SINGULAR EQUIVARIANT ASYMPTOTICS AND THE MOMENTUM MAP. RESIDUE FORMULAE IN EQUIVARIANT COHOMOLOGY

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Abstract. Let M be a smooth manifold and G a compact connected Lie group acting on M by isometries. In this paper, we study the equivariant cohomology of $\mathbf{X} = T^*M$, and relate it to the cohomology of the Marsden-Weinstein reduced space via certain residue formulae. In case that $\mathbf{X}$ is a compact symplectic manifold with a Hamiltonian G-action, similar residue formulae were derived by Jeffrey, Kirwan et al. [26, 25].

Contents

1. Introduction

Let $\mathbf{X}$ be a symplectic manifold carrying a Hamiltonian action of a compact, connected Lie group G with Lie algebra $\mathfrak{g}$, and denote the corresponding momentum map by $\mathbb{J} : \mathbf{X} \to \mathfrak{g}^*$. In case that $\mathbf{X}$ is compact and 0 a regular value of the momentum map, the cohomology of the Marsden-Weinstein reduced space $\mathbf{X}_{red} = \mathbb{J}^{-1}(0)/G$ was expressed by Jeffrey and Kirwan [26] in terms of the equivariant cohomology of $\mathbf{X}$ via certain residue formulae. If 0 is not a regular value, similar residue formulae were derived by them and their collaborators [25] for nonsingular, connected, complex projective varieties $\mathbf{X}$. These formulae rely on the localization theorem for compact group actions of Berline-Vergne [4, 3], and are related to the non-Abelian localization theorem of Witten [40]. The intention of this paper is to extend their results to non-compact situations, and derive similar residue formulae in case that $\mathbf{X}$ is given by the cotangent bundle of a G-manifold.

Date: October 7, 2013.
Key words and phrases. Equivariant cohomology, residue formulae, momentum map, symplectic quotients, stationary phase principle, resolution of singularities.

Let $\mathbf{X}$ be a smooth manifold carrying a smooth action of a connected Lie group G. According to Cartan [11], its equivariant cohomology can be defined by replacing the algebra $\Lambda(\mathbf{X})$ of smooth differential forms on $\mathbf{X}$ by the algebra $(S(\mathfrak{g}^*)\otimes\Lambda(\mathbf{X}))^G$ of G-equivariant polynomial mappings

$$\varrho : \mathfrak{g} \ni X \longmapsto \varrho(X) \in \Lambda(\mathbf{X}),$$

where $\mathfrak{g}$ denotes the Lie algebra of G. Let $\widetilde{X}$ denote the fundamental vector field on $\mathbf{X}$ generated by an element $X \in \mathfrak{g}$. Defining *equivariant exterior differentiation* by

$$D\varrho(X) = d(\varrho(X)) - \iota_{\widetilde{X}}(\varrho(X)), \qquad X \in \mathfrak{g},\ \varrho \in (S(\mathfrak{g}^*)\otimes\Lambda(\mathbf{X}))^G,$$

where d and ι denote the usual exterior differentiation and contraction, the *equivariant cohomology* of the G-action on $\mathbf{X}$ is given by the quotient

$$H^*_G(\mathbf{X}) = \operatorname{Ker} D/\operatorname{Im} D,$$

which is canonically isomorphic to the topological equivariant cohomology introduced in [2] in case that G is compact, an assumption that we will make from now on. The main difference between ordinary and equivariant cohomology is that the latter has a larger coefficient ring, namely $S(\mathfrak{g}^*)$, and that it depends on the orbit structure of the underlying G-action. Let us now assume that $\mathbf{X}$ admits a symplectic structure ω which is left invariant by G. By Cartan's homotopy formula,

$$0 = \mathcal{L}_{\widetilde{X}}\omega = d\circ\iota_{\widetilde{X}}\omega + \iota_{\widetilde{X}}\circ d\omega = d\circ\iota_{\widetilde{X}}\omega,$$

where $\mathcal{L}$ denotes the Lie derivative with respect to a vector field, implying that $\iota_{\widetilde{X}}\omega$ is closed for each $X\in\mathfrak{g}$. G is said to act on $\mathbf{X}$ in a *Hamiltonian fashion*, if this form is even exact, meaning that there exists a linear function $J:\mathfrak{g}\to \mathrm{C}^\infty(\mathbf{X})$ such that for each $X\in\mathfrak{g}$, the fundamental vector field $\widetilde{X}$ is equal to the Hamiltonian vector field of $J(X)$, so that

$$d(J(X)) + \iota_{\widetilde{X}}\omega = 0.$$

An immediate consequence of this is that for any equivariantly closed form ϱ the form given by $e^{i(J(X)-\omega)}\varrho(X)$ is equivariantly closed, too. Following Souriau and Kostant, one defines the momentum map of a Hamiltonian action as the equivariant map

$$\mathbb{J}:\mathbf{X}\longrightarrow\mathfrak{g}^*, \quad \mathbb{J}(\eta)(X) = J(X)(\eta).$$

Assume next that $0\in\mathfrak{g}^*$ is a regular value of $\mathbb{J}$, which is equivalent to the assumption that the stabilizer of each point of $\mathbb{J}^{-1}(0)$ is finite. In this case, $\mathbb{J}^{-1}(0)$ is a smooth manifold, and the corresponding *Marsden-Weinstein reduced space*, or *symplectic quotient*

$$\mathbf{X}_{red} = \mathbb{J}^{-1}(0)/G$$

is an orbifold with a unique symplectic form ω_{red} determined by the identity $\iota^*\omega = \pi^*\omega_{red}$, where $\pi:\mathbb{J}^{-1}(0)\to\mathbf{X}_{red}$ and $\iota:\mathbb{J}^{-1}(0)\hookrightarrow\mathbf{X}$ denote the canoncial projection and inclusion, respectively. Furthermore, π^* induces an isomorphism between $H^*(\mathbf{X}_{red})$ and $H^*_G(\mathbb{J}^{-1}(0))$. Consider now the map

$$\mathcal{K}: H^*_G(\mathbf{X}) \xrightarrow{\iota^*} H^*_G(\mathbb{J}^{-1}(0)) \xrightarrow{(\pi^*)^{-1}} H^*(\mathbf{X}_{red}),$$

and assume that $\mathbf{X}$ is compact and oriented. In this case, Kirwan [28] showed that $\mathcal{K}$ defines a surjective homomorphism, so that the cohomology of $\mathbf{X}_{red}$ should be computable from the equivariant cohomology of $\mathbf{X}$. This is the content of the residue formula of Jeffrey and Kirwan [26], which for any $\varrho\in H^*_G(\mathbf{X})$ expresses the integral

$$\int_{\mathbf{X}_{red}} e^{-i\omega_{red}}\mathcal{K}(\varrho) = \int_{\mathbf{X}_{red}} \sum_{k=0}^{\dim\mathbf{X}_{red}/2} \frac{(-i\omega_{red})^k}{k!}\mathcal{K}(\varrho)_{[\dim\mathbf{X}_{red}-2k]} \tag{1}$$

in terms of data of $\mathbf{X}$. More precisely, let $T \subset G$ be a maximal torus, and $\mathbf{X}^T$ its fixed point set. Then (1) is given by a sum over the components F of $\mathbf{X}^T$ of certain residues involving the restriction of ϱ to the G-orbit $G \cdot F$ and the equivariant Euler form χ_{NF} of the normal bundle NF of F. The departing point of their work is the observation that the integral (1) should be given by the $\mathfrak{g}$-Fourier transform of the tempered distribution

$$\mathfrak{g} \ni X \mapsto \int_{\mathbf{X}} e^{i(J(X)-\omega)} \varrho(X)$$

evaluated at $0 \in \mathfrak{g}^*$. The mentioned formula of Jeffrey and Kirwan is then essentially a consequence of the localization formula of Berline and Vergne [4]. In case that $0 \in \mathfrak{g}^*$ is not a regular value, analogous residue formulae were derived in [25] for nonsingular, connected, complex projective varieties $\mathbf{X}$ within the framework of geometric invariant theoretic quotients, under some weak assumptions about the group action. In this situation, there is no longer a surjection from equivariant cohomology onto the cohomology of the corresponding quotient, whose singularities are worse than in the orbifold case. Nevertheless, their is still a surjection onto its intersection cohomology, which is a direct summand of the ordinary cohomology of any resolution of singularities of the quotient. Using a canonical desingularization procedure for such quotients developed by Kirwan [29] in combination with certain residue operations established by Guillemin and Kalkman [21], residue formulae for intersection pairings can then be derived.

Historically, the Berline-Vergne localization formula emerged as a generalization of a result of Duistermaat and Heckman [17] concerning the pushforward of the Liouville measure of a compact, symplectic manifold carrying a Hamiltonian torus action along the momentum map. As it turns out, this pushforward is a piecewise polynomial measure, or equivalently, its inverse Fourier transform is exactly given by the leading term in the stationary phase approximation. The study of the pushforward of the Liouville measure was motivated by attempts of finding an asymptotic approximation to the Kostant multiplicity formula [31] in order to examine the partition function occuring in that formula, which otherwise is very difficult to evaluate [22]. On the other side, the origin of the Berline-Vergne localization formula can be traced back to a residue formula for holomorphic vector fields derived by Bott [7], which was inspired by the generalized Lefschetz formula of Atiyah and Bott [1].

In this paper, we shall prove a residue formula in case that $\mathbf{X} = T^*M$ is given by the cotangent bundle of a smooth manifold M on which a compact, connected Lie group G acts by general isometries. For this, we shall determine the asymptotic behavior of integrals of the form

$$I_\varsigma(\mu) = \int_{\mathfrak{g}} \left[\int_{\mathbf{X}} e^{i(\mathbb{J}(\eta)-\varsigma)(X)/\mu} a(\eta, X)\, d\eta \right] dX, \qquad \mu \to 0^+,$$

via the stationary phase principle, where $\varsigma \in \mathfrak{g}^*$, $a \in \mathrm{C}^\infty_c(\mathbf{X} \times \mathfrak{g})$ is an amplitude, $d\eta$ the Liouville measure on $\mathbf{X}$, and dX denotes an Euclidean measure on $\mathfrak{g}$ given by an $\mathrm{Ad}\,(G)$-invariant inner product on $\mathfrak{g}$. While asymptotics for $I_\varsigma(\mu)$ can be easily obtained for free group actions, one meets with serious difficulties when singular orbits are present. The reason is that, when trying to examine these integrals in case that $\varsigma \in \mathfrak{g}^*$ is not a regular value of the momentum map, the critical set of $(\mathbb{J}(\eta)-\varsigma)(X)$ is no longer smooth, so that, a priori, the stationary phase principle can not be applied in this case. Instead, we shall circumvent this obstacle in the case $\varsigma = 0$ by partially resolving the singularities of the critical set of the momentum map, and then apply the stationary phase theorem in a suitable resolution space. By this we are able to obtain asymptotics for $I_0(\mu)$ with remainder estimates in the case of singular group actions. This approach was developed first in [13, 36] to describe the spectrum of an invariant elliptic operator on a compact G-manifold, where similar integrals occur, and used in the derivation of equivariant heat asymptotics in [35]. The asymptotic description of $I_\varsigma(\mu)$ in a neighborhood of $\varsigma = 0$ then allows us to derive the following residue formula. Let $\varrho \in H^*_G(T^*M)$ be of the form $\varrho(X) = \alpha + D\beta(X)$, where α is a closed, basic differential form on T^*M of compact support, and β is an equivariant differential form of compact support. Fix a maximal torus $T \subset G$, and denote the corresponding root system by $\Delta(\mathfrak{g}, \mathfrak{t})$. Assume that the dimension κ of a principal G-orbit is equal to $d = \dim \mathfrak{g}$, and denote the product of the positive roots by Φ. Let further W be the Weyl group and H a principal isotropy group of the G-action. Denote the principal stratum of $\mathbb{J}^{-1}(0)$ by $\operatorname{Reg} \mathbb{J}^{-1}(0)$, and put $\operatorname{Reg} \mathbf{X}_{red} = \operatorname{Reg} \mathbb{J}^{-1}(0)/G$. Also, let $r : \Lambda^*(\mathbf{X}) \to \Lambda^{*-\kappa}(\operatorname{Reg} \mathbb{J}^{-1}(0))$ be the natural restriction map, and write $\widetilde{\mathcal{K}} = (\pi^*)^{-1} \circ r$. Then, by Theorem 7,

$$(2\pi)^d \int_{\operatorname{Reg} \mathbf{X}_{red}} \widetilde{\mathcal{K}}(e^{-i\omega}\alpha) = \frac{|H|}{|W| \operatorname{vol} T} \operatorname{Res}\Big(\Phi^2 \sum_{F \in \mathcal{F}} u_F\Big),$$

where $\mathcal{F}$ denotes the set of components of the fixed point set of the T-action on $\mathbf{X} = T^*M$, and the u_F are rational functions on $\mathfrak{t}$ given by

$$u_F : \mathfrak{t} \ni Y \longmapsto (-2\pi)^{\operatorname{rk} F/2} e^{iJ_Y(F)} \int_F \frac{e^{-i\omega}\varrho(Y)}{\chi_{NF}(Y)},$$

$J_Y(F)$ being the constant value of $J(Y)$ on F. The definition of the residue operation, given in Section 2, relies on the fact that the Fourier transform of u_F is a piecewise polynomial measure. Our approach is in many respects similar to the one of Jeffrey, Kirwan et al., but differs from their's in that it is based on the stationary phase principle, which suggests that it should be possible to find a new proof of their results, and extend them to general symplectic manifolds.

Acknowledgements. The author wishes to thank Michèle Vergne for pointing out to him that the results in [36] could be related to equivariant cohomology, and teaching him many things about the field. This research was financed in its beginnings by the grant RA 1370/2-1 of the German Research Foundation (DFG).

2. Localization in equivariant cohomology

Let $\mathbf{X}$ be a $2n$-dimensional, paracompact, symplectic manifold with symplectic form ω and Riemannian metric g. Since ω is non-degenerate, $\omega^n/n!$ yields a volume form on $\mathbf{X}$ called the *Liouville form*, whose existence is equivalent to the fact that $\mathbf{X}$ is orientable. Define a bundle morphism $\mathcal{J} : T\mathbf{X} \to T\mathbf{X}$ by setting

$$g_\eta(\mathcal{J}\mathfrak{X}, \mathfrak{Y}) = \omega_\eta(\mathfrak{X}, \mathfrak{Y}), \qquad \mathfrak{X}, \mathfrak{Y} \in T_\eta\mathbf{X},$$

and assume that $\mathcal{J}$ is normed in such a way that $\mathcal{J}^2 = -1$, which defines $\mathcal{J}$ uniquely. $\mathcal{J}$ constitutes an almost-complex structure that is compatible with ω, meaning that

$$\omega_\eta(\mathcal{J}\mathfrak{X}, \mathcal{J}\mathfrak{Y}) = \omega_\eta(\mathfrak{X}, \mathfrak{Y}), \qquad \omega_\eta(\mathfrak{X}, \mathcal{J}\mathfrak{X}) > 0.$$

Furthermore, $g_\eta(\mathcal{J}\mathfrak{X}, \mathcal{J}\mathfrak{Y}) = g_\eta(\mathfrak{X}, \mathfrak{Y})$. $(\mathbf{X}, \mathcal{J}, g)$ is consequently an almost-Hermitian manifold. Next, assume that $\mathbf{X}$ carries a Hamiltonian action of a compact, connected Lie group G of dimension d, and denote the corresponding Kostant-Souriau momentum map by

$$\mathbb{J} : \mathbf{X} \to \mathfrak{g}^*, \quad \mathbb{J}(\eta)(X) = J_X(\eta) = J(X)(\eta).$$

By definition, $dJ_X + \iota_{\widetilde{X}}\omega = 0$ for all $X \in \mathfrak{g}$, where $\widetilde{X}$ denotes the vector field on $\mathbf{X}$ given by

$$(\widetilde{X}f)(\eta) = \frac{d}{dt} f(e^{-tX} \cdot \eta)_{|t=0}, \qquad X \in \mathfrak{g}, \quad f \in \mathrm{C}^\infty(\mathbf{X}).$$

By this choice, the mapping $X \mapsto \widetilde{X}$ becomes a Lie-algebra homomorphism, so that in particular $\widetilde{[X, Y]} = [\widetilde{X}, \widetilde{Y}]$. Also note that $\mathbb{J}$ is G-equivariant in the sense that $\mathbb{J}(g^{-1}\eta) = \operatorname{Ad}^*(g)\mathbb{J}(\eta)$.

In what follows, we assume that $\mathfrak{g}$ is endowed with an $\mathrm{Ad}\,(G)$-invariant inner product, which allows us to identify $\mathfrak{g}^*$ with $\mathfrak{g}$. Let further dX and $d\xi$ be corresponding measures on $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively, and denote by

$$\mathcal{F}_\mathfrak{g} : \mathcal{S}(\mathfrak{g}^*) \to \mathcal{S}(\mathfrak{g}), \qquad \mathcal{F}_\mathfrak{g} : \mathcal{S}'(\mathfrak{g}) \to \mathcal{S}'(g^*)$$

the $\mathfrak{g}$-Fourier transform on the Schwartz space and the space of tempered distributions, respectively. In this paper, we intend to relate the equivariant cohomology $H^*_G(\mathbf{X})$ of $\mathbf{X}$ to the cohomology of the symplectic quotient

$$\mathbf{X}_{red} = \Omega_0/G, \qquad \Omega_\varsigma = \mathbb{J}^{-1}(\varsigma).$$

Following [40] and [26], we consider for this the map

$$X \quad \mapsto \quad L_\alpha(X) = \int_\mathbf{X} e^{iJ_X}\alpha, \qquad X \in \mathfrak{g}, \qquad \alpha \in \Lambda_c(\mathbf{X}),$$

regarded as a tempered distribution in $\mathcal{S}'(\mathfrak{g})$, where $\Lambda_c(\mathbf{X})$ denotes the algebra of differential forms on $\mathbf{X}$ of compact support. If $(\mathbf{X}, \omega)$ is a compact symplectic manifold, G a torus, and $\alpha = \sigma^n/n!$ the Liouville measure, L_α is the Duistermaat-Heckman integral, and corresponds to the inverse $\mathfrak{g}$-Fourier transform of the pushforward $\mathbb{J}_*(\sigma^n/n!)$ of the Liouville form along the momentum map. In this case, the $\mathfrak{g}$-Fourier transform of L_α is exactly $\mathbb{J}_*(\sigma^n/n!)$ and a piecewise polynomial measure on $\mathfrak{g}^*$ [17].

We are therefore interested in the $\mathfrak{g}$-Fourier transform $\mathcal{F}_\mathfrak{g} L_\alpha$ of L_α in general, and particularly, in its description near $0 \in \mathfrak{g}^*$. Take an $\mathrm{Ad}\,^*(G)$-invariant function $\varphi \in \mathrm{C^\infty_c}(\mathfrak{g}^*)$ with total integral equal to one and $\mathfrak{g}$-Fourier transform $\hat\varphi(X) = (\mathcal{F}_\mathfrak{g}\varphi)(X) = \int_{\mathfrak{g}^*} e^{-i\langle \xi, X\rangle}\varphi(\xi)\, d\xi$, where we wrote $\xi(X) = \langle \xi, X\rangle$. Then $\varphi_\varepsilon(\xi) = \varphi(\varepsilon^{-1}\xi)/\varepsilon^d$, $\varepsilon > 0$, constitutes an approximation of the δ-distribution in $\mathfrak{g}^*$ at 0 as $\varepsilon \to 0$, and we consider the limit

$$\lim_{\varepsilon\to 0} \langle \mathcal{F}_\mathfrak{g} L_\alpha, \varphi_\varepsilon\rangle = \lim_{\varepsilon\to 0} \int_\mathfrak{g} L_\alpha(X)\hat\varphi(\varepsilon X)\, dX = \lim_{\varepsilon\to 0}\int_\mathfrak{g}\int_\mathbf{X} e^{iJ_X/\varepsilon}\alpha\, \hat\varphi(X)\frac{dX}{\varepsilon^d}, \tag{2}$$

where we took into account that $\hat\varphi_\varepsilon(X) = \hat\varphi(\varepsilon X)$. Next, fix a maximal torus $T \subset G$ of dimension d_T with Lie algebra $\mathfrak{t}$, and consider the root space decomposition

$$\mathfrak{g}^\mathbb{C} = \mathfrak{t}^\mathbb{C} \oplus \bigoplus_{\gamma\in\Delta} \mathfrak{g}_\gamma,$$

where $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ denotes the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$, and $\mathfrak{g}_\gamma$ are the corresponding root spaces. Since $\dim_\mathbb{C} \mathfrak{g}_\gamma = 1$, the decomposition implies $d - d_T = \dim_\mathbb{R} \mathfrak{g} - \dim_\mathbb{R} \mathfrak{t} = |\Delta|$. Assume that α is such that L_α is $\mathrm{Ad}\,(G)$-invariant. Using Weyl's integration formula [26, Lemma 3.1], (2) can be rewritten as

$$\lim_{\varepsilon\to 0} \langle \mathcal{F}_\mathfrak{g} L_\alpha, \varphi_\varepsilon\rangle = \frac{\mathrm{vol}\, G}{|W| \mathrm{vol}\, T} \lim_{\varepsilon\to 0} \int_\mathfrak{t} \left[\int_\mathbf{X} e^{iJ_Y}\alpha\right] \hat\varphi(\varepsilon Y)\Phi^2(Y) dY, \tag{3}$$

where $\Phi(Y) = \prod_{\gamma\in\Delta_+}\gamma(Y)$ and Δ_+ is the set of positive roots, while $W = W(\mathfrak{g}, \mathfrak{t})$ denotes the Weyl group. Here $\mathrm{vol}\, G$ and $\mathrm{vol}\, T$ stand for the volumes of G and T with respect to the corresponding volume forms on G and T induced by the invariant inner product on $\mathfrak{g}$ and its restriction to $\mathfrak{t}$, respectively. In what follows, we shall express this limit in terms of the set

$$F^T = \{\eta \in \mathbf{X} : t\cdot\eta = \eta \quad \forall\, t \in T\}$$

of fixed points of the underlying T-action. The connected components of F^T are smooth submanifolds of possibly different dimensions, and we denote the set of these components by $\mathcal{F}$. Let $F \in \mathcal{F}$ be fixed, and consider the normal bundle NF of F. As can be shown, the real vector bundle NF can be given a complex structure, and splits into a direct sum of two-dimensional real bundles P^F_q, which can be regarded as complex line bundles over F. For each $\eta \in F$, the fibers $(P^F_q)_\eta$

are T-invariant, and endowing them with the standard complex structure, the action of $\mathfrak{t}$ can be written as

$$(P_q^F)_\eta \ni v \mapsto i\lambda_q^F(Y)v \in (P_q^F)_\eta, \qquad Y \in \mathfrak{t},$$

where the $\lambda_q^F \in \mathfrak{t}^*$ are the weights of the torus action [20]. They do not depend on η. Now, if ϱ is an equivariantly closed form, $L_{e^{-i\omega}\varrho(Y)}(Y)$ can be computed using

Theorem 1 (Localization formula of Berline-Vergne)**.** *Let $\mathbf{X}$ be a smooth n-dimensional manifold acted on by a compact Lie group G, and ϱ an equivariantly closed form on $\mathbf{X}$ with compact support. For $Y \in \mathfrak{g}$, let $\mathbf{X}_0$ denote the zero set of Y. Then $\varrho(Y)_{[n]}$ is exact outside $\mathbf{X}_0$, and*

$$\int_{\mathbf{X}} \varrho(Y) = \int_{\mathbf{X}_0} (-2\pi)^{\operatorname{rk} N\mathbf{X}_0/2} \frac{\varrho(Y)}{\chi_{N\mathbf{X}_0}(Y)},$$

where $N\mathbf{X}_0$ denotes the normal bundle of $\mathbf{X}_0$, which has been endowed with an orientation compatible with the one of $\mathbf{X}_0$, and $\chi_{N\mathbf{X}_0}$ is the equivariant Euler form of the normal bundle.

Proof. The proof is the same as the proof of [3, Theorem 7.13], which consists essentially in a local computation, except for [3, Lemma 7.14] which, nevertheless, can be easily generalized to equivariantly closed forms with compact support on non-compact manifolds. □

To apply this theorem in our context, recall that an element $Y \in \mathfrak{t}$ is called *regular*, if the set $\{\exp(sY) : s \in \mathbb{R}\}$ is dense in T. The set of regular elements, in the following denoted by $\mathfrak{t}'$, is dense in $\mathfrak{t}$, and

$$\left\{\eta \in \mathbf{X} : \widetilde{Y}_\eta = 0\right\} = F^T, \qquad Y \in \mathfrak{t}'. \tag{4}$$

We then have the following

Corollary 1. *Let $\varrho \in H_G^*(\mathbf{X})$ be an equivariantly closed form on $\mathbf{X}$ of compact support, and $Y \in \mathfrak{t}'$. Then*

$$L_{e^{-i\omega}\varrho(Y)}(Y) = \int_{\mathbf{X}} e^{i(J_Y-\omega)}\varrho(Y) = \sum_{F\in\mathcal{F}} u_F(Y),$$

where the u_F are rational functions on $\mathfrak{t}$ given by

$$u_F : \mathfrak{t} \ni Y \longmapsto (-2\pi)^{\operatorname{rk} NF/2} e^{iJ_Y(F)} \int_F \frac{e^{-i\omega}\varrho(Y)}{\chi_{NF}(Y)}, \tag{5}$$

$J_Y(F)$ being the constant value of J_Y on F.

