

EQUIVARIANT LEFSCHETZ FORMULAE AND HEAT ASYMPTOTICS

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ABSTRACT. We prove an equivariant Lefschetz formula for elliptic complexes over a compact manifold carrying the action of a compact Lie group of isometries via heat equation methods.

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1. INTRODUCTION

The computation of the Lefschetz number of an endomorphism of an elliptic complex constitutes a generalization of the index problem for an elliptic operator. For geometric endomorphisms arising from transversal mappings, this computation was accomplished by Atiyah and Bott in [2], generalizing the classical Lefschetz fixed point theorem. In this paper, we shall prove a local formula for the equivariant Lefschetz number of an elliptic complex over a compact manifold carrying the action of a compact Lie group of isometries.

To explain our result, let M be a compact Riemannian manifold of dimension n , and G a compact Lie group acting effectively and isometrically on M . Consider further a family E_0, \dots, E_N of C^∞ -vector bundles over M , and let

$$C^\infty(E_0) \xrightarrow{P_0} C^\infty(E_1) \xrightarrow{P_1} \dots \xrightarrow{P_{N-1}} C^\infty(E_N)$$

be an elliptic complex \mathcal{E} on M . Assume that for every $g \in G$ and $0 \leq j \leq N$ there exist smooth bundle homomorphisms $\Phi_j(g) : g^*E_j \rightarrow E_j$, and define the linear maps

$$T_j(g) : C^\infty(E_j) \xrightarrow{g^*} C^\infty(g^*E_j) \xrightarrow{\Phi_j(g)} C^\infty(E_j), \quad T_j(g)s(x) = \Phi_j(g)[s(gx)].$$

If $P_j T_j(g) = T_{j+1}(g) P_j$ for each $g \in G$ and $0 \leq j \leq N-1$, the maps $T_j(g)$ constitute a geometric endomorphism $T(g)$ of the complex \mathcal{E} , and we denote the corresponding endomorphisms on the cohomology groups $H^j(\mathcal{E}, \mathbb{C})$ by $\mathcal{T}_j(g)$. Since the cohomology groups are finite-dimensional, one

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can define the Lefschetz number for each of the $T(g)$ by

$$L(T(g)) = \sum_{j=0}^N (-1)^j \operatorname{tr} \mathcal{T}_j(g)|_{H^j(\mathcal{E}, \mathbb{C})}, \quad g \in G.$$

In case that $g : M \rightarrow M$ has only simple fixed points, the Lefschetz fixed point theorem of Atiyah and Bott expresses $L(T(g))$ as a sum over fixed points of g . Consider now a unitary irreducible representation (π_ρ, V_ρ) of G associated to a character $\rho \in \hat{G}$, and write $T = \{T(g)\}_{g \in G}$. We then define the ρ -equivariant Lefschetz number of T as

$$\mathcal{L}_\rho(T) = \frac{1}{\operatorname{vol} G} \int_G L(T(g)) \overline{\rho(g)} d_G(g),$$

where d_G denotes a Haar measure on G . Note that if G is trivial, this simply reduces to the Euler–Poincaré characteristic of \mathcal{E} . Our aim is to prove a local formula for $\mathcal{L}_\rho(T)$, based on asymptotics of the heat equation. For this we shall approximate the heat operator by pseudodifferential operators. This leads to the problem of determining the asymptotic behavior of certain oscillatory integrals which have been examined before in [16] during the study of the spectrum of an invariant elliptic operator. The existence of such local formulae for $\mathcal{L}_\rho(T)$ suggests that, using invariance theory, it should be possible to find global expressions for $\mathcal{L}_\rho(T)$ in terms of characteristic classes. This will be the subject of a subsequent paper.

The original proof of Atiyah and Bott of the Lefschetz fixed point theorem relies on the theory of pseudodifferential operators, and an extension of the trace of a finite rank operator to a larger class of maps including geometric endomorphisms. They also gave an alternative proof based on work of Seeley [17] on the zeta–function of an elliptic operator, going back to work of Minakshisundaram and Pleijel [14]. It was first pointed out by Hörmander that heat equation techniques can be used instead of fractional powers to obtain a local formula for the Lefschetz number of a geometric endomorphism. Following this approach, Kotake gave another proof of the Atiyah–Bott fixed point theorem in [12]. Based on work of McKean and Singer [13], Patodi [15] and Gilkey [7], this development finally culminated in a proof of the index theorem by Atiyah, Bott and Patodi using heat equation methods [1]. These methods were then applied to derive generalized Lefschetz fixed point formulae for the classical complexes. In the case of the signature complex, Donnelly–Patodi [6] and Kawasaki [11] gave a new proof of the G -signature theorem of Atiyah–Singer, while the other complexes were treated in Gilkey [8].

The paper is structured as follows. In Section 2 we review the Lefschetz formula of Atiyah and Bott for elliptic complexes, and state the main result of this paper. Section 3 introduces the heat equation, and explains how it is related to the index problem. The crucial observation here, which is due to Bott, is that the Lefschetz number of an elliptic complex can be expressed as an alternating sum of heat traces. In Section 4 the heat operator is approximated by pseudodifferential operators, obtaining an expansion for the equivariant heat trace and for $\mathcal{L}_\rho(T)$ in terms of oscillatory integrals. Their asymptotic behavior is described in Section 5 using the stationary phase theorem and resolution of singularities. A local formula for $\mathcal{L}_\rho(T)$ is then derived in Section 6, while an outlook is given in Section 7.

2. EQUIVARIANT LEFSCHETZ FORMULAE FOR ELLIPTIC COMPLEXES

We begin by reviewing the classical Atiyah–Bott fixed point formula for elliptic complexes following [2]. Let M be a compact C^∞ -manifold of dimension n , and E and F complex vector bundles over M . Denote the corresponding spaces of smooth sections by $C^\infty(E)$ and $C^\infty(F)$, respectively, and consider a differential operator

$$P : C^\infty(E) \longrightarrow C^\infty(F)$$

of order d between E and F , which is a linear map given locally by a matrix of partial differential operators with smooth coefficients. Let T^*M be the cotangent bundle of M , and $\pi : T^*M \rightarrow M$ the canonical projection. The terms of order d of D define in an invariant manner a bundle homomorphism

$$p_d : \pi^*E \longrightarrow \pi^*F$$

over the cotangent space T^*M called the *principal symbol* of D . If p_d is an isomorphism away from the zero section of T^*M , the operator D is called *elliptic*. Let now E_0, \dots, E_N be a family of C^∞ -vector bundles over M . Then a sequence

$$(1) \quad C^\infty(E_0) \xrightarrow{P_0} C^\infty(E_1) \xrightarrow{P_1} \dots \xrightarrow{P_{N-1}} C^\infty(E_N)$$

of differential operators is called an *elliptic complex* if $P_j P_{j-1} = 0$ for all $1 \leq j \leq N-1$, and the sequence of corresponding principal symbols

$$0 \longrightarrow \pi^*E_0 \xrightarrow{P_{0,d_0}} \pi^*E_1 \xrightarrow{P_{1,d_1}} \dots \xrightarrow{P_{N-1,d_{N-1}}} \pi^*E_N \longrightarrow 0$$

is exact outside the zero section. The complex (1) is denoted by \mathcal{E} , and its cohomology groups are defined as usual according to

$$H^j(\mathcal{E}, \mathbb{C}) = \ker P_j / \text{Im } P_{j-1}.$$

As one can show, these cohomology groups are all finite-dimensional for an elliptic complex. In particular, the Euler–Poincaré characteristic

$$\chi(\mathcal{E}) = \sum_{j=0}^N (-1)^j \dim H^j(\mathcal{E}, \mathbb{C})$$

is well defined.

