. In particular, a simple fixed point is an isolated fixed point. Let now U be an open subset of $\mathbb{R}^n$, V open in U, and consider a smooth map $\alpha : V \to U$ with a simple fixed point at x_0. We choose V so small, that $x \mapsto x - \alpha(x)$ defines a diffeomorphism of V onto its image. Let $\Lambda : V \to U \times U$ be the map $\Lambda(x) = (\alpha(x), x)$, and assume that $A \in \mathrm{L}^{-\infty}(U)$ is a smooth operator with symbol $a(x, \xi)$. The kernel K_A of A is a smooth function on $U \times U$, and its restriction $\Lambda^* K_A$ to the graph of α defines a distribution on V according to

(11)
$$\langle \Lambda^* K_A, v \rangle = \int \int e^{i(\alpha(x) - x) \cdot \xi} a(\alpha(x), \xi) v(x)\, d\xi\, dx$$
$$= \int \int e^{-iy \cdot \xi} \frac{a(\alpha(x(y)), \xi) v(x(y))}{|\det(\mathbf{1} - d\alpha(x(y)))|}\, dy\, d\xi, \qquad v \in \mathrm{C}_c^{\infty}(V),$$

where we made the substitution $y = x - \alpha(x)$, and the change in order of integration is permissible because $a(x, \xi) \in \mathrm{S}^{-\infty}(U)$. Now, for $a(x, \xi) \in \mathrm{S}^l(U)$, we observe that by differentiating

$$\int e^{-iy \cdot \xi} a(\alpha(x(y)), \xi) \frac{v(x(y))}{|\det(\mathbf{1} - d\alpha(x(y)))|}\, dy$$

with respect to ξ, and integrating by parts with respect to y, we obtain the estimate

$$|\partial_\xi^\gamma \int e^{-iy\cdot\xi} a(\alpha(x(y)), \xi) \frac{v(x(y))}{|\det(\mathbf{1} - d\alpha(x(y)))|} dy| \leq C \langle \xi \rangle^{l-|\beta|}$$

for arbitrary multi-indices γ and β and some constant $C > 0$. Thus, as an oscillatory integral, the last expression in (11) defines a distribution on V for any $a(x, \xi) \in S^l(U)$. The distribution $\Lambda^* K_A$ is called *the transversal trace of* $A \in L^l(U)$. If, in particular, $a(x, \xi) = a(x)$ is a polynomial of degree zero in ξ, one computes that

$$(12) \qquad \Lambda^* K_A = \frac{a(x_0)\delta_{x_0}}{|\det(1 - d\alpha(x_0))|}.$$

This diccussion can be globalized. Let $\mathbf{X}$ be a smooth manifold, E a vector bundle over $\mathbf{X}$, $\alpha : \mathbf{X} \to \mathbf{X}$ a C^∞-map with only simple fixed points, and

$$A : \Gamma_c(\alpha^* E) \longrightarrow \Gamma(E)$$

a pseudodifferential operator of order l between smooth sections. Denote the density bundle on $\mathbf{X}$ by Ω, put $F = \alpha^* E$, and define $F' = F^* \otimes \Omega$. The kernel K_A is then a distributional section of $E \boxtimes F'$. In other words, $K_A \in \mathcal{D}'(E \boxtimes F') = \mathcal{D}'(\mathbf{X} \times \mathbf{X}, E \boxtimes F')$. Similarly, one has $K_{\alpha^* A} \in \mathcal{D}'(\mathbf{X} \times \mathbf{X}, F \boxtimes F')$, where $\alpha^* A$ denotes the composition

$$\alpha^* A : \Gamma_c(F) \xrightarrow{A} \Gamma(E) \xrightarrow{\alpha^*} \Gamma(F).$$

If $A \in L^{-\infty}(F, E)$, K_A is a smooth section on $\mathbf{X} \times \mathbf{X}$, and $K_A(x, y) \in E_x \otimes F'_y$. In this case, $K_{\alpha^* A}(x, y) = K_A(\alpha(x), y)$, so that one deduces $K_{\alpha^* A}(x, x) \in E_{\alpha(x)} \otimes F'_x = F_x \otimes (F^* \otimes \Omega)_x \simeq \mathcal{L}(F_x, F_x) \otimes \Omega_x$. As a consequence, $\mathrm{Tr}\, K_{\alpha^* A}(x, x)$ becomes a section of Ω, where Tr denotes the bundle homomorphism

$$(13) \qquad \mathrm{Tr} : F \otimes F' \longrightarrow \Omega.$$

Hence, if $\mathbf{X}$ is compact, one can define the trace of $\alpha^* A$ as

$$\mathrm{Tr}\, \alpha^* A = \int_{\mathbf{X}} \mathrm{Tr}\, K_{\alpha^* A}(x, x).$$

This trace can be extended to arbitrary $A \in L^l(\mathbf{X})$. Indeed, let Δ be the diagonal in $\mathbf{X} \times \mathbf{X}$, and denote the canonical isomorphism $\Delta \simeq \mathbf{X}$ also by Δ. The foregoing local considerations imply that the map $\Theta : \mathcal{L}(\mathcal{E}'(F), \Gamma(E)) \to \Gamma(F \otimes F')$ given by $A \mapsto \Delta^* K_{\alpha^* A} = K_{\alpha^* A}(x, x)$ has an extension

$$\Theta : L^l(F, E) \longrightarrow \mathcal{D}'(F \otimes F')$$

which is continuous with respect to the strong operator topology on bounded sets of $L^l(F, E)$, see [2], Proposition 5.3. Since the bundle homomorphism (13) induces continuous linear maps

$$\mathrm{Tr} : \Gamma(F \otimes F') \longrightarrow \Gamma(\Omega), \qquad \mathrm{Tr} : \mathcal{D}'(F \otimes F') \longrightarrow \mathcal{D}'(\Omega),$$