Proof. Since $Y \mapsto e^{i(J_Y-\omega)}\varrho(Y)$ defines an equivariantly closed form, the assertion follows immediately from the previous theorem and (4). □

In the last corollary, the equivariant Euler class is given by

$$\chi_{NF}(Y) = \prod_q (c_1(P_q^F) + \lambda_q^F(Y)),$$

where $c_1(P_q^F) \in H^2(F)$ denotes the first Chern class of the complex line bundle P_q^F. Thus,

$$\frac{1}{\chi_{NF}(Y)} = \frac{1}{\prod_q \lambda_q^F(Y)} \prod_q \Big(1 + \frac{c_1(P_q^F)}{\lambda_q^F(Y)}\Big)^{-1} = \frac{1}{\prod_q \lambda_q^F(Y)} \prod_q \sum_{0\le r_q} (-1)^{r_q} \Big(\frac{c_1(P_q^F)}{\lambda_q^F(Y)}\Big)^{r_q}.$$

Note that the sum in the last expression is finite, since $c_1(P_q^F)/\lambda_q^F(Y)$ is nilpotent. Consequently, the inverse makes sense. Let us also note that the set of critical points of J_X is given by

$$\operatorname{Crit} J_X = \left\{\eta \in \mathbf{X} : \widetilde{X}_\eta = 0\right\}, \qquad X \in \mathfrak{g},$$

and is clean in the sense of Bott. Indeed, $\mathrm{Crit}\, J_X$ is a smooth submanifold consisting of possibly several components of different dimension. On the other hand, the Hessian of J_X is given by the symmetric bilinear form

$$\mathrm{Hess}\, J_X : T_\eta(\mathbf{X}) \times T_\eta(\mathbf{X}) \longrightarrow \mathbb{R}, \quad (\mathfrak{X}_1, \mathfrak{X}_2) \mapsto (\tilde{\mathfrak{X}}_1)_\eta(\tilde{\mathfrak{X}}_2(J_X)), \qquad \eta \in \mathrm{Crit}\, J_X,$$

where $\tilde{\mathfrak{X}}_2(J_X) = dJ_X(\tilde{\mathfrak{X}}_2) = -\iota_{\widetilde{X}}\omega(\tilde{\mathfrak{X}}_2)$, and $\tilde{\mathfrak{X}}$ denotes the extension of a vector $\mathfrak{X} \in T_\eta(\mathbf{X})$ to a vector field. Now,

$$\begin{aligned}\widetilde{\mathfrak{X}}_i(\omega(\widetilde{X}, \widetilde{\mathfrak{X}}_j)) &= \mathcal{L}_{\widetilde{\mathfrak{X}}_i}(\iota_{\widetilde{X}}\iota_{\widetilde{\mathfrak{X}}_j}\omega) = \iota_{\mathcal{L}_{\widetilde{\mathfrak{X}}_i}\widetilde{X}}\,\iota_{\widetilde{\mathfrak{X}}_j}\omega + \iota_{\widetilde{X}}\,\mathcal{L}_{\widetilde{\mathfrak{X}}_i}(\iota_{\widetilde{\mathfrak{X}}_j}\omega) \\ &= \iota_{\mathcal{L}_{\widetilde{\mathfrak{X}}_i}\widetilde{X}}\,\iota_{\widetilde{\mathfrak{X}}_j}\omega + \iota_{\widetilde{X}}\,\iota_{\mathcal{L}_{\widetilde{\mathfrak{X}}_i}\widetilde{\mathfrak{X}}_j}\omega + \iota_{\widetilde{X}}\,\iota_{\widetilde{\mathfrak{X}}_j}\mathcal{L}_{\widetilde{\mathfrak{X}}_i}(\omega),\end{aligned} \tag{6}$$

so that at a point $\eta \in \mathrm{Crit}\, J_X$ one computes

$$-\mathrm{Hess}\, J_X(\mathfrak{X}_1, \mathfrak{X}_2) = \widetilde{\mathfrak{X}}_1(\omega(\widetilde{X}, \tilde{\mathfrak{X}}_2)) = -\omega([\widetilde{X}, \widetilde{\mathfrak{X}}_1], \widetilde{\mathfrak{X}}_2), \tag{7}$$

since $\widetilde{X}$ vanishes on $\mathrm{Crit}\, J_X$. But the Lie derivative $\mathfrak{X} \mapsto (\mathcal{L}_{\widetilde{X}}\widetilde{\mathfrak{X}})_\eta = [\widetilde{X}, \widetilde{\mathfrak{X}}]_\eta$ defines an invertible endomorphism of $N_\eta \mathrm{Crit}\, J_X$. Consequently, the Hessian of J_X is transversally non-degenerate and $\mathrm{Crit}\, J_X$ is clean.

We would like to compute (3) using Corollary 1, but since the rational functions (5) are not locally integrable on $\mathfrak{t}$, we cannot proceed directly. Instead note that, since Φ^2 and $\hat{\varphi}$ have analytic continuations to $\mathfrak{t}^{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}$, Cauchy's integral theorem yields for arbitrary $Z \in \mathfrak{t}$

$$\int_{\mathfrak{t}} \left[\int_{\mathbf{X}} e^{i(J_Y - \omega)}\varrho(Y)\right](\hat{\varphi}_\varepsilon \Phi^2)(Y)dY = \int_{\mathfrak{t}} \left[\int_{\mathbf{X}} e^{i(J_{Y+iZ} - \omega)}\varrho(Y+iZ)\right](\hat{\varphi}_\varepsilon \Phi^2)(Y+iZ)dY.$$

Here we took into account that by the Theorem of Paley-Wiener-Schwartz [24, Theorem 7.3.1] $\hat{\varphi}_\varepsilon(Y+iZ)$ is rapidly falling in Y. Let now Λ be a proper cone in the complement of all the hyperplanes $\{Y \in \mathfrak{t} : \lambda_q^F(Y) = 0\}$, so that $Y \in \Lambda$ necessarily implies $\lambda_q^F(Y) \neq 0$ for alle q and F. By the foregoing considerations, u_F defines a holomorphic function on $\mathfrak{t} + i\Lambda$, and for arbitrary compacta $M \subset \mathrm{Int}\,\Lambda$, there is an estimate of the form

$$|u_F(\zeta)| \leq C(1 + |\zeta|)^N, \qquad \zeta = Y + iZ, \quad \mathrm{Im}\,\zeta \in M,$$

for some $N \in \mathbb{N}$. The functions $u_F\Phi^k$, $k = 0, 1, 2, \dots$, are holomorphic on $\mathfrak{t} + i\Lambda$, too, and satisfy similar bounds. Then, by [24, Theorem 7.4.2], there exists for each k a distribution $U_F^{\Phi^k} \in \mathcal{D}'(\mathfrak{t}^*)$ such that

$$e^{-\langle \cdot, Z\rangle} U_F^{\Phi^k} \in \mathcal{S}'(\mathfrak{t}^*), \qquad \mathcal{F}_{\mathfrak{t}}^{-1}(e^{-\langle \cdot, Z\rangle} U_F^{\Phi^k}) = (u_F \Phi^k)(\cdot + iZ), \qquad Z \in \Lambda. \tag{8}$$

We therefore obtain with Corollary 1 for arbitrary $Z \in \Lambda$ and $\varsigma \in \mathfrak{t}^*$ the equality

$$\begin{aligned}\int_{\mathfrak{t}} \left[\int_{\mathbf{X}} e^{i(J_Y - \omega)}\varrho(Y)\right](e^{-i\langle\varsigma,\cdot\rangle}\hat{\varphi}_\varepsilon \Phi^2)(Y)dY &= \sum_{F \in \mathcal{F}} \left\langle (u_F\Phi^2)(\cdot + iZ), (e^{-i\langle\varsigma,\cdot\rangle}\hat{\varphi}_\varepsilon)(\cdot + iZ)\right\rangle \\ &= \sum_{F \in \mathcal{F}} \left\langle e^{-\langle\cdot,Z\rangle} U_F^{\Phi^2}, \mathcal{F}_{\mathfrak{t}}^{-1}\big((e^{-i\langle\varsigma,\cdot\rangle}\hat{\varphi}_\varepsilon)(\cdot + iZ)\big)\right\rangle \\ &= \sum_{F \in \mathcal{F}} \left\langle U_F^{\Phi^2}, \mathcal{F}_{\mathfrak{t}}^{-1}\big(e^{-i\langle\varsigma,\cdot\rangle}\hat{\varphi}_\varepsilon\big)\right\rangle.\end{aligned} \tag{9}$$

Remark 1. Let us mention that for arbitrary $\varsigma \in \mathfrak{t}^*$

$$\mathcal{F}_{\mathfrak{t}}^{-1}(e^{-i\langle\varsigma,\cdot\rangle}\hat{\varphi}_\varepsilon)(\xi) = \frac{1}{\varepsilon^{d_T}}(\mathcal{F}_{\mathfrak{t}}^{-1}\hat{\varphi})\Big(\frac{\xi - \varsigma}{\varepsilon}\Big), \qquad \xi \in \mathfrak{t}^*,$$

constitutes an approximation of the δ-distribution in $\mathfrak{t}^*$ at ς, since for arbitrary $v \in \mathrm{C_c^\infty}(\mathfrak{t}^*)$

$$\left\langle \mathcal{F}_\mathfrak{t}^{-1}(e^{-i\langle \varsigma, \cdot \rangle} \hat{\varphi}_\varepsilon), v \right\rangle = \int_{\mathfrak{t}^*} (\mathcal{F}_\mathfrak{t}^{-1} \hat{\varphi})(\xi) v(\varepsilon \xi + \varsigma) \, d\xi \to v(\varsigma) \hat{\varphi}(0) = v(\varsigma), \qquad \varepsilon \to 0.$$

Remark 2. Alternatively, each of the summands in (9) can be expressed as

$$\begin{aligned} &\left\langle (u_F \Phi)(\cdot + iZ), (e^{-i\langle \varsigma, \cdot \rangle} \Phi \hat{\varphi}_\varepsilon)(\cdot + iZ) \right\rangle \\ &= (2\pi)^{|\Delta_+|} \left\langle \mathcal{F}_\mathfrak{t}^{-1}(e^{-\langle \cdot, Z \rangle} U_F^\Phi), (e^{-i\langle \varsigma, \cdot \rangle} \mathcal{F}_\mathfrak{t}(\Phi \varphi_\varepsilon))(\cdot + iZ) \right\rangle = (2\pi)^{|\Delta_+|} \left\langle U_F^\Phi, (\Phi \varphi_\varepsilon)(\cdot - \varsigma) \right\rangle, \end{aligned}$$

where we used the equality $\Phi \hat{\varphi}_\varepsilon = \Phi \mathcal{F}_\mathfrak{g}(\varphi_\varepsilon) = (2\pi)^{|\Delta_+|} \mathcal{F}_\mathfrak{t}(\Phi \varphi_\varepsilon)$, see [26, Lemma 3.4], and the fact that $(e^{-i\langle \varsigma, \cdot \rangle} \mathcal{F}_\mathfrak{t}(\varphi_\varepsilon \Phi))(\cdot + iZ) = \mathcal{F}_\mathfrak{t}(e^{\langle \cdot, Z \rangle} (\varphi_\varepsilon \Phi)(\cdot - \varsigma))$, or as

$$\begin{aligned} &\left\langle u_F(\cdot + iZ), (e^{-i\langle \varsigma, \cdot \rangle} \Phi^2 \hat{\varphi}_\varepsilon)(\cdot + iZ) \right\rangle \\ &= (2\pi)^{|\Delta_+|} \left\langle \mathcal{F}_\mathfrak{t}^{-1}(e^{-\langle \cdot, Z \rangle} U_F), (e^{-i\langle \varsigma, \cdot \rangle} \mathcal{F}_\mathfrak{t}(D_\Phi(\Phi \varphi_\varepsilon)))(\cdot + iZ) \right\rangle = (2\pi)^{|\Delta_+|} \left\langle U_F, D_\Phi(\Phi \varphi_\varepsilon)(\cdot - \varsigma) \right\rangle, \end{aligned}$$

where D_Φ denotes the differential operator such that $\mathcal{F}_\mathfrak{t}(D_\Phi(\Phi \varphi_\varepsilon)) = \Phi \mathcal{F}_\mathfrak{t}(\Phi \varphi_\varepsilon)$.

As a consequence of equations (2), (3), and (9) we arrive at

Proposition 1. *Let ϱ be an equivariantly closed differential form. Then*

$$\begin{aligned} \lim_{\varepsilon \to 0} \left\langle \mathcal{F}_\mathfrak{g}\Big(L_{e^{-i\omega}\varrho(\cdot)}(\cdot)\Big), \varphi_\varepsilon \right\rangle &= \lim_{\varepsilon \to 0} \int_\mathfrak{g} \int_{\mathbf{X}} e^{i(J_X/\varepsilon - \omega)} \varrho(X/\varepsilon) \, \hat{\varphi}(X) \frac{dX}{\varepsilon^d} \\ &= \frac{vol\ G}{|W| vol\ T} \lim_{\varepsilon \to 0} \sum_{F \in \mathcal{F}} \left\langle U_F^{\Phi^2}, \mathcal{F}_\mathfrak{t}^{-1}(\hat{\varphi}_\varepsilon) \right\rangle. \end{aligned}$$

□

In order to further investigate the distributions $U_F^{\Phi^k}$, note that the functions $u_F \Phi^k$ are given by a linear combination of terms of the form

$$\frac{e^{iJ_Y(F)}}{\Pi_q \lambda_q^F(Y)^{r_q}} P(Y), \qquad P \in \mathbb{C}[\mathfrak{t}^*].$$

The crucial observation is now that, due to this fact, the $u_F \Phi^k$ are tempered distributions whose $\mathfrak{t}$-Fourier transforms are *piecewise polynomial* measures [26, Proposition 3.6]. By the continuity of the Fourier transform in $\mathcal{S}'$ we therefore have

$$\mathcal{F}_\mathfrak{t}(u_F \Phi^k) = \mathcal{F}_\mathfrak{t}\Big(\lim_{t \to 0} u_F \Phi^k(\cdot + itZ)\Big) = \lim_{t \to 0} \mathcal{F}_\mathfrak{t}(u_F \Phi^k(\cdot + itZ)) = \lim_{t \to 0} e^{-\langle \cdot, tZ \rangle} U_F^{\Phi^k} = U_F^{\Phi^k}.$$

Thus, $U_F^{\Phi^k} \in \mathcal{S}'(\mathfrak{t}^*)$ is the $\mathfrak{t}$-Fourier transform of $u_F \Phi^k$, and, in particular, a piecewise polynomial measure. Motivated by Proposition 1, we are interested in the behavior of $U_F^{\Phi^k}$ near the orgin, which leads us to the following

Definition 1. *Let $\varsigma \in \mathfrak{t}^*$ be such that for all $F \in \mathcal{F}$ the Fourier transforms $U_F^{\Phi^k}$ are smooth on the segment $t\varsigma$, $t \in (0, \delta)$. We then define the so-called* residues

$$\mathrm{Res}^{\Lambda, \varsigma}(u_F \Phi^k) = \lim_{t \to 0} U_F^{\Phi^k}(t\varsigma).$$

Note that the limit defining $\mathrm{Res}^{\Lambda,\varsigma}(u_F\Phi^k)$ certainly exists, but does depend on ς (and Λ) as $U_F^{\Phi^k}$ is not continuous at the origin. Furthermore, for arbitrary $Z \in \Lambda$,

$$\begin{aligned}\mathrm{Res}^{\Lambda,\varsigma}(u_F\Phi^k) &= \lim_{t\to 0}\lim_{\varepsilon\to 0}\int_{\mathfrak{t}^*} U_F^{\Phi^k}(\xi)\mathcal{F}_{\mathfrak{t}}^{-1}(e^{-i\langle t\varsigma,\cdot\rangle}\hat{\varphi}_\varepsilon)(\xi)\,d\xi \\ &= \lim_{t\to 0}\lim_{\varepsilon\to 0}\Big\langle \mathcal{F}_{\mathfrak{t}}^{-1}\big(U_F^{\Phi^k}e^{-\langle\cdot,Z\rangle}\big),\big(e^{-i\langle t\varsigma,\cdot\rangle}\hat{\varphi}_\varepsilon\big)(\cdot+iZ)\Big\rangle \\ &= \lim_{t\to 0}\lim_{\varepsilon\to 0}\int_{\mathfrak{t}}(u_F\Phi^k)(Y+iZ)e^{-i\langle t\varsigma,Y+iZ\rangle}\hat{\varphi}_\varepsilon(Y+iZ)dY,\end{aligned}$$

in concordance with the definition of the residues in [26, Section 8]. In particular, this implies

$$\sum_{F\in\mathcal{F}}\mathrm{Res}^{\Lambda,\varsigma}(u_F\Phi^k) = \lim_{t\to 0}\lim_{\varepsilon\to 0}\int_{\mathfrak{t}}\left[\int_{\mathbf{X}} e^{i(J-t\varsigma)(Y)}e^{-i\omega}\varrho(Y)\right]\Phi^k(Y)\hat{\varphi}(\varepsilon Y)\,dY. \tag{10}$$

Similarly,

$$\sum_{F\in\mathcal{F}} U_F^{\Phi^k}(\varsigma) = \lim_{\varepsilon\to 0}\int_{\mathfrak{t}}\left[\int_{\mathbf{X}} e^{i(J-\varsigma)(Y)}e^{-i\omega}\varrho(Y)\right]\Phi^k(Y)\hat{\varphi}(\varepsilon Y)\,dY.$$

For a deeper understanding of the residues and the limits in Proposition 1, we are therefore led to a systematic study of the asymptotic behavior of integrals of the form

$$I_\varsigma(\mu) = \int_{\mathfrak{g}}\left[\int_{\mathbf{X}} e^{i\psi_\varsigma(\eta,X)/\mu}a(\eta,X)\,d\eta\right]dX, \qquad \mu\to 0^+, \tag{11}$$

where $\mathfrak{g}$ is the Lie algebra of an arbitrary connected, compact Lie group G, $a \in \mathrm{C}_c^\infty(\mathbf{X}\times\mathfrak{g})$ is an amplitude, $d\eta = \omega^n/n!$ the Liouville measure on $\mathbf{X}$, and dX an Euclidean measure on $\mathfrak{g}$ given by an $\mathrm{Ad}\,(G)$-invariant inner product on $\mathfrak{g}$, while

$$\psi_\varsigma(\eta,X) = \mathbb{J}(\eta)(X) - \varsigma(X), \qquad \varsigma\in\mathfrak{g}^*. \tag{12}$$

This will occupy us in the next sections.

3. The stationary phase theorem and resolution of singularities

In what follows, we shall describe the asymptotic behavior of the integrals $I_\varsigma(\mu)$ defined in (11) by means of the stationary phase principle. As we shall see, the critical set of the corresponding phase function is in general not smooth. We shall therefore first partially resolve its singularities, and then apply the stationary phase principle in a suitable resolution space. We begin by recalling

Theorem A (Stationary phase theorem for vector bundles)**.** *Let M be an n-dimensional, oriented manifold, and $\pi : E \to M$ an oriented vector bundle of rang l. Let further $\alpha \in \Lambda^q_{cv}(E)$ be a differential form on E with compact support along the fibers, $\tau \in \Lambda_c^{n+l-q}(M)$ a differential form on M of compact support, $\psi \in \mathrm{C}^\infty(E)$, and consider the integral*

$$I(\mu) = \int_E e^{i\psi/\mu}(\pi^*\tau)\wedge\alpha, \qquad \mu > 0. \tag{13}$$

Let $\iota : M \hookrightarrow E$ denote the zero section. Assume that the critical set of ψ coincides with $\iota(M)$, and that the transversal Hessian of ψ is non-degenerate along $\iota(M)$. Then, for each $N\in\mathbb{N}$, $I(\mu)$ possesses an asymptotic expansion of the form

$$I(\mu) = e^{i\psi_0/\mu}e^{i\frac{\pi}{4}\sigma_\psi}(2\pi\mu)^{\frac{l}{2}}\sum_{j=0}^{N-1}\mu^j Q_j(\psi;\alpha,\tau) + R_N(\mu),$$

where ψ_0 and σ_ψ denote the value of ψ and the signature of the transversal Hessian along $\iota(M)$, respectively. The coefficients Q_j are given by measures supported on M, and can be computed explicitly, as well as the remainder term $R_N(\mu) = O(\mu^{l/2+N})$.

Proof. See Appendix A. □

If the critical set of the phase function is not smooth, the stationary phase principle can not be applied a priori, and one faces serious difficulties in describing the asymptotic behavior of oscillatory integrals. We shall therefore first partially resolve the singularities of the critical set, and then apply the stationary phase principle in a suitable resolution space. To explain our approach, let $\mathcal{M}$ be a smooth variety, $\mathcal{O}_\mathcal{M}$ the structure sheaf of rings of $\mathcal{M}$, and $I \subset \mathcal{O}_\mathcal{M}$ an ideal sheaf. The aim in the theory of resolution of singularities is to construct a birational morphism $\Pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ such that $\widetilde{\mathcal{M}}$ is smooth, and the inverse image ideal sheaf $\Pi^* I$ is locally principal. This is called the *principalization* of I, and implies resolution of singularities. That is, for every quasi-projective variety $\mathcal{X}$, there is a smooth variety $\widetilde{\mathcal{X}}$, and a birational and projective morphism $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$. Vice versa, resolution of singularities implies principalization. If $\Pi^*(I)$ is monomial, that is, if for every $\tilde{x} \in \widetilde{\mathcal{M}}$ there are local coordinates σ_i and natural numbers c_i such that

$$\Pi^*(I) \cdot \mathcal{O}_{\tilde{x},\widetilde{\mathcal{M}}} = \prod_i \sigma_i^{c_i} \cdot \mathcal{O}_{\tilde{x},\widetilde{\mathcal{M}}},$$

one obtains strong resolution of singularities, which means that, in addition to the properties stated above, π is an isomorphism over the smooth locus of $\mathcal{X}$, and $\pi^{-1}(\operatorname{Sing}\mathcal{X})$ a divisor with simple normal crossings. Consider next the derivative $D(I)$ of I, which is the sheaf ideal that is generated by all derivatives of elements of I. Let further $Z \subset \mathcal{M}$ be a smooth subvariety, and $\pi : B_Z\mathcal{M} \to \mathcal{M}$ the corresponding monoidal transformation with center Z and exceptional divisor $F \subset B_Z\mathcal{M}$. Assume that (I, m) is a marked ideal sheaf with $m \leq \operatorname{ord}_Z I$. The *total transform* $\pi^* I$ vanishes along F with multiplicity $\operatorname{ord}_Z I$, and by removing the ideal sheaf $\mathcal{O}_{B_Z\mathcal{M}}(-\operatorname{ord}_Z I \cdot F)$ from $\pi^* I$ we obtain the *birational, or weak transform* $\pi_*^{-1} I$ of I. Take local coordinates $(x_1, \dots, x_n)$ on $\mathcal{M}$ such that $Z = (x_1 = \dots = x_r = 0)$. As a consequence,

$$y_1 = \frac{x_1}{x_r}, \dots, y_{r-1} = \frac{x_{r-1}}{x_r}, y_r = x_r, \dots, y_n = x_n$$

define local coordinates on $B_Z\mathcal{M}$, and for $(f, m) \in (I, m)$ one has

$$\pi_*^{-1}(f(x_1, \dots, x_n), m) = (y_r^{-m} f(y_1 y_r, \dots y_{r-1} y_r, y_r, \dots, y_n), m).$$

By the work of Hironaka [23], resolutions are known to exist, and we refer the reader to [30] for a detailed exposition.

Consider now an oscillatory integral of the form (13) in case that the critical set $\mathcal{C} = \iota(M) \subset E = \mathcal{M}$ of the phase function ψ is not clean. Let $I_\mathcal{C}$ be the ideal sheaf of $\mathcal{C}$, and $I_\psi = (\psi)$ the ideal sheaf generated by the phase function ψ. Then $D(I_\psi) = D_\mathcal{C}$. The essential idea behind our approach to singular asymptotics is to construct a partial monomialization

$$\Pi^*(I_\psi) \cdot \mathcal{O}_{\tilde{x},\widetilde{\mathcal{M}}} = \sigma_1^{c_1} \cdots \sigma_k^{c_k} \, \Pi_*^{-1}(I_\psi) \cdot \mathcal{O}_{\tilde{x},\widetilde{\mathcal{M}}}, \qquad \tilde{x} \in \widetilde{\mathcal{M}},$$

of the ideal sheaf $I_\psi = (\psi)$ via a suitable resolution $\Pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ in such a way that $D(\Pi_*^{-1}(I_\psi))$ is a resolved ideal sheaf. As a consequence, the phase function factorizes locally according to $\psi \circ \Pi \equiv \sigma_1^{c_1} \cdots \sigma_k^{c_k} \cdot \tilde{\psi}^{wk}$, and we show that the corresponding weak transforms $\tilde{\psi}^{wk} = \Pi_*^{-1}(\psi)$ have clean critical sets in the sense of Bott [6]. Here $\sigma_1, \dots, \sigma_k$ are local variables near each $\tilde{x} \in \widetilde{\mathcal{M}}$ and c_i are natural numbers. This enables one to apply the stationary phase theorem in the resolution space $\widetilde{\mathcal{M}}$ to the weak transforms $\tilde{\psi}^{wk}$ with the variables $\sigma_1, \dots, \sigma_k$ as parameters. Note that by

Hironaka's theorem, I_ψ can always be monomialized. But in general, this monomialization would not be explicit enough to allow an application of the stationary phase theorem.

4. Equivariant asymptotics and the momentum map

We commence now with our study of the asymptotic behavior of the integrals (11) by means of the generalized stationary phase theorem. To determine the critical set of the phase function $\psi_\varsigma(\eta, X)$, let $\{X_1, \dots, X_d\}$ be a basis of $\mathfrak{g}$, and write $X = \sum_{i=1}^d s_i X_i$. Due to the linear dependence of J_X in X,

$$\partial_{s_i} \psi_\varsigma(\eta, X) = J_{X_i}(\eta) - \varsigma(X_i),$$

and because of the non-degeneracy of ω,

$$J_{X,*} = 0 \quad \Longleftrightarrow \quad dJ_X = -\iota_{\widetilde{X}}\omega = 0 \quad \Longleftrightarrow \quad \widetilde{X} = 0.$$

Hence,

$$\mathrm{Crit}(\psi_\varsigma) = \{(\eta, X) \in \mathbf{X} \times \mathfrak{g} : \psi_{\varsigma,*}(\eta, X) = 0\} = \left\{(\eta, X) \in \Omega_\varsigma \times \mathfrak{g} : \widetilde{X}_\eta = 0\right\}, \tag{14}$$

where $\Omega_\varsigma = \mathbb{J}^{-1}(\varsigma)$ is the ς-level of the momentum map. Now, the major difficulty in applying the generalized stationary phase theorem in our setting stems from the fact that, due to the singular orbit structure of the underlying group action, Ω_ς and, consequently, the considered critical set $\mathrm{Crit}(\psi_\varsigma)$, are in general singular varieties. In fact, if the G-action on $\mathbf{X}$ is not free, Ω_ς and the symplectic quotients $\Omega_\varsigma/G_\varsigma$ are no longer smooth for general $\varsigma \in \mathfrak{g}^*$, where G_ς denotes the stabilizer of ς under the co-adjoint action. Nevertheless, both Ω_ς and $\Omega_\varsigma/G_\varsigma$ have Whitney stratifications into smooth submanifolds, see Lerman-Sjamaar [37], and Ortega-Ratiu [34, Theorems 8.3.1 and 8.3.2], which correspond to the stratification of $\mathbf{X}$ into orbit types, see Duistermaat-Kolk [18]. In particular, one has the following

Lemma 1. *Ω_ς has a principal stratum $\mathrm{Reg}\,\Omega_\varsigma$, which is an open and dense subset of Ω_ς, and a smooth submanifold in $\mathbf{X}$ of codimension equal to the dimension κ of a principal G-orbit in $\mathbf{X}$. Furthermore,*

$$T_\eta(\mathrm{Reg}\ \Omega_\varsigma) = [T_\eta(G \cdot \eta)]^\omega = (\mathfrak{g} \cdot \eta)^\omega, \qquad \eta \in \mathrm{Reg}\ \Omega_\varsigma, \tag{15}$$

where we denoted the symplectic complement of a subspace $V \subset T_\eta \mathbf{X}$ by V^ω, and wrote $\mathfrak{g} \cdot \eta = \{\widetilde{X}_\eta : X \in \mathfrak{g}\}$.