Consider next an *endomorphism* T of the elliptic complex (1), by which one means a sequence of linear maps $T_j : C^\infty(E_j) \rightarrow C^\infty(E_j)$ such that $P_j T_j = T_{j+1} P_j$. Since both $\ker P_j$ and $\text{Im } P_{j-1}$ are left invariant by T_j , such an endomorphism induces endomorphisms \mathcal{T}_j on the cohomology groups $H^j(\mathcal{E}, \mathbb{C})$. Since the latter are finite-dimensional, one can define the *Lefschetz number* of T as

$$L(T) = \sum_{j=0}^N (-1)^j \text{tr } \mathcal{T}_j|_{H^j(\mathcal{E}, \mathbb{C})}.$$

Clearly, if T is the identity, $L(T)$ just reduces to the Euler–Poincaré characteristic $\chi(\mathcal{E})$ of the complex \mathcal{E} . In case that $N = 1$, $\chi(\mathcal{E})$ is just the index of the elliptic operator P_0 . Therefore, the computation of the Lefschetz number $L(T)$ constitutes a generalization of the index problem for an elliptic operator, which was solved by Atiyah–Singer in [3].

Let now $g : M \rightarrow M$ be a smooth map, so that for each $0 \leq j \leq N$ we have the induced bundles g^*E_j over M , together with the linear maps

$$g^* : C^\infty(E_j) \longrightarrow C^\infty(g^*E_j), \quad (g^*s)(x) = s(gx).$$

In addition, assume that we are given smooth bundle homomorphisms $\Phi_j(g) : g^*E_j \rightarrow E_j$. We can then define the linear maps

$$(2) \quad T_j(g) : C^\infty(E_j) \xrightarrow{g^*} C^\infty(g^*E_j) \xrightarrow{\Phi_j(g)} C^\infty(E_j), \quad T_j s(x) = \Phi_j(g)[s(gx)].$$

If $P_j T_j(g) = T_{j+1}(g) P_j$, the system consisting of g and the linear maps $T_j(g) : C^\infty(E_j) \rightarrow C^\infty(E_j)$ is called a *geometric endomorphism* of \mathcal{E} . If g has only *simple* fixed points, meaning that $\det(\mathbf{1} - dg_x) \neq 0$ for each fixed point $x \in M$, the mapping g is called *transversal*. In this case, each fixed point is isolated so that, M being compact, the set of fixed points $\text{Fix}(g)$ of g is finite. Note that at a fixed point $x \in \text{Fix}(g)$, $\Phi_j(g)_x$ is an endomorphism of the fiber $E_{j,x}$, so that its trace $\text{tr } \Phi_j(g)_x$ is defined. After these preparations, we can state

Theorem 1 (Atiyah–Bott–Lefschetz fixed point theorem). *Consider a geometric endomorphism $T(g)$ of an elliptic complex (1), given by a transversal mapping $g : M \rightarrow M$, and bundle homomorphisms $\Phi_j(g) : g^*E_j \rightarrow E_j$. Then the Lefschetz number $L(T(g))$ of $T(g)$ is given by*

$$L(T(g)) = \sum_{x \in \text{Fix}(g)} \sum_{j=0}^N \frac{(-1)^j \text{tr } \Phi_j(g)_x}{|\det(\mathbf{1} - dg_x)|}.$$

Proof. See Atiyah–Bott [2]. □

The classical example for an elliptic complex is the De–Rham complex. In this case, the j^{th} exterior powers of dg yield a geometric endomorphism, and the above theorem reduces to the classical Lefschetz fixed point formula.

Consider now a compact Lie group G , acting effectively and isometrically on M . Let us assume that for every $g \in G$ and $0 \leq j \leq N$ there exist smooth bundle homomorphisms $\Phi_j(g) : g^*E_j \rightarrow E_j$, so that we can define the linear maps (2). In addition, we shall assume that $P_j T_j(g) = T_{j+1}(g) P_j$ for each $g \in G$ and $0 \leq j \leq N - 1$. Under these conditions, the mappings $T_j(g)$ define geometric endomorphisms $T(g)$ of \mathcal{E} for each $g \in G$, and we write $T = \{T(g)\}_{g \in G}$. The Lefschetz number of $T(g)$ is given by

$$(3) \quad L(T(g)) = \sum_{j=0}^N (-1)^j \text{tr } \mathcal{T}_j(g)_{|H^j(\mathcal{E}, \mathbb{C})},$$

where the $\mathcal{T}_j(g)$ denote the endomorphisms induced by the maps $T_j(g)$ on the cohomology groups $H^j(\mathcal{E}, \mathbb{C})$. In what follows, we shall consider the following generalization of the Euler–Poincaré characteristic of \mathcal{E} . Let (π_ϱ, V_ϱ) be a unitary irreducible representation of G associated to the character $\varrho \in \hat{G}$. We then define the ϱ -equivariant Lefschetz number of T as

$$(4) \quad \mathcal{L}_\varrho(T) = \frac{1}{\text{vol } G} \int_G L(T(g)) \overline{\varrho(g)} d_G(g),$$

where d_G is a Haar measure on G . Clearly, if G is trivial, this just reduces to $\chi(\mathcal{E})$. The main result of this paper is the following local formula for $\mathcal{L}_\varrho(T)$.

Theorem 2. *Let M be a compact Riemannian manifold of dimension n , and G a compact Lie group acting effectively and isometrically on M . Let $\mathbb{J} : T^*M \rightarrow \mathfrak{g}^*$ be the momentum map of the induced Hamiltonian action on the cotangent bundle T^*M , and put $\Xi = \mathbb{J}^{-1}(0)$. Consider further an elliptic complex \mathcal{E} on M , together with a family of geometric endomorphisms $T = \{T(g)\}_{g \in G}$ of \mathcal{E} defined by the isometries $g : M \rightarrow M$, and bundle homomorphisms $\Phi_j(g) : g^*E_j \rightarrow E_j$, and denote by Δ_j the associated Laplacians. Let $\{(\kappa_\gamma, U^\gamma)\}$ be an atlas of M , $\{f_\gamma\}$ a subordinated partition of unity, and $\{\varphi_{E_j}^\gamma\}$ corresponding trivializations of the bundles E_j .*

(1) *For each $\varrho \in \hat{G}$, the ϱ -equivariant Lefschetz number $\mathcal{L}_\varrho(T)$ of T is given by the local formula*

$$\mathcal{L}_\varrho(T) = \frac{(2\pi)^{\kappa-n}}{\text{vol } G} \sum_{j=0}^N (-1)^j \sum_{\gamma} \left[\mathcal{L}_{j, n-\kappa, \gamma} + \mathcal{R}_{j, \gamma} \right],$$

where κ is the dimension of a principal G -orbit in M , and

$$\mathcal{L}_{j, k, \gamma} = \int_{\text{Reg } \mathcal{C}} \frac{f_\gamma(x) \cdot \text{tr} \left[\Phi_j(g)_x \circ (\varphi_{E_j}^\gamma)_x^{-1} \circ e_k^\gamma(1, \kappa_\gamma(x), \eta, \Delta_j) \circ (\varphi_{E_j}^\gamma)_x \right] \cdot \overline{\varrho(g)}}{|\det \Phi_\gamma''(x, \eta, g)_{N(x, \eta, g) \text{Reg } \mathcal{C}}|^{1/2}} d(\text{Reg } \mathcal{C})(x, \eta, g).$$

$\text{Reg } \mathcal{C}$ denotes the regular part of the critical set $\mathcal{C} = \{(x, \xi, g) \in \Xi \times G : g \cdot (x, \xi) = (x, \xi)\}$ of the phase functions $\Phi_\gamma(x, \eta, g) = (\kappa_\gamma(gx) - \kappa_\gamma(x)) \cdot \eta$, and $d(\text{Reg } \mathcal{C})$ the induced volume

density. The $e_k^\gamma(1, \kappa_\gamma(x), \eta, \Delta_j)$, where $0 \leq k \leq n - \kappa$, are local symbols, and the remainder terms $\mathcal{R}_{j,\gamma}$ are given in terms of local symbols up to order $n - \kappa - 1$.