where $\mathcal{D}'(\Omega) = \mathcal{D}'(\mathbf{X}, \Omega) = \Gamma_c(\Omega^* \otimes \Omega)' = \Gamma_c(1)' = C_c^\infty(\mathbf{X})'$ is the space of distribution densities on $\mathbf{X}$, we see that $\mathrm{Tr}\,\Theta(A)$ can be defined for any $A \in L^l(F, E)$ in a unique way. Consequently, for compact $\mathbf{X}$, the map $L^{-\infty}(F, E) \to \mathbb{C}, A \to \mathrm{Tr}\, \alpha^* A$ has a unique continuous extension

$$\mathrm{Tr}_\alpha : L^l(F, E) \longrightarrow \mathbb{C}, \qquad A \mapsto \mathrm{Tr}_\alpha A = \langle \mathrm{Tr}\,\Theta(A), 1 \rangle,$$

called the *transversal trace of* A. In the case that A is induced by a bundle homomorphism φ, it follows from (12) that

$$(14) \qquad \mathrm{Tr}_\alpha A = \sum_{x \in \mathrm{Fix}(\alpha)} \nu_x(A), \qquad \nu_x(A) = \frac{\mathrm{Tr}\,\varphi_x}{|\det(\mathbf{1} - d\alpha(x))|},$$

the sum being over the fixed points of α on $\mathbf{X}$, see [2], Corollary 5.4.

In the context of representation theory, this trace was employed by Atiyah and Bott in [3] to compute the global character of an induced representation. Thus, let G be a Lie group, H a closed subgroup of G, and ϱ a representation of H on a finite dimensional vector space V. The representation of G induced by ϱ is a geometric endomorphism in the space of sections over G/H with values in the homogeneous vector bundle $G \times_H V$, and shall be denoted by $T(g) = (\iota_* \varrho)(g)$. Assume that G/H is compact, and let d_G be a Haar measure on G. Consider a compactly supported smooth function $f \in \mathrm{C}_c^\infty(G)$, and the corresponding convolution operator $T(f) = \int_G f(g) T(g) d_G(g)$. It is a smooth operator, and, since G/H is compact, has a well defined trace. Consequently, the map

$$\Theta_T : \mathrm{C}_c^\infty(G) \ni f \longmapsto \operatorname{Tr} T(f) \in \mathbb{C}$$

defines a distribution on G called the *distribution character of the induced representation* T. On the other hand, assume that $g \in G$ is such that $l_{g^{-1}} : G/H \to G/H, xH \mapsto g^{-1}xH$, has only simple fixed points. In this case, a transversal trace $\operatorname{Tr}^\flat T(g)$ of $T(g)$ can be defined according to

$$\operatorname{Tr}^\flat T(g) = \operatorname{Tr}_{l_{g^{-1}}}(\Gamma(\varphi_g)),$$

where $\varphi_g : l_{g^{-1}}^*(G \times_H V) \to G \times_H V$ is the endomorphism associated to $T(g)$ such that

$$T(g) = \varphi_g \circ l_{g^{-1}}^*,$$

and $\Gamma(\varphi_g) : \Gamma(l_{g^{-1}}^*(G \times_H V)) \to \Gamma(G \times_H V)$. $\operatorname{Tr}^\flat T(g)$ is given by a sum over fixed points of g, and one can show that, on an open set $G_T \subset G$,

$$(15) \qquad \Theta_T(f) = \int_{G_T} f(g) \operatorname{Tr}^\flat T(g) d_G(g), \qquad f \in \mathrm{C}_c^\infty(G_T).$$

Thus, the distribution character of a parabolically induced representation of a Lie group G is represented on G_T by the transversal trace of the corresponding geometric endomorphism. If G is compact, the Lefschetz theorem reduces to the Hermann–Weyl formula by the theory of Borel and Weil. It can be interpreted as expressing the character of a finite dimensional representation as an alternating sum of characters of infinite dimensional representations. In what follows, we shall prove similar formulae for the distributions Θ_π and Θ_π^s defined in the previous section, after reviewing some largely known facts about group actions on homogeneous spaces.

5. Fixed point actions on homogeneous spaces

Let G be a Lie group with Lie algebra $\mathfrak{g}$, $H \subset G$ a closed subgroup with Lie algebra $\mathfrak{h}$, and $\pi : G \to G/H$ the canonical projection. For an element $g \in G$, consider the natural left action $l_g : G/H \to G/H$ given by $l_g(xH) = gxH$. Let Ad^G denote the adjoint action of G on $\mathfrak{g}$. We begin with two well-known lemmata, see e.g. [3], page 463.

Lemma 1. $l_{g^{-1}} : G/H \to G/H$ *has a fixed point if and only if* $g \in \bigcup_{x \in G} xHx^{-1}$. *Moreover, to every fixed point* xH *one can associate a unique conjugacy class* $h(g, xH)$ *in* H.

Proof. Clearly,

$$l_{g^{-1}}(xH) = xH \iff g^{-1}xH = xH \iff (g^{-1}x)^{-1}x \in H \iff x^{-1}gx = h(g, x),$$

where $h(g, x) \in H$. So $l_{g^{-1}}$ has a fixed point xH if, and only if, $g \in \bigcup_{x \in G} xHx^{-1}$. Now, if $y \in G$ is such that $xH = yH$, then $y = xh$ for some $h \in H$. This gives us that $h(g, y) = y^{-1}gy = (xh)^{-1}g(xh) = h^{-1}(x^{-1}gx)h = h^{-1}h(g, x)h$. Thus, as x varies over representatives of the coset xH, $h(g, x)$ varies over a conjugacy class $h(g, xH)$ in H. $\qquad \square$

Lemma 2. *Let xH be a fixed point of $l_{g^{-1}}$ and let $h \in h(g, xH)$. Then*

$$\det\left(\mathbf{1} - dl_{g^{-1}}\right)_{xH} = \det\left(\mathbf{1} - \operatorname{Ad}_H^G(h)\right),$$

where $\operatorname{Ad}_H^G : H \to \operatorname{Aut}(\mathfrak{g}/\mathfrak{h})$ is the isotropy action of H on $\mathfrak{g}/\mathfrak{h}$.