Proof. Let $\mathrm{Reg}\,\mathbf{X}$ denote the union of all orbits of principal type in $\mathbf{X}$, so that $\mathrm{Reg}\,\Omega_\varsigma = \Omega_\varsigma \cap \mathrm{Reg}\,\mathbf{X}$. By the principal orbit theorem, $\mathrm{Reg}\,\mathbf{X}$ is open and dense, and the assertion follows with [34, Corollary 4.6.2 and (5.5.7)]. ☐

Let us consider first the case when $\varsigma \in \mathfrak{g}^*$ is a regular value of the momentum map, which is equivalent to the fact that G acts *locally freely* on Ω_ς, meaning that

$$\widetilde{X}_\eta \neq 0 \qquad \text{for all } \eta \in \Omega_\varsigma,\ 0 \neq X \in \mathfrak{g}. \tag{16}$$

Consequently, all stabilizers G_η of points $\eta \in \Omega_\varsigma$ are finite, and therefore either of principal or exceptional type. In this case, both Ω_ς and $\mathrm{Crit}(\psi_\varsigma) = \Omega_\varsigma \times \{0\}$ are smooth, and $\dim \mathfrak{g} \cdot \eta = \kappa$ for all $\eta \in \Omega_\varsigma$, where κ is the dimension of a principal G-orbit. Furthermore, (16) implies that $\kappa = \dim \mathfrak{g}$. We then have the following

Proposition 2. *Let $\mathbf{X}$ be a paracompact, symplectic manifold of dimension $2n$ with a Hamiltonian action of a compact Lie group G of dimension d. Assume that $\varsigma \in \mathfrak{g}^*$ is a regular value of the* *momentum map* $\mathbb{J} : \mathbf{X} \to \mathfrak{g}^*$*, and let* $I_\varsigma(\mu)$ *be defined as in* (11)*. Then, for each* $N \in \mathbb{N}$*, there exists a constant* $C_{N,\psi_\varsigma,a}$ *such that*

$$\Big| I_\varsigma(\mu) - (2\pi\mu)^\kappa \sum_{j=0}^{N-1} \mu^j Q_j(\psi_\varsigma, a) \Big| \le C_{N,\psi_\varsigma,a}\, \mu^N,$$

where the coefficients Q_j *are given explicitly in terms of measures on* Ω_ς*.*

Proof. As already noted, $\mathcal{C}_\varsigma = \mathrm{Crit}(\psi_\varsigma) = \Omega_\varsigma \times \{0\}$ is a smooth manifold of dimension 2κ, and due to (15) we have

$$T_{(\eta,0)}\mathcal{C}_\varsigma \simeq T_\eta \Omega_\varsigma = (\mathfrak{g}\cdot\eta)^\omega, \qquad N_{(\eta,0)}\mathcal{C}_\varsigma = \mathcal{J}(\mathfrak{g}\cdot\eta) \times \mathbb{R}^d,$$

where $\mathcal{J} : T\mathbf{X} \to T\mathbf{X}$ denotes the bundle homomorphism introduced in Section 2. By definition, the Hessian of ψ_ς at $(\eta, 0) \in \mathcal{C}_\varsigma$ is given by the symmetric bilinear form

$$\mathrm{Hess}\,\psi_\varsigma : T_{(\eta,0)}(\mathbf{X}\times\mathfrak{g}) \times T_{(\eta,0)}(\mathbf{X}\times\mathfrak{g}) \to \mathbb{C}, \qquad (v_1, v_2) \mapsto \tilde v_1(\tilde v_2(\psi_\varsigma))(\eta, 0).$$

Let $\{\widetilde{\mathfrak{X}}_1, \dots, \widetilde{\mathfrak{X}}_{2n}\}$ be a local orthonormal frame in $T\mathbf{X}$ and $\{e_1, \dots, e_d\}$ the standard basis in $\mathbb{R}^d$ corresponding to an orthonormal basis $\{A_1, \dots, A_d\}$ of $\mathfrak{g}$. In the basis

$$((\widetilde{\mathfrak{X}}_i)_\eta; 0), \qquad (0; e_j), \qquad i = 1, \dots, 2n, \quad j = 1, \dots, d,$$

of $T_{(\eta,X)}(\mathbf{X}\times\mathfrak{g}) = T_\eta\mathbf{X} \times \mathbb{R}^d$, $\mathrm{Hess}\,\psi_\varsigma$ is then given by the matrix

$$\mathcal{A} = -\begin{pmatrix} 0 & \omega_\eta(\widetilde{A}_j, \widetilde{\mathfrak{X}}_i) \\ \omega_\eta(\widetilde{A}_i, \widetilde{\mathfrak{X}}_j) & 0 \end{pmatrix} = -\begin{pmatrix} 0 & g_\eta(\mathcal{J}\widetilde{A}_j, \widetilde{\mathfrak{X}}_i) \\ g_\eta(\mathcal{J}\widetilde{A}_i, \widetilde{\mathfrak{X}}_j) & 0 \end{pmatrix}.$$

Indeed, for arbitrary $X \in \mathfrak{g}$ one has $\widetilde{\mathfrak{X}}_i(J_X) = dJ_X(\widetilde{\mathfrak{X}}_i) = -\iota_{\widetilde{X}}\omega(\widetilde{\mathfrak{X}}_i)$, and with (6) we obtain $(\widetilde{\mathfrak{X}}_i)_\eta(\omega(\widetilde{0}, \widetilde{\mathfrak{X}}_j)) = 0$. In order to compute the transversal Hessian of ψ_ς, we have to exhibit a basis for $N_{(\eta,0)}\mathcal{C}_\varsigma$. Let therefore $\{B_1, \dots, B_\kappa\}$ be another basis of $\mathfrak{g} = \mathfrak{g}_\eta^\perp$ such that $\{(\widetilde{B}_1)_\eta, \dots, (\widetilde{B}_\kappa)_\eta\}$ is an orthonormal basis of $\mathfrak{g}\cdot\eta$, where we remind the reader that $\kappa = d$. It is then easy to see that

$$\mathcal{B}_k = (\mathcal{J}(\widetilde{B}_k)_\eta; 0), \qquad \mathcal{B}'_k = (0; g_\eta(\widetilde{A}_1, \widetilde{B}_k), \dots, g_\eta(\widetilde{A}_\kappa, \widetilde{B}_k)), \qquad k = 1, \dots, \kappa,$$

constitutes a basis of $N_{(\eta,0)}\mathcal{C}_\varsigma$ with $\langle \mathcal{B}_k, \mathcal{B}_l\rangle = \delta_{kl}$, $\mathcal{B}_k \perp \mathcal{B}'_l$, and $\langle \mathcal{B}'_k, \mathcal{B}'_l\rangle = (\Xi)_{kl}$, where Ξ is given by the linear transformation

$$\Xi : \mathfrak{g}\cdot\eta \longrightarrow \mathfrak{g}\cdot\eta : \mathfrak{X} \mapsto \sum_{j=1}^{\kappa} g_\eta(\mathfrak{X}, \widetilde{A}_j)(\widetilde{A}_j)_\eta. \tag{17}$$

With these definitions one computes

$$\begin{aligned} \mathcal{A}(\mathcal{B}_k) =& \Big(0; -\sum_{j=1}^{2n} g_\eta(\mathcal{J}\widetilde{A}_1, \widetilde{\mathfrak{X}}_j) g_\eta(\mathcal{J}\widetilde{B}_k, \widetilde{\mathfrak{X}}_j), \dots\Big) \\ =& (0; -g_\eta(\mathcal{J}\widetilde{A}_1, \mathcal{J}\widetilde{B}_k), \dots, -g_\eta(\mathcal{J}\widetilde{A}_\kappa, \mathcal{J}\widetilde{B}_k)) = -\mathcal{B}'_k, \\ \mathcal{A}(\mathcal{B}'_k) =& \Big(-\Big(\sum_{j=1}^{\kappa} g_\eta(\mathcal{J}\widetilde{A}_j, \widetilde{\mathfrak{X}}_1) g_\eta(\widetilde{A}_j, \widetilde{B}_k), \dots\Big); 0\Big) = ((g_\eta(\Xi(\widetilde{B}_k)_\eta, \mathcal{J}\widetilde{\mathfrak{X}}_1), \dots); 0). \end{aligned}$$

Since the $\{\mathcal{J}(\widetilde{B}_1)_\eta, \dots, \mathcal{J}(\widetilde{B}_\kappa)_\eta\}$ form an orthonormal basis of $\mathcal{J}(\mathfrak{g}\cdot\eta)$, we obtain

$$\mathcal{A}(\mathcal{B}'_k) = -(\mathcal{J}\Xi(\widetilde{B}_k)_\eta; 0) = -\sum_{j=1}^{\kappa} g_\eta(\mathcal{J}\Xi(\widetilde{B}_k)_\eta, \mathcal{J}(\widetilde{B}_j)_\eta)\, \mathcal{B}_j.$$

Thus, the transversal Hessian $\mathrm{Hess}\,\psi_\varsigma(\eta,0)_{|N_{(\eta,0)}\mathcal{C}_\varsigma}$ is given by the non-degenerate matrix

$$\mathcal{A}_{\text{trans}} = \begin{pmatrix} 0 & -\mathbf{1}_\kappa \\ -\Xi_{|\mathfrak{g}\cdot\eta} & 0 \end{pmatrix}. \tag{18}$$

By the non-stationary principle, we can choose the support of the amplitude a in the integral $I_\varsigma(\mu)$ close to $\mathcal{C}_\varsigma$. Identifying a tubular neighborhood of $\mathcal{C}_\varsigma$ with a neighborhood of the zero section in $N\mathcal{C}_\varsigma$, the assertion now follows with Theorem A by integrating along the fibers of $\nu : N\mathcal{C}_\varsigma \to \mathcal{C}_\varsigma$. The exact form of the coefficients can be read off from (62), in which ψ'' corresponds to $\mathcal{A}_{\text{trans}}$. Note that the submersion $P_\varsigma : \mathcal{C}_\varsigma \to \Omega_\varsigma, (\eta,0) \mapsto \eta$ is simply the identity, so that measures on $\mathcal{C}_\varsigma$ are identical with measures on Ω_ς. □

Let us resume the considerations in Section 2, the notation being the one introduced previously, and consider the following, more specific oscillatory integrals.

Lemma 2. *Let $\varrho = D\beta$ be an equivariantly exact form on $\mathbf{X}$ of compact support, $\varsigma \in \mathfrak{g}^*$, and $\varepsilon > 0$. Then*

$$\int_{\mathfrak{g}} \left[\int_{\mathbf{X}} e^{i(J-\varsigma)(X)} e^{-i\omega} \varrho(X)\right] \hat{\varphi}_\varepsilon(X) dX = 0.$$

Proof. The proof is essentially an elaboration of an argument given in [26, Equation (8.20)]. In what follows, write $\bar{\omega}(X) = \omega - J_X$ for the extension of the symplectic form to an equivariantly closed form, and assume that $\beta = \sum \theta_j \beta_j$, $\theta_j \in S^j(\mathfrak{g}^*)$, where the β_j are differential forms of compact support. Let further $\varphi \in \mathrm{C}^\infty_c(\mathfrak{g}^*)$ and $\delta = \delta(\varepsilon) > 0$ be such that $\operatorname{supp} \varphi_\varepsilon \subset B(0,\delta)$. Define $\Delta_\delta = \{\eta \in \mathbf{X} : |\mathbb{J}(\eta) - \varsigma| < \delta\}$, and let $\Delta_\delta \subset \Delta'_\delta$ be a smooth domain with smooth boundary $\partial\,\Delta'_\delta$. Since $D\sigma(X)_{[2n]} = d(\sigma(X)_{[2n-1]})$ for any equivariant differential form σ, one computes

$$\begin{aligned}
\int_{\mathfrak{g}} \left[\int_{\mathbf{X}} e^{-i\bar{\omega}(X)} \varrho(X)\right] e^{-i\varsigma(X)} \hat{\varphi}_\varepsilon(X) dX &= \int_{\mathfrak{g}} \left[\int_{\mathbf{X}} D\Big(e^{-i\bar{\omega}}\beta\Big)(X)\right] e^{-i\varsigma(X)} \hat{\varphi}_\varepsilon(X)\, dX \\
= \int_{\mathfrak{g}} \left[\int_{\mathbf{X}} d\Big((e^{-i\bar{\omega}}\beta)(X)\Big)\right] e^{-i\varsigma(X)} \hat{\varphi}_\varepsilon(X)\, dX &= \int_{\mathbf{X}} d\left(\int_{\mathfrak{g}} e^{-i\varsigma(X)} \hat{\varphi}_\varepsilon(X) (e^{-i\bar{\omega}}\beta)(X)\, dX\right) \\
&= \sum_j \int_{\mathbf{X}} d\left(\int_{\mathfrak{g}} e^{i(J-\varsigma)(X)} \hat{\varphi}_\varepsilon(X) \theta_j(X)\, dX e^{-i\omega} \beta_j\right) \\
&= \sum_j \int_{\mathbf{X}} d\left(\int_{\mathfrak{g}} e^{i(J-\varsigma)(X)} \mathcal{F}_{\mathfrak{g}}(\theta_j(-i\,\partial_\xi)\varphi_\varepsilon)(X)\, dX e^{-i\omega} \beta_j\right) \\
&= (2\pi)^d \sum_j \int_{\Delta'_\delta} d\Big([(\theta_j(-i\,\partial_\xi)\varphi_\varepsilon) \circ (\mathbb{J} - \varsigma)] e^{-i\omega} \beta_j\Big) \\
&= (2\pi)^d \sum_j \int_{\partial\,\Delta'_\delta} [(-i\theta_j(\partial_\xi)\varphi_\varepsilon) \circ (\mathbb{J} - \varsigma)] e^{-i\omega} \beta_j = 0
\end{aligned}$$

since $\varphi_\varepsilon \circ (\mathbb{J} - \varsigma)$ vanishes on $\partial\,\Delta'_\delta$. Hereby we used the Theorem of Stokes for differential forms with compact support, see [38, page 119]. □

Proposition 3. *Let $\varsigma \in \mathfrak{g}^*$ be a regular value of $\mathbb{J} : \mathbf{X} \to \mathfrak{g}^*$, $\alpha \in \Lambda_c(\mathbf{X})$, and $\theta \in S^r(\mathfrak{g}^*)$. Then*

$$\lim_{\varepsilon\to 0} \int_{\mathfrak{g}} \left[\int_{\mathbf{X}} e^{i(J-\varsigma)(X)} \alpha\right] \theta(X) \hat{\varphi}(\varepsilon X)\, dX = \frac{(2\pi)^d vol\, G}{|H_G|} \int_{\mathbb{J}^{-1}(\varsigma)} \frac{\iota^*_\varsigma(F)}{vol\, \mathcal{O}_G}$$

for some form $F \in \Lambda_c(\mathbf{X})$ explicitly given in terms of $\mathbb{J}$, α and θ, where H_G denotes a principal isotropy group of the G-action, and $\mathcal{O}_G(\eta) = G \cdot \eta$ the G-orbit through a point $\eta \in \mathbf{X}$, while $\iota_\varsigma : \mathbb{J}^{-1}(\varsigma) \hookrightarrow \mathbf{X}$ is the inclusion.

Proof. Let $\psi_\varsigma(\eta, X) = (\mathbb{J}(\eta) - \varsigma)(X)$, so that the limit in question reads

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{d+r}} \int_{\mathfrak{g}} \left[\int_{\mathbf{X}} e^{i\psi_\varsigma/\varepsilon}\alpha\right] \theta\, \hat\varphi\, dX.$$

Proposition 2 yields for the integral above an asymptotic expansion with leading power ε^d and coefficients $Q_{r,j}$ given by measures on $\mathcal{C}_\varsigma = \mathrm{Crit}(\psi_\varsigma) = \Omega_\varsigma \times \{0\} \equiv \Omega_\varsigma$. In order to compute them, let $\{\mathcal{B}_k, \mathcal{B}'_l\}$ be the basis of $N_{(\eta,0)}\mathcal{C}_\varsigma$ introduced in the proof of Proposition 2, and let $\{s_k, s'_l\}$ be corresponding coordinates in $N_{(\eta,0)}\mathcal{C}_\varsigma$. The transversal Hessian of ψ_ς is given by the matrix (18). By the non-stationary principle, we can choose the support of α close to Ω_ς. Identify a tubular neighborhood of Ω_ς with a neighborhood of the zero section in $N\Omega_\varsigma$. Integrating along the fibers of $\nu : N\mathcal{C}_\varsigma \simeq N\Omega_\varsigma \times \mathfrak{g} \to \mathcal{C}_\varsigma$ then yields

$$\int_{\mathfrak{g}} \left[\int_{\mathbf{X}} e^{i\psi_\varsigma/\varepsilon}\alpha\right] \theta\, \hat\varphi\, dX = \int_{N\mathcal{C}_\varsigma} e^{i\psi_\varsigma/\varepsilon}\theta\, \hat\varphi\, \alpha\, dX = \int_{\mathcal{C}_\varsigma} \nu_*\Big(e^{i\psi_\varsigma/\varepsilon}\theta\, \hat\varphi\, \alpha\, dX\Big).$$

Assume now that with respect to the trivialization of ν given by the frame $\{\mathcal{B}_k, \mathcal{B}'_l\}$ we have

$$\alpha\, dX \equiv f\, \nu^*(\beta) \wedge ds \wedge ds', \qquad \beta \in \Lambda_c(\Omega_\varsigma),$$

for some smooth function f. Applying (62) we obtain for arbitrary large $N \in \mathbb{N}$ an expansion of the form

$$\begin{aligned} &\nu_*\Big(e^{i\psi_\varsigma/\varepsilon}\theta\, \hat\varphi\, \alpha\, dX\Big) \\ &= \frac{\beta}{\det\left(\mathcal{A}_{\mathrm{trans}}(\eta,0)/2\pi i\varepsilon\right)^{1/2}} \sum_{p-q<N} \sum_{2p\geq 3q} \frac{\varepsilon^{p-q}}{p!\, q!\, i^j\, 2^p} \left\langle \mathcal{A}_{\mathrm{trans}}^{-1} D, D\right\rangle^p (\theta\, \hat\varphi\, f\, H^q)(\eta, 0) + R_N, \end{aligned} \tag{19}$$

where $\eta \in \Omega_\varsigma$, $D = -i(\partial_{s_1}, \dots, \partial_{s_\kappa}, \partial_{s'_1}, \dots, \partial_{s'_\kappa})$, $(\theta\, \hat\varphi)(\eta, s, s') = (\theta\, \hat\varphi)(X(s'))$, and

$$H(\eta, s, s') = \psi_\varsigma(\eta, s, s') - \left\langle \mathcal{A}_{\mathrm{trans}}\begin{pmatrix} s \\ s' \end{pmatrix}, \begin{pmatrix} s \\ s' \end{pmatrix}\right\rangle \Big/ 2, \qquad \psi_\varsigma(\eta, s, s') = J_{X(s')}(\eta, s) - \varsigma(X(s')),$$

is a smooth function vanishing at $(\eta, 0)$ of order 3. The inner sum with $p - q = j$ therefore corresponds to a differential operator of order $2j$ acting on $\theta\, \hat\varphi\, f$, since in this case $2p - 3q = 2j - q$, the maximal order being attained for $p = j$ and $q = 0$. Now, since $\psi_\varsigma(\eta, X)$ depends linearly on X, derivatives at $s' = 0$ of $\psi_\varsigma(\eta, s, s')$, and consequently of $H(\eta, s, s')$, of order greater or equal 3 vanish, unless exactly one s'-derivative occurs. On the other hand, θ vanishes at $X(s') = 0$ of order r. Furthermore, due to the particular form of $\mathcal{A}_{\mathrm{trans}}$ in (18),

$$\left\langle \mathcal{A}_{\mathrm{trans}}^{-1} D, D\right\rangle \equiv \sum c_{kl}\, \partial_{s_k}\, \partial_{s'_l}$$

is a differential operator of first order in the s'-variables. Consequently, the inner sums in (19) with $p < r + q$ must vanish, and for $N = p - q = r$, only terms proportional to $\hat\varphi(0)$ occur. Summing up we have shown that

$$Q_{r,j} = 0, \qquad \text{for all } j = 0, \dots, r-1,$$

the leading term being of order ε^{d+r}, and we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{d+r}} \int_{\mathfrak{g}} \left[\int_{\mathbf{X}} e^{i(J-\varsigma)(X)/\varepsilon}\alpha\right] \theta(X)\hat\varphi(X)\, dX \\ &= (2\pi)^d \hat\varphi(0) \int_{\mathbb{J}^{-1}(\varsigma)} \frac{i_\varsigma^*(F)}{|\det \Xi|^{1/2}} = \frac{(2\pi)^d \hat\varphi(0) \operatorname{vol} G}{|H_G|} \int_{\mathbb{J}^{-1}(\varsigma)} \frac{i_\varsigma^*(F)}{\operatorname{vol} \mathcal{O}_G}, \end{aligned}$$

where $F \in \Lambda_c(\mathbf{X})$ is explicitly given in terms of α, $\mathbb{J}$ and θ. Here we took into account that $|\det \Xi_{|\mathfrak{g}\cdot\eta}|^{1/2} = \operatorname{vol}(G\cdot\eta)\, |G_\eta| / \operatorname{vol} G$ for $\eta \in \Omega_\varsigma$, [12, Lemma 3.6]. Since $\hat\varphi(0) = 1$, the assertion follows. $\square$

Let $T \subset G$ be a maximal torus, and consider next the composition $\mathbb{J}_T : \mathbf{X} \to \mathfrak{t}^*$ of the momentum map $\mathbb{J}$ with the restriction map from $\mathfrak{g}^*$ to $\mathfrak{t}^*$, which yields a momentum map for the T-action on $\mathbf{X}$. Then $\mathbb{J}_T^{-1}(\varsigma)/T_\varsigma \simeq \mathbb{J}_T^{-1}(\varsigma)/T$. Also, define

$$\mathcal{K}_\varsigma^T : H_T^*(\mathbf{X}) \xrightarrow{\iota_{\varsigma,T}^*} H_T^*(\mathbb{J}_T^{-1}(\varsigma)) \xrightarrow{(\pi_{\varsigma,T}^*)^{-1}} H^*(\mathbb{J}_T^{-1}(\varsigma)/T),$$

$\iota_{\varsigma,T} : \mathbb{J}_T^{-1}(\varsigma) \hookrightarrow \mathbf{X}$ being the inclusion, and $\pi_{\varsigma,T} : \mathbb{J}_T^{-1}(\varsigma) \to \mathbb{J}_T^{-1}(\varsigma)/T$ the canonical projection. In what follows, we shall also write $\Omega_\varsigma^T = \mathbb{J}_T^{-1}(\varsigma)$. We then have the following

Proposition 4. *Consider the segment $\{t\varsigma : 0 < t < 1, \varsigma \in \mathfrak{t}^*\}$, and assume that it consists of regular values of $\mathbb{J}_T : \mathbf{X} \to \mathfrak{t}^*$ and that all $U_F^{\Phi^2}$ are smooth on the segment. Then, if $\varrho \in H_G^*(\mathbf{X})$ is an equivariantly closed form of compact support,*

$$\sum_{F \in \mathcal{F}} \operatorname{Res}^{\varsigma,\Lambda}(u_F \Phi^2) = \frac{(2\pi)^{d_T}\, vol\, T}{|H_T|} \int_{\operatorname{Reg}\Omega_0^T/T} \mathcal{K}_0^T(F),$$

where $\mathcal{K}_0^T = (\pi_{0,T}^*)^{-1} \circ i_{0,T}^*$ is defined over $\operatorname{Reg}\Omega_0^T/T$, and F is explicitly given in terms of $e^{-i\omega}\varrho$, Φ, and $\mathbb{J}$. In particular, the sum of the residues is independent of ς and Λ, and will be denoted by

$$\operatorname{Res}\Big(\Phi^2 \sum_{F \in \mathcal{F}} u_F\Big).$$

Proof. By (10) and the previous proposition,

$$\sum_{F \in \mathcal{F}} \operatorname{Res}^{\varsigma,\Lambda}(u_F \Phi^2) = \frac{(2\pi)^{d_T} \operatorname{vol} T}{|H_T|} \lim_{t \to 0} \int_{\Omega_{t\varsigma}^T/T} \mathcal{K}_{t\varsigma}^T(F), \tag{20}$$

where $d = \dim \mathfrak{g} = \dim \mathfrak{t} + |\Delta| = d_T + 2|\Delta_+|$. We now assert that for sufficiently small $t > 0$ there exists a birational map

$$\Xi_{t\varsigma} : \Omega_{t\varsigma}^T/T \longrightarrow \Omega_0^T/T$$

which is a diffeomorphism over $\operatorname{Reg}\Omega_0^T/T$. To see this, consider an embedded resolution $\Pi : \widetilde{\mathbf{X}} \to \mathbf{X}$ of Ω_0^T [5]. By the functoriality of the resolution, the strict transform $\widetilde{\Omega}_0^T$ is a T-invariant submanifold of the resolution space $\widetilde{\mathbf{X}}$, and there exists an invariant tubular neighborhood $\widetilde{W}$ of $\widetilde{\Omega}_0^T$. Let $\tilde{p} : \widetilde{W} \to \widetilde{\Omega}_0^T$ be the canonical projection. For sufficiently small $t > 0$, $\Pi^{-1}(\Omega_{t\varsigma}^T)$ is contained in $\widetilde{W}$. Since $\Omega_{t\varsigma}^T$ is diffeomorphic to $\Pi^{-1}(\Omega_{t\varsigma}^T)$, which by Lemma 1 is diffeomorphic to $\widetilde{\Omega}_0^T$, we obtain the birational map

$$\Omega_{t\varsigma}^T \xrightarrow{\Pi^{-1}} \Pi^{-1}(\Omega_{t\varsigma}^T) \xrightarrow{\tilde{p}} \widetilde{\Omega}_0^T \xrightarrow{\Pi} \Omega_0^T.$$

Dividing by T then yields the desired map $\Xi_{t\varsigma}$. As a consequence, we obtain with Lebesgue's theorem as $t \to 0$

$$\int_{\Omega_{t\varsigma}^T/T} \mathcal{K}_{t\varsigma}^T(F) = \int_{\Xi_{t\varsigma}^{-1}(\operatorname{Reg}\Omega_0^T/T)} \mathcal{K}_{t\varsigma}^T(F) = \int_{\operatorname{Reg}\Omega_0^T/T} (\Xi_{t\varsigma}^{-1})^*(\mathcal{K}_{t\varsigma}^T(F)) \to \int_{\operatorname{Reg}\Omega_0^T/T} \mathcal{K}_0^T(F),$$

and the assertion follows with (20). □

Corollary 2. *Let the notation be as in Section 2, and $\varrho \in H_G^*(\mathbf{X})$ an equivariantly closed differential form. Then*

$$\lim_{\varepsilon \to 0} \Big\langle \mathcal{F}_{\mathfrak{g}}\Big(L_{e^{-i\omega}\varrho(\cdot)}(\cdot)\Big), \varphi_\varepsilon \Big\rangle = \frac{vol\, G}{|W|\, vol\, T} \operatorname{Res}\Big(\Phi^2 \sum_{F \in \mathcal{F}} u_F\Big).$$

Proof. Since $U_F^{\Phi^2}$ is a piecewise polynomial measure, and $\mathcal{F}_{\mathfrak{t}}^{-1}(\hat{\varphi}_\varepsilon) \in \mathcal{S}(\mathfrak{t}^*)$,

$$\left\langle U_F^{\Phi^2}, \mathcal{F}_{\mathfrak{t}}^{-1}(\hat{\varphi}_\varepsilon)\right\rangle = \int_{\mathfrak{t}^*} U_F^{\Phi^2}(\varepsilon\varsigma)(\mathcal{F}_{\mathfrak{t}}^{-1}\hat{\varphi})(\varsigma)\, d\varsigma.$$

Furthermore, for $0 < \varepsilon \leq 1$ and almost every $\varsigma \in \mathfrak{t}^*$ we have the estimate $|U_F^{\Phi^2}(\varepsilon\varsigma)(\mathcal{F}_{\mathfrak{t}}^{-1}\hat{\varphi})(\varsigma)| \leq C(1+|\varsigma|)^N |(\mathcal{F}_{\mathfrak{t}}^{-1}\hat{\varphi})(\varsigma)|$ for some $C, N > 0$. Taking into account Remark 1 and the previous proposition, an application of Lebesgue's theorem on bounded convergence then yields

$$\lim_{\varepsilon\to 0} \sum_{F\in\mathcal{F}} \left\langle U_F^{\Phi^2}, \mathcal{F}_{\mathfrak{t}}^{-1}(\hat{\varphi}_\varepsilon)\right\rangle = \lim_{\varepsilon\to 0} \int_{\mathfrak{t}^*} \sum_{F\in\mathcal{F}} U_F^{\Phi^2}(\varepsilon\varsigma)(\mathcal{F}_{\mathfrak{t}}^{-1}\hat{\varphi})(\varsigma)\, d\varsigma = \hat{\varphi}(0)\operatorname{Res}\Big(\Phi^2 \sum_{F\in\mathcal{F}} u_F\Big),$$

and the assertion follows with Proposition 1. $\square$

Thus, in order to derive the residue formula mentioned in the introduction, we are left with the task of evaluating the limit $\lim_{\varepsilon\to 0}\left\langle \mathcal{F}_{\mathfrak{g}} L_{e^{-i\omega}\varrho(\cdot)}(\cdot), \varphi_\varepsilon\right\rangle$ in terms of the reduced space $\mathbf{X}_{red}$. This amounts to an examination of the asymptotic behavior of the integrals (11) in case that $\varsigma \in \mathfrak{g}^*$, and in particular $\varsigma = 0$, is a singular value of the momentum map, in which case $\operatorname{Crit}(\psi_\varsigma)$ is a singular variety. From now on, we will only be considering the case $\varsigma = 0$, and simply write ψ for ψ_0, $I(\mu)$ for $I_0(\mu)$, and so on. As explained in the previous section, we shall partially resolve the singularities of the critical set $\operatorname{Crit}(\psi)$ first, and then make use of the stationary phase principle in a suitable resolution space. Partial desingularizations of the zero level set $\Omega = \mathbb{J}^{-1}(0)$ of the momentum map and the symplectic quotient Ω/G have been obtained by Meinrenken-Sjamaar [32] for compact symplectic manifolds with a Hamiltonian compact Lie group action by performing blowing-ups along minimal symplectic suborbifolds containing the strata of maximal depth in Ω. In the context of geometric invariant-theoretic quotients, partial desingularizations were studied in [29] and [25].