(2) If the endomorphisms $\Phi_j(g)_x$ act trivially on the fibers $E_{j,x}$,

$$\mathcal{L}_\varrho(T) = \frac{(2\pi)^{\kappa-n} [\pi_{\varrho|H} : 1]}{\text{vol } G} \sum_{j=0}^N \sum_{\gamma} (-1)^j \cdot \left[\int_{\text{Reg } \Xi} f_\gamma(x) \cdot \text{tr} \left[(\varphi_{E_j}^\gamma)_x^{-1} \circ e_{n-\kappa}^\gamma(1, \kappa_\gamma(x), \eta, \Delta_j) \circ (\varphi_{E_j}^\gamma)_x \right] \frac{d(\text{Reg } \Xi)(x, \eta)}{\text{vol } \mathcal{O}_{(x,\eta)}} + \mathcal{R}_{j,\gamma} \right],$$

where $H \subset G$ a principal isotropy group, and $[\pi_{\varrho|H} : 1]$ the multiplicity of the trivial representation in the restriction of π_ϱ to H , while $\mathcal{O}_{(x,\eta)}$ denotes the G -orbit in T^*M through (x, η) .

In contrast to the Atiyah–Bott fixed point theorem, higher dimensional fixed point sets are now involved. The proof of Theorem 2 will therefore require the more elaborate techniques of fractional powers and the heat equation, which were not needed in the original proof of Theorem 1.

3. HEAT EQUATION AND LEFSCHETZ NUMBERS

Let M be a closed n -dimensional Riemannian manifold, dM its volume density, and E a complex C^∞ -vector bundle over M endowed with a smooth Hermitian metric h . Under these assumptions, $C^\infty(E)$ becomes a Pre–Hilbert space with inner product

$$(s, s')_{L^2} = \int_M h(s(x), s'(x)) dM, \quad s, s' \in C^\infty(E).$$

Its completion is given by the Hilbert space $L^2(E)$ of square integrable sections of E . Denote by Ω the density bundle on M , which is the line bundle associated to the tangent bundle TM via the representation $A \rightarrow |\det A|$ of $\text{GL}(n, \mathbb{R})$. Consider further E^* , the dual bundle of E , and set $E' = E^* \otimes \Omega$. Let $C^\infty(E)'$ be the dual topological vector space of $C^\infty(E)$. An element of $\mathcal{D}'(E) = C^\infty(E)'$ is called a *distributional section* of E . In general, if

$$A : C^\infty(E) \longrightarrow C^\infty(F)$$

is a continuous linear operator, its Schwartz kernel K_A is a distributional section on $M \times M$ of the bundle $F \boxtimes E'$. Here $F \boxtimes E'$ denotes the exterior tensor product of F and E' , which is the smooth bundle over $M \times M$ with fibers $F_x \otimes E'_y$, $x, y \in M$. Suppose now that

$$(5) \quad P : C^\infty(E) \longrightarrow L^2(E)$$

is an elliptic differential operator of order m on E , regarded as an operator in $L^2(E)$ with domain $C^\infty(E)$, and assume that P is symmetric and positive¹. Then P has discrete spectrum, and there exists an orthonormal basis of $L^2(E)$ consisting of smooth sections $\{e_j\}$ such that $Pe_j = \lambda_j e_j$, $|\lambda_j| \rightarrow \infty$. Associated to P , we consider the heat equation

$$(\partial_t + P)h(x, t) = 0, \quad \lim_{t \rightarrow 0} h(x, t) = f(x), \quad t > 0,$$

with initial condition $f \in C^\infty(E)$. It is a parabolic differential equation, and its solution is given by $h(x, t) = e^{-tP} f(x)$, where

$$(6) \quad e^{-tP} = \frac{1}{2\pi i} \int_\Gamma e^{-t\lambda} (P - \lambda \mathbf{1})^{-1} d\lambda$$

¹The positivity of P means that, outside the zero section of T^*M , the principal symbol is given by a positive definite matrix.

is the corresponding heat operator. Here Γ is a suitable path in \mathbb{C} coming from infinity and going to infinity such that $(P - \lambda \mathbf{1})$ is invertible for $\lambda \in \Gamma$. The heat operator has a smooth kernel $K_{e^{-tP}} \in C^\infty(E \boxtimes E')$, which for each $x, y \in M$ defines an element $K(t, x, y, P) dM(y) \in \text{Hom}(E_y, E_x) \otimes \Omega_y$. As Seeley showed in [17], e^{-tP} is of L^2 -trace class, its trace being given by

$$\text{tr}_{L^2}(e^{-tP}) = \sum_j (e^{-tP} e_j, e_j)_{L^2} = \sum_j e^{-t\lambda_j} = \int_M \text{tr} K(t, x, x, P) dM(x).$$

Let next \mathcal{E} be an elliptic complex over M as in (1), where each of the bundles E_j is equipped with a smooth Hermitian metric. For simplicity, we shall assume that all the P_j have the same order. Consider the adjoint complex

$$C^\infty(E_0) \xleftarrow{P_0^*} C^\infty(E_1) \xleftarrow{P_1^*} \dots \xleftarrow{P_{N-1}^*} C^\infty(E_N),$$

where the P_j^* are differential operators determined uniquely by the condition $(P_j s, s')_{L^2} = (s, P_j^* s')_{L^2}$ for all $s \in C^\infty(E_j)$, $s' \in C^\infty(E_{j+1})$, and define the associated Laplacians

$$\Delta_j = P_{j-1} P_{j-1}^* + P_j^* P_j.$$

Then $\Delta_j : C^\infty(E) \rightarrow L^2(E)$ is an elliptic, symmetric and positive operator.

Suppose now that a compact Lie group G acts effectively and isometrically on M , and that for every $g \in G$ and $0 \leq j \leq N$ there exist smooth bundle homomorphisms $\Phi_j(g) : g^* E_j \rightarrow E_j$, so that we can define the linear maps (2). Assume that $P_j T_j(g) = T_{j+1}(g) P_j$ for each $g \in G$ and $0 \leq j \leq N-1$, and let $L(T(g))$ be the Lefschetz number of the geometric endomorphism $T(g)$ defined in (3). The following algebraic observation is due to Bott, and is a direct consequence of the Hodge decomposition theorem. It is crucial for the heat equation approach to the index problem.