Proof. Let L_g and R_g be the left and right translations, respectively, of $g \in G$ on G. We begin with the observation that

(16)
$$\pi \circ L_{g^{-1}} = l_{g^{-1}} \circ \pi,$$

where π is the natural map from G to G/H. Let e be the identity in G, and $T_{\pi(e)}(G/H)$ the tangent space to G/H at the point $\pi(e)$. The derivative $d\pi : \mathfrak{g} \to T_{\pi(e)}(G/H)$ is a surjective linear map with kernel $\mathfrak{h}$, and therefore induces an isomorphism between $\mathfrak{g}/\mathfrak{h}$ and $T_{\pi(e)}(G/H)$, which we shall again denote by $d\pi$. Notice also that, for $h \in H$, $\operatorname{Ad}^G(h)$ leaves $\mathfrak{h}$ invariant and so induces a map $\operatorname{Ad}_H^G(h) : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$. Now, let xH be a fixed point of $l_{g^{-1}}$, and take $h \in h(g, xH)$. Choose x in the coset xH such that $g^{-1}x = xh$. For $y \in G$ one computes

$$(\pi \circ L_{g^{-1}} \circ R_{h^{-1}})(y) = \pi(g^{-1}yh^{-1}) = g^{-1}yH = l_{g^{-1}}(yH) = (l_{g^{-1}} \circ \pi)(y),$$

so that

(17)
$$\pi \circ L_{g^{-1}} \circ R_{h^{-1}} = l_{g^{-1}} \circ \pi.$$

Observe, additionally, that $L_{g^{-1}} \circ R_{h^{-1}}$ fixes x. We therefore see that $L_{g^{-1}} \circ R_{h^{-1}} \circ L_x = L_x \circ L_h \circ R_{h^{-1}}$, which, together with equations (16) and (17), leads us to

(18)
$$l_x \circ \pi \circ L_h \circ R_{h^{-1}} = l_{g^{-1}} \circ l_x \circ \pi.$$

Differentiating this, and using the identification $dl_x \circ d\pi : \mathfrak{g}/\mathfrak{h} \to T_{\pi(x)}(G/H)$, we obtain the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{g}/\mathfrak{h} & \xrightarrow{\operatorname{Ad}_H^G(h)} & \mathfrak{g}/\mathfrak{h} \\
{\scriptstyle dl_x \circ d\pi} \downarrow & & {\scriptstyle dl_x \circ d\pi} \downarrow \\
T_{\pi(x)}(G/H) & \xrightarrow{dl_{g^{-1}}} & T_{\pi(x)}(G/H)
\end{array}
$$

thus proving the lemma. $\qquad \square$

Consider now the case when G is a connected, real, semi-simple Lie group with finite centre, θ a Cartan involution of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. Further, let K be the maximal compact subgroup of G associated to $\mathfrak{k}$, and consider the corresponding Riemannian symmetric space $\mathbb{X} = G/K$ which is assumed to be of non-compact type. By definition, θ is an involutive automorphism of $\mathfrak{g}$ such that the bilinear form $\langle \cdot, \cdot \rangle_\theta$ is strictly positive definite. In particular, $\langle \cdot, \cdot \rangle_{\theta | \mathfrak{p} \times \mathfrak{p}}$ is a symmetric, positive-definite, bilinear form, yielding a left-invariant metric on G/K. Endowed with this metric, G/K becomes a complete, simply connected, Riemannian manifold with non-positive sectional curvature. Such manifolds are called *Hadamard manifolds*. Furthermore, for each $g \in G$, $l_{g^{-1}} : G/K \to G/K$ is an isometry on G/K with respect to this left-invariant metric. Note that Riemannian symmetric spaces of non-compact type are precisely the simply connected Riemannian symmetric spaces with sectional curvature $\kappa \le 0$ and with no Euclidean de Rham factor. We then have the following

Lemma 3. *Let $g \in G$ be such that $l_{g^{-1}} : G/K \to G/K$ is transversal. Then $l_{g^{-1}}$ has a unique fixed point in G/K.*

Proof. Let M be a Hadamard manifold, and φ an isometry on M that leaves two distinct points $x, y \in M$ fixed. By general theory, there is a unique minimal geodesic $\gamma : \mathbb{R} \to M$ joining x and y. Let $\gamma(0) = x$ and $\gamma(1) = y$, so that $\varphi \circ \gamma(0) = \varphi(x) = x$ and $\varphi \circ \gamma(1) = \varphi(y) = y$. Since isometries take geodesics to geodesics, $\varphi \circ \gamma$ is a geodesic in M, joining x and y. By the uniqueness of γ we therefore conclude that $\varphi \circ \gamma = \gamma$. This means that an isometry on a Hadamard manifold with two distinct fixed points also fixes the unique geodesic joining them point by point. Since, by assumption, $l_{g^{-1}} : G/K \to G/K$ has only isolated fixed points, the lemma follows. $\square$

In what follows, we shall call an element $g \in G$ *transversal relative to a closed subgroup H* if $l_{g^{-1}} : G/H \to G/H$ is transversal, and denote the set of all such elements by $G(H)$.