From now on, we will restrict ourselves to the case where $\mathbf{X}$ is given by the cotangent bundle of a Riemannian manifold. For a general symplectic manifold, the desingularization process should be similar, but more involved, and we intend to deal with this case at some other occasion. Thus, let M be a Riemannian manifold of dimension n, $\gamma : T^*M \to M$ its cotangent bundle, and $\tau : T(T^*M) \to T^*M$ the tangent bundle, endowed with corresponding Riemannian structures [33]. Define on T^*M the Liouville form

$$\Theta_\eta(\mathfrak{X}) = \tau(\mathfrak{X})[\gamma_*(\mathfrak{X})], \qquad \mathfrak{X} \in T_\eta(T^*M).$$

We then regard T^*M as a symplectic manifold with symplectic form $\omega = d\Theta$ and Riemannian metric g. Assume now that M carries an isometric action of a compact, connected Lie group G with Lie algebra $\mathfrak{g}$, and define for every $X \in \mathfrak{g}$ the function

$$J_X : T^*M \longrightarrow \mathbb{R}, \quad \eta \mapsto \Theta(\widetilde{X})(\eta).$$

Note that $\Theta(\widetilde{X})(\eta) = \eta(\widetilde{X}_{\pi(\eta)})$. The function J_X is linear in X, and due to the invariance of the Liouville form [10] one has

$$\mathcal{L}_{\widetilde{X}}\Theta = dJ_X + \iota_{\widetilde{X}}\omega = 0, \qquad \forall X \in \mathfrak{g},$$

where $\mathcal{L}$ denotes the Lie derivative. Hence, the infinitesimal action of $X \in \mathfrak{g}$ on T^*M is given by the Hamiltonian vector field defined by J_X, which means that G acts on T^*M in a Hamiltonian way. The corresponding symplectic momentum map is then given by

$$\mathbb{J} : T^*M \to \mathfrak{g}^*, \quad \mathbb{J}(\eta)(X) = J_X(\eta).$$

Note that

$$\eta \in \Omega \quad \Longleftrightarrow \quad \eta_m \in \operatorname{Ann}(T_m(G\cdot m)) \quad \forall m \in M, \tag{21}$$

where $\mathrm{Ann}\,(V_m) \subset T^*_m M$ denotes the annihilator of a vector subspace $V_m \subset T_m M$.

Example 1. In case that $M = \mathbb{R}^n$, let $(q_1, \dots, q_n, p_1, \dots p_n)$ denote the canonical coordinates on $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$. Let further $G \subset \mathrm{GL}(n, \mathbb{R})$ be a closed subgroup acting on $\mathrm{T}^*\mathbb{R}^n$ by $g \cdot (q,p) = (g\,q,\, {}^T g^{-1}\, p)$. The symplectic form reads $\omega = d\theta = \sum_{i=1}^n dp_i \wedge\, dq_i$, where $\theta = \sum p_i\, dq_i$ is the Liouville form, and the corresponding momentum map is given by

$$\mathbb{J} : T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n \to \mathfrak{g}^*, \quad \mathbb{J}(q,p)(X) = \theta(\widetilde{X})(q,p) = \langle Xq, p\rangle\,,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$. In this case, for $\varsigma \in \mathfrak{g}^*$,

$$\mathrm{Crit}(\psi_\varsigma) = \big\{(q,p,X) \in \Omega_\varsigma \times \mathfrak{g} : X \in \mathfrak{g}_{(q,p)}\big\}\,,$$

where $\Omega_\varsigma = \{(q,p) \in T^*\mathbb{R}^n : \langle Aq, p\rangle - \varsigma(A) = 0 \text{ for all } A \in \mathfrak{g}\}$ and $\mathfrak{g}_{(q,p)}$ is given by the set of all $X \in \mathfrak{g}$ such that $Xq = 0$, $Xp = 0$.

By Lemma 1, Ω has a principal stratum $\mathrm{Reg}\,\Omega$, which is an open and dense subset of Ω, and a smooth submanifold in T^*M of codimension equal to the dimension κ of a principal G-orbit in T^*M. Furthermore, $T_\eta(\mathrm{Reg}\;\Omega) = [T_\eta(G\cdot\eta)]^\omega = (\mathfrak{g}\cdot\eta)^\omega$, $\eta \in \mathrm{Reg}\;\Omega$. We describe next the smooth part of the critical set (14) for the phase function $\psi(\eta)(X) = \mathbb{J}(\eta)(X)$.

Lemma 3. *The smooth part of* $\mathrm{Crit}(\psi)$ *corresponds to*

$$\mathrm{Reg}\,\mathrm{Crit}(\psi) = \{(\eta, X) \in \mathrm{Reg}\,\Omega \times \mathfrak{g} : X \in \mathfrak{g}_\eta\}\,, \tag{22}$$

and constitutes a submanifold of codimension 2κ*. Furthermore,*

$$T_{(\eta,X)}\mathrm{Reg}\,\mathrm{Crit}(\psi) = \left\{(\mathfrak{X}, w) \in (\mathfrak{g}\cdot\eta)^\omega \times \mathbb{R}^d : \sum_{i=1}^d w_i(\widetilde{X}_i)_\eta = [\widetilde{X}, \widetilde{\mathfrak{X}}]_\eta\right\}, \tag{23}$$

where $\widetilde{\mathfrak{X}}$ *denotes an extension of* $\mathfrak{X}$ *to a vector field* [1].

Proof. Since the Lie algebra of G_η is given by $\mathfrak{g}_\eta = \{X \in \mathfrak{g} : \widetilde{X}_\eta = 0\}$, the first assertion is clear from (14). To see the second, let $(\eta(t), X(t))$ be a smooth curve in $\mathrm{Reg}\;\Omega \times \mathfrak{g}$. Writing $X(t) = \sum s_j(t) X_j$ with respect to a basis $\{X_1, \dots, X_d\}$ of $\mathfrak{g}$, one computes for any $f \in \mathrm{C}^\infty(\mathrm{Reg}\;\Omega)$

$$\begin{aligned}\frac{d}{dt}\widetilde{X(t)}_{\eta(t)} f_{|t=t_0} &= \sum_{j=1}^d \frac{d}{dt}\Big(s_j(t)(\widetilde{X_j})_{\eta(t)} f\Big)_{|t=t_0} \\ &= \sum_{j=1}^d \dot{s}_j(t_0)(\widetilde{X_j} f)(\eta(t_0)) + \sum_{j=1}^d s_j(t_0)\frac{d}{dt}(\widetilde{X_j} f)(\eta(t))_{|t=t_0}.\end{aligned}$$

Writing $\mathfrak{X} = \dot{\eta}(t_0) \in T_{\eta(t_0)}\mathrm{Reg}\;\Omega$, one has $\frac{d}{dt}(\widetilde{X_j} f)(\eta(t))_{|t=t_0} = \widetilde{\mathfrak{X}}_{\eta(t_0)}(\widetilde{X_j} f)$, so that if $(\eta(t), X(t))$ is a curve in $\mathrm{Reg}\,\mathrm{Crit}(\psi)$ one obtains

$$\sum_{j=1}^d \dot{s}_j(t_0)(\widetilde{X_j})_{\eta(t_0)} f + \sum_{j=1}^d s_j(t_0)[\widetilde{\mathfrak{X}}, \widetilde{X_j}]_{\eta(t_0)} f = 0,$$

since $\widetilde{X}(t_0)_{\eta(t_0)}(\widetilde{\mathfrak{X}} f) = 0$, and the assertion follows with (15). □

Before we start with the actual desingularization process of the phase function ψ, let us mention the following

Proposition 5. *The mapping* $P : \mathrm{Reg}\;\mathrm{Crit}(\psi) \to \mathrm{Reg}\;\Omega, (\eta, X) \mapsto \eta$ *is a submersion.*

[1] In the proposition below, we shall actually see that $[\widetilde{X}, \widetilde{\mathfrak{X}}]_\eta \in \mathfrak{g}\cdot\eta$ for $X \in \mathfrak{g}_\eta$ and $\mathfrak{X} \in (\mathfrak{g}\cdot\eta)^\omega$.

Proof. Let $\eta \in \text{Reg}\, \Omega$ and $X \in \mathfrak{g}_\eta$. We show that $[\widetilde{\mathfrak{X}}, \widetilde{X}]_\eta \in \mathfrak{g}\cdot\eta$ for all $\mathfrak{X} \in T_\eta \text{Reg}\, \Omega$. To begin, note that $\pi_G : \text{Reg}\, \Omega \to \text{Reg}\, \Omega/G$ is a submersion and a principal fiber bundle with $\ker(\pi_G)_{*,\eta} = \mathfrak{g}\cdot\eta$ [34, Theorem 8.1.1]. If therefore $\eta(t) \in \text{Reg}\, \Omega$ denotes a curve with $\eta(0) = \eta$, $\dot\eta(0) = \mathfrak{X}$, and $g \in G_\eta$, differentiation of $\pi_G(g\cdot\eta(t)) = \pi_G(\eta(t))$ yields $\mathfrak{X} - g_{*,\eta}(\mathfrak{X}) \in \ker(\pi_G)_{*,\eta} = \mathfrak{g}\cdot\eta$. Consequently,

$$\frac{d}{dt}(e^{-tX})_{*,\eta}\mathfrak{X}_{|t=0} = \lim_{t\to 0} t^{-1}\left[(e^{-tX})_{*,\eta}\mathfrak{X} - \mathfrak{X}\right] \in \mathfrak{g}\cdot\eta, \tag{24}$$

where we made the identification $T_{\mathfrak{X}}(T_\eta \text{Reg}\, \Omega) \simeq T_\eta \text{Reg}\, \Omega$. Now, for arbitrary $Y \in \mathfrak{g}$ [34, Proposition 4.2.2],

$$\omega_\eta([\widetilde{\mathfrak{X}}, \widetilde{X}], \widetilde{Y}) = -\omega_\eta([\widetilde{X}, \widetilde{Y}], \widetilde{\mathfrak{X}}) - \omega_\eta([\widetilde{Y}, \widetilde{\mathfrak{X}}], \widetilde{X}) = 0,$$

since $\widetilde{X}_\eta = 0$, and $\widetilde{\mathfrak{X}}_\eta = \mathfrak{X} \in (\mathfrak{g}\cdot\eta)^\omega$. Hence, $[\widetilde{\mathfrak{X}}, \widetilde{X}]_\eta \in (\mathfrak{g}\cdot\eta)^\omega$. Furthermore, for arbitrary $f \in \mathrm{C}^\infty(T^*M)$,

$$[\widetilde{\mathfrak{X}}, \widetilde{X}]_\eta f = \widetilde{\mathfrak{X}}_\eta(\widetilde{X} f) = \frac{d}{ds}(\widetilde{X} f)(\eta(s))_{|s=0} = \frac{d}{dt}\left(\frac{d}{ds} f(e^{-tX}\cdot\eta(s))_{|s=0}\right)_{|t=0} = \frac{d}{dt}\big(\widetilde{(e^{-tX})_{*,\eta}\mathfrak{X}_{|t=0}}\big)_\eta f,$$

so that with (24)

$$[\widetilde{\mathfrak{X}}, \widetilde{X}]_\eta = \frac{d}{dt}(e^{-tX})_{*,\eta}\mathfrak{X}_{|t=0} \in \mathfrak{g}\cdot\eta. \tag{25}$$

The previous lemma then implies that $P_{*,(\eta,X)} : T_{(\eta,X)}\text{Reg}\,\text{Crit}(\psi) \to T_\eta \text{Reg}\, \Omega, (\mathfrak{X}, w) \mapsto \mathfrak{X}$ is a surjection, and the assertion follows. $\square$

Remark 3. Note that for $\eta \in \text{Reg}\, \Omega$, and $X \in \mathfrak{g}_\eta$, the previous proposition implies that the Lie derivative defines a homomorphism

$$L_X : \mathfrak{g}\cdot\eta \ni \mathfrak{X} \longmapsto \mathcal{L}_{\widetilde{X}}(\widetilde{\mathfrak{X}})_\eta = [\widetilde{X}, \widetilde{\mathfrak{X}}]_\eta \in \mathfrak{g}\cdot\eta. \tag{26}$$

5. The desingularization process in the case $\mathbf{X} = T^*M$, $\varsigma = 0$

We shall now proceed to a partial desingularization of the critical set of the phase function (12) for $\mathbf{X} = T^*M$, $\varsigma = 0$, and derive an asymptotic description of the integral (11) in this case. An analogous desingularization process was already implemented in [36] to describe the asymptotic distribution of eigenvalues of an invariant elliptic operator. The desingularization employed here constitutes a local version of the latter, and for this reason is slightly simpler. Indeed, the phase function considered in [36] is a global analogue of $\psi(\eta, X) = \mathbb{J}(\eta)(X)$. It should be noticed, however, that these phase functions are not equivalent in the sense of Duistermaat [16], so that the corresponding desingularization processes can not be reduced to each other [2]. To begin, we shall need a suitable G-invariant covering of M. In its construction, we shall follow Kawakubo [27], Theorem 4.20. For a more detailed survey on compact group actions, we refer the reader to [36], Section 3. Thus, let $(H_1), \dots, (H_L)$ denote the isotropy types of M, and arrange them in such a way that

$$H_j \text{ is conjugate to a subgroup of } H_i \quad \Rightarrow \quad i \le j.$$

Let $H \subset G$ be a closed subgroup, and $M(H)$ the union of all orbits of type G/H. Then M has a stratification into orbit types according to

$$M = M(H_1) \cup \dots \cup M(H_L).$$

[2] Observe that a similar phenomenon occurs in [19].

By the principal orbit theorem, the set $M(H_L)$ is open and dense in M, while $M(H_1)$ is a G-invariant submanifold. Denote by ν_1 the normal G-vector bundle of $M(H_1)$, and by $f_1 : \nu_1 \to M$ a G-invariant tubular neighbourhood of $M(H_1)$ in M. Take a G-invariant metric on ν_1, and put

$$D_t(\nu_1) = \{v \in \nu_1 : \|v\| \leq t\}, \qquad t > 0.$$

We then define the G-invariant submanifold with boundary

$$M_2 = M - f_1(\overset{\circ}{D}_{1/2}(\nu_1)),$$

on which the isotropy type (H_1) no longer occurs, and endow it with a G-invariant Riemannian metric with product form in a G-invariant collar neighborhood of ∂M_2 in M_2. Consider now the union $M_2(H_2)$ of orbits in M_2 of type G/H_2, a G-invariant submanifold of M_2 with boundary, and let $f_2 : \nu_2 \to M_2$ be a G-invariant tubular neighborhood of $M_2(H_2)$ in M_2, which exists due to the particular form of the metric on M_2. Taking a G-invariant metric on ν_2, we define

$$M_3 = M_2 - f_2(\overset{\circ}{D}_{1/2}(\nu_2)),$$

which constitutes a G-invariant submanifold with corners and isotropy types $(H_3), \dots (H_L)$. Continuing this way, one finally obtains for M the decomposition

$$M = f_1(D_{1/2}(\nu_1)) \cup \dots \cup f_L(D_{1/2}(\nu_L)),$$

where we identified $f_L(D_{1/2}(\nu_L))$ with M_L. This leads to the covering

$$M = f_1(\overset{\circ}{D}_1(\nu_1)) \cup \dots \cup f_L(\overset{\circ}{D}_1(\nu_L)), \qquad f_L(\overset{\circ}{D}_1(\nu_L)) = \overset{\circ}{M}_L .$$

Let us now start resolving the singularities of the critical set $\mathrm{Crit}(\psi)$. For this, we will set up an iterative desingularization process along the strata of the underlying G-action, where each step in our iteration will consist of a decomposition, a monoidal transformation, and a reduction. For simplicity, we shall assume that at each iteration step the set of maximally singular orbits is connected. Otherwise each of the connected components, which might even have different dimensions, has to be treated separately.

First decomposition. Take $1 \leq k \leq L-1$. As before, let $f_k : \nu_k \to M_k$ be an invariant tubular neighborhood of $M_k(H_k)$ in

$$M_k = M - \bigcup_{i=1}^{k-1} f_i(\overset{\circ}{D}_{1/2}(\nu_i)),$$

a manifold with corners on which G acts with the isotropy types $(H_k), (H_{k+1}), \dots, (H_L)$, and put $W_k = f_k(\overset{\circ}{D}_1(\nu_k))$, $W_L = \overset{\circ}{M}_L$, so that $M = W_1 \cup \dots \cup W_L$. Write further $S_k = \{v \in \nu_k : \|v\| = 1\}$. Introduce a partion of unity $\{\chi_k\}_{k=1,\dots,L}$ subordinate to the covering $\{W_k\}$, and with the notation of (11) define

$$I_k(\mu) = \int_{T^*W_k} \int_{\mathfrak{g}} e^{i\psi(\eta,X)/\mu} (a\chi_k)(\eta, X)\, dX\, d\eta,$$

so that $I(\mu) = I_1(\mu) + \dots + I_L(\mu)$. As will be explained in Lemma 6, the critical set of ψ is clean on the support of $a\chi_L$, so that we can apply directly the stationary phase theorem to compute the integral $I_L(\mu)$. But if $k \in \{1, \dots, L-1\}$, the sets

$$\Omega_k = \Omega \cap T^*W_k,$$

$$\mathrm{Crit}_k(\psi) = \left\{(\eta, X) \in \Omega_k \times \mathfrak{g} : \widetilde{X}_\eta = 0\right\}$$

are no longer smooth manifolds, so that the stationary phase theorem can not a priori be applied in this situation. Instead, we shall resolve the singularities of $\mathrm{Crit}_k(\psi)$, and after this apply the principle of the stationary phase in a suitable resolution space. For this, introduce for each $x^{(k)} \in M_k(H_k)$ the decomposition

$$\mathfrak{g} = \mathfrak{g}_{x^{(k)}} \oplus \mathfrak{g}_{x^{(k)}}^\perp,$$

where $\mathfrak{g}_{x^{(k)}}$ denotes the Lie algebra of the stabilizer $G_{x^{(k)}}$ of $x^{(k)}$, and $\mathfrak{g}_{x^{(k)}}^\perp$ its orthogonal complement with respect to some $\mathrm{Ad}\,(G)$-invariant inner product in $\mathfrak{g}$. Let further $A_1(x^{(k)}), \dots, A_{d^{(k)}}(x^{(k)})$ be an orthonormal basis of $\mathfrak{g}_{x^{(k)}}^\perp$, and $B_1(x^{(k)}), \dots, B_{e^{(k)}}(x^{(k)})$ an orthonormal basis of $\mathfrak{g}_{x^{(k)}}$. Consider the isotropy algebra bundle over $M_k(H_k)$

$$\mathfrak{iso}\, M_k(H_k) \to M_k(H_k),$$

as well as the canonical projection

$$\pi_k : W_k \to M_k(H_k), \qquad m = f_k(x^{(k)}, v^{(k)}) \mapsto x^{(k)}, \qquad x^{(k)} \in M_k(H_k),\, v^{(k)} \in (\nu_k)_{x^{(k)}},$$

where $f_k(x^{(k)}, v^{(k)}) = (\exp_{x^{(k)}} \circ \gamma^{(k)})(v^{(k)})$, and

$$\gamma^{(k)}(v^{(k)}) = \frac{F_k(x^{(k)})}{(1 + \left\| v^{(k)} \right\|)^{1/2}} v^{(k)}$$

is an equivariant diffeomorphism from $(\nu_k)_{x^{(k)}}$ onto its image, $F_k : M_k(H_k) \to \mathbb{R}$ being a smooth, G-invariant positive function, see Bredon [9, pages 306-307]. We consider then the induced bundle

$$\pi_k^* \mathfrak{iso}\, M_k(H_k) = \left\{ (f_k(x^{(k)}, v^{(k)}), X) \in W_k \times \mathfrak{g} : X \in \mathfrak{g}_{x^{(k)}} \right\},$$

and denote by

$$\Pi_k : W_k \times \mathfrak{g} \to \pi_k^* \mathfrak{iso}\, M_k(H_k)$$

the canonical projection which is obtained by considering geodesic normal coordinates around $\pi_k^*\, \mathfrak{iso} M_k(H_k)$, and identifying $W_k \times \mathfrak{g}$ with a neighborhood of the zero section in the normal bundle $N\, \pi_k^*\, \mathfrak{iso}\, M_k(H_k)$. Note that the fiber of the normal bundle to $\pi^* \mathfrak{iso}\, M_k(H_k)$ at a point $(f_k(x^{(k)}, v^{(k)}), X)$ can be identified with $\mathfrak{g}_{x^{(k)}}^\perp$. Integrating along the fibers of the normal bundle to $\pi_k^*\, \mathfrak{iso} M_k(H_k)$ we therefore obtain for $I_k(\mu)$ the expression

(27)
$$\int_{\pi_k^*\, \mathfrak{iso} M_k(H_k)} \left[\int_{\Pi_k^{-1}(m, B^{(k)}) \times T_m^* W_k} e^{i\psi/\mu} a \chi_k\, \Phi_k\; d(T_m^* W_k)\, dA^{(k)} \right] dB^{(k)}\, dm$$
$$= \int_{M_k(H_k)} \Big[\int_{\mathfrak{g} \times \pi_k^{-1}(x^{(k)})} \Big[\int_{T^*_{\exp_{x^{(k)}} v^{(k)}} W_k} e^{i\psi/\mu} a \chi_k\, \Phi_k\; d(T^*_{\exp_{x^{(k)}} v^{(k)}} W_k) \Big] dA^{(k)}\, dB^{(k)}\, dv^{(k)} \Big] dx^{(k)},$$

where

$$\gamma^{(k)}\big(\overset{\circ}{D}_1\; (\nu_k)_{x^{(k)}} \big) \times \mathfrak{g}_{x^{(k)}}^\perp \times \mathfrak{g}_{x^{(k)}} \ni (v^{(k)}, A^{(k)}, B^{(k)}) \mapsto (\exp_{x^{(k)}} v^{(k)}, A^{(k)} + B^{(k)}) = (m, X)$$

are coordinates on $\mathfrak{g} \times \pi_k^{-1}(x^{(k)})$, while dm, $dx^{(k)}$, $dA^{(k)}, dB^{(k)}, dv^{(k)}$, and $d(T_m^* W_k)$ are suitable measures on W_k, $M_k(H_k)$, $\mathfrak{g}_{x^{(k)}}^\perp$, $\mathfrak{g}_{x^{(k)}}$, $\gamma^{(k)}(\overset{\circ}{D}_1\; (\nu_k)_{x^{(k)}})$, and $T_m^* W_k$, respectively, such that

$$dX\, d\eta \equiv \Phi_k\, d(T^*_{\exp_{x^{(k)}} v^{(k)}} W_k)(\eta) dA^{(k)}\, dB^{(k)}\, dv^{(k)}\, dx^{(k)},$$

where Φ_k is a Jacobian.