Lemma 1. *Let \mathcal{E} be an elliptic complex, and Δ_j the associated Laplacians. For each $g \in G$, let $T(g)$ be a geometric endomorphism determined by the action of g on M , and smooth bundle homomorphisms $\Phi_j(g)$. Then*

$$L(T(g)) = \sum_{j=0}^N (-1)^j \text{tr}_{L^2} (T_j(g) e^{-t\Delta_j})$$

for any $t > 0$.

Proof. See Atiyah–Bott [2], Section 8, Kotake [12], Lemma 3, or Gilkey, [9] Lemma 1.10.1. \square

Next, let (π_ϱ, V_ϱ) be a unitary irreducible representation of G associated to the character $\varrho \in \hat{G}$. In what follows, we shall use Lemma 1 to prove a local formula for the ϱ -equivariant Lefschetz number $L_\varrho(T)$ introduced in (4). For this, we shall require an asymptotic expansion for

$$\int_G \text{tr}_{L^2} (T_j(g) e^{-t\Delta_j}) \overline{\varrho(g)} d_G(g), \quad t \rightarrow 0^+,$$

which will be derived in the next sections.

4. PSEUDODIFFERENTIAL OPERATORS AND EQUIVARIANT HEAT ASYMPTOTICS

Our aim is to give a local formula for the ϱ -equivariant Lefschetz number $\mathcal{L}_\varrho(T)$ using the alternating sum formula of Lemma 1, and asymptotics of the heat equation. For this, we shall first construct an approximation of the heat operator by pseudodifferential operators. Let \tilde{U} be an open set in \mathbb{R}^n . Recall that a continuous linear operator

$$A : C_c^\infty(\tilde{U}) \longrightarrow C^\infty(\tilde{U})$$

is called a *pseudodifferential operator* if it can be written in the form

$$(7) \quad Au(\tilde{x}) = \int e^{i\tilde{x}\cdot\xi} a(\tilde{x}, \xi) \hat{u}(\xi) d\xi,$$

where $\hat{u} = \mathcal{F}(u)$ denotes the Fourier transform of u , $d\xi = (2\pi)^{-n} d\xi$, and $a(\tilde{x}, \xi) \in C^\infty(\tilde{U} \times \mathbb{R}^n)$ is an amplitude with the following property. There is an $l \in \mathbb{R}$ such that for any multiindices α, β , and any compact set $K \subset \tilde{U}$, there exist constants $C_{\alpha, \beta, K}$ for which

$$|\partial_\xi^\alpha \partial_{\tilde{x}}^\beta a(\tilde{x}, \xi)| \leq C_{\alpha, \beta, K} (1 + |\xi|^2)^{(l - |\alpha|)/2}, \quad \tilde{x} \in K, \quad \xi \in \mathbb{R}^n,$$

where $|\alpha| = \alpha_1 + \dots + \alpha_n$. The class of all such functions $a(\tilde{x}, \xi)$ is denoted by $S^l(\tilde{U} \times \mathbb{R}^n)$, and the class of operators of the form (7) with $a(\tilde{x}, \xi) \in S^l(\tilde{U} \times \mathbb{R}^n)$, by $L^l(\tilde{U})$. In particular, one puts $S^{-\infty}(\tilde{U} \times \mathbb{R}^n) = \bigcap_{l \in \mathbb{R}} S^l(\tilde{U} \times \mathbb{R}^n)$.

Consider next an n -dimensional C^∞ -manifold M , and let $\{(\kappa_\gamma, U^\gamma)\}$ be an atlas for M . Write $\tilde{U}^\gamma = \kappa_\gamma(U^\gamma) \subset \mathbb{R}^n$. If $\pi_E : E \rightarrow M$ and $\pi_F : F \rightarrow M$ are smooth vector bundles over M trivialized by

$$\alpha_E^\gamma : E|_{U^\gamma} \longrightarrow U^\gamma \times \mathbb{C}^e, e \mapsto (\pi_E(e), \varphi_E^\gamma(e)), \quad \alpha_F^\gamma : F|_{U^\gamma} \longrightarrow U^\gamma \times \mathbb{C}^f, f \mapsto (\pi_F(f), \varphi_F^\gamma(f)),$$

then a continuous linear operator

$$A : C_c^\infty(E) \longrightarrow C^\infty(F)$$

is called a *pseudodifferential operator between sections of E and F of order l* , if for any U^γ there is a $f \times e$ -matrix of pseudodifferential operators $\tilde{A}_{ij} \in L^l(\tilde{U}^\gamma)$ such that

$$(\varphi_F^\gamma \circ (Av)|_{U^\gamma})_i = \sum_j A_{ij}(\varphi_E^\gamma \circ v)_j, \quad v \in C_c^\infty(U^\gamma; E),$$

where the A_{ij} are defined by the relations $A_{ij}u = [\tilde{A}_{ij}(u \circ \kappa_\gamma^{-1})] \circ \kappa_\gamma$, $u \in C_c^\infty(U^\gamma)$. In this case we write $A \in L^l(M; E, F)$, or simply $L^l(E, F)$. As explained before, the Schwartz kernel K_A of A is a distribution section on $M \times M$ of the bundle $F \boxtimes E'$. For an introduction into the theory of pseudodifferential operators, the reader is referred to [18] or [10].

Suppose now that M is a closed Riemannian manifold, and E a complex smooth vector bundle over M with a smooth Hermitian metric, and let $P : C^\infty(E) \rightarrow L^2(E)$ be an elliptic differential operator as in (5). Consider the heat operator associated to P , and let $\Gamma \subset \mathbb{C}$ be the path specified in (6). P is locally given by a matrix of differential operators $P_{ij}^\gamma \in L^l(\tilde{U}^\gamma)$ with symbols $p_{ij}^\gamma(\tilde{x}, \xi) \in S^l(\tilde{U}^\gamma \times \mathbb{R}^n)$. On each chart U^γ , the symbol of P is represented by the matrix $p^\gamma(\tilde{x}, \xi) = (p_{ij}^\gamma(\tilde{x}, \xi))_{ij}$, and we decompose the latter into its homogeneous components

$$p^\gamma(\tilde{x}, \xi) = p_m^\gamma(\tilde{x}, \xi) + \dots + p_0^\gamma(\tilde{x}, \xi),$$

where m is the order of P . The positivity of P means that $p_m^\gamma(\tilde{x}, \xi)$ is a positive definite matrix for $\xi \neq 0$, and together with the ellipticity and the symmetry of P this implies that $p_m^\gamma(\tilde{x}, \xi) - \lambda$ is invertible for $\lambda \in \Gamma$. Write $(\kappa_\gamma^{-1})^* dM = \beta_\gamma d\tilde{y}$. We now recursively define the local symbols

$$r_0^\gamma(\tilde{x}, \xi, \lambda, P) = (p_m^\gamma(\tilde{x}, \xi) - \lambda)^{-1},$$

$$r_k^\gamma(\tilde{x}, \xi, \lambda, P) = -r_0^\gamma(\tilde{x}, \xi, \lambda, P) \left(\sum_{|\beta|+m+l'-l=k, l'<k} \frac{(-i)^{|\beta|}}{\beta!} (\partial_\xi^\beta p_l^\gamma)(\tilde{x}, \xi) \cdot (\partial_{\tilde{x}}^\beta r_{l'}^\gamma)(\tilde{x}, \xi, \lambda, P) \right),$$

as well as

$$e_k^\gamma(t, \tilde{x}, \xi, P) = \frac{1}{2\pi i} \int_\Gamma e^{-t\lambda} r_k^\gamma(\tilde{x}, \xi, \lambda, P) d\lambda, \quad t > 0,$$