Proposition 3. *Let G be a connected, real, semi-simple Lie group with finite centre, and K a maximal compact subgroup of G. Suppose $\mathrm{rank}(G) = \mathrm{rank}(K)$. Then any regular element of G is transversal relative to K. In other words, $G' \subset G(K)$, where G' denotes the set of regular elements in G.*

Proof. If a regular element g is such that $l_{g^{-1}} : G/K \to G/K$ has no fixed points, it is of course transversal. Let, therefore, $g \in G'$ be such that $l_{g^{-1}}$ has a fixed point $x_0 K$. By Lemma 1, g must be conjugate to an element $k(g, x_0)$ in K. Consider now a maximal family of mutually non-conjugate Cartan subgroups $J_1, \ldots, J_r$ in G, and put $J_i' = J_i \cap G'$ for $i \in \{1, \ldots, r\}$. A result of Harish Chandra then implies that

$$G' = \bigcup_{i=1}^{r} \bigcup_{x \in G} x J_i' x^{-1},$$

see [9], Theorem 1.4.1.7. From this we deduce that

$$g = x k(g, x_0) x^{-1} = y j y^{-1} \quad \text{for some } x, y \in G, j \in J_i' \text{ for some } i.$$

Hence, $k(g, x_0)$ must be regular. Now, let T be a maximal torus of K. It is a Cartan subgroup of K, and the assumption that $\mathrm{rank}(G) = \mathrm{rank}(K)$ implies that that T is also Cartan in G. Let $k(g, x_0 K)$ be the conjugacy class in K associated to $x_0 K$, as in Lemma 1. As K is compact, the maximal torus T intersects every conjugacy class in K. Varying x_0 over the coset $x_0 K$, we can therefore assume that $k(g, x_0) \in k(g, x_0 K) \cap T$. Thus, we conclude that $k(g, x_0) \in T \cap G'$. Note that, in particular, we can choose $J_i = T$ by the maximality of the $J_1, \ldots, J_r$. Now, for a regular element $h \in G$ belonging to a Cartan subgroup H one necessarily has $\det(\mathbf{1} - \mathrm{Ad}_H^G(h)) \neq 0$, compare the proof of Proposition 1.4.2.3 in [9]. Therefore $\det(\mathbf{1} - \mathrm{Ad}_T^G(k(g, x_0))) \neq 0$, and consequently, $\det(\mathbf{1} - \mathrm{Ad}_K^G(k(g, x_0))) \neq 0$. The assertion of the proposition now follows from Lemma 2. $\square$

Corollary 1. *Let G be a connected, real, semi-simple Lie group with finite centre, K a maximal compact subgroup of G, and suppose that $\mathrm{rank}(G) = \mathrm{rank}(K)$. Then the set of transversal elements $G(K)$ is open and dense in G.*

Proof. Clearly, $G(K)$ is open. Since the set of regular elements G' is dense in G, the corollary follows from the previous proposition. $\square$

Remark 3. To close this section, let us remark that with G as above, and P a parabolic subgroup of G, it is a classical result that $G' \subset G(P)$, see [5], page 51.

6. Character formulae

Let the notation be as before. We are now in a position to describe the distributions Θ_π^s and Θ_π introduced in Section 3. Thus, let $(\pi, C(\widetilde{\mathbb{X}}))$ be the regular representation of G on the Oshima compactification $\widetilde{\mathbb{X}}$ of the Riemannian symmetric space $\mathbb{X} = G/K$ of non-compact type, and denote

by $\Phi_g(\tilde{x}) = g \cdot \tilde{x}$ the G-action on $\widetilde{\mathbb{X}}$. Let further $G(\widetilde{\mathbb{X}}) \subset G$ be the set of elements g in G acting transversally on $\widetilde{\mathbb{X}}$.

Remark 4. The set $G(\widetilde{\mathbb{X}})$ is open. Corollary 1 and Remark 3 imply that $G(\widetilde{\mathbb{X}})$ is dense if $\operatorname{rank}(G/K) = 1$, and non-empty if $\operatorname{rank}(G/K) = 2$, and $\operatorname{rank}(G) = \operatorname{rank}(K)$.

Theorem 2. *Let $f \in \mathrm{C}_{\mathrm{c}}^\infty(G)$ have support in $G(\widetilde{\mathbb{X}})$, and $s \in \mathbb{C}$, $\operatorname{Re} s > -1$. Then*

$$(19) \qquad \operatorname{Tr}_s \pi(f) = \int_{G(\widetilde{\mathbb{X}})} f(g) \left(\sum_{\tilde{x} \in \mathrm{Fix}(g)} \sum_{\gamma} \frac{\alpha_\gamma(\tilde{x}) |x_{k+1}(\kappa_\gamma^{-1}(\tilde{x})) \cdots x_{k+l}(\kappa_\gamma^{-1}(\tilde{x}))|^{s+1}}{|\det(1 - d\Phi_{g^{-1}}(\tilde{x}))|} \right) d_G(g),$$

where $\mathrm{Fix}(g)$ *denotes the set of fixed points of g on $\widetilde{\mathbb{X}}$. In particular, $\Theta_\pi^s : \mathrm{C}_{\mathrm{c}}^\infty(G) \ni f \to \operatorname{Tr}_s \pi(f) \in \mathbb{C}$ is regular on $G(\widetilde{\mathbb{X}})$.*