First monoidal transformation. Let now $k \in \{1, \dots, L-1\}$ be fixed. For the further analysis of the integral $I_k(\mu)$, we shall sucessively resolve the singularities of $\mathrm{Crit}_k(\psi)$, until we are in position to apply the principle of the stationary phase in a suitable resolution space. To begin with, we perform a monoidal transformation

$$\zeta_k : B_{Z_k}(W_k \times \mathfrak{g}) \longrightarrow W_k \times \mathfrak{g}$$

in $W_k \times \mathfrak{g}$ with center $Z_k = \mathfrak{iso}\, M_k(H_k)$. For this, let us write $A^{(k)}(x^{(k)}, \alpha^{(k)}) = \sum \alpha_i^{(k)} A_i^{(k)}(x^{(k)}) \in \mathfrak{g}^{\perp}_{x^{(k)}}$, $B^{(k)}(x^{(k)}, \beta^{(k)}) = \sum \beta_i^{(k)} B_i^{(k)}(x^{(k)}) \in \mathfrak{g}_{x^{(k)}}$, and

$$\gamma^{(k)}(v^{(k)}) = \sum_{i=1}^{c^{(k)}} \theta_i^{(k)} v_i^{(k)}(x^{(k)}) \in \gamma^{(k)}\Big(\overset{\circ}{D}_1\, (\nu_k)_{x^{(k)}} \Big),$$

where $\{v_1^{(k)}, \dots, v_{c^{(k)}}^{(k)}\}$ denotes an orthonormal frame in ν_k. With respect to these coordinates we have $Z_k = \big\{ T^{(k)} = (\alpha^{(k)}, \theta^{(k)}) = 0 \big\}$, so that

$$B_{Z_k}(W_k \times \mathfrak{g}) = \Big\{ (m, X, [t]) \in W_k \times \mathfrak{g} \times \mathbb{RP}^{c^{(k)}+d^{(k)}-1} : T_i^{(k)} t_j = T_j^{(k)} t_i, \Big\},$$
$$\zeta_k : (m, X, [t]) \longmapsto (m, X).$$

Let us now cover $B_{Z_k}(W_k \times \mathfrak{g})$ with charts $\{(\varphi_k^{\varrho}, U_k^{\varrho})\}$, where $U_k^{\varrho} = B_{Z_k}(W_k \times \mathfrak{g}) \cap (W_k \times \mathfrak{g} \times V_\varrho)$, $V_\varrho = \Big\{ [t] \in \mathbb{RP}^{c^{(k)}+d^{(k)}-1} : t_\varrho \neq 0 \Big\}$, and φ_k^{ϱ} is given by the canonical coordinates on V_ϱ. As a consequence, we obtain for ζ_k in each of the $\theta^{(k)}$-charts $\{U_k^{\varrho}\}_{1 \leq \varrho \leq c^{(k)}}$ the expressions

$$\begin{aligned} \zeta_k^{\varrho} = \zeta_k \circ \varphi_k^{\varrho} : (x^{(k)}, \tau_k, {}^{\varrho}\tilde{v}^{(k)}, A^{(k)}, B^{(k)}) &\xmapsto{{}'\zeta_k^{\varrho}} (x^{(k)}, \tau_k\, {}^{\varrho}\tilde{v}^{(k)}, \tau_k A^{(k)}, B^{(k)}) \\ \longmapsto (\exp_{x^{(k)}} \tau_k\, {}^{\varrho}\tilde{v}^{(k)}, \tau_k A^{(k)} + B^{(k)}) &\equiv (m, X), \end{aligned} \tag{28}$$

where $\tau_k \in (-1, 1)$,

$${}^{\varrho}\tilde{v}^{(k)}(x^{(k)}, \theta^{(k)}) = \gamma^{(k)}\Big(\big(v_\varrho^{(k)}(x^{(k)}) + \sum_{i \neq \varrho}^{c^{(k)}} \theta_i^{(k)} v_i^{(k)}(x^{(k)})\big) \Big/ \sqrt{1 + \sum_{i \neq \varrho} (\theta_i^{(k)})^2} \Big) \in \gamma^{(k)}({}^{\varrho}S_k^+)_{x^{(k)}},$$

and

$${}^{\varrho}S_k^+ = \Big\{ v \in \nu_k : v = \sum s_i v_i, s_\varrho > 0, \|v\| = 1 \Big\}.$$

Note that for each $1 \leq \varrho \leq c^{(k)}$,

$$W_k \simeq f_k({}^{\varrho}S_k^+ \times (-1, 1))$$

up to a set of measure zero. Now, for given $m \in M$, let $Z_m \subset T_m M$ be a neighborhood of zero such that $\exp_m : Z_m \longrightarrow M$ is a diffeomorphism onto its image. Then

$$(\exp_m)_{*,v} : T_v Z_m \longrightarrow T_{\exp_m v} M, \quad v \in Z_m,$$

and $g \cdot \exp_m v = L_g(\exp_m v) = \exp_{L_g(m)} (L_g)_{*,m}(v)$. As a consequence, since $B^{(k)} \in \mathfrak{g}_{x^{(k)}}$, we obtain

$$\begin{aligned} \widetilde{B^{(k)}}_{\exp_{x^{(k)}} \tau_k\, {}^{\varrho}\tilde{v}^{(k)}} &= \frac{d}{dt} \exp_{x^{(k)}} \big(L_{\mathrm{e}^{-tB^{(k)}}} \big)_{*,x^{(k)}} (\tau_k\, {}^{\varrho}\tilde{v}^{(k)})_{|t=0} = (\exp_{x^{(k)}})_{*,\tau_k\, {}^{\varrho}\tilde{v}^{(k)}} \big(\lambda(B^{(k)})(\tau_k\, {}^{\varrho}\tilde{v}^{(k)}) \big) \\ &= \tau_k (\exp_{x^{(k)}})_{*,\tau_k\, {}^{\varrho}\tilde{v}^{(k)}} \big(\lambda(B^{(k)})({}^{\varrho}\tilde{v}^{(k)}) \big), \end{aligned}$$

where we denoted by

$$\lambda : \mathfrak{g}_{x^{(k)}} \longrightarrow \mathfrak{gl}(\nu_{k,x^{(k)}}), \quad B^{(k)} \mapsto \frac{d}{dt} (L_{\mathrm{e}^{-tB^{(k)}}})_{*,x^{(k)}|t=0}$$

the linear representation of $\mathfrak{g}_{x^{(k)}}$ in $\nu_{k,x^{(k)}}$, and made the canonical identification $T_v(\nu_{k,x^{(k)}}) \equiv \nu_{k,x^{(k)}}$ for any $v \in (\nu_k)_{x^{(k)}}$. With $\pi(\eta) = \exp_{x^{(k)}} \tau_k \, {}^{\varrho}\tilde{v}^{(k)}$ we therefore obtain for the phase function the factorization

$$\begin{aligned}\psi(\eta, X) &= \eta(\tilde{X}_{\pi(\eta)}) = \eta\big((\tau_k \widetilde{A^{(k)} + B^{(k)}})_{\exp_{x^{(k)}} \tau_k \, {}^{\varrho}\tilde{v}^{(k)}}\big)\\ &= \tau_k \Big[\eta\big(\widetilde{A^{(k)}}_{\exp_{x^{(k)}} \tau_k \, {}^{\varrho}\tilde{v}^{(k)}}\big) + \eta\big((\exp_{x^{(k)}})_{*,\tau_k \, {}^{\varrho}\tilde{v}^{(k)}}[\lambda(B^{(k)})^{\varrho}\tilde{v}^{(k)}]\big)\Big].\end{aligned}$$

Similar considerations hold for ζ_k in the $\alpha^{(k)}$-charts $\{U_k^{\varrho}\}_{c^{(k)}+1 \leq \varrho \leq c^{(k)}+d^{(k)}}$, so that we get on the resolution space

$$\psi \circ (\mathrm{id}_{\,fiber} \otimes \zeta_k) = {}^{(k)}\tilde{\psi}^{tot} = \tau_k \cdot {}^{(k)}\tilde{\psi}^{wk},$$

${}^{(k)}\tilde{\psi}^{tot}$ and ${}^{(k)}\tilde{\psi}^{wk}$ being the *total* and *weak transform* of the phase function ψ, respectively.

Example 2. In the case $M = T^*\mathbb{R}^n$ and $G \subset \mathrm{GL}(n, \mathbb{R})$ a closed subgroup, the phase function factorizes with respect to the canonical coordinates $\eta = (q, p)$ according to

$$\begin{aligned}\psi(q, p, X) &= \langle Xq, p\rangle = \Big\langle \big(\tau_k A^{(k)} + B^{(k)}\big) \exp_{x^{(k)}} \tau_k \, {}^{\varrho}\tilde{v}^{(k)}, p \Big\rangle\\ &= \tau_k \left[\Big\langle A^{(k)} x^{(k)} + B^{(k)} \, {}^{\varrho}\tilde{v}^{(k)}, p \Big\rangle + \tau_k \Big\langle A^{(k)} \, {}^{\varrho}\tilde{v}^{(k)}, p \Big\rangle\right],\end{aligned}$$

where we took into account that in $\mathbb{R}^n$ the exponential map is given by $\exp_{x^{(k)}} v^{(k)} = x^{(k)} + v^{(k)}$.

Introducing a partition $\{u_k^{\varrho}\}$ of unity subordinated to the covering $\{U_k^{\varrho}\}$ now yields

$$I_k(\mu) = \sum_{\varrho=1}^{c^{(k)}} I_k^{\varrho}(\mu) + \sum_{\varrho=c^{(k)}+1}^{d^{(k)}} \tilde{I}_k^{\varrho}(\mu),$$

where the integrals $I_k^{\varrho}(\mu)$ and $\tilde{I}_k^{\varrho}(\mu)$ are given by the expressions

$$\int_{B_{Z_k}(W_k \times \mathfrak{g})} u_k^{\varrho}(\mathrm{id}_{\,fiber} \otimes \zeta_k)^*(e^{i\psi/\mu} a\chi_k dX d\eta).$$

As we shall see in Section 9, the weak transform ${}^{(k)}\tilde{\psi}^{wk}$ has no critical points in the $\alpha^{(k)}$-charts, which implies that the integrals $\tilde{I}_k(\mu)^{\varrho}$ contribute to $I(\mu)$ only with higher order terms. In what follows, we shall therefore restrict ourselves to the examination of the integrals $I_k^{\varrho}(\mu)$. Setting $a_k^{\varrho} = (u_k^{\varrho} \circ \varphi_k^{\varrho})[(a\chi_k) \circ (\mathrm{id}_{\,fiber} \otimes \zeta_k^{\varrho})]$ we obtain with (27) and (28)

$$\begin{aligned}I_k^{\varrho}(\mu) = \int_{M_k(H_k) \times (-1,1)} \Big[\int_{\gamma^{(k)}((S_k)_{x^{(k)}}) \times \mathfrak{g}_{x^{(k)}} \times \mathfrak{g}^{\perp}_{x^{(k)}}} \Big[\int_{T^*_{\exp_{x^{(k)}} \tau_k \tilde{v}^{(k)}} W_k} e^{i\frac{\tau_k}{\mu} \, {}^{(k)}\tilde{\psi}^{wk}} a_k^{\varrho} \, \tilde{\Phi}_k^{\varrho}\\ d(T^*_{\exp_{x^{(k)}} \tau_k \tilde{v}^{(k)}} W_k)\Big] dA^{(k)} \, dB^{(k)} \, d\tilde{v}^{(k)}\Big] \, d\tau_k \, dx^{(k)},\end{aligned}$$

where $d\tilde{v}^{(k)}$ is a suitable measure on the set $\gamma^{(k)}((S_k)_{x^{(k)}})$ such that

$$dX \, d\eta \equiv \tilde{\Phi}_k^{\varrho} \; d(T^*_{\exp_{x^{(k)}} \tau_k \tilde{v}^{(k)}} W_k) \, dA^{(k)} \, dB^{(k)} \, d\tilde{v}^{(k)} \, d\tau_k \, dx^{(k)},$$

$\tilde{\Phi}_k^{\varrho}$ being a Jacobian. Furthermore, a computation shows that $\tilde{\Phi}_k^{\varrho} = |\tau_k|^{c^{(k)}+d^{(k)}-1} \, \Phi_k \circ {}' \zeta_k^{\varrho}$.

First reduction. Let us now assume that there exists a $m \in W_k$ with orbit type G/H_j, and let $x^{(k)} \in M_k(H_k), v^{(k)} \in (\nu_k)_{x^{(k)}}$ be such that $m = f_k(x^{(k)}, v^{(k)})$. Since we can assume that m lies in a slice at $x^{(k)}$ around the G-orbit of $x^{(k)}$, we have $G_m \subset G_{x^{(k)}}$, see Kawakubo [27, pages 184-185], and Bredon [9, page 86]. Hence, $H_j \simeq G_m$ must be conjugate to a subgroup of $H_k \simeq G_{x^{(k)}}$. Now, G acts on M_k with the isotropy types $(H_k), (H_{k+1}), \dots, (H_L)$. The isotropy types occuring in W_k are therefore those for which the corresponding isotropy groups $H_k, H_{k+1}, \dots, H_L$ are conjugate to a subgroup of H_k, and we shall denote them by

$$(H_k) = (H_{i_1}), (H_{i_2}), \dots, (H_L).$$

Now, for every $x^{(k)} \in M_k(H_k)$, $(\nu_k)_{x^{(k)}}$ is an orthogonal $G_{x^{(k)}}$-space; therefore $G_{x^{(k)}}$ acts on $(S_k)_{x^{(k)}}$ with isotropy types $(H_{i_2}), \dots, (H_L)$, cp. Donnelly [15, pp. 34]. Furthermore, by the invariant tubular neighborhood theorem, one has the isomorphism

$$W_k/G \simeq (\nu_k)_{x^{(k)}}/G_{x^{(k)}},$$

so that G acts on $S_k = \{v \in \nu_k : \|v\| = 1\}$ with isotropy types $(H_{i_2}), \dots, (H_L)$ as well. As will turn out, if G acted on S_k only with type (H_L), the critical set of ${}^{(k)}\tilde{\psi}^{wk}$ would be clean in the sense of Bott, and we could proceed to apply the stationary phase theorem to compute $I_k(\mu)$. But in general this will not be the case, and we are forced to continue with the iteration.

Second decomposition. Let now $x^{(k)} \in M_k(H_k)$ be fixed. Since $\gamma^{(k)} : \nu_k \to \nu_k$ is an equivariant diffeomorphism onto its image, $\gamma^{(k)}((S_k)_{x^{(k)}})$ is a compact $G_{x^{(k)}}$-manifold, and we consider the covering

$$\gamma^{(k)}((S_k)_{x^{(k)}}) = W_{ki_2} \cup \dots \cup W_{kL}, \qquad W_{ki_j} = f_{ki_j}(\overset{\circ}{D}_1 (\nu_{ki_j})), \quad W_{kL} = \mathrm{Int}(\gamma^{(k)}((S_k)_{x^{(k)}})_L),$$

where $f_{ki_j} : \nu_{ki_j} \to \gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}$ is an invariant tubular neighborhood of $\gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})$ in

$$\gamma^{(k)}((S_k)_{x^{(k)}})_{i_j} = \gamma^{(k)}((S_k)_{x^{(k)}}) - \bigcup_{r=2}^{j-1} f_{ki_r}(\overset{\circ}{D}_{1/2} (\nu_{ki_r})), \qquad j \geq 2,$$

and $f_{ki_j}(x^{(i_j)}, v^{(i_j)}) = (\exp_{x^{(i_j)}} \circ \gamma^{(i_j)})(v^{(i_j)})$, $x^{(i_j)} \in \gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})$, $v^{(i_j)} \in (\nu_{ki_j})_{x^{(i_j)}}$, $\gamma^{(i_j)} : \nu_{ki_j} \to \nu_{ki_j}$ being an equivariant diffeomorphism onto its image. Let further $\{\chi_{ki_j}\}$ denote a partition of unity subordinated to the covering $\{W_{ki_j}\}$, and define

$$I^{\varrho}_{ki_j}(\mu) = \int_{M_k(H_k) \times (-1,1)} \Big[\int_{\gamma^{(k)}((S_k)_{x^{(k)}}) \times \mathfrak{g}_{x^{(k)}} \times \mathfrak{g}^{\perp}_{x^{(k)}}} \Big[\int_{T^*_{\exp_{x^{(k)}} \tau_k \tilde{v}^{(k)}} W_k} e^{i \frac{\tau_k}{\mu} {}^{(k)}\tilde{\psi}^{wk}} a^{\varrho}_k$$
$$\chi_{ki_j} \tilde{\Phi}^{\varrho}_k \, d(T^*_{\exp_{x^{(k)}} \tau_k \tilde{v}^{(k)}} W_k) \Big] \, dA^{(k)} \, dB^{(k)} \, d\tilde{v}^{(k)} \Big] \, d\tau_k \, dx^{(k)},$$

so that $I^{\varrho}_k(\mu) = I^{\varrho}_{ki_2}(\mu) + \dots + I^{\varrho}_{kL}(\mu)$. It is important to note that the partition functions χ_{ki_j} depend smoothly on $x^{(k)}$ as a consequence of the tubular neighborhood theorem, by which in particular $\gamma^{(k)}(S_k)/G \simeq \gamma^{(k)}((S_k)_{x^{(k)}})/G_{x^{(k)}}$, and the smooth dependence in $x^{(k)}$ of the induced Riemannian metric on $\gamma^{(k)}((S_k)_{x^{(k)}})$, and the metrics on the normal bundles ν_{ki_j}. Since $G_{x^{(k)}}$ acts on W_{kL} only with type (H_L), the iteration process for $I^{\varrho}_{kL}(\mu)$ ends here. For the remaining integrals $I^{\varrho}_{ki_j}(\mu)$ with $k < i_j < L$, let us denote by

$$\mathfrak{iso}\, \gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j}) \to \gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})$$

the isotropy algebra bundle over $\gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})$, and by $\pi_{ki_j} : W_{ki_j} \to \gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})$ the canonical projection. For $x^{(i_j)} \in \gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})$, consider the decomposition

$$\mathfrak{g} = \mathfrak{g}_{x^{(k)}} \oplus \mathfrak{g}^{\perp}_{x^{(k)}} = (\mathfrak{g}_{x^{(i_j)}} \oplus \mathfrak{g}^{\perp}_{x^{(i_j)}}) \oplus \mathfrak{g}^{\perp}_{x^{(k)}}.$$

Let further $A_1^{(i_j)}, \dots, A^{(i_j)}_{d^{(i_j)}}$ be an orthonormal basis in $\mathfrak{g}^\perp_{x^{(i_j)}}$, as well as $B_1^{(i_j)}, \dots, B^{(i_j)}_{e^{(i_j)}}$ be an orthonormal basis in $\mathfrak{g}_{x^{(i_j)}}$, and $\{v_1^{(ki_j)}, \dots, v^{(ki_j)}_{c^{(i_j)}}\}$ an orthonormal frame in ν_{ki_j}. Integrating along the fibers in a neighborhood of $\pi^*_{ki_j} \mathfrak{iso}\, \gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j}) \subset W_{ki_j} \times \mathfrak{g}_{x^{(k)}}$ then yields for $I^\varrho_{ki_j}(\mu)$ the expression

$$\int_{M_k(H_k)\times(-1,1)} \Big[\int_{\gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})} \Big[\int_{\pi^{-1}_{ki_j}(x^{(i_j)})\times \mathfrak{g}_{x^{(k)}}\times \mathfrak{g}^\perp_{x^{(k)}}} \Big[\int_{T^*_{\exp_{x^{(k)}} \tau_k \exp_{x^{(i_j)}} v^{(i_j)}} W_k} e^{i \frac{\tau_k}{\mu} {}^{(k)}\tilde{\psi}^{wk}}$$

$$\times a^\varrho_k \chi_{ki_j}\, \Phi^\varrho_{ki_j}\; d(T^*_{\exp_{x^{(k)}} \tau_k \exp_{x^{(i_j)}} v^{(i_j)}} W_k) \Big] dA^{(k)}\, dA^{(i_j)}\, dB^{(i_j)}\, dv^{(i_j)} \Big] dx^{(i_j)} \Big] d\tau_k dx^{(k)},$$

where $\Phi^\varrho_{ki_j}$ is a Jacobian, and

$$\gamma^{(i_j)}\big(\overset{\circ}{D}_1\, (\nu_{ki_j})_{x^{(i_j)}}\big) \times \mathfrak{g}^\perp_{x^{(i_j)}} \times \mathfrak{g}_{x^{(i_j)}} \ni (v^{(i_j)}, A^{(i_j)}, B^{(i_j)}) \mapsto (\exp_{x^{(i_j)}} v^{(i_j)}, A^{(i_j)} + B^{(i_j)}) = (\tilde{v}^{(k)}, B^{(k)})$$

are coordinates on $\pi^{-1}_{ki_j}(x^{(i_j)}) \times \mathfrak{g}_{x^{(k)}}$, while $dx^{(i_j)}$, and $dA^{(i_j)}, dB^{(i_j)}, dv^{(i_j)}$ are suitable measures in the spaces $\gamma^{(k)}((S_k)_{x^{(k)}})_{i_j}(H_{i_j})$, and $\mathfrak{g}^\perp_{x^{(i_j)}}$, $\mathfrak{g}_{x^{(i_j)}}$, $\gamma^{(i_j)}\big(\overset{\circ}{D}_1\, (\nu_{ki_j})_{x^{(i_j)}}\big)$, respectively, such that we have the equality $\tilde{\Phi}^\varrho_k\, dB^{(k)}\, d\tilde{v}^{(k)} \equiv \Phi^\varrho_{ki_j}\, dA^{(i_j)}\, dB^{(i_j)}\, dv^{(i_j)}\, dx^{(i_j)}$.

Second monoidal transformation. Let us fix an l such that $k < l < L$, $(H_l) \leq (H_k)$, and consider in $B_{Z_k}(W_k \times \mathfrak{g})$ a monoidal transformation

$$\zeta_{kl} : B_{Z_{kl}}(B_{Z_k}(W_k \times \mathfrak{g})) \longrightarrow B_{Z_k}(W_k \times \mathfrak{g})$$

with center

$$Z_{kl} \simeq \bigcup_{x^{(k)} \in M_k(H_k)} (-1,1) \times \mathfrak{iso}\, \gamma^{(k)}((S_k)_{x^{(k)}})_l(H_l).$$

Let $A^{(l)} \in \mathfrak{g}^\perp_{x^{(l)}}$ and $B^{(l)} \in \mathfrak{g}_{x^{(l)}}$ be arbitrary and write $A^{(l)}(x^{(k)}, x^{(l)}, \alpha^{(l)}) = \sum \alpha_i^{(l)} A_i^{(l)}(x^{(k)}, x^{(l)}) \in \mathfrak{g}^\perp_{x^{(l)}}$, $B^{(l)}(x^{(k)}, x^{(l)}, \beta^{(l)}) = \sum \beta_i^{(l)} B_i^{(l)}(x^{(l)}) \in \mathfrak{g}_{x^{(l)}}$, as well as

$$\gamma^{(l)}(v^{(l)})(x^{(k)}, x^{(l)}, \theta^{(l)}) = \sum_{i=1}^{c^{(l)}} \theta_i^{(l)} v_i^{(kl)}(x^{(k)}, x^{(l)}).$$

Then $Z_{kl} \simeq \big\{\alpha^{(k)} = 0,\, \alpha^{(l)} = 0,\, \theta^{(l)} = 0\big\}$ locally, which in particular shows that Z_{kl} is a manifold. If we now cover $B_{Z_{kl}}(B_{Z_k}(W_k \times \mathfrak{g}))$ with the standard charts, we shall see again in Section 9 that modulo higher order terms the main contributions to $I^\varrho_{kl}(\mu)$ come from the $(\theta^{(k)}, \theta^{(l)})$-charts. Therefore it suffices to examine ζ_{kl} in one of these charts, in which it reads

$$\zeta^{\varrho\sigma}_{kl} : (x^{(k)}, \tau_k, x^{(l)}, \tau_l, \tilde{v}^{(l)}, A^{(k)}, A^{(l)}, B^{(l)}) \xmapsto{{}'\zeta^{\varrho\sigma}_{kl}} (x^{(k)}, \tau_k, x^{(l)}, \tau_l \tilde{v}^{(l)}, \tau_l A^{(k)}, \tau_l A^{(l)}, B^{(l)})$$
$$\longmapsto (x^{(k)}, \tau_k, \exp_{x^{(l)}} \tau_l \tilde{v}^{(l)}, \tau_l A^{(k)}, \tau_l A^{(l)} + B^{(l)}) \equiv (x^{(k)}, \tau_k, \tilde{v}^{(k)}, A^{(k)}, B^{(k)}),$$

where

$$\tilde{v}^{(l)}(x^{(k)}, x^{(l)}, \theta^{(l)}) \in \gamma^{(l)}\big((S^+_{kl})_{x^{(l)}}\big).$$

Note that Z_{kl} has normal crossings with the exceptional divisor $E_k = \zeta_k^{-1}(Z_k) = \{\tau_k = 0\}$, and that

$$W_{kl} \simeq f_{kl}(S^+_{kl} \times (-1,1))$$

up to a set of measure zero, where S_{kl} denotes the sphere subbundle in ν_{kl}, and we set $S_{kl}^+ = \Big\{v \in S_{kl} : v = \sum v_i v_i^{(kl)},\, v_\sigma > 0\Big\}$ for some σ. Consequently, the phase function factorizes according to

$$\psi \circ (\mathrm{id}_{\,fiber} \otimes (\zeta_k^\varrho \circ \zeta_{kl}^{\varrho\sigma})) = {}^{(kl)}\tilde{\psi}^{tot} = \tau_k\, \tau_l \cdot {}^{(kl)}\tilde{\psi}^{wk},$$

which in the given charts reads

$$\begin{aligned}\psi(\eta, X) = \tau_k &\Big[\eta\Big(\widetilde{\tau_l A^{(k)}}_{\exp_{x^{(k)}} \tau_k \exp_{x^{(l)}} \tau_l \tilde{v}^{(l)}}\Big) \\ &+\eta\Big((\exp_{x^{(k)}})_{*,\tau_k \exp_{x^{(l)}} \tau_l \tilde{v}^{(l)}} [\lambda(\tau_l A^{(l)} + B^{(l)}) \exp_{x^{(l)}} \tau_l \tilde{v}^{(l)}]\Big)\Big] \\ = \tau_k \tau_l &\Big[\eta\Big(\widetilde{A^{(k)}}_{\exp_{x^{(k)}} \tau_k \exp_{x^{(l)}} \tau_l \tilde{v}^{(l)}}\Big) + \eta\Big((\exp_{x^{(k)}})_{*,\tau_k \exp_{x^{(l)}} \tau_l \tilde{v}^{(l)}} [\lambda(A^{(l)}) \exp_{x^{(l)}} \tau_l \tilde{v}^{(l)}]\Big) \\ &+\eta\Big((\exp_{x^{(k)}})_{*,\tau_k \exp_{x^{(l)}} \tau_l \tilde{v}^{(l)}} \big[(\exp_{x^{(l)}})_{*,\tau_l \tilde{v}^{(l)}} [(\lambda(B^{(l)}) \tilde{v}^{(l)}]]\Big)\Big]\end{aligned}$$

where we took into account that

$$\lambda(B^{(l)}) \exp_{x^{(l)}} \tau_l \tilde{v}^{(l)} = \frac{d}{dt} \exp_{x^{(l)}} \big(L_{\,\mathrm{e}^{-tB^{(l)}}}\big)_{*,x^{(k)}} \tau_l \tilde{v}^{(l)}_{|t=0} = (\exp_{x^{(l)}})_{*,\tau_l \tilde{v}^{(l)}} \big(\lambda(B^{(l)}) \tau_l \tilde{v}^{(l)}\big).$$

Since the weak transforms ${}^{kl}\tilde{\psi}^{wk}$ have no critical points in the $(\theta^{(k)}, \alpha^{(l)})$-charts, modulo lower order terms, $I_{kl}^\varrho(\mu)$ is given by a sum of integrals of the form

$$\begin{aligned}I_{kl}^{\varrho\sigma}(\mu) = \int_{M_k(H_k)\times(-1,1)} &\Big[\int_{\gamma^{(k)}((S_k)_{x^{(k)}})_l(H_l)\times(-1,1)} \Big[\int_{\gamma^{(l)}((S_{kl})_{x^{(l)}})\times \mathfrak{g}_{x^{(l)}} \times \mathfrak{g}^\perp_{x^{(l)}} \times \mathfrak{g}^\perp_{x^{(k)}}} \Big[\int_{T^*_{m^{(kl)}} W_k} \\ &\times e^{i\frac{\tau_k \tau_l}{\mu}\, {}^{(kl)}\tilde{\psi}^{wk}} a_{kl}^{\varrho\sigma}\, \tilde{\Phi}_{kl}^{\varrho\sigma}\; d(T^*_{m^{(kl)}} W_k)\Big] dA^{(k)}\, dA^{(l)}\, dB^{(l)}\, d\tilde{v}^{(l)}\Big] d\tau_l\, dx^{(l)}\Big]\, d\tau_k\, dx^{(k)},\end{aligned}$$

where we wrote $m^{(kl)} = \exp_{x^{(k)}} \tau_k \exp_{x^{(l)}} \tau_l \tilde{v}^{(l)}$, $a_{kl}^{\varrho\sigma}$ are smooth amplitudes with compact support in a $(\theta^{(k)}, \theta^{(l)})$-chart labeled by the indices ϱ, σ, and $d\tilde{v}^{(l)}$ is a suitable measure in $\gamma^{(l)}((S_{kl})_{x^{(l)}})$ such that we have the equality

$$dX\, d\eta \equiv \tilde{\Phi}_{kl}^{\varrho\sigma}\; d(T^*_{m^{(kl)}} W_k)\, dA^{(k)}\, dA^{(l)}\, dB^{(l)}\, d\tilde{v}^{(l)}\, d\tau_l\, dx^{(l)}\, d\tau_k\, dx^{(k)}.$$

Furthermore, $\tilde{\Phi}_{kl}^{\varrho\sigma} = |\tau_l|^{c^{(l)} + d^{(k)} + d^{(l)} - 1} \Phi_{kl}^\varrho \circ {}'\zeta_{kl}^{\varrho\sigma}$.