and consider the corresponding pseudodifferential operators

$$\begin{aligned} [\tilde{R}_k^\gamma(\lambda, P)v](\tilde{x}) &= \int e^{i\tilde{x}\cdot\eta} r_k^\gamma(\tilde{x}, \eta, \lambda, P) \widehat{v\beta_\gamma}(\eta) d\eta, \\ [\tilde{E}_k^\gamma(t, P)v](\tilde{x}) &= \int e^{i\tilde{x}\cdot\eta} e_k^\gamma(t, \tilde{x}, \eta, P) \widehat{v\beta_\gamma}(\eta) d\eta, \end{aligned}$$

where $v \in C_c^\infty(\tilde{U}^\gamma; \mathbb{C}^e)$. With these definitions, set

$$\begin{aligned} R_k^\gamma(\lambda, P)u &= (\varphi_E^\gamma)^{-1} \circ [\tilde{R}_k^\gamma(\lambda, P)(\varphi_E^\gamma \circ u \circ \kappa_\gamma^{-1})] \circ \kappa_\gamma, \\ E_k^\gamma(t, P)u &= (\varphi_E^\gamma)^{-1} \circ [\tilde{E}_k^\gamma(t, P)(\varphi_E^\gamma \circ u \circ \kappa_\gamma^{-1})] \circ \kappa_\gamma, \end{aligned}$$

where $u \in C_c^\infty(U^\gamma; E)$. Let $\{f_\gamma\}$ be a partition of unity subordinated to the atlas $\{(\kappa_\gamma, U^\gamma)\}$, and $\bar{f}_\gamma \in C_c^\infty(U^\gamma)$ test functions satisfying $\bar{f}_\gamma \equiv 1$ on $\text{supp } f_\gamma$. Denote by F_γ and \bar{F}_γ the multiplication operators corresponding to f_γ and \bar{f}_γ , respectively, and define on M

$$(8) \quad R_K(\lambda, P) = \sum_{k=0}^K \sum_{\gamma} F_\gamma R_k^\gamma(\lambda, P) \bar{F}_\gamma, \quad E_K(t, P) = \sum_{k=0}^K \sum_{\gamma} F_\gamma E_k^\gamma(t, P) \bar{F}_\gamma.$$

Explicitly, one computes for $u \in C_c^\infty(E)$

$$\begin{aligned} (9) \quad F_\gamma E_k^\gamma(t, P) \bar{F}_\gamma u(x) &= f_\gamma(x) \left[(\varphi_E^\gamma)^{-1} \circ [\tilde{E}_k^\gamma(t, P)(\varphi_E^\gamma \circ \bar{f}_\gamma u \circ \kappa_\gamma^{-1})] \circ \kappa_\gamma \right](x) \\ &= f_\gamma(x) (\varphi_E^\gamma)^{-1} \left[\int e^{i\kappa_\gamma(x)\cdot\eta} e_k^\gamma(t, \kappa_\gamma(x), \eta, P) \mathcal{F}((\varphi_E^\gamma \circ \bar{f}_\gamma u \circ \kappa_\gamma^{-1})\beta_\gamma)(\eta) d\eta \right] \\ &= f_\gamma(x) (\varphi_E^\gamma)^{-1} \left[\int_{\tilde{U}^\gamma} \int e^{i[\kappa_\gamma(x)-\tilde{y}]\cdot\eta} e_k^\gamma(t, \kappa_\gamma(x), \eta, P) (\varphi_E^\gamma \circ \bar{f}_\gamma u)(\kappa_\gamma^{-1}(\tilde{y})) \beta_\gamma(\tilde{y}) d\tilde{\eta} d\tilde{y} \right] \\ &= f_\gamma(x) (\varphi_E^\gamma)^{-1} \left[\int_{U^\gamma} \int e^{i[\kappa_\gamma(x)-\kappa_\gamma(y)]\cdot\eta} \bar{f}_\gamma(y) e_k^\gamma(t, \kappa_\gamma(x), \eta, P) (\varphi_E^\gamma \circ u)(y) d\eta dM(y) \right]. \end{aligned}$$

The operators $R_K(\lambda, P)$ and $E_K(t, P)$ are approximations of the resolvent $(P - \lambda \mathbf{1})^{-1}$ and the heat operator e^{-tP} , respectively, as $K \rightarrow \infty$. More precisely, if $Q : H^s(E) \rightarrow H^{s'}(E)$ is an operator between Sobolev spaces of sections, define the operator norms

$$\|Q\|_{s, s'} = \sup_{u \in C^\infty, u \neq 0} \|Qu\|_{s'} \|u\|_s^{-1}.$$

Then, one has the following

Lemma 2.

(1) For every $K \in \mathbb{N}$, we have

$$(P - \lambda \mathbf{1})R_K(\lambda, P) - \mathbf{1} \sim_K 0, \quad R_K(\lambda, P)(P - \lambda \mathbf{1}) - \mathbf{1} \sim_K 0.$$

(2) The operators $E_K(t, P)$ have smooth kernels and for every $l \in \mathbb{N}$ there exists a $K(l) \in \mathbb{N}$ such that for $0 < t < 1$

$$\|e^{-tP} - E_K(t, P)\|_{-l, l} \leq C t^l$$

for all $K \geq K(l)$.

Proof. See Gilkey, [9], Lemmata 1.7.2 and 1.8.1. \square

Let now $T_j(g)$ and Δ_j be as in Lemma 1, and $l \in \mathbb{N}$. Assertion (2) of the preceding lemma implies that $E_K(t, P)$ is of L^2 -trace class, and for every $K \geq K(l)$

$$(10) \quad \text{tr}_{L^2}(T_j(g)e^{-t\Delta_j}) = \text{tr}_{L^2}(T_j(g)E_K(t, \Delta_j)) + O(t^l).$$

Since $T_j(g) = \Phi_j(g) \circ g^*$, (8) and (9) imply that

$$\begin{aligned}
(11) \quad \text{tr}_{L^2} (T_j(g) E_K(t, \Delta_j)) &= \sum_{k=0}^K \sum_{\gamma} \int_M \int e^{i[\kappa_{\gamma}(gx) - \kappa_{\gamma}(x)] \cdot \eta} \bar{f}_{\gamma}(x) f_{\gamma}(gx) \\
&\cdot \text{tr} \left[\Phi_j(g)_{gx} \circ (\varphi_{E_j}^{\gamma})_{gx}^{-1} \circ e_k^{\gamma}(t, \kappa_{\gamma}(gx), \eta, \Delta_j) \circ (\varphi_{E_j}^{\gamma})_x \right] d\eta dM(x) \\
&= \sum_{k=0}^K t^{\frac{k-n}{m}} \sum_{\gamma} \int_M \int e^{i t^{1/m} [\kappa_{\gamma}(gx) - \kappa_{\gamma}(x)] \cdot \eta} \bar{f}_{\gamma}(x) f_{\gamma}(gx) \\
&\cdot \text{tr} \left[\Phi_j(g)_{gx} \circ (\varphi_{E_j}^{\gamma})_{gx}^{-1} \circ e_k^{\gamma}(1, \kappa_{\gamma}(gx), \eta, \Delta_j) \circ (\varphi_{E_j}^{\gamma})_x \right] d\eta dM(x),
\end{aligned}$$

where we took into account that $e_k^{\gamma}(t, \kappa_{\gamma}(gx), \eta, \Delta_j) = t^{k/m} e_k^{\gamma}(1, \kappa_{\gamma}(gx), t^{1/m} \cdot \eta, \Delta_j)$. As a consequence of (10), we obtain the following