Proof. By Proposition 1,

$$\operatorname{Tr}_s \pi(f) = \sum_\gamma \int_{W_\gamma} (\alpha_\gamma \circ \varphi_\gamma)(x) |x_{k+1} \cdots x_{k+l}|^s \widetilde{A}_f^\gamma(x, 0) dx$$

is a meromorphic function in s with possible poles at $-1, -3, \ldots$. Assume that $\operatorname{Re} s > -1$. Since $\alpha_\gamma \in \mathrm{C}_{\mathrm{c}}^\infty(\widetilde{W}_\gamma)$, and $\widetilde{A}_f^\gamma(x, 0) = \int \tilde{a}_f^\gamma(x, \xi) d\!\!\!/\xi$, where $\tilde{a}_f^\gamma(x, \xi) \in \mathrm{S}_{la}^{-\infty}(W_\gamma \times \mathbb{R}^{k+l})$ is rapidly decaying in ξ by Theorem 1, we can interchange the order of integration to obtain

$$\operatorname{Tr}_s \pi(f) = \sum_\gamma \int \int_{W_\gamma} (\alpha_\gamma \circ \varphi_\gamma)(x) |x_{k+1} \cdots x_{k+l}|^s \tilde{a}_f^\gamma(x, \xi) dx \, d\!\!\!/\xi.$$

Let $\chi \in \mathrm{C}_{\mathrm{c}}^\infty(\mathbb{R}^{k+l}, \mathbb{R}^+)$ be equal 1 in a neighborhood of 0, and $\varepsilon > 0$. Then, by Lebesgue's theorem on bounded convergence,

$$\operatorname{Tr}_s \pi(f) = \lim_{\varepsilon \to 0} I_\varepsilon,$$

where we defined

$$I_\varepsilon = \sum_\gamma \int \int_{W_\gamma} (\alpha_\gamma \circ \varphi_\gamma)(x) |x_{k+1} \cdots x_{k+l}|^s \tilde{a}_f^\gamma(x, \xi) \chi(\varepsilon \xi) \, dx \, d\!\!\!/\xi.$$

Taking into account (3), and interchanging the order of integration once more, one sees that

$$I_\varepsilon = \int_G f(g) \sum_\gamma \int \int_{W_\gamma} e^{i\Psi_\gamma(g,x)\cdot\xi} c_\gamma(x, g) (\alpha_\gamma \circ \varphi_\gamma)(x) |x_{k+1} \cdots x_{k+l}|^s \chi(\varepsilon \xi) dx \, d\!\!\!/\xi \, d_G(g),$$

everything in sight being absolutely convergent. Let us now set

$$I_\varepsilon(g) = f(g) \sum_\gamma \int \int_{W_\gamma} e^{i\Psi_\gamma(g,x)\cdot\xi} c_\gamma(x, g) (\alpha_\gamma \circ \varphi_\gamma)(x) |x_{k+1} \cdots x_{k+l}|^s \chi(\varepsilon \xi) dx \, d\!\!\!/\xi,$$

so that $I_\varepsilon = \int_G I_\varepsilon(g) \, d_G(g)$. We would like to pass to the limit under the integral, for which we are going to show that $\lim_{\varepsilon \to 0} I_\varepsilon(g)$ is an integrable function on G. For this, let us fix an arbitrary $g \in G(\widetilde{\mathbb{X}})$. By definition, g acts only with simple fixed points on $\widetilde{\mathbb{X}}$. Since each of them is isolated, g can have at most finitely many fixed points on $\widetilde{\mathbb{X}}$. Consider therefore a cut–off function $\beta_g \in \mathrm{C}^\infty(\widetilde{\mathbb{X}}, \mathbb{R}^+)$ which is equal 1 in a small neighborhood of each fixed point of g, and whose support decomposes into a union of connected components each of them containing only one fixed point of g. By choosing the support of β_g sufficiently close to the fixed points we can, in addition, assume that

$$(20) \qquad\qquad \det(d\Phi_g(\tilde{x}) - \mathbf{1}) \neq 0 \quad \text{on } \operatorname{supp} \beta_g.$$

Since the action of G is real analytic, we obtain a family of functions $\beta_g(\tilde{x})$ depending analytically on $g \in G(\widetilde{\mathbb{X}})$. Multiplying the integrand of $I_\varepsilon(g)$ with $\beta_g \circ \varphi_\gamma(x)$, and $1 - \beta_g \circ \varphi_\gamma(x)$, respectively, we obtain the decomposition

$$I_\varepsilon(g) = I_\varepsilon^{(1)}(g) + I_\varepsilon^{(2)}(g).$$

Let us first examine what happens away from the fixed points. Integrating by parts $2N$ times with respect to ξ yields

$$I_\varepsilon^{(2)}(g) = f(g) \sum_\gamma \int \int_{W_\gamma} e^{i\Psi_\gamma(g,x)\cdot\xi} c_\gamma(x,g)(\alpha_\gamma(1-\beta_g))(\varphi_\gamma(x))|x_{k+1}\cdots x_{k+l}|^s \chi(\varepsilon\xi) dx \, đ\xi$$

$$= f(g) \sum_\gamma \int \int_{W_\gamma} \frac{e^{i\Psi_\gamma(g,x)\cdot\xi}}{|\Psi_\gamma(g,x)|^{2N}} \Delta_\xi^N[\chi(\varepsilon\xi)] c_\gamma(x,g)(\alpha_\gamma(1-\beta_g))(\varphi_\gamma(x))|x_{k+1}\cdots x_{k+l}|^s dx \, đ\xi,$$

where $\Delta_\xi = \partial_{\xi_1}^2 + \cdots + \partial_{\xi_{k+l}}^2$. Now, for arbitrary N,

$$|\Delta_\xi^N[\chi(\varepsilon\xi)]| \le C_N(1+|\xi|^2)^{-N},$$

where C_N does not depend on ε for $0 < \varepsilon \le 1$. Furthermore, there exists a constant $M_f > 0$ such that $|\Psi_\gamma(g,x)|^{2N} \ge M_f$ on the support of $1 - \beta_g \circ \varphi_\gamma$ for all $g \in \operatorname{supp} f$ and γ. By Lebesgue's theorem, we may therefore pass to the limit under the integral, and obtain