Second reduction. Now, the group $G_{x^{(k)}}$ acts on $\gamma^{(k)}((S_k)_{x^{(k)}})_l$ with the isotropy types $(H_l) = (H_{i_j}), (H_{i_{j+1}}), \dots, (H_L)$. By the same arguments given in the first reduction, the isotropy types occuring in W_{kl} constitute a subset of these types, and we shall denote them by

$$(H_l) = (H_{i_{r_1}}), (H_{i_{r_2}}), \dots, (H_L).$$

Consequently, $G_{x^{(k)}}$ acts on S_{kl} with the isotropy types $(H_{i_{r_2}}), \dots, (H_L)$. Again, if G acted on S_{kl} only with type (H_L), we shall see later that the critical set of ${}^{(kl)}\tilde{\psi}^{wk}$ would be clean. However, in general this will not be the case, and we have to continue with the iteration.

N-th decomposition. Denote by $\Lambda \leq L$ the maximal number of elements that a totally ordered subset of the set of isotropy types can have. Assume that $3 \leq N < \Lambda$, and let $\{(H_{i_1}), \dots, (H_{i_N})\}$ be a totally ordered subset of the set of isotropy types with $i_1 < \dots < i_N < L$. Let f_{i_1}, $f_{i_1 i_2}$, S_{i_1}, $S_{i_1 i_2}$, as well as $x^{(i_1)} \in M_{i_1}(H_{i_1})$, $\quad x^{(i_2)} \in \gamma^{(i_1)}\big((S_{i_1}^+)_{x^{(i_1)}}\big)_{i_2}(H_{i_2})$ be defined as in the first two iteration steps. Let now $j < N$, and assume that $f_{i_1 \dots i_j}$, $S_{i_1 \dots i_j}$,... have already been defined.

For each $x^{(i_{N-1})}$, let $\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}$ be the submanifold with corners of the $G_{x^{(i_{N-1})}}$-manifold $\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})$ from which all the isotropy types less than (H_{i_N}) have been removed. Consider the invariant tubular neighborhood

$$f_{i_1\dots i_N} = \exp\circ\gamma^{(i_N)} : \nu_{i_1\dots i_N} \to \gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}$$

of the set of maximal singular orbits $\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}(H_{i_N})$, and define $S_{i_1\dots i_N}$ as the sphere subbundle in $\nu_{i_1\dots i_N}$ over $\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}(H_{i_N})$. Put further $W_{i_1\dots i_N} = f_{i_1\dots i_N}(\overset{\circ}{D}_1(\nu_{i_1\dots i_N}))$ and denote the corresponding integral in the decomposition of $I^{\varrho_{i_1}\dots\varrho_{i_{N-1}}}_{i_1\dots i_{N-1}}(\mu)$ by $I^{\varrho_{i_1}\dots\varrho_{i_{N-1}}}_{i_1\dots i_N}(\mu)$. For a point $x^{(i_N)} \in \gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}(H_{i_N})$ we then consider the decomposition

$$\mathfrak{g}_{x^{(i_{N-1})}} = \mathfrak{g}_{x^{(i_N)}} \oplus \mathfrak{g}^{\perp}_{x^{(i_N)}},$$

and set $d^{(i_N)} = \dim\mathfrak{g}^{\perp}_{x^{(i_N)}}$, $e^{(i_N)} = \dim\mathfrak{g}_{x^{(i_N)}}$, yielding the decomposition

$$\mathfrak{g} = \mathfrak{g}_{x^{(i_1)}} \oplus \mathfrak{g}^{\perp}_{x^{(i_1)}} = (\mathfrak{g}_{x^{(i_2)}} \oplus \mathfrak{g}^{\perp}_{x^{(i_2)}}) \oplus \mathfrak{g}^{\perp}_{x^{(i_1)}} = \dots = \mathfrak{g}_{x^{(i_N)}} \oplus \mathfrak{g}^{\perp}_{x^{(i_N)}} \oplus \dots \oplus \mathfrak{g}^{\perp}_{x^{(i_1)}}. \tag{29}$$

Denote by $\{A^{(i_N)}_r(x^{(i_1)},\dots,x^{(i_N)})\}$ a basis of $\mathfrak{g}^{\perp}_{x^{(i_N)}}$, and by $\{B^{(i_N)}_r(x^{(i_1)},\dots,x^{(i_N)})\}$ a basis of $\mathfrak{g}_{x^{(i_N)}}$. For arbitrary elements $A^{(i_N)} \in \mathfrak{g}^{\perp}_{x^{(i_N)}}$ and $B^{(i_N)} \in \mathfrak{g}_{x^{(i_N)}}$ write

$$A^{(i_N)} = \sum_{r=1}^{d^{(i_N)}} \alpha^{(i_N)}_r A^{(i_N)}_r(x^{(i_1)},\dots,x^{(i_N)}), \qquad B^{(i_N)} = \sum_{r=1}^{e^{(i_N)}} \beta^{(i_N)}_r B^{(i_N)}_r(x^{(i_1)},\dots,x^{(i_N)}),$$

and put

$$\tilde{v}^{(i_N)}(x^{(i_N)},\theta^{(i_N)}) = \gamma^{(i_N)}\left(\left(v^{(i_1\dots i_N)}_{\varrho}(x^{(i_N)}) + \sum_{r\neq\varrho}^{c^{(i_N)}} \theta^{(i_N)}_r v^{(i_1\dots i_N)}_r(x^{(i_N)})\right)\Big/\sqrt{1+\sum_{r\neq\varrho}(\theta^{(i_N)}_r)^2}\right)$$

for some ϱ, where $\left\{v^{(i_1\dots i_N)}_r\right\}$ is an orthonormal frame in $\nu_{i_1\dots i_N}$. Finally, we shall use the notations

$$m^{(i_j\dots i_N)} = \exp_{x^{(i_j)}}[\tau_{i_j}\exp_{x^{(i_{j+1})}}[\tau_{i_{j+1}}\exp_{x^{(i_{j+2})}}[\dots[\tau_{i_{N-2}}\exp_{x^{(i_{N-1})}}[\tau_{i_{N-1}}\exp_{x^{(i_N)}}[\tau_{i_N}\tilde{v}^{(i_N)}]]]\dots]]],$$
$$X^{(i_j\dots i_N)} = \tau_{i_j}\cdots\tau_{i_N}A^{(i_j)} + \tau_{i_{j+1}}\cdots\tau_{i_N}A^{(i_{j+1})} + \dots + \tau_{i_{N-1}}\tau_{i_N}A^{(i_{N-1})} + \tau_{i_N}A^{(i_N)} + B^{(i_N)},$$

where $j = 1,\dots,N$.

N-th monoidal transformation. Let the monoidal transformations ζ_{i_1} and $\zeta_{i_1 i_2}$ be defined as in the first two iteration steps, and assume that $\zeta_{i_1\dots i_j}$ have already been defined for $j < N$. Consider the monoidal transformation

$$\zeta_{i_1\dots i_N} : B_{Z_{i_1\dots i_N}}(B_{Z_{i_1\dots i_{N-1}}}(\dots B_{Z_{i_1}}(W_k\times\mathfrak{g})\dots)) \longrightarrow B_{Z_{i_1\dots i_{N-1}}}(\dots B_{Z_{i_1}}(W_k\times\mathfrak{g})\dots)$$

with center

$$Z_{i_1\dots i_N} \simeq \bigcup_{x^{(i_1)},\dots,x^{(i_{N-1})}} (-1,1)^{N-1}\times\mathfrak{iso}\,\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}(H_{i_N}).$$

Denote by $\zeta^{\varrho_{i_1}}_{i_1}\circ\dots\circ\zeta^{\varrho_{i_1}\dots\varrho_{i_N}}_{i_1\dots i_N}$ a local realization of the sequence of monoidal transformations $\zeta_{i_1}\circ\dots\circ\zeta_{i_1\dots i_N}$ in a set of $(\theta^{(i_1)},\dots,\theta^{(i_N)})$-charts labeled by the indices $\varrho_{i_1},\dots,\varrho_{i_N}$. Now, for an arbitrary element $B^{(i_1)} \in \mathfrak{g}_{i_1}$ one computes

$$\begin{aligned}(\tilde{B}^{i_1})_{m^{(i_1\dots i_N)}} &= \frac{d}{dt}\,\mathrm{e}^{-tB^{(i_1)}}\cdot m^{(i_1\dots i_N)}_{|t=0} = \frac{d}{dt}\exp_{x^{(i_1)}}\left[(\,\mathrm{e}^{-tB^{(i_1)}}\,)_{*,x^{(i_1)}}[\tau_{i_1}m^{(i_2\dots i_N)}]\right]_{|t=0}\\ &= (\exp_{x^{(i_1)}})_{*,\tau_{i_1}m^{(i_2\dots i_N)}}[\lambda(B^{(i_1)})\tau_{i_1}m^{(i_2\dots i_N)}].\end{aligned} \tag{30}$$

By iteration we obtain for arbitrary $A^{(i_j)} \in \mathfrak{g}_{i_j}^\perp$, $2 \leq j \leq N$,

$$(\tilde A^{i_j})_{m^{(i_1\dots i_N)}} = \frac{d}{dt} \exp_{x^{(i_1)}} \big[\tau_{i_1} \exp_{x^{(i_2)}}[\dots[\tau_{i_{j-1}}(\mathrm{e}^{-tA^{(i_j)}})_{*,x^{(i_1)}} m^{(i_j\dots i_N)}]\dots]\big]_{|t=0}$$
$$= (\exp_{x^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}} \big[\tau_{i_1} (\exp_{x^{(i_2)}})_{*,\tau_{i_2} m^{(i_3\dots i_N)}}[\dots[\tau_{i_{j-1}} \lambda(A^{(i_j)}) m^{(i_j\dots i_N)}]\dots]\big], \tag{31}$$

and similarly

$$(\tilde B^{i_N})_{m^{(i_1\dots i_N)}} = (\exp_{x^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}} \big[\tau_{i_1} (\exp_{x^{(i_2)}})_{*,\tau_{i_2} m^{(i_3\dots i_N)}}[\dots[\tau_{i_N} \lambda(B^{(i_N)}) \tilde v^{(i_N)}]\dots]\big]. \tag{32}$$

As a consequence, the phase function factorizes locally according to

$${}^{(i_1\dots i_N)}\tilde\psi^{tot} = \psi \circ (\mathrm{id}_{fiber} \otimes (\zeta_{i_1}^{\varrho_{i_1}} \circ \dots \circ \zeta_{i_1\dots i_N}^{\varrho_{i_1}\dots\varrho_{i_N}})) = \mathbb{J}(\eta_{m^{(i_1\dots i_N)}})(X^{(i_1\dots i_N)}) = \tau_{i_1} \cdots \tau_{i_N} \, {}^{(i_1\dots i_N)}\tilde\psi^{wk},$$

where in the given charts ${}^{(i_1\dots i_N)}\tilde\psi^{wk}$ is given by

$$\eta_{m^{(i_1\dots i_N)}} \Big(\widetilde{A^{(i_1)}}_{m^{(i_1\dots i_N)}}\Big) + \sum_{j=2}^N \eta_{m^{(i_1\dots i_N)}} \Big((\exp_{x^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}}$$
$$\big[(\exp_{x^{(i_2)}})_{*,\tau_{i_2} m^{(i_3\dots i_N)}}\big[\dots(\exp_{x^{(i_{j-1})}})_{*,\tau_{i_{j-1}} m^{(i_j\dots i_N)}}[\lambda(A^{(i_j)}) m^{(i_j\dots i_N)}]\dots\big]\big]\Big) \tag{33}$$
$$+ \eta_{m^{(i_1\dots i_N)}} \Big((\exp_{x^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}} \big[(\exp_{x^{(i_2)}})_{*,\tau_{i_2} m^{(i_3\dots i_N)}}\big[\dots$$
$$(\exp_{x^{(i_N)}})_{*,\tau_{i_N} \tilde v^{(i_N)}}[\lambda(B^{(i_N)}) \tilde v^{(i_N)}]\dots\big]\big]\Big).$$

Modulo lower order terms, $I(\mu)$ is then given by a sum of integrals of the form

$$I_{i_1\dots i_N}^{\varrho_{i_1}\dots\varrho_{i_N}}(\mu)$$
$$= \int_{M_{i_1}(H_{i_1})\times(-1,1)} \Big[\int_{\gamma^{(i_1)}((S_{i_1})_{x^{(i_1)}})_{i_2}(H_{i_2})\times(-1,1)} \dots \Big[\int_{\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}(H_{i_N})\times(-1,1)}$$
$$\Big[\int_{\gamma^{(i_N)}((S_{i_1\dots i_N})_{x^{(i_N)}})\times\mathfrak{g}_{x^{(i_N)}}\times\mathfrak{g}^\perp_{x^{(i_N)}}\times\dots\times\mathfrak{g}^\perp_{x^{(i_1)}}\times T^*_{m^{(i_1\dots i_N)}}W_{i_1}} e^{i\frac{\tau_1\dots\tau_N}{\mu}\,{}^{(i_1\dots i_N)}\tilde\psi^{wk}} a_{i_1\dots i_N}^{\varrho_{i_1}\dots\varrho_{i_N}} \tilde\Phi_{i_1\dots i_N}^{\varrho_{i_1}\dots\varrho_{i_N}}$$
$$d(T^*_{m^{(i_1\dots i_N)}}W_{i_1})\, dA^{(i_1)} \dots dA^{(i_N)}\, dB^{(i_N)}\, d\tilde v^{(i_N)}\Big] d\tau_{i_N}\, dx^{(i_N)} \dots\Big] d\tau_{i_2}\, dx^{(i_2)}\Big] d\tau_{i_1}\, dx^{(i_1)}. \tag{34}$$

Here $a_{i_1\dots i_N}^{\varrho_{i_1}\dots\varrho_{i_N}}$ are amplitudes with compact support in a system of $(\theta^{(i_1)},\dots,\theta^{(i_N)})$-charts labelled by the indices $\varrho_{i_1}\dots\varrho_{i_N}$, while

$$\tilde\Phi_{i_1\dots i_N}^{\varrho_{i_1}\dots\varrho_{i_N}} = \prod_{j=1}^N |\tau_{i_j}|^{c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-1} \Phi_{i_1\dots i_N}^{\varrho_{i_1}\dots\varrho_{i_N}},$$

where $\Phi_{i_1\dots i_N}^{\varrho_{i_1}\dots\varrho_{i_N}}$ are smooth functions which do not depend on the variables τ_{i_j}.

N-th reduction. For each $x^{(i_{N-1})}$, the isotropy group $G_{x^{(i_{N-1})}}$ acts on $\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}$ by the types $(H_{i_N}),\dots,(H_L)$. The types occuring in $W_{i_1\dots i_N}$ constitute a subset of these, and $G_{x^{(i_{N-1})}}$ acts on the sphere bundle $S_{i_1\dots i_N}$ over the submanifold $\gamma^{(i_{N-1})}((S_{i_1\dots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}(H_{i_N}) \subset W_{i_1\dots i_N}$ with one type less.

End of iteration. As before, let $\Lambda \leq L$ be the maximal number of elements of a totally ordered subset of the set of isotropy types. After maximally $N = \Lambda - 1$ steps, the end of the iteration is reached.

6. Phase analysis of the weak transforms. Smoothness of the critical sets

We shall now prove the smoothness of the critical sets of the weak transforms. We continue with the notation of the previous sections, and consider a sequence of local monoidal transformations $\zeta_{i_1}^{\varrho_{i_1}} \circ \cdots \circ \zeta_{i_1\dots i_N}^{\varrho_{i_1}\dots\varrho_{i_N}}$ corresponding to a totally ordered subset $\{(H_{i_1}), \dots, (H_{i_N})\}$ of non-principal isotropy types that are maximal in the sense that, if there is an isotropy type $(H_{i_{N+1}})$ with $i_N < i_{N+1}$ such that $\big\{(H_{i_1}), \dots, (H_{i_{N+1}})\big\}$ is a totally ordered subset, then $(H_{i_{N+1}}) = (H_L)$. For later purposes, let us define certain geometric distributions $E^{(i_j)}$ and $F^{(i_N)}$ on M by setting

$$
\begin{aligned}
&E^{(i_1)}_{m^{(i_1\dots i_N)}} = \mathrm{Span}\{\tilde Y_{m^{(i_1\dots i_N)}} : Y \in \mathfrak{g}^\perp_{x^{(i_1)}}\},\\
&E^{(i_j)}_{m^{(i_1\dots i_N)}} = (\exp_{x^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}} \dots (\exp_{x^{(i_{j-1})}})_{*,\tau_{i_{j-1}} m^{(i_j\dots i_N)}}[\lambda(\mathfrak{g}^\perp_{x^{(i_j)}}) m^{(i_j\dots i_N)}],\\
&F^{(i_N)}_{m^{(i_1\dots i_N)}} = (\exp_{x^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}} \dots (\exp_{x^{(i_N)}})_{*,\tau_{i_N} \tilde v^{(i_N)}}[\lambda(\mathfrak{g}_{x^{(i_N)}}) \tilde v^{(i_N)}],
\end{aligned}
\tag{35}
$$

where $2 \le j \le N$. Note that by (29), (31) and (32) we have

$$
T_{m^{(i_1\dots i_N)}}(G\cdot m^{(i_1\dots i_N)}) = E^{(i_1)}_{m^{(i_1\dots i_N)}} \oplus \bigoplus_{j=2}^N \tau_{i_1}\dots\tau_{i_{j-1}} E^{(i_j)}_{m^{(i_1\dots i_N)}} \oplus \tau_{i_1}\dots\tau_{i_N} F^{(i_N)}_{m^{(i_1\dots i_N)}}. \tag{36}
$$

By construction, for $\tau_{i_j} \neq 0$, $1 \le j \le N$, the G-orbit through $m^{(i_1\dots i_N)}$ is of principal type G/H_L, which amounts to the fact that $G_{x^{(i_{N-1})}}$ acts on $S_{i_1\dots i_N}$ only with the isotropy type (H_L), where we understand that $G_{x^{(i_0)}} = G$. We then have the following

Theorem 2. *Let $\{(H_{i_1}), \dots, (H_{i_N})\}$ be a maximal, totally ordered subset of non-principal isotropy types, and $\zeta_{i_1}^{\varrho_{i_1}} \circ \cdots \circ \zeta_{i_1\dots i_N}^{\varrho_{i_1}\dots\varrho_{i_N}}$ a corresponding sequence of local monoidal transformations in a set of $(\theta^{(i_1)}, \dots, \theta^{(i_N)})$-charts labeled by the indices $\varrho_{i_1}, \dots, \varrho_{i_N}$. Let $\eta_{m^{(i_1\dots i_N)}} \in \pi^{-1}(m^{(i_1\dots i_N)})$, and consider the factorization*

$$
\mathbb{J}(\eta_{m^{(i_1\dots i_N)}})(X^{(i_1\dots i_N)}) = {}^{(i_1\dots i_N)}\tilde\psi^{tot} = \tau_{i_1}\cdots\tau_{i_N}\, {}^{(i_1\dots i_N)}\tilde\psi^{wk,\,pre}
$$

of the phase function ψ after N iteration steps, where ${}^{(i_1\dots i_N)}\tilde\psi^{wk,pre}$ is given by (33).[3] *Let further*

$$
{}^{(i_1\dots i_N)}\tilde\psi^{wk}
$$

denote the pullback of ${}^{(i_1\dots i_N)}\tilde\psi^{wk,\,pre}$ along the substitution $\tau = \delta_{i_1\dots i_N}(\sigma)$ given by the sequence of monoidal transformations

$$
\begin{aligned}
\delta_{i_1\dots i_N} : (\sigma_{i_1}, \dots \sigma_{i_N}) &\mapsto \sigma_{i_1}(1, \sigma_{i_2}, \dots, \sigma_{i_N}) = (\sigma'_{i_1}, \dots, \sigma'_{i_N}) \mapsto \sigma'_{i_2}(\sigma'_{i_1}, 1, \dots, \sigma'_{i_N}) = (\sigma''_{i_1}, \dots, \sigma''_{i_N})\\
&\mapsto \sigma''_{i_3}(\sigma''_{i_1}, \sigma''_{i_2}, 1, \dots, \sigma''_{i_N}) = \cdots \mapsto \cdots = (\tau_{i_1}, \dots, \tau_{i_N}).
\end{aligned}
$$

Then the critical set $\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde\psi^{wk})$ of ${}^{(i_1\dots i_N)}\tilde\psi^{wk}$ is given by all points

$$
(\sigma_{i_1}, \dots, \sigma_{i_N}, x^{(i_1)}, \dots, x^{(i_N)}, \tilde v^{(i_N)}, A^{(i_1)}, \dots, A^{(i_N)}, B^{(i_N)}, \eta_{m^{(i_1\dots i_N)}})
$$

satisfying the conditions

- (I) $A^{(i_j)} = 0$ *for all* $j = 1, \dots, N$, *and* $\lambda(B^{(i_N)})\tilde v^{(i_N)} = 0$;
- (II) $\eta_{m^{(i_1\dots i_N)}} \in \mathrm{Ann}\big(E^{(i_j)}_{m^{(i_1\dots i_N)}}\big)$ *for all* $j = 1, \dots, N$;
- (III) $\eta_{m^{(i_1\dots i_N)}} \in \mathrm{Ann}\big(F^{(i_N)}_{m^{(i_1\dots i_N)}}\big)$.

Furthermore, $\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde\psi^{wk})$ is a C^∞-submanifold of codimension 2κ, where $\kappa = \dim G/H_L$ is the dimension of a principal orbit.

[3] Note that ${}^{(i_1\dots i_N)}\tilde\psi^{wk,pre}$ was denoted in (33) by ${}^{(i_1\dots i_N)}\tilde\psi^{wk}$.