Proposition 1. *Let $g \in G$ be fixed and $t > 0$. For every $l \in \mathbb{N}$ there exists a $K(l) \in \mathbb{N}$ such that*

$$\begin{aligned}
\text{tr}_{L^2} (T_j(g) e^{-t\Delta_j}) &= \sum_{k=0}^K t^{\frac{k-n}{m}} \sum_{\gamma} \int_M \int e^{i t^{1/m} [\kappa_{\gamma}(gx) - \kappa_{\gamma}(x)] \cdot \eta} \bar{f}_{\gamma}(x) f_{\gamma}(gx) \\
&\cdot \text{tr} \left[\Phi_j(g)_{gx} \circ (\varphi_{E_j}^{\gamma})_{gx}^{-1} \circ e_k^{\gamma}(1, \kappa_{\gamma}(gx), \eta, \Delta_j) \circ (\varphi_{E_j}^{\gamma})_x \right] d\eta dM(x) + O(t^l)
\end{aligned}$$

for all $K \geq K(l)$. □

Now, Lemma 1 implies that for any $t > 0$

$$\mathcal{L}_{\varrho}(T) = \frac{1}{\text{vol } G} \sum_{j=0}^N (-1)^j \int_G \text{tr}_{L^2} (T_j(g) e^{-t\Delta_j}) \overline{\varrho(g)} d_G(g).$$

With Proposition 1 we therefore obtain

Theorem 3. *Let $\varrho \in \hat{G}$. For every $l \in \mathbb{N}$ there exists a $K(l) \in \mathbb{N}$ such that*

$$\begin{aligned}
(12) \quad \mathcal{L}_{\varrho}(T) &= \frac{1}{\text{vol } G} \sum_{j=0}^N (-1)^j \int_G \sum_{k=0}^K t^{\frac{k-n}{m}} \sum_{\gamma} \int_M \int e^{i t^{1/m} [\kappa_{\gamma}(gx) - \kappa_{\gamma}(x)] \cdot \eta} \bar{f}_{\gamma}(x) f_{\gamma}(gx) \\
&\cdot \text{tr} \left[\Phi_j(g)_{gx} \circ (\varphi_{E_j}^{\gamma})_{gx}^{-1} \circ e_k^{\gamma}(1, \kappa_{\gamma}(gx), \eta, \Delta_j) \circ (\varphi_{E_j}^{\gamma})_x \right] d\eta dM(x) \overline{\varrho(g)} d_G(g) + O(t^l)
\end{aligned}$$

for all $K \geq K(l)$ and any $t > 0$. □

The left-hand side of (12) does not depend on $t > 0$. In order to find a local formula for $\mathcal{L}_{\varrho}(T)$, we have to find an asymptotic expansion of the right-hand side, and determine the constant term. We are therefore left with the task of examining the asymptotic behavior of integrals of the form

$$(13) \quad I(\mu) = \int_{T^*U} \int_G e^{i\Phi(x, \xi, g)/\mu} a(gx, x, \xi, g) d_G(g) d(T^*U)(x, \xi), \quad \mu \rightarrow 0^+,$$

via the generalized stationary phase theorem, where (κ, U) are local coordinates on M , $d(T^*U)(x, \xi)$ is the canonical volume density on T^*U , and $d_G(g)$ is the volume density of a left invariant metric on G , while $a \in C_c^{\infty}(U \times T^*U \times G)$ is an amplitude which does not depend on μ , and

$$(14) \quad \Phi(x, \xi, g) = (\kappa(gx) - \kappa(x)) \cdot \xi.$$

This will be done in the next section.

5. SINGULAR EQUIVARIANT ASYMPTOTICS AND RESOLUTION OF SINGULARITIES

To examine the asymptotic behavior of the integrals (13) by means of the stationary phase principle, we have to study the critical set of the phase function (14). Consider for this the cotangent bundle $\pi : T^*M \rightarrow M$, as well as the tangent bundle $\tau : T(T^*M) \rightarrow T^*M$, and define on T^*M the Liouville form

$$\Theta(\mathfrak{X}) = \tau(\mathfrak{X})[\pi_*(\mathfrak{X})], \quad \mathfrak{X} \in T(T^*M).$$

Regard T^*M as a symplectic manifold with symplectic form $\omega = d\Theta$, and define for any element X in the Lie algebra \mathfrak{g} of G the function

$$J_X : T^*M \longrightarrow \mathbb{R}, \quad \eta \mapsto \Theta(\tilde{X})(\eta),$$

where \tilde{X} denotes the fundamental vector field on T^*M , respectively M , generated by X . G acts on T^*M in a Hamiltonian way, and the corresponding symplectic momentum map is given by

$$\mathbb{J} : T^*M \rightarrow \mathfrak{g}^*, \quad \mathbb{J}(\eta)(X) = J_X(\eta).$$

Let (κ, U) be local coordinates on M as in (13), and write $\kappa(x) = (\tilde{x}_1, \dots, \tilde{x}_n)$, $\eta = \sum \xi_i (d\tilde{x}_i)_x \in T_x^*U$. One computes then for any $X \in \mathfrak{g}$

$$\begin{aligned} \frac{d}{dt} \Phi(x, \xi, e^{-tX})|_{t=0} &= \frac{d}{dt} \left(\kappa(e^{-tX} x) \cdot \xi \right)|_{t=0} = \sum \xi_i \tilde{X}_x(\tilde{x}_i) = \sum \xi_i (d\tilde{x}_i)_x(\tilde{X}_x) \\ &= \eta(\tilde{X}_x) = \Theta(\tilde{X})(\eta) = \mathbb{J}(\eta)(X). \end{aligned}$$

Therefore Φ represents the global analogue of the momentum map. Further, one has

$$\partial_{\tilde{x}} \Phi(\kappa^{-1}(\tilde{x}), \xi, g) = [{}^T(\kappa \circ g \circ \kappa^{-1})_{*, \tilde{x}} - \mathbf{1}] \xi = (g_x^* - \mathbf{1}) \cdot \xi,$$

so that $\partial_x \Phi(x, \xi, g) = 0$ amounts precisely to the condition $g^* \xi = \xi$. Since $\partial_\xi \Phi(x, \xi, g) = 0$ if, and only if $gx = x$, one obtains

$$\begin{aligned} (15) \quad \mathcal{C} = \text{Crit}(\Phi) &= \{(x, \xi, g) \in T^*U \times G : (\Phi_*)_{(x, \xi, g)} = 0\} \\ &= \{(x, \xi, g) \in (\Xi \cap T^*U) \times G : g \cdot (x, \xi) = (x, \xi)\}, \end{aligned}$$

where $\Xi = \mathbb{J}^{-1}(0)$ is the zero level of the momentum map. If G acts on M only with one orbit type, the critical set of the phase function $\Phi(x, \xi, g)$ is clean. In this case, the stationary phase method can directly be applied to yield an asymptotic expansion of the integrals $I(\mu)$.