$$\lim_{\varepsilon \to 0} I_\varepsilon^{(2)}(g) = 0.$$

Hence, as $\varepsilon \to 0$, the main contributions to $I_\varepsilon(g)$ originate from the fixed points of g. To examine these contributions, note that condition (20) implies that $x \mapsto \varphi_\gamma^g(x) - x$ defines a diffeomorphism on each of the connected components of $\operatorname{supp}(\alpha_\gamma \beta_g) \circ \varphi_\gamma$ onto their respective images. Performing the change of variables $y = x - \varphi_\gamma^g(x)$ we get

$$I_\varepsilon^{(1)}(g) = f(g) \sum_\gamma \int \int_{W_\gamma} e^{i\Psi_\gamma(g,x)\cdot\xi} c_\gamma(x,g)(\alpha_\gamma\beta_g)(\varphi_\gamma(x))|x_{k+1}\cdots x_{k+l}|^s \chi(\varepsilon\xi) dx \, đ\xi$$

$$= f(g) \sum_\gamma \int \int e^{-i(\mathbf{1}_k \otimes T_{x(y)}^{-1})y\cdot\xi} |x_{k+1}(y)\cdots x_{k+l}(y)|^s \frac{(\alpha_\gamma\beta_g)(\varphi_\gamma(x(y)))c_\gamma(x(y),g)}{|\det(\mathbf{1} - d\varphi_\gamma^g(x(y)))|} \chi(\varepsilon\xi) dy \, đ\xi$$

$$= f(g) \sum_\gamma \int |x_{k+1}(y)\cdots x_{k+l}(y)|^s c_\gamma(x(y),g) \frac{(\alpha_\gamma\beta_g)(\varphi_\gamma(x(y)))\hat{\chi}((\mathbf{1}_k \otimes T_{x(y)}^{-1})y/\varepsilon)}{(2\pi)^{k+l}\varepsilon^{k+l}|\det(\mathbf{1} - d\varphi_\gamma^g(x(y)))|} dy$$

$$= f(g) \sum_\gamma \int |x_{k+1}(\varepsilon y)\cdots x_{k+l}(\varepsilon y)|^s c_\gamma(x(\varepsilon y),g) \frac{(\alpha_\gamma\beta_g)(\varphi_\gamma(x(\varepsilon y)))\hat{\chi}((\mathbf{1}_k \otimes T_{x(\varepsilon y)}^{-1})y)}{(2\pi)^{k+l}|\det(\mathbf{1} - d\varphi_\gamma^g(x(\varepsilon y)))|} dy.$$

Since in a neighborhood of a fixed point $\tilde{x}$ of g the Jacobian of the singular change of coordinates $z = (\mathbf{1}_k \otimes T_{x(\varepsilon y)}^{-1})y$ converges to the expression $|x_{k+1}(\kappa_\gamma^{-1}(\tilde{x}))\cdots x_{k+l}(\kappa_\gamma^{-1}(\tilde{x}))|^{-1}$ as $\varepsilon \to 0$, we finally obtain with $(2\pi)^{-k-l}\int \hat{\chi}(y) dy = \chi(0) = 1$ that

$$\lim_{\varepsilon \to 0} I_\varepsilon^{(1)}(g) = \lim_{\varepsilon \to 0} f(g)$$

$$\cdot \sum_\gamma \int |x_{k+1}(\varepsilon y(z))\cdots x_{k+l}(\varepsilon y(z))|^s c_\gamma(x(\varepsilon y(z)),g) \frac{(\alpha_\gamma\beta_g)(\varphi_\gamma(x(\varepsilon y(z))))|\partial y/\partial z|}{(2\pi)^{k+l}|\det(\mathbf{1} - d\varphi_\gamma^g(x(\varepsilon y(z))))|} \hat{\chi}(z) dz$$

$$= f(g) \sum_{\tilde{x}\in\operatorname{Fix}(g)} \sum_\gamma \frac{\alpha_\gamma(\tilde{x})|x_{k+1}(\kappa_\gamma^{-1}(\tilde{x}))\cdots x_{k+l}(\kappa_\gamma^{-1}(\tilde{x}))|^{s+1}}{|\det(\mathbf{1} - d\Phi_{g^{-1}}(\tilde{x}))|},$$

since $\bar{\alpha}_\gamma \equiv 1$ on $\operatorname{supp}\alpha_\gamma$, and $\beta_g(\tilde{x}) = 1$. The limit function $\lim_{\varepsilon\to 0} I_\varepsilon(g)$ is therefore clearly integrable on G for $\operatorname{Re} s > -1$, so that by passing to the limit under the integral one computes

$$\operatorname{Tr}_s \pi(f) = \lim_{\varepsilon\to 0} I_\varepsilon = \lim_{\varepsilon\to 0} \int_G I_\varepsilon(g)\, d_G(g) = \int_G \lim_{\varepsilon\to 0} \left(I_\varepsilon^{(1)} + I_\varepsilon^{(2)}\right)(g) d_G(g)$$

$$= \int_G f(g) \sum_{\tilde{x}\in\operatorname{Fix}(g)} \sum_\gamma \frac{\alpha_\gamma(\tilde{x})|x_{k+1}(\kappa_\gamma^{-1}(\tilde{x}))\cdots x_{k+l}(\kappa_\gamma^{-1}(\tilde{x}))|^{s+1}}{|\det(1 - d\Phi_{g^{-1}}(\tilde{x}))|}\, d_G(g),$$

yielding the desired description of Θ_π^s. $\qquad\square$

As an immediate consequence of the previous theorem, we see that if $f \in \mathrm{C}_c^\infty(G(\widetilde{\mathbb{X}}))$, $\operatorname{Tr}_s \pi(f)$ is not singular at $s = -1$. This observation leads to the following

Corollary 2. *Let $f \in \mathrm{C}_c^\infty(G)$ have support in $G(\widetilde{\mathbb{X}})$. Then*

$$\operatorname{Tr}_{reg} \pi(f) = \operatorname{Tr}_{-1} \pi(f) = \int_{G(\widetilde{\mathbb{X}})} f(g) \sum_{\tilde{x}\in\operatorname{Fix}(g)} \frac{1}{|\det(1 - d\Phi_{g^{-1}}(\tilde{x}))|}\, d_G(g).$$

In particular, the distribution $\Theta_\pi : f \to \operatorname{Tr}_{reg}(f)$ is regular on $G(\widetilde{\mathbb{X}})$.