Proof. To begin with, let $\sigma_{i_1}\cdots\sigma_{i_N}\neq 0$, so that all τ_{i_j} are non-zero. In this case, the sequence of monoidal transformations $\zeta_{i_1}^{\varrho_{i_1}}\circ\cdots\circ\zeta_{i_1\dots i_N}^{\varrho_{i_1}\cdots\varrho_{i_N}}\circ\delta_{i_1\dots i_N}$ constitutes a diffeomorphism, so that

$$\begin{aligned}\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{tot})_{\sigma_{i_1}\cdots\sigma_{i_N}\neq 0}=\{&(\sigma_{i_1},\dots,\sigma_{i_N},x^{(i_1)},\dots,x^{(i_N)},\tilde{v}^{(i_N)},A^{(i_1)},\dots,A^{(i_N)},B^{(i_N)},\eta_{m^{(i_1\dots i_N)}}):\\ &(\eta_{m^{(i_1\dots i_N)}},X^{(i_1\dots i_N)})\in\mathrm{Crit}(\psi),\quad \sigma_{i_1}\cdots\sigma_{i_N}\neq 0\}.\end{aligned}$$

Now,

$$(\eta_{m^{(i_1\dots i_N)}},X^{(i_1\dots i_N)})\in\mathrm{Crit}(\psi)\quad\Leftrightarrow\quad \eta_{m^{(i_1\dots i_N)}}\in\Omega,\quad \tilde{X}^{(i_1\dots i_N)}_{\eta_{m^{(i_1\dots i_N)}}}=0.$$

Furthermore, $\tilde{X}_\eta=0$ clearly implies $\tilde{X}_{\pi(\eta)}=\pi_*(\tilde{X}_\eta)=0$. Since the point $m^{(i_1\dots i_N)}$ lies in a slice at $x^{(i_1)}$, the condition $\tilde{X}^{(i_1\dots i_N)}_{m^{(i_1\dots i_N)}}=0$ means that the vector field $\tilde{X}^{(i_1\dots i_N)}$ must vanish at $x^{(i_1)}$ as well. Hence, $X^{(i_1\dots i_N)}\in\mathfrak{g}_{x^{(i_1)}}$, since

$$\mathfrak{g}_m=\mathrm{Lie}(G_m)=\left\{X\in\mathfrak{g}:\tilde{X}_m=0\right\},\qquad m\in M.$$

Now

$$\mathfrak{g}_{x^{(i_N)}}\subset\mathfrak{g}_{x^{(i_{N-1})}}\subset\cdots\subset\mathfrak{g}_{x^{(i_1)}}$$

and $\mathfrak{g}^\perp_{x^{(i_{j+1})}}\subset\mathfrak{g}_{x^{(i_j)}}$ imply

$$\tilde{X}^{(i_1\dots i_N)}_{x^{(i_1)}}=\tau_{i_1}\dots\tau_{i_N}\sum\alpha_r^{(i_1)}(\tilde{A}_r^{(i_1)})_{x^{(i_1)}}=0.$$

Thus we conclude $\alpha^{(i_1)}=0$, which gives $X^{(i_2\dots i_N)}=X^{(i_1\dots i_N)}\in\mathfrak{g}_{m^{(i_1\dots i_N)}}$, and consequently $X^{(i_2\dots i_N)}\in\mathfrak{g}_{m^{(i_2\dots i_N)}}$ by (30). A repetition of the above argument yields that the condition $\tilde{X}^{(i_1\dots i_N)}_{m^{(i_1\dots i_N)}}=0$ is equivalent to (I) in the case that all σ_{i_j} are different from zero. Actually, the same argument shows that for $\sigma_{i_j}\neq 0$

$$\mathfrak{g}_{m^{(i_1\dots i_N)}}=\mathfrak{g}_{\tilde{v}^{(i_N)}},\tag{37}$$

since $\mathfrak{g}_{\tilde{v}^{(i_N)}}\subset\mathfrak{g}_{x^{(i_N)}}$. Next, $\eta_{m^{(i_1\dots i_N)}}\in\Omega$ means that

$$\mathbb{J}(\eta_{m^{(i_1\dots i_N)}})(X)=\eta_{m^{(i_1\dots i_N)}}(\tilde{X}_{m^{(i_1\dots i_N)}})=0\qquad\forall X\in\mathfrak{g},$$

which by (21) is equivalent to $\eta_{m^{(i_1\dots i_N)}}\in\mathrm{Ann}(T_{m^{(i_1\dots i_N)}}(G\cdot m^{(i_1\dots i_N)}))$. If $\sigma_{i_j}\neq 0$ for all $j=1,\dots,N$, (II) and (III) imply that

$$\eta_{m^{(i_1\dots i_N)}}\Big((\exp_{x^{(i_1)}})_{*,\tau_{i_1}m^{(i_2\dots i_N)}}[\dots(\exp_{x^{(i_{j-1})}})_{*,\tau_{i_{N-1}}m^{(i_N)}}[\lambda(\mathfrak{g}_{x^{(i_{N-1})}})m^{(i_N)}]\dots]\Big)=0,$$

since $\mathfrak{g}_{x^{(i_{N-1})}}=\mathfrak{g}_{x^{(i_N)}}\oplus\mathfrak{g}^\perp_{x^{(i_N)}}$. By repeatedly using this argument, we conclude with (36) that for $\sigma_{i_j}\neq 0$

$$\text{(II), (III)}\quad\Longleftrightarrow\quad\eta_{m^{(i_1\dots i_N)}}\in\mathrm{Ann}(T_{m^{(i_1\dots i_N)}}(G\cdot m^{(i_1\dots i_N)})).\tag{38}$$

Taking everything together therefore gives

$$\begin{aligned}&\mathrm{Crit}({}^{(i_1\dots i_N)}\psi^{tot})_{\sigma_{i_1}\cdots\sigma_{i_N}\neq 0}\\ &\qquad=\{(\sigma_{i_1},\dots,\sigma_{i_N},x^{(i_1)},\dots,x^{(i_N)},\tilde{v}^{(i_N)},A^{(i_1)},\dots,A^{(i_N)},B^{(i_N)},\eta_{m^{(i_1\dots i_N)}}):\\ &\qquad\sigma_{i_1}\cdots\sigma_{i_N}\neq 0,\ \text{(I)-(III) are fulfilled and } \tilde{B}^{(i_N),\mathrm{v}}_{\eta_{m^{(i_1\dots i_N)}}}=0\}.\end{aligned}\tag{39}$$

Here $\mathfrak{X}^{\mathrm{v}}_\eta$ denotes the vertical component of a vector field $\mathfrak{X}\in T(T^*M)$ with respect to the decomposition $T_\eta(T^*M)=T^{\mathrm{v}}\oplus T^{\mathrm{h}}$, T^{v} being the tangent space to the fiber, and T^{h} the tangent space to the zero section at η. We now assert that

$$\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk})=\overline{\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{tot})_{\sigma_{i_1}\cdots\sigma_{i_N}\neq 0}}.$$

To show this, let $(\kappa, \mathcal{O})$ be a chart on M with coordinates $\kappa(m) = (q_1, \dots, q_n)$, and introduce on $T^*\mathcal{O}$ the coordinates

$$\eta_m = \sum p_i (dq_i)_m, \qquad \tilde\kappa(\eta) = (q_1, \dots, q_n, p_1, \dots, p_n), \quad \eta \in T^*\mathcal{O}.$$

Write $\eta_{m^{(i_1\dots i_N)}} = \sum p_i(\, dq_i)_{m^{(i_1\dots i_N)}}$, and still assume that all σ_{i_j} are different from zero. Then all τ_{i_j} are different from zero, too, and $\partial_p \, {}^{(i_1\dots i_N)}\tilde\psi^{wk} = 0$ is equivalent to

$$\partial_p \, \mathbb{J}(\eta_{m^{(i_1\dots i_N)}})(X^{(i_1\dots i_N)}) = (\, dq_1(\tilde X^{(i_1\dots i_N)}_{m^{(i_1\dots i_N)}}), \dots, \, dq_n(\tilde X^{(i_1\dots i_N)}_{m^{(i_1\dots i_N)}})) = 0,$$

which gives us the condition $\tilde X^{(i_1\dots i_N)}_{m^{(i_1\dots i_N)}} = 0$. By (37) we therefore obtain condition I) in the case that all σ_{i_j} are different from zero. Let next $N_{x^{(i_1)}}(G\cdot x^{(i_1)})$ be the normal space in $T_{x^{(i_1)}}M$ to the orbit $G \cdot x^{(i_1)}$, on which $G_{x^{(i_1)}}$ acts, and define $N_{x^{(i_{j+1})}}(G_{x^{(i_j)}} \cdot x^{(i_{j+1})})$ successively as the normal space to the orbit $G_{x^{(i_j)}} \cdot x^{(i_{j+1})}$ in the $G_{x^{(i_j)}}$-space $N_{x^{(i_j)}}(G_{x^{(i_{j-1})}} \cdot x^{(i_j)})$, where we understand that $G_{x^{(i_0)}} = G$. By Bredon [9, page 308], these actions can be assumed to be orthogonal. Set

$$V^{(i_1\dots i_j)} = \bigcap_{r=1}^{j} N_{x^{(i_r)}}(G_{x^{(i_{r-1})}} \cdot x^{(i_r)}) = N_{x^{(i_j)}}(G_{x^{(i_{j-1})}} \cdot x^{(i_j)}). \tag{40}$$

With the identification $T_0(T_mM) \simeq T_mM$ one has

$$(\exp_m)_{*,0} : T_0(T_mM) \longrightarrow T_mM, \qquad (\exp_m)_{*,0} \simeq \mathrm{id}\,, \tag{41}$$

and similarly $(\exp_{x^{(i_j)}})_{*,0} \simeq \mathrm{id}$ for all $j = 2, \dots, N$. Therefore, if $\tau_{i_j} = 0$ for all j, then $E^{(i_1)}_{x^{(i_1)}} = T_{x^{(i_1)}}(G \cdot x^{(i_1)})$, and

$$E^{(i_j)}_{x^{(i_1)}} \simeq T_{x^{(i_j)}}(G_{x^{(i_{j-1})}} \cdot x^{(i_j)}) \subset V^{(i_1\dots i_{j-1})}, \qquad 2 \leq j \leq N,$$

while $F^{(i_N)}_{x^{(i_1)}} \simeq T_{\tilde v^{(i_N)}}(G_{x^{(i_N)}} \cdot \tilde v^{(i_N)}) \subset V^{(i_1\dots i_N)}$. Therefore $E^{(i_j)}_{x^{(i_1)}} \cap V^{(i_1\dots i_j)} = \{0\}$, so that we obtain the direct sum of vector spaces

$$E^{(i_1)}_{x^{(i_1)}} \oplus E^{(i_2)}_{x^{(i_1)}} \oplus \dots \oplus E^{(i_N)}_{x^{(i_1)}} \oplus F^{(i_N)}_{x^{(i_1)}} \subset T_{x^{(i_1)}}M. \tag{42}$$

Let now one of the σ_{i_j} be equal to zero, so that all τ_{i_j} are zero. With the identification (41) one has

$$ {}^{(i_1\dots i_N)}\tilde\psi^{wk} = \sum p_i \, dq_i\Big(\widetilde{A^{(i_1)}}_{x^{(i_1)}} + \sum_{j=2}^N \lambda(A^{(i_j)})x^{(i_j)} + \lambda(B^{(i_N)})\tilde v^{(i_N)}\Big), \tag{43}$$

and $\partial_p \, {}^{(i_1\dots i_N)}\tilde\psi^{wk} = 0$ is equivalent to

$$\widetilde{A^{(i_1)}}_{x^{(i_1)}} + \sum_{j=2}^N \lambda(A^{(i_j)})x^{(i_j)} + \lambda(B^{(i_N)})\tilde v^{(i_N)} = 0.$$

Since $x^{(i_j)} \in \gamma^{(i_{j-1})}((S_{i_1\dots i_{j-1}})_{x^{(i_{j-1})}}) \subset V^{(i_1\dots i_{j-1})}$, we see that for every $j = 2, \dots, N$

$$\lambda\Big(\sum_r \alpha_r^{(i_j)} A_r^{(i_j)}\Big)\, x^{(i_j)} \in T_{x^{(i_j)}}(G_{x^{(i_{j-1})}} \cdot x^{(i_j)}) \subset V^{(i_1\dots i_{j-1})}.$$

In addition, $(\tilde A_r^{(i_1)})_{x^{(i_1)}} \in T_{x^{(i_1)}}(G \cdot x^{(i_1)})$, and $\lambda\Big(\sum_r \beta_r^{(i_N)} B_r^{(i_N)}\Big)\tilde v^{(i_N)} \in V^{(i_1\dots i_N)}$, so that taking everything together we obtain with (42) for arbitrary σ_{i_j}

$$\partial_p \, {}^{(i_1\dots i_N)}\tilde\psi^{wk} = 0 \quad \Longleftrightarrow \quad \text{(I)}.$$

In particular, one concludes that ${}^{(i_1\dots i_N)}\tilde\psi^{wk}$ must vanish on its critical set. Since

$$d(\, {}^{(i_1\dots i_N)}\tilde\psi^{tot}) = d(\tau_{i_1} \dots \tau_{i_N}) \cdot \, {}^{(i_1\dots i_N)}\tilde\psi^{wk} + \tau_{i_1} \dots \tau_{i_N} \, d\,({}^{(i_1\dots i_N)}\tilde\psi^{wk}),$$

one sees that

$$\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}) \subset \mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{tot}).$$

In turn, the vanishing of ψ on its critical set implies

$$\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk})_{\sigma_{i_1}\cdots\sigma_{i_N}\neq 0} = \mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{tot})_{\sigma_{i_1}\dots\sigma_{i_N}\neq 0}.$$

Therefore, by continuity,

$$\overline{\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{tot})_{\sigma_{i_1}\cdots\sigma_{i_N}\neq 0}} \subset \mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}). \tag{44}$$

In order to see the converse inclusion, let us consider next the α-derivatives. Clearly,

$$\partial_{\alpha^{(i_1)}}\, {}^{(i_1\dots i_N)}\tilde{\psi}^{wk} = 0 \quad \iff \quad \eta_{m^{(i_1\dots i_N)}}(\tilde{Y}_{m^{(i_1\dots i_N)}}) = 0 \quad \forall\, Y \in \mathfrak{g}^{\perp}_{x^{(i_1)}}.$$

For the remaining derivatives one computes

$$\partial_{\alpha_r^{(i_j)}}\, {}^{(i_1\dots i_N)}\tilde{\psi}^{wk}$$
$$= \eta_{m^{(i_1\dots i_N)}}\Big((\exp_{x^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}}\big[\dots(\exp_{x^{(i_{j-1})}})_{*,\tau_{i_{j-1}} m^{(i_j\dots i_N)}}[\lambda(A_r^{(i_j)})m^{(i_j\dots i_N)}]\dots\big]\Big),$$

from which one deduces that for $j = 2,\dots,N$

$$\partial_{\alpha^{(i_j)}}\, {}^{(i_1\dots i_N)}\tilde{\psi}^{wk} = 0 \quad \iff \quad \forall\, Y \in \mathfrak{g}^{\perp}_{x^{(i_j)}}$$
$$\eta_{m^{(i_1\dots i_N)}}\Big((\exp_{x^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}}\big[\dots(\exp_{x^{(i_{j-1})}})_{*,\tau_{i_{j-1}} m^{(i_j\dots i_N)}}[\lambda(Y)m^{(i_j\dots i_N)}]\dots\big]\Big) = 0.$$

In a similar way,

$$\partial_{\beta^{(i_j)}}\, {}^{(i_1\dots i_N)}\tilde{\psi}^{wk} = 0 \quad \iff \quad \forall\, Z \in \mathfrak{g}_{x^{(i_N)}}$$
$$\eta_{m^{(i_1\dots i_N)}}\Big((\exp_{x^{(i_1)}})_{*,\tau_{i_1} m^{(i_2\dots i_N)}}\big[\dots(\exp_{x^{(i_N)}})_{*,\tau_{i_N} \tilde{v}^{(i_N)}}[\lambda(Z)\tilde{v}^{(i_N)}]\dots\big]\Big) = 0.$$

by which the necessity of the conditions (I)–(III) is established. In order to see their sufficiency, let them be fulfilled, and assume again that $\sigma_{i_j} \neq 0$ for all $j = 1,\dots,N$. Then (38) implies that $\eta_{m^{(i_1\dots i_N)}} \in \mathrm{Ann}(T_{m^{(i_1\dots i_N)}}(G\cdot m^{(i_1\dots i_N)}))$. Now, if $\sigma_{i_j} \neq 0$, $G\cdot m^{(i_1\dots i_N)}$ is of principal type G/H_L in M, so that the isotropy group of $m^{(i_1\dots i_N)}$ must act trivially on $N_{m^{(i_1\dots i_N)}}(G\cdot m^{(i_1\dots i_N)})$, compare Bredon [9, page 181]. If therefore $\mathfrak{X} = \mathfrak{X}_T + \mathfrak{X}_N$ denotes an arbitrary element in $T_{m^{(i_1\dots i_N)}}M = T_{m^{(i_1\dots i_N)}}(G\cdot m^{(i_1\dots i_N)})) \oplus N_{m^{(i_1\dots i_N)}}(G\cdot m^{(i_1\dots i_N)}))$, and $g \in G_{m^{(i_1\dots i_N)}}$, one computes

$$\begin{aligned} g\cdot\eta_{m^{(i_1\dots i_N)}}(\mathfrak{X}) &= [(L_{g^{-1}})^*_{gm^{(i_1\dots i_N)}}\eta_{m^{(i_1\dots i_N)}}](\mathfrak{X}) = \eta_{m^{(i_1\dots i_N)}}((L_{g^{-1}})_{*,m^{(i_1\dots i_N)}}(\mathfrak{X}_N)) \\ &= \eta_{m^{(i_1\dots i_N)}}(\mathfrak{X}_N) = \eta_{m^{(i_1\dots i_N)}}(\mathfrak{X}). \end{aligned}$$

In view of $\lambda(B^{(i_N)})\tilde{v}^{(i_N)} = 0$ and (37) we therefore get the condition $\tilde{B}^{(i_N),\mathrm{v}}_{\eta_{m^{(i_1\dots i_N)}}} = 0$. Let us now assume that one of the σ_{i_j} equals zero. Then

$$\text{(II), (III)} \quad \Leftrightarrow \quad \begin{cases} \eta_{x^{(i_1)}} \in \mathrm{Ann}(T_{x^{(i_j)}}(G_{x^{(i_{j-1})}}\cdot x^{(i_j)})) & \forall\, j = 1,\dots,N, \\ \eta_{x^{(i_1)}} \in \mathrm{Ann}(T_{\tilde{v}^{(i_N)}}(G_{x^{(i_N)}}\cdot \tilde{v}^{(i_N)})). \end{cases} \tag{45}$$

Lemma 4. *The orbit of the point $\tilde{v}^{(i_N)}$ in the $G_{x^{(i_N)}}$-space $V^{(i_1\dots i_N)}$ is of principal type.*

Proof of the lemma. By assumption, for $\sigma_{i_j} \neq 0$, $1 \leq j \leq N$, the G-orbit of $m^{(i_1\dots i_N)}$ is of principal type G/H_L in M. The theory of compact group actions then implies that this is equivalent to the fact that $m^{(i_2\dots i_N)} \in V^{(i_1)}$ is of principal type in the $G_{x^{(i_1)}}$-space $V^{(i_1)}$, see Bredon [9, page 181], which in turn is equivalent to the fact that $m^{(i_3\dots i_N)} \in V^{(i_1 i_2)}$ is of principal type in the $G_{x^{(i_2)}}$-space $V^{(i_1 i_2)}$, and so forth. Thus, $m^{(i_j\dots i_N)} \in V^{(i_1\dots i_{j-1})}$ must be of principal type in the $G_{x^{(i_{j-1})}}$-space $V^{(i_1\dots i_{j-1})}$ for all $j = 1,\dots N$, and the assertion follows. □

As a consequence of the previous lemma, the stabilizer of $\tilde{v}^{(i_N)}$ must act trivially on $N_{\tilde{v}^{(i_N)}}(G_{x^{(i_N)}} \cdot \tilde{v}^{(i_N)})$. If therefore $\mathfrak{X} = \mathfrak{X}_T + \mathfrak{X}_N$ denotes an arbitrary element in

$$T_{x^{(i_1)}} M \simeq \bigoplus_{j=1}^N T_{x^{(i_j)}}(G_{x^{(i_{j-1})}} \cdot x^{(i_j)}) \oplus T_{\tilde{v}^{(i_N)}}(G_{x^{(i_N)}} \cdot \tilde{v}^{(i_N)}) \oplus N_{\tilde{v}^{(i_N)}}(G_{x^{(i_N)}} \cdot \tilde{v}^{(i_N)}),$$

we obtain with (45)

$$\begin{aligned} g \cdot \eta_{x^{(i_1)}}(\mathfrak{X}) &= [(L_{g^{-1}})^*_{gx^{(i_1)}} \eta_{x^{(i_1)}}](\mathfrak{X}) = \eta_{x^{(i_1)}}((L_{g^{-1}})_{*,x^{(i_1)}}(\mathfrak{X}_N)) \\ &= \eta_{x^{(i_1)}}(\mathfrak{X}_N) = \eta_{x^{(i_1)}}(\mathfrak{X}), \qquad g \in G_{\tilde{v}^{(i_N)}}. \end{aligned}$$

Collecting everything together we have shown for arbitrary σ_{i_j} that

$$\partial_{p,\alpha^{(i_1)},\dots,\alpha^{(i_N)},\beta^{(i_N)}} \; {}^{(i_1\dots i_N)}\tilde{\psi}^{wk} = 0 \iff \text{(I), (II), (III)} \implies \tilde{B}^{(i_N),\mathrm{v}}_{\eta_{m^{(i_1\dots i_N)}}} = 0. \tag{46}$$

By (39) and (44) we therefore conclude

$$\overline{\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{tot})_{\sigma_{i_1}\cdots\sigma_{i_N}\neq 0}} = \mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}). \tag{47}$$

Thus we have computed the critical set of ${}^{(i_1\dots i_N)}\tilde{\psi}^{wk}$, and it remains to show that it is a C^∞-submanifold of codimension 2κ. By our previous considerations, we have the characterization

$$\begin{aligned} &\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}) \\ &= \Big\{ A^{(i_j)} = 0, \quad \lambda(B^{(i_N)})\tilde{v}^{(i_N)} = 0, \quad \eta_{m^{(i_1\dots i_N)}} \in \mathrm{Ann}\Big(\bigoplus_{j=1}^N E^{(i_j)}_{m^{(i_1\dots i_N)}} \oplus F^{(i_N)}_{m^{(i_1\dots i_N)}} \Big) \Big\}. \end{aligned} \tag{48}$$

Note that the condition $\tilde{B}^{(i_N),\mathrm{v}}_{\eta_{m^{(i_1\dots i_N)}}} = 0$ is already implied by the others. Now, $\dim E^{(i_j)}_{m^{(i_1\dots i_N)}} = \dim G_{x^{(i_{j-1})}} \cdot x^{(i_j)}$. Since for $\sigma_{i_1}\cdots\sigma_{i_N} \neq 0$ the G-orbit of $m^{(i_1\dots i_N)}$ is of principal type G/H_L in M, one computes in this case with (36)

$$\begin{aligned} \kappa &= \dim G \cdot m^{(i_1\dots i_N)} = \dim T_{m^{(i_1\dots i_N)}}(G \cdot m^{(i_1\dots i_N)}) \\ &= \dim[E^{(i_1)}_{m^{(i_1\dots i_N)}} \oplus \bigoplus_{j=2}^N \tau_{i_1}\dots\tau_{i_{j-1}} E^{(i_j)}_{m^{(i_1\dots i_N)}} \oplus \tau_{i_1}\dots\tau_{i_N} F^{(i_N)}_{m^{(i_1\dots i_N)}}] \\ &= \sum_{j=1}^N \dim E^{(i_j)}_{m^{(i_1\dots i_N)}} + \dim F^{(i_N)}_{m^{(i_1\dots i_N)}}. \end{aligned}$$

But since the dimension of the spaces $E^{(i_j)}_{m^{(i_1\dots i_N)}}$ and $F^{(i_N)}_{m^{(i_1\dots i_N)}}$ does not depend on the variables σ_{i_j}, we obtain the equality

$$\kappa = \sum_{j=1}^N \dim E^{(i_j)}_{m^{(i_1\dots i_N)}} + \dim F^{(i_N)}_{m^{(i_1\dots i_N)}} \tag{49}$$

for arbitrary $m^{(i_1\dots i_N)}$. Note that, in contrast, the dimension of $T_{m^{(i_1\dots i_N)}}(G \cdot m^{(i_1\dots i_N)})$ collapses, as soon as one of the τ_{i_j} becomes zero. Since the annihilator of a subspace of $T_m M$ is itself a linear subspace of $T^*_m M$, we arrive at a vector bundle with $(n-\kappa)$-dimensional fiber that is locally given by the trivialization

$$\Big((\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}), \mathrm{Ann}\Big(\bigoplus_{j=1}^N E^{(i_j)}_{m^{(i_1\dots i_N)}} \oplus F^{(i_N)}_{m^{(i_1\dots i_N)}} \Big) \Big) \mapsto (\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}).$$

Consequently, by equation (48) we see that $\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk})$ is equal to the total space of the fiber product of the mentioned vector bundle with the isotropy algebra bundle given by the local trivialization

$$(\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}, \mathfrak{g}_{\tilde{v}^{(i_N)}}) \mapsto (\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}).$$

Lastly, since by equation (37) we have $\mathfrak{g}_{\tilde{v}^{(i_N)}} = \mathfrak{g}_{m^{(i_1,\dots,i_N)}}$ in case that all σ_{i_j} are different from zero, we necessarily have $\dim \mathfrak{g}_{\tilde{v}^{(i_N)}} = d - \kappa$, which concludes the proof of the theorem. □

7. Phase analysis of the weak transforms. Non-degeneracy of the transversal Hessians

In this section, we prove the non-degeneracy of the transversal Hessians of the weak transforms. To begin with, let M be a n-dimensional Riemannian manifold, and C the critical set of a function $\psi \in \mathrm{C}^\infty(M)$, which is assumed to be a smooth submanifold in a chart $\mathcal{O} \subset M$. Let further

$$\alpha : (x, y) \mapsto m, \qquad \beta : (q_1, \dots, q_n) \mapsto m, \qquad m \in \mathcal{O},$$

be two systems of local coordinates on $\mathcal{O}$, such that $\alpha(x, y) \in C$ if and only if $y = 0$. As one computes, the transversal Hessian is given by

$$\partial_{y_k} \partial_{y_l}(\psi \circ \alpha)(x, 0) = \mathrm{Hess}\, \psi_{|\alpha(x,0)}(\alpha_{*,(x,0)}(\partial_{y_k}), \alpha_{*,(x,0)}(\partial_{y_l})), \tag{50}$$

Let us now write $x = (x', x'')$, and consider the restriction of ψ onto the C^∞-submanifold

$$M_{c'} = \{m \in \mathcal{O} : m = \alpha(c', x'', y)\}.$$

We write $\psi_{c'} = \psi_{|M_{c'}}$, and denote the critical set of $\psi_{c'}$ by $C_{c'}$, which contains $C \cap M_{c'}$ as a subset. Introducing on $M_{c'}$ the local coordinates $\alpha' : (x'', y) \mapsto \alpha(c', x'', y)$, we obtain

$$\partial_{y_k} \partial_{y_l}(\psi_{c'} \circ \alpha')(x'', 0) = \mathrm{Hess}\, \psi_{c'|\alpha(x'',0)}(\alpha'_{*,(x'',0)}(\partial_{y_k}), \alpha'_{*,(x'',0)}(\partial_{y_l})).$$

Let us now assume $C_{c'} = C \cap M_{c'}$, a transversal intersection. Then $C_{c'}$ is a submanifold of $M_{c'}$, and the normal space to $C_{c'}$ as a submanifold of $M_{c'}$ at a point $\alpha'(x'', 0)$ is spanned by the vector fields $\alpha'_{*,(x'',0)}(\partial_{y_k})$. Since clearly

$$\partial_{y_k} \partial_{y_l}(\psi_{c'} \circ \alpha')(x'', 0) = \partial_{y_k} \partial_{y_l}(\psi \circ \alpha)(x, 0), \qquad x = (c', x''),$$

we thus have proven the following

Lemma 5. *Assume that $C_{c'} = C \cap M_{c'}$. Then the restriction*

$$\mathrm{Hess}\, \psi(\alpha(c', x'', 0))_{|N_{\alpha(c',x'',0)}C}$$

of the Hessian of ψ to the normal space $N_{\alpha(c',x'',0)}C$ defines a non-degenerate quadratic form if, and only if the restriction

$$\mathrm{Hess}\, \psi_{c'}(\alpha'(x'', 0))_{|N_{\alpha'(x'',0)}C_{c'}}$$

of the Hessian of $\psi_{c'}$ to the normal space $N_{\alpha'(x'',0)}C_{c'}$ defines a non-degenerate quadratic form.