Proposition 2. *Let M be a connected, closed Riemannian manifold, and G a compact, connected Lie group of isometries acting on M with one orbit type. Consider further an oscillatory integral $I(\mu)$ of the form (13). We then have the asymptotic expansion*

$$I(\mu) \sim (2\pi\mu)^\kappa \sum_{j=0}^{\infty} \mu^j Q_j(\Phi; a)$$

as $\mu \rightarrow 0^+$, where κ denotes the dimension of an orbit of principal type, and the coefficients $Q_j(\Phi; a)$ can be computed explicitly. In particular, one has

$$Q_0(\Phi; a) = \int_{\mathcal{C}} \frac{a(m)}{|\det \Phi''(m)|_{N_m \mathcal{C}}|^{1/2}} d\sigma_{\mathcal{C}}(m),$$

where $d\sigma_{\mathcal{C}}$ is the induced volume density on the critical set $\mathcal{C} = \text{Crit}(\Phi)$ given by (15).

Proof. It is not hard to see that under the assumption that G acts on M only with one orbit type, $\Phi(x, \xi, g)$ has a clean critical set, meaning that

- (I) \mathcal{C} is a smooth submanifold of M of codimension 2κ ;

(II) at each point $x \in \mathcal{C}$, the Hessian $\Phi''(x)$ of Φ is transversally non-degenerate, i.e. non-degenerate on $T_x M / T_x \mathcal{C} \simeq N_x \mathcal{C}$, where $N_x \mathcal{C}$ denotes the normal space to \mathcal{C} at x .

The generalized stationary phase theorem [16], Theorem 5, then implies that for all $N \in \mathbb{N}$, there exists a constant $C_{N, \Phi, a} > 0$ such that

$$\left| I(\mu) - e^{i\Phi_0/\mu} (2\pi\mu)^\kappa \sum_{j=0}^{N-1} \mu^j Q_j(\Phi; a) \right| \leq C_{N, \Phi, a} \mu^N,$$

where Φ_0 is the constant value of Φ on \mathcal{C} . Furthermore, the $Q_j(\Phi; a)$ can be computed explicitly, and for each j there exists a constant $\tilde{C}_{j, \Phi, a} > 0$ such that

$$|Q_j(\Phi; a)| \leq \tilde{C}_{j, \Phi, a}.$$

In particular,

$$Q_0(\Phi; a) = \int_{\mathcal{C}} \frac{a(m)}{|\det \Phi''(m)|_{N_m \mathcal{C}}|^{1/2}} d\sigma_{\mathcal{C}}(m) e^{i\frac{\Phi_0}{2}\sigma_{\Phi''}},$$

where $d\sigma_{\mathcal{C}}$ is the induced volume density on \mathcal{C} , and $\sigma_{\Phi''}$ the constant value of the signature of the transversal Hessian $\Phi''(m)|_{N_m \mathcal{C}}$ on \mathcal{C} . Since $\Phi_0 = 0$, one computes for arbitrary $N \in \mathbb{N}$

$$\begin{aligned} \left| I(\mu) - (2\pi\mu)^\kappa \sum_{j=0}^{N-1} \mu^j Q_j(\Phi; a) \right| &\leq \left| I(\mu) - (2\pi\mu)^\kappa \sum_{j=0}^{\kappa+N-1} \mu^j Q_j(\Phi; a) \right| + \left| (2\pi\mu)^\kappa \sum_{j=N}^{\kappa+N-1} \mu^j Q_j(\Phi; a) \right| \\ &\leq C_{\kappa+N, \Phi, a} \mu^{\kappa+N} + (2\pi\mu)^\kappa \sum_{j=N}^{\kappa+N-1} \mu^j \tilde{C}_{j, \Phi, a} = O(\mu^{\kappa+N}), \end{aligned}$$

yielding the proposition. \square

In general, the major difficulty resides in the fact that, unless the G -action on T^*M is free, the considered momentum map is not a submersion, so that Ξ and $\mathcal{C} = \text{Crit}(\Phi)$ are not smooth manifolds. The stationary phase theorem can therefore not immediately be applied to the integrals $I(\mu)$. Nevertheless, it was shown in [16] that by resolving the singularities of the critical set \mathcal{C} , and applying the stationary phase theorem in a suitable resolution space, an asymptotic description of $I(\mu)$ can be obtained. More precisely, one has the following

Theorem 4. *Let M be a connected, closed Riemannian manifold, and G a compact, connected Lie group G acting isometrically and effectively on M . Consider the oscillatory integral*

$$I(\mu) = \int_{T^*U} \int_G e^{i\Phi(x, \xi, g)/\mu} a(gx, x, \xi, g) d_G(g) d(T^*U)(x, \xi), \quad \mu \rightarrow 0^+,$$

where (κ, U) are local coordinates on M , $d(T^*U)(x, \xi)$ is the canonical volume density on T^*U , and $d_G(g)$ the volume density on G with respect to some left invariant metric on G , while $a \in C_c^\infty(U \times T^*U \times G)$ is an amplitude, and $\Phi(x, \xi, g) = (\kappa(gx) - \kappa(x)) \cdot \xi$. Then $I(\mu)$ has the asymptotic expansion

$$I(\mu) = (2\pi\mu)^\kappa \mathcal{L}_0 + O(\mu^{\kappa+1} (\log \mu^{-1})^{\Lambda-1}), \quad \mu \rightarrow 0^+.$$

Here κ is the dimension of an orbit of principal type in M , Λ the maximal number of elements of a totally ordered subset of the set of isotropy types, and the leading coefficient is given by

$$(16) \quad \mathcal{L}_0 = \int_{\text{Reg } \mathcal{C}} \frac{a(gx, x, \xi, g)}{|\det \Phi''(x, \xi, g)|_{N_{(x, \xi, g)} \text{Reg } \mathcal{C}}|^{1/2}} d(\text{Reg } \mathcal{C})(x, \xi, g),$$

where $\text{Reg } \mathcal{C}$ denotes the regular part of $\mathcal{C} = \{(x, \xi, g) \in \Xi \times G : g \cdot (x, \xi) = (x, \xi)\}$, and $d(\text{Reg } \mathcal{C})$ the induced volume density. In particular, the integral over $\text{Reg } \mathcal{C}$ exists.

Proof. See [16], Theorem 11. □

6. A LOCAL FORMULA FOR $\mathcal{L}_\varrho(T)$

We are now able to derive a local formula for $\mathcal{L}_\varrho(T)$. Let us begin with the non-singular case.

Proposition 3. *Let M be a connected, closed Riemannian manifold, and G a compact, connected Lie group G of isometries acting on M with one orbit type. Take $\varrho \in \hat{G}$, and let $\mathcal{L}_\varrho(T)$ be the ϱ -equivariant Lefschetz number defined in (4). Then*

$$\mathcal{L}_\varrho(T) = \frac{(2\pi)^{\kappa-n}}{\text{vol } G} \sum_{j=0}^N \sum_{k+q=n-\kappa} \sum_{\gamma} (-1)^j Q_q(\Phi; a_{j,k,\gamma}),$$

where the coefficients $Q_q(\Phi; a_{j,k,\gamma})$ can be computed explicitly.