Proof. Consider the Laurent expansion of $\Theta_\pi^s(f)$ at $s = -1$ given by

$$\operatorname{Tr}_s \pi(f) = \left\langle |x_{k+1}\cdots x_{k+l}|^s, \sum_\gamma (\alpha_\gamma \circ \varphi_\gamma)\widetilde{A}_f^\gamma(\cdot, 0)\right\rangle = \sum_{j=-q}^\infty S_j\left(\sum_\gamma (\alpha_\gamma \circ \varphi_\gamma)\widetilde{A}_f^\gamma(\cdot, 0)\right)(s+1)^j,$$

where $S_k \in \mathcal{D}'(\mathbb{R}^{k+l})$. Since by (19), $\operatorname{Tr}_s \pi(f)$ has no pole at $s = -1$, we necessarily must have

$$S_j\left(\sum_\gamma (\alpha_\gamma \circ \varphi_\gamma)\widetilde{A}_f^\gamma(\cdot, 0)\right) = 0 \qquad \text{for } j < 0,$$

so that

$$\operatorname{Tr}_{-1} \pi(f) = \left\langle S_0, \sum_\gamma (\alpha_\gamma \circ \varphi_\gamma)\widetilde{A}_f^\gamma(\cdot, 0)\right\rangle = \operatorname{Tr}_{reg} \pi(f).$$

The assertion now follows with the previous theorem. $\qquad\square$

In particular, Corollary 2 implies that $\operatorname{Tr}_{reg} \pi(f)$ is invariantly defined. Now, interpreting $\pi(g)$ as a geometric endomorphism on the trivial bundle $E = \widetilde{\mathbb{X}} \times \mathbb{C}$ over the Oshima compactification $\widetilde{\mathbb{X}}$, a transversal trace $\operatorname{Tr}^\flat \pi(g)$ of $\pi(g)$ can be defined according to

$$\operatorname{Tr}^\flat \pi(g) = \operatorname{Tr}_{\Phi_{g^{-1}}}(\Gamma(\varphi_g)),$$

where $\varphi_g : \Phi_{g^{-1}}^* E \to E$ is the associated bundle homomorphism which identifies the fiber $E_{\Phi_{g^{-1}}(\tilde{x})}$ with $E_{\tilde{x}}$, and satisfies $(\operatorname{Tr}\varphi_g)_{|\tilde{x}} = 1$ at each fixed point $\tilde{x}$ of g. Taking into account (14), the previous corollary can be reformulated, and we finally deduce the following character formula for the distribution character of π.

Theorem 3. *On the set of transversal elements $G(\widetilde{\mathbb{X}})$, the distribution $\Theta_\pi : f \to \operatorname{Tr}_{reg}(f)$ is given by*

$$\operatorname{Tr}_{reg} \pi(f) = \int_{G(\widetilde{\mathbb{X}})} f(g) \operatorname{Tr}^\flat \pi(g) d_G(g), \qquad f \in \mathrm{C}_c^\infty(G(\widetilde{\mathbb{X}})),$$

where

$$\operatorname{Tr}^\flat \pi(g) = \sum_{\tilde{x}\in\operatorname{Fix}(g)} \frac{1}{|\det(1 - d\Phi_{g^{-1}}(\tilde{x}))|},$$

the sum being over the (simple) fixed points of $g \in G(\widetilde{\mathbb{X}})$ on $\widetilde{\mathbb{X}}$.

$\square$

7. The case $\mathbb{X} = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$

We shall finish this paper by describing in detail the Oshima compactification of the Riemannian symmetric space $\mathbb{X} = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$. Thus, let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ be the Lie algebra of G. A Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ is given by $X \mapsto -X^t$, where X^t denotes the transpose of X, and the corresponding Cartan decomposition of $\mathfrak{g}$ reads $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \{X \in \mathfrak{sl}(3, \mathbb{R}) : X^t = -X\}$, and $\mathfrak{p} = \{X \in \mathfrak{sl}(3, \mathbb{R}) : X^t = X\}$. Next, let

$$\mathfrak{a} = \{D(a_1, a_2, a_3) : a_1, a_2, a_3 \in \mathbb{R}, a_1 + a_2 + a_3 = 0\},$$

where $D(a_1, a_2, a_3)$ denotes the diagonal matrix with diagonal elements a_1, a_2 and a_3. Then $\mathfrak{a}$ is a maximal Abelian subalgebra in $\mathfrak{p}$. Define $e_i : \mathfrak{a} \to \mathbb{R}$ by $D(a_1, a_2, a_3) \mapsto a_i$, $i = 1, 2, 3$. The set of roots Σ of $(\mathfrak{g}, \mathfrak{a})$ is given by $\Sigma = \{\pm(e_i - e_j) : 1 \le i < j \le 3\}$. We order the roots such that the positive roots are $\Sigma^+ = \{e_1 - e_2, e_2 - e_3, e_1 - e_3\}$, and obtain $\Delta = \{e_1 - e_2, e_2 - e_3\}$ as the set of simple roots. The root space corresponding to the root $e_1 - e_2$ is given by

$$\mathfrak{g}^{e_1-e_2} = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\},$$

and similar computations show that

$$\mathfrak{g}^{e_2-e_3} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : z \in \mathbb{R} \right\}, \quad \mathfrak{g}^{e_1-e_3} = \left\{ \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : y \in \mathbb{R} \right\}.$$

For a subset $\Theta \subset \Delta$, let $\langle \Theta \rangle$ denote those elements of Σ that are given as linear combinations of the roots in Θ. Write $\langle \Theta \rangle^\pm$ for $\Sigma^\pm \cap \langle \Theta \rangle$. Put $\mathfrak{n}^\pm(\Theta) = \sum_{\lambda \in \langle \Theta \rangle^\pm} \mathfrak{g}^\lambda$, and $\mathfrak{n}_\Theta^+ = \sum_{\lambda \in \Sigma^+ - \langle \Theta \rangle^+} \mathfrak{g}^\lambda$. Let $\mathfrak{n}_\Theta^- = \theta(\mathfrak{n}_\Theta^+)$. Consider now the case $\Theta = \{e_1 - e_2\}$. Then $\mathfrak{n}^+(e_1 - e_2) = \mathfrak{g}^{e_1-e_2}$, and $\mathfrak{n}_{e_1-e_2}^+ = \mathfrak{g}^{e_2-e_3} \oplus \mathfrak{g}^{e_1-e_3}$. In other words,

$$\mathfrak{n}_{e_1-e_2}^+ = \left\{ \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : y, z \in \mathbb{R} \right\}.$$

Exponentiating, we find that the corresponding analytic subgroups are given by

$$N^+(e_1 - e_2) = \left\{ \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}, \quad N_{e_1-e_2}^+ = \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : y, z \in \mathbb{R} \right\}.$$

In a similar fashion, we obtain that

$$\mathfrak{n}^-(e_1 - e_2) = \mathfrak{g}^{e_2-e_1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\},$$

$$\mathfrak{n}_{e_1-e_2}^- = \theta(\mathfrak{n}_{e_1-e_2}^+) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y & z & 0 \end{pmatrix} : y, z \in \mathbb{R} \right\},$$

and that the corresponding analytic subgroups read

$$N^-(e_1 - e_2) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}, \quad N_{e_1-e_2}^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & z & 1 \end{pmatrix} : y, z \in \mathbb{R} \right\}.$$

The Cartan-Killing form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is given by $(X, Y) \mapsto \mathrm{Tr}(XY)$, and the modified Cartan-Killing form by $\langle X, Y \rangle_\theta := -\mathrm{Tr}(X\theta(Y)) = -\mathrm{Tr}(X(-Y^t)) = \mathrm{Tr}(XY^t)$. Next, let $\mathfrak{a}(\Theta) = \sum_{\lambda \in \langle \Theta \rangle_+} \mathbb{R} Q_\lambda$, where $Q_\lambda = [\theta X, X]$ for $X \in \mathfrak{g}^\lambda$ such that $\langle X, X \rangle_\theta = 1$. Also, let $\mathfrak{a}_\Theta$ be the orthogonal complement of $\mathfrak{a}(\Theta)$ with respect to $\langle \cdot, \cdot \rangle_\theta$. Again, suppose that $\Theta = \{e_1 - e_2\}$. We find

$$Q_{e_1 - e_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that

$$\mathfrak{a}(e_1 - e_2) = \mathbb{R} Q_{e_1 - e_2} = \left\{ \begin{pmatrix} r & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{pmatrix} : r \in \mathbb{R} \right\}.$$

This in turn gives us that

$$\mathfrak{a}_{e_1 - e_2} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} : a \in \mathbb{R} \right\}.$$

Exponentiation then shows that the corresponding analytic subgroups are

$$A(e_1 - e_2) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} : a \in \mathbb{R}^+ \right\}, A_{e_1 - e_2} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-2} \end{pmatrix} : a \in \mathbb{R}^+ \right\}.$$

Take $K = \mathrm{SO}(3)$ as a maximal compact subgroup of $\mathrm{SL}(3, \mathbb{R})$, and denote by $M_\Theta(K)$ the centralizer of $\mathfrak{a}_\Theta$ in K. Observing that the adjoint action of a matrix group G is just the matrix conjugation, we see that

$$M_{e_1 - e_2}(K) = Z_K(\mathfrak{a}_{e_1 - e_2}) = \begin{pmatrix} \mathrm{SO}(2) & 0 \\ 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{SO}(2) & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice that $M_{e_1 - e_2}(K)$ has 2 connected components. Put $M = Z_K(A)$, and let $P = MAN^+$ be the minimal parabolic subgroup given by the ordering of the roots of $(\mathfrak{g}, \mathfrak{a})$. For $G = \mathrm{SL}(3, \mathbb{R})$ one computes

$$M = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}.$$

As Θ varies over the subsets of Δ, we get all the parabolic subgroups P_Θ of G containing P, and we write $P_\Theta = M_\Theta(K)AN^+$. By definition, $P_\Theta(K) = M_\Theta(K)A_\Theta N_\Theta^+$ so that, in particular,

$$P_{e_1 - e_2}(K) = \left(\begin{pmatrix} \mathrm{SO}(2) & 0 \\ 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{SO}(2) & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-2} \end{pmatrix} : a \in \mathbb{R}^+ \right\}$$

$$\cdot \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\}.$$

The orbital decomposition of the Oshima compactification $\widetilde{\mathbb{X}}$ of $\mathbb{X} = \mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ is therefore given by

$$\widetilde{\mathbb{X}} = G/P \sqcup 2 \cdot G/P_{e_1 - e_2}(K) \sqcup 2 \cdot G/P_{e_2 - e_3}(K) \sqcup 2^2 \cdot G/K.$$

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