Proof. As an immediate consequence of Theorem 3 and Proposition 2, for any l and $L \in \mathbb{N}$ there exists a $K(l)$ such that for all $K \geq K(l)$ one has the expansion

$$\mathcal{L}_\varrho(T) = \frac{1}{\text{vol } G} t^{\frac{\kappa-n}{m}} \sum_{j=0}^N (-1)^j \sum_{k=0}^K t^{\frac{k}{m}} \sum_{\gamma} \left[(2\pi)^{\kappa-n} \sum_{q=0}^{L-1} t^{\frac{q}{m}} Q_q(\Phi; a_{j,k,\gamma}) + O(t^{\frac{L}{m}}) \right] + O(t^l)$$

for any $t > 0$, where

$$(17) \quad a_{j,k,\gamma}(x, \eta, g) = \bar{f}_\gamma(x) f_\gamma(gx) \cdot \text{tr} \left[\Phi_j(g)_{gx} \circ (\varphi_{E_j}^\gamma)_{gx}^{-1} \circ e_k^\gamma(1, \kappa_\gamma(gx), \eta, \Delta_j) \circ (\varphi_{E_j}^\gamma)_x \right] \cdot \overline{\varrho(g)}.$$

Choose $l, L > n - \kappa$. Since $\mathcal{L}_\varrho(T)$ is independent of t , it is equal to the constant term in this expansion, while all other terms must vanish. The assertion now follows. □

We come now to the general case and to the proof of the main result.

Proof of Theorem 2. By Theorems 3 and 4, for any l there exists a $K(l)$ such that

$$(18) \quad \mathcal{L}_\varrho(T) = \frac{1}{\text{vol } G} t^{\frac{\kappa-n}{m}} \sum_{j=0}^N (-1)^j \sum_{k=0}^K t^{\frac{k}{m}} \sum_{\gamma} \left[(2\pi)^{\kappa-n} \mathcal{L}_{j,k,\gamma} + O(t^{\frac{1}{m}} (\log t^{-1})^{\Lambda-1}) \right]$$

up to terms of order $O(t^l)$ for all $K \geq K(l)$ and any $t > 0$, where

$$\mathcal{L}_{j,k,\gamma} = \int_{\text{Reg } \mathcal{C}} \frac{a_{j,k,\gamma}(x, \eta, g)}{|\det \Phi_\gamma''(x, \eta, g)_{N(x, \eta, g) \text{Reg } \mathcal{C}}|^{1/2}} d(\text{Reg } \mathcal{C})(x, \eta, g),$$

$\Phi_\gamma(x, \eta, g) = (\kappa_\gamma(gx) - \kappa_\gamma(x)) \cdot \eta$, and $a_{j,k,\gamma}(x, \eta, g)$ restricted to \mathcal{C} is given by

$$a_{j,k,\gamma}(x, \eta, g) = f_\gamma(x) \cdot \text{tr} \left[\Phi_j(g)_x \circ (\varphi_{E_j}^\gamma)_x^{-1} \circ e_k^\gamma(1, \kappa_\gamma(x), \eta, \Delta_j) \circ (\varphi_{E_j}^\gamma)_x \right] \cdot \overline{\varrho(g)}.$$

Note that at a fixed point x , $\Phi_j(g)_x$ is an endomorphism of $E_{j,x}$, so that the above trace is well defined. Choose $l > n - \kappa$. Since $\mathcal{L}_\varrho(T)$ must be equal to the constant term in the expansion (18), one finally obtains the equality

$$\mathcal{L}_\varrho(T) = \frac{(2\pi)^{\kappa-n}}{\text{vol } G} \sum_{j=0}^N (-1)^j \sum_{\gamma} \left[\mathcal{L}_{j,n-\kappa,\gamma} + \mathcal{R}_{j,\gamma} \right],$$

where the remainder terms $\mathcal{R}_{j,\gamma}$ do depend on amplitudes $a_{j,k,\gamma}$ with $0 \leq k \leq n - \kappa - 1$. This proves Assertion (1) of Theorem 2. Assume now that $\Phi_j(g)_x$ acts trivially on $E_{j,x}$, and recall that for any smooth, compactly supported function α on $\Xi \cap T^*U^\gamma$ one has the formula

$$\int_{\text{Reg } \mathcal{C}} \frac{\overline{\varrho(g)} \alpha(x, \eta)}{|\det \Phi_\gamma''(x, \eta, g)_{N(x, \eta, g) \text{Reg } \mathcal{C}}|^{1/2}} d(\text{Reg } \mathcal{C})(x, \eta, g) = [\pi_{\varrho|H} : 1] \int_{\text{Reg } \Xi} \alpha(x, \eta) \frac{d(\text{Reg } \Xi)(x, \eta)}{\text{vol } \mathcal{O}(x, \eta)},$$

where H is a principal isotropy group, and $[\pi_{\varrho|H} : 1]$ denotes the multiplicity of the trivial representation in the restriction of π_{ϱ} to H , while $\mathcal{O}_{(x,\eta)}$ is the orbit in T^*M through (x, η) , compare [4], Lemma 7. In this case,

$$\mathcal{L}_{j,k,\gamma} = [\pi_{\varrho|H} : 1] \int_{\text{Reg } \Xi} f_{\gamma}(x) \cdot \text{tr} \left[(\varphi_{E_j}^{\gamma})_x^{-1} \circ e_k^{\gamma}(1, \kappa_{\gamma}(x), \eta, \Delta_j) \circ (\varphi_{E_j}^{\gamma})_x \right] \frac{d(\text{Reg } \Xi)(x, \eta)}{\text{vol } \mathcal{O}_{(x,\eta)}},$$

and we obtain Assertion (2) of Theorem 2. \square

7. OUTLOOK

A few years after the index theorem was proved by heat equation methods, the same techniques were employed to derive generalized Lefschetz fixed point formulae. Thus, for fixed $g \in G$, asymptotic expansions for $\text{tr}_{L^2}(T_j(g)e^{-t\Delta_j})$ were obtained by Gilkey and Lee, see [9], Lemma 1.10.2, and also by Donnelly [5]. As they showed,

$$(19) \quad \text{tr}_{L^2}(T_j(g)e^{-t\Delta_j}) \sim \sum_i \sum_k t^{\frac{k-n_{g,i}}{m}} \int_{N_{g,i}} a_k(x, \Delta_j, T_j(g)) dN_{g,i}(x),$$

where the $N_{g,i}$ are the connected components of dimension $n_{g,i}$ of the fixed point set of $g : M \rightarrow M$, and the $a_k(x, \Delta_j, T_j(g))$ are scalar invariants depending functorially on the symbol of Δ_j and on $T_j(g)$. The existence of such expansions strongly suggested new proofs of the Atiyah–Singer–Lefschetz fixed point formulae for compact group actions. In the case of isolated fixed points, Kotake gave an expansion of $\text{tr}_{L^2}(T_j(g)e^{-t\Delta_j})$ in [12], which was sufficient to give a new proof of Theorem 1 by heat equation methods. In general, as a consequence of the expansion (19) and Lemma 1, one has the local formulae

$$L(T(g)) = \sum_i \sum_{j=0}^N (-1)^j \int_{N_{g,i}} a_{n_i}(x, \Delta_j, T_j(g)) dN_{g,i}(x),$$

and using invariance theory, the terms in these formulae can be identified as characteristic classes. In this way, Donnelly–Patodi [6] and Kawasaki [11] gave a new proof of the G -signature theorem of Atiyah–Singer in the case of the signature complex, while the other classical complexes were treated in Gilkey [8]. In the same way, Theorem 2 suggests that it should be possible to find global expressions for $\mathcal{L}_{\varrho}(T)$ in terms of characteristic classes using invariance theory. This will be pursued in a subsequent paper, and should lead to topological formulae relating characteristic classes of fixed point sets on M to characteristic classes of the symplectic quotient Ξ/G .

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