□

We can now state the main result of this section, the notation being the same as in the previous ones.

Theorem 3. *Let $\{(H_{i_1}), \dots, (H_{i_N})\}$ be a maximal, totally ordered subset of non-principal isotropy types of the G-action on M, and $\zeta_{i_1}^{\varrho_{i_1}} \circ \dots \circ \zeta_{i_1 \dots i_N}^{\varrho_{i_1} \dots \varrho_{i_N}}$ a corresponding sequence of local monoidal transformations labeled by the indices $\varrho_{i_1}, \dots, \varrho_{i_N}$. Consider the corresponding factorization*

$${}^{(i_1\dots i_N)}\tilde{\psi}^{tot} = \tau_{i_1} \dots \tau_{i_N}\, {}^{(i_1\dots i_N)}\tilde{\psi}^{wk,\,pre} = \tau_{i_1}(\sigma) \dots \tau_{i_N}(\sigma)\, {}^{(i_1\dots i_N)}\tilde{\psi}^{wk}$$

of the phase function (12). *Then, for each point of the critical manifold* $\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk})$, *the restriction of*

$$\mathrm{Hess}\,{}^{(i_1\dots i_N)}\tilde{\psi}^{wk}$$

to the normal space to $\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde{\psi}^{wk})$ *at the given point defines a non-degenerate symmetric bilinear form.*

Note that by construction, for $\tau_{i_j} \neq 0$, $1 \leq j \leq N$, the G-orbit through $m^{(i_1\dots i_N)}$ is of principal type G/H_L. For the proof of Theorem 3 we need the following

Lemma 6. *Let* $(\eta, X) \in \mathrm{Crit}(\psi)$, *and* $\pi(\eta) \in M(H_L)$. *Then* $(\eta, X) \in \mathrm{Reg}\,\mathrm{Crit}(\psi)$. *Furthermore, the restriction of the Hessian of* ψ *at the point* (η, X) *to the normal space* $N_{(\eta,X)}\mathrm{Reg}\,\mathrm{Crit}(\psi)$ *defines a non-degenerate quadratic form.*

Proof. The first assertion is clear from (15) and (22), since

$$\eta \in \Omega, \quad G_{\pi(\eta)} \sim H_L \quad \Rightarrow \quad G_\eta = G_{\pi(\eta)}.$$

To see the second, note that by the last implication

$$\eta \in \Omega \cap T^*M(H_L), \tilde{X}_{\pi(\eta)} = 0 \quad \Longrightarrow \quad \tilde{X}_\eta = 0. \tag{51}$$

Let now $\{q_1, \dots, q_n\}$ be local coordinates on M, $\pi(\eta) = m = m(q)$, and write $\eta_m = \sum p_i (dq_i)_m$, $X = \sum s_i X_i$, where $\{X_1, \dots, X_d\}$ denotes a basis of $\mathfrak{g}$. Then

$$\psi(\eta, X) = \sum p_i (dq_i)_m(\tilde{X}_m),$$

and

$$\partial_p\, \psi(\eta, X) = 0 \quad \Longleftrightarrow \quad \tilde{X}_m = 0, \qquad \partial_s\, \psi(\eta, X) = 0 \quad \Longleftrightarrow \quad \eta \in \Omega.$$

As a consequence of (51), on $T^*M(H_L) \times \mathfrak{g}$ we get

$$\partial_{p,s}\, \psi(\eta, X) = 0 \quad \Longrightarrow \quad \partial_q\, \psi(\eta, X) = 0.$$

Let $\psi_q(p, s)$ denote the phase function regarded as a function of the coordinates p, s alone, while q is regarded as a parameter. Lemma 5 then implies that on $T^*M(H_L) \times \mathfrak{g}$ the study of the transversal Hessian of ψ can be reduced to the study of the transversal Hessian of ψ_q. Now, with respect to the coordinates s, p, the Hessian of ψ_q is given by

$$\begin{pmatrix} 0 & (dq_i)_m((\tilde{X}_j)_m) \\ (dq_j)_m((\tilde{X}_i)_m) & 0 \end{pmatrix}.$$

A computation shows that the kernel of the corresponding linear transformation is isomorphic to

$$T_{p,s}(\mathrm{Crit}\,\psi_q) \simeq \left\{ (\tilde{p}, \tilde{s}) \in \mathbb{R}^n \times \mathbb{R}^d : \sum \tilde{p}_j (dq_j)_{m(q)} \in \mathrm{Ann}(T_{m(q)}(G \cdot m(q))), \sum \tilde{s}_j X_j \in \mathfrak{g}_{m(q)} \right\}.$$

The lemma then follows with the following general observation. Let $\mathcal{B}$ be a symmetric bilinear form on an n-dimensional $\mathbb{K}$-vector space V, and $B = (B_{ij})_{i,j}$ the corresponding Gramsian matrix with respect to a basis $\{v_1, \dots, v_n\}$ of V such that

$$\mathcal{B}(u, w) = \sum_{i,j} u_i w_j B_{ij}, \qquad u = \sum u_i v_i, \quad w = \sum w_i v_i.$$

We denote the linear operator given by B with the same letter, and write

$$V = \ker B \oplus W.$$

Consider the restriction $\mathcal{B}_{|W \times W}$ of $\mathcal{B}$ to $W \times W$, and assume that $\mathcal{B}_{|W\times W}(u, w) = 0$ for all $u \in W$, but $w \neq 0$. Since the Euclidean scalar product in V is non-degenerate, we necessarily must have $Bw = 0$, and consequently $w \in \ker B \cap W = \{0\}$, which is a contradiction. Therefore $\mathcal{B}_{|W\times W}$ defines a non-degenerate symmetric bilinear form. ∎

Proof of Theorem 3. As before, let $m = m(q_1, \dots, q_n)$ be local coordinates on M, and write $\eta_m = \sum p_i (dq_i)_m$. For $\sigma_{i_1} \cdots \sigma_{i_N} \neq 0$, the sequence of monoidal transformations $\zeta_{i_1}^{\varrho_{i_1}} \circ \dots \circ \zeta_{i_1 \dots i_N}^{\varrho_{i_1} \dots \varrho_{i_N}} \circ \delta_{i_1 \dots i_N}$ constitutes a diffeomorphism, so that by the previous lemma the restriction of

$$\mathrm{Hess}\,{}^{(i_1 \dots i_N)} \tilde{\psi}^{tot}(\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}, \alpha^{(i_j)}, \beta^{(i_N)}, p)$$

to the normal space of

$$\mathrm{Crit}(\,{}^{(i_1 \dots i_N)} \psi^{tot})_{\sigma_{i_1} \cdots \sigma_{i_N} \neq 0}$$

defines a non-degenerate quadratic form. Next, one computes for the Hessian of the total transform

$$\left(\frac{\partial^2 \; {}^{(i_1 \dots i_N)} \tilde{\psi}^{tot}}{\partial \gamma_k \, \partial \gamma_l} \right)_{k,l} = \tau_{i_1}(\sigma) \cdots \tau_{i_N}(\sigma) \left(\frac{\partial^2 \; {}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}}{\partial \gamma_k \, \partial \gamma_l} \right)_{k,l}$$
$$+ \begin{pmatrix} \left(\frac{\partial^2 (\tau_{i_1}(\sigma) \cdots \tau_{i_N}(\sigma))}{\partial \sigma_{i_r} \sigma_{i_s}} \right)_{r,s} & 0 \\ 0 & 0 \end{pmatrix} {}^{(i_1 \dots i_N)} \tilde{\psi}^{wk} + R,$$

where R is a matrix whose entries contain first order derivatives of ${}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}$ as factors. But since ${}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}$ vanishes along its critical set, and

$$\mathrm{Crit}(\,{}^{(i_1 \dots i_N)} \tilde{\psi}^{tot})_{\sigma_{i_1} \cdots \sigma_{i_N} \neq 0} = \mathrm{Crit}({}^{(i_1 \dots i_N)} \tilde{\psi}^{wk})_{|\sigma_{i_1} \cdots \sigma_{i_N} \neq 0},$$

we conclude that the transversal Hessian of ${}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}$ does not degenerate along the manifold $\mathrm{Crit}({}^{(i_1 \dots i_N)} \tilde{\psi}^{wk})_{|\sigma_{i_1} \cdots \sigma_{i_N} \neq 0}$. Therefore, it remains to study the transversal Hessian of ${}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}$ in the case that any of the σ_{i_j} vanishes. Now, the proof of Theorem 2, in particular (46), showed that

$$\partial_{p, \alpha^{(i_1)}, \dots, \alpha^{(i_N)}, \beta^{(i_N)}} \; {}^{(i_1 \dots i_N)} \tilde{\psi}^{wk} = 0 \quad \Longrightarrow \quad \partial_{\sigma_{i_1}, \dots \sigma_{i_N}, x^{(i_1)}, \dots, x^{(i_N)}, \tilde{v}^{(i_N)}} \; {}^{(i_1 \dots i_N)} \tilde{\psi}^{wk} = 0.$$

If therefore

$${}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}_{\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}} (\alpha^{(i_j)}, \beta^{(i_N)}, p)$$

denotes the weak transform of the phase function ψ regarded as a function of the variables $(\alpha^{(i_1)}, \dots, \alpha^{(i_N)}, \beta^{(i_N)}, p)$ alone, while the variables $(\sigma_{i_1}, \dots, \sigma_{i_N}, x^{(i_1)}, \dots, x^{(i_N)}, \tilde{v}^{(i_N)})$ are kept fixed,

$$\mathrm{Crit}\big(\,{}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}_{\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}}\big) = \mathrm{Crit}\big(\,{}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}\big) \cap \Big\{ \sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)} = \text{ constant} \Big\},$$

a transversal intersection. Thus, the critical set of ${}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}_{\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}}$ is equal to the fiber over $(\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)})$ of the vector bundle

$$\Big((\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}), \mathfrak{g}_{\tilde{v}^{(i_N)}} \times \mathrm{Ann}\big(\bigoplus_{j=1}^{N} E^{(i_j)}_{m^{(i_1 \dots i_N)}} \oplus F^{(i_N)}_{m^{(i_1 \dots i_N)}} \big) \Big) \mapsto (\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}),$$

and in particular a smooth submanifold. Lemma 5 then implies that the study of the transversal Hessian of ${}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}$ can be reduced to the study of the transversal Hessian of ${}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}_{\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}}$.

The crucial fact is now contained in the following

Proposition 6. *Assume that $\sigma_{i_1} \cdots \sigma_{i_N} = 0$. Then*

$$\ker \mathrm{Hess}\,{}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}_{\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}} (0, \dots, 0, \beta^{(i_N)}, p) \simeq T_{(0, \dots, 0, \beta^{(i_N)}, p)} \mathrm{Crit}\big(\,{}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}_{\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}}\big)$$

for all $(0, \dots, 0, \beta^{(i_N)}, p) \in \mathrm{Crit}\big(\,{}^{(i_1 \dots i_N)} \tilde{\psi}^{wk}_{\sigma_{i_j}, x^{(i_j)}, \tilde{v}^{(i_N)}}\big)$, and arbitrary $x^{(i_j)}$, $\tilde{v}^{(i_j)}$.

Proof. Let $\sigma_{i_1}\cdots\sigma_{i_N}=0$. With (33), or directly from (43) one computes or the second derivatives of the weak transform at a critical point $(0,\dots,0,\beta^{(i_N)},p)$

$$\begin{aligned}\partial_{\alpha_s^{(i_1)}}\partial_{p_r}\, {}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde{v}^{(i_N)}} &= dq_r((\tilde{A}_s^{(i_1)})_{x^{(i_1)}}),\\ \partial_{\alpha_s^{(i_j)}}\partial_{p_r}\, {}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde{v}^{(i_N)}} &= dq_r(\lambda(A_s^{(i_j)})x^{(i_j)}),\\ \partial_{\beta_s^{(i_N)}}\partial_{p_r}\, {}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde{v}^{(i_N)}} &= dq_r(\lambda(B_s^{(i_N)})\tilde{v}^{(i_N)}),\end{aligned}$$

while all other second derivatives vanish. Thus, for $\sigma_{i_1}\cdots\sigma_{i_j}=0$, the Hessian of the function ${}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde{v}^{(i_N)}}$ with respect to the coordinates $p,\alpha^{(i_j)},\beta^{(i_j)}$ is given on its critical set by the matrix

$$\begin{pmatrix} 0 & dq_r((\tilde{A}_s^{(i_1)})_{x^{(i_1)}}) & \dots & dq_r(\lambda(A_s^{(i_N)})x^{(i_j)}) & dq_r(\lambda(B_s^{(i_N)})\tilde{v}^{(i_N)})\\ dq_s((\tilde{A}_r^{(i_1)})_{x^{(i_1)}}) & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ dq_s(\lambda(A_r^{(i_N)})x^{(i_j)}) & 0 & \dots & 0 & 0\\ dq_s(\lambda(B_r^{(i_N)})\tilde{v}^{(i_N)}) & 0 & \dots & 0 & 0\end{pmatrix}.$$

Let us now compute the kernel of the linear transformation corresponding to this matrix. Cleary, the vector $(\tilde{p},\tilde{\alpha}^{(i_1)},\dots,\tilde{\alpha}^{(i_N)},\tilde{\beta}^{(i_N)})$ lies in the kernel if and only if

(a) $\sum\tilde{\alpha}_s^{(i_1)}(\tilde{A}_s^{(i_1)})_{x^{(i_1)}}+\dots+\sum\tilde{\alpha}_s^{(i_N)}\lambda(A_s^{(i_N)})x^{(i_N)}+\sum\tilde{\beta}_s^{(i_N)}\lambda(B_s^{(i_N)})\tilde{v}^{(i_N)}=0$;
(b) $\sum\tilde{p}_s dq_s((\tilde{Y}^{(i_1)})_{x^{(i_1)}})=0$ for all $Y^{(i_1)}\in\mathfrak{g}^\perp_{x^{(i_1)}}$, $\sum\tilde{p}_s dq_s(\lambda(\mathfrak{g}^\perp_{x^{(i_j)}})x^{(i_j)})=0$, $2\le j\le N$;
(c) $\sum\tilde{p}_s dq_s(\lambda(\mathfrak{g}_{x^{(i_N)}})\tilde{v}^{(i_N)})=0$.

Let $E^{(i_j)}$, $F^{(i_N)}$, and $V^{(i_1\dots i_N)}$ be defined as in (35) and (40). Then

$$\sum\tilde{\alpha}_r^{(i_j)}(\tilde{A}_r^{(i_1)})_{x^{(i_1)}}+\dots+\sum\tilde{\alpha}_r^{(i_N)}\lambda(A_r^{(i_N)})x^{(i_N)}+\sum\tilde{\beta}_r^{(i_N)}\lambda(B_r^{(i_N)})\tilde{v}^{(i_N)}\in\bigoplus_{j=1}^N E^{(i_j)}_{x^{(i_1)}}\oplus F^{(i_N)}_{x^{(i_1)}},$$

so that for condition (a) to hold, it is necessary and sufficient that

$$\tilde{\alpha}^{(i_j)}=0,\quad 1\le j\le N,\qquad \sum\tilde{\beta}_r^{(i_N)}\lambda(B_r^{(i_N)})\tilde{v}^{(i_N)}=0.$$

Condition (b) is equivalent to $\sum\tilde{p}_s(dq_s)_{x^{(i_1)}}\in\mathrm{Ann}(E^{(i_j)}_{x^{(i_1)}})$ for al $j=1,\dots,N$. Similarly, condition (c) is equivalent to $\sum\tilde{p}_s(dq_s)_{x^{(i_1)}}\in\mathrm{Ann}(F^{(i_N)}_{x^{(i_1)}})$. On the other hand, by (48),

$$\begin{aligned}T_{(0,\dots,0,\beta^{(i_N)},p)}\mathrm{Crit}\big({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde{v}^{(i_N)}}\big)=\Big\{&(\tilde{\alpha}^{(i_1)},\dots,\tilde{\alpha}^{(i_N)},\tilde{\beta}^{(i_N)},\tilde{p}):\tilde{\alpha}^{(i_j)}=0,\\ &\sum\tilde{\beta}_r^{(i_N)}\lambda(B_r^{(i_N)})\in\mathfrak{g}_{\tilde{v}^{(i_N)}},\ \sum\tilde{p}_s(dq_s)_{x^{(i_1)}}\in\mathrm{Ann}\Big(\bigoplus_{j=1}^N E^{(i_j)}_{x^{(i_1)}}\oplus F^{(i_N)}\Big)\Big\},\end{aligned}$$

and the proposition follows. ☐

The previous proposition implies that for $\sigma_{i_1}\cdots\sigma_{i_N}=0$

$$\mathrm{Hess}\,{}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde{v}^{(i_N)}}(0,\dots,0,\beta^{(i_N)},p)_{|N_{(0,\dots,0,\beta^{(i_N)},p)}\mathrm{Crit}\big({}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde{v}^{(i_N)}}\big)}$$

defines a non-degenerate symmetric bilinear form for all points $(0,\dots,0,\beta^{(i_N)},p)$ lying in the critical set of ${}^{(i_1\dots i_N)}\tilde{\psi}^{wk}_{\sigma_{i_j},x^{(i_j)},\tilde{v}^{(i_N)}}$, and Theorem 3 follows with Lemma 5. ☐

8. Asymptotics in the resolution space

We are now in position to give an asymptotic description of the integrals $I^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\ldots i_N}(\mu)$ defined in (34). Since the considered integrals are absolutely convergent, we can interchange the order of integration by Fubini, and write

$$I^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\ldots i_N}(\mu)=\int_{(-1,1)^N}\hat{J}^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\ldots i_N}\Big(\frac{\mu}{\tau_{i_1}\cdots\tau_{i_N}}\Big)\prod_{j=1}^N|\tau_{i_j}|^{c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-1}\,d\tau_{i_N}\ldots d\tau_{i_1},$$

where we set

$$\hat{J}^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\ldots i_N}(\nu)=\int_{M_{i_1}(H_{i_1})}\Big[\int_{\gamma^{(i_1)}((S_{i_1})_{x^{(i_1)}})_{i_2}(H_{i_2})}\cdots\Big[\int_{\gamma^{(i_{N-1})}((S_{i_1\cdots i_{N-1}})_{x^{(i_{N-1})}})_{i_N}(H_{i_N})}$$
$$\Big[\int_{\gamma^{(i_N)}((S_{i_1\ldots i_N})_{x^{(i_N)}})\times\mathfrak{g}_{x^{(i_N)}}\times\mathfrak{g}^\perp_{x^{(i_N)}}\times\cdots\times\mathfrak{g}^\perp_{x^{(i_1)}}\times T^*_{m^{(i_1\ldots i_N)}}W_{i_1}}e^{i^{(i_1\ldots i_N)}\tilde{\psi}^{wk,pre}/\nu}\,a^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\ldots i_N}\,\Phi^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\ldots i_N}$$
$$d(T^*_{m^{(i_1\cdots i_N)}}W_{i_1})\,dA^{(i_1)}\ldots dA^{(i_N)}\,dB^{(i_N)}\,d\tilde{v}^{(i_N)}\Big]\,d\tau_{i_N}\,dx^{(i_N)}\ldots\Big]\,d\tau_{i_2}\,dx^{(i_2)}\Big]\,d\tau_{i_1}\,dx^{(i_1)},$$

and introduced the new parameter

$$\nu=\frac{\mu}{\tau_{i_1}\cdots\tau_{i_N}}.$$

Now, for an arbitrary $0<\varepsilon<T$ to be chosen later we define

$${}^1I^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\ldots i_N}(\mu)=\int_{((-1,1)\setminus(-\varepsilon,\varepsilon))^N}\hat{J}^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\ldots i_N}\Big(\frac{\mu}{\tau_{i_1}\cdots\tau_{i_N}}\Big)\prod_{j=1}^N|\tau_{i_j}|^{c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-1}\,d\tau_{i_N}\ldots d\tau_{i_1},$$

$${}^2I^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\ldots i_N}(\mu)=\int_{(-\varepsilon,\varepsilon)^N}\hat{J}^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\ldots i_N}\Big(\frac{\mu}{\tau_{i_1}\cdots\tau_{i_N}}\Big)\prod_{j=1}^N|\tau_{i_j}|^{c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-1}\,d\tau_{i_N}\ldots d\tau_{i_1}.$$

Lemma 7. *One has $c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-1\geq\kappa$ for arbitrary $j=1,\ldots,N$.*

Proof. We first note that for $j=1,\ldots,N-1$

$$c^{(i_j)}=\dim(\nu_{i_1\ldots i_j})_{x^{(i_j)}}\geq\dim G_{x^{(i_j)}}\cdot m^{(i_{j+1}\ldots i_N)}+1.$$

Indeed, $(\nu_{i_1\ldots i_j})_{x^{(i_j)}}$ is an orthogonal $G_{x^{(i_j)}}$-space, so that the dimension of the $G_{x^{(i_j)}}$-orbit of $m^{(i_{j+1}\ldots i_N)}\in\gamma^{(i_j)}((S_{i_1\ldots i_j})_{x^{(i_j)}})$ can be at most $c^{(i_j)}-1$. Now, under the assumption $\sigma_{i_1}\cdots\sigma_{i_N}\neq 0$, (29), (31) and (32) imply

$$\begin{aligned}T_{m^{(i_{j+1}\cdots i_N)}}(G_{x^{(i_j)}}\cdot m^{(i_{j+1}\ldots i_N)})&\simeq T_{m^{(i_1\cdots i_N)}}(G_{x^{(i_j)}}\cdot m^{(i_1\ldots i_N)})\\&=E^{(i_{j+1})}_{m^{(i_1\cdots i_N)}}\oplus\bigoplus_{k=j+2}^N\tau_{i_{j+1}}\ldots\tau_{i_{k-1}}E^{(i_k)}_{m^{(i_1\cdots i_N)}}\oplus\tau_{i_{j+1}}\ldots\tau_{i_N}F^{(i_N)}_{m^{(i_1\cdots i_N)}},\end{aligned}$$

where the distributions $E^{(i_j)}$, $F^{(i_N)}$ where defined in (35). On then computes

$$\begin{aligned}\dim G_{x^{(i_j)}}\cdot m^{(i_{j+1}\ldots i_N)}&=\dim T_{m^{(i_{j+1}\cdots i_N)}}(G_{x^{(i_j)}}\cdot m^{(i_{j+1}\ldots i_N)})\\&=\sum_{l=j+1}^N\dim E^{(i_l)}_{m^{(i_1\cdots i_N)}}+\dim F^{(i_N)}_{m^{(i_1\cdots i_N)}},\end{aligned}$$

which implies

$$c^{(i_j)}\geq\sum_{l=j+1}^N\dim E^{(i_l)}_{m^{(i_1\cdots i_N)}}+\dim F^{(i_N)}_{m^{(i_1\cdots i_N)}}+1$$

for arbitrary σ_{i_j}. On the other hand, one has

$$d^{(i_j)} = \dim \mathfrak{g}^\perp_{x^{(i_j)}} = \dim[\lambda(\mathfrak{g}^\perp_{x^{(i_j)}}) \cdot x^{(i_j)}] = \dim[\lambda(\mathfrak{g}^\perp_{x^{(i_j)}}) \cdot m^{(i_j \dots i_N)}] = \dim E^{(i_j)}_{m^{(i_1 \dots i_N)}}.$$

For $j = 1, \dots, N-1$, the assertion of the lemma now follows with (49). Since

$$c^{(i_N)} = \dim(\nu_{i_1 \dots i_N})_{x^{(i_N)}} \geq \dim G_{x^{(i_N)}} \cdot \tilde{v}^{(i_N)} + 1,$$

a similar argument yields the assertion for $j = N$. □

As a consequence of the lemma, we obtain for ${}^2 I^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \dots i_N}(\mu)$ the estimate

$$\begin{aligned} {}^2 I^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \dots i_N}(\mu) &\leq C \int_{(-\varepsilon,\varepsilon)^N} \prod_{j=1}^N |\tau_{i_j}|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1} \, d\tau_{i_N} \dots d\tau_{i_1} \\ &\leq C \int_{(-\varepsilon,\varepsilon)^N} \prod_{j=1}^N |\tau_{i_j}|^\kappa \, d\tau_{i_N} \dots d\tau_{i_1} = \frac{2C}{\kappa+1} \varepsilon^{N(\kappa+1)} \end{aligned} \tag{52}$$

for some $C > 0$. Let us now turn to the integral ${}^1 I^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \dots i_N}(\mu)$. After performing the change of variables $\delta_{i_1 \dots i_N}$ one obtains

$${}^1 I^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \dots i_N}(\mu) = \int\limits_{\varepsilon < |\tau_{i_j}(\sigma)| < 1} J^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \dots i_N}\Big(\frac{\mu}{\tau_{i_1}(\sigma) \cdots \tau_{i_N}(\sigma)}\Big) \prod_{j=1}^N |\tau_{i_j}(\sigma)|^{c^{(i_j)} + \sum_{r=1}^j d^{(i_r)} - 1} |\det D\delta_{i_1 \dots i_N}(\sigma)| \, d\sigma,$$

where $J^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \dots i_N}(\nu)$ is defined like $\hat{J}^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \dots i_N}(\nu)$, but with ${}^{(i_1 \dots i_N)}\tilde{\psi}^{wk,pre}$ being replaced by ${}^{(i_1 \dots i_N)}\tilde{\psi}^{wk}_\sigma$, which denotes the weak transform of the phase function ψ as a function of the variables $x^{(i_j)}$, $\tilde{v}^{(i_N)}, \alpha^{(i_j)}, \beta^{(i_N)}, p$ alone, while the variables $\sigma = (\sigma_{i_1}, \dots \sigma_{i_N})$ are regarded as parameters. The idea is now to make use of the principle of the stationary phase to give an asymptotic expansion of $J^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \dots i_N}(\nu)$.

Theorem 4. *Let $\sigma = (\sigma_{i_1}, \dots, \sigma_{i_N})$ be a fixed set of parameters. Then, for every $\tilde{N} \in \mathbb{N}$ there exists a constant $C_{\tilde{N}, {}^{(i_1 \cdots i_N)}\tilde{\psi}^{wk}_\sigma} > 0$ such that*

$$|J^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \dots i_N}(\nu) - (2\pi|\nu|)^\kappa \sum_{j=0}^{\tilde{N}-1} |\nu|^j Q_j({}^{(i_1 \dots i_N)}\tilde{\psi}^{wk}_\sigma; a_{i_1 \dots i_N} \Phi_{i_1 \dots i_N})| \leq C_{\tilde{N}, {}^{(i_1 \cdots i_N)}\tilde{\psi}^{wk}_\sigma} |\nu|^{\tilde{N}},$$

with explicit expressions and estimates for the coefficients Q_j. Moreover, the constants $C_{\tilde{N}, {}^{(i_1 \cdots i_N)}\tilde{\psi}^{wk}_\sigma}$ and the coefficients Q_j have uniform bounds in σ.

Proof. As a consequence of Theorems 2 and 3, together with Lemma 5, the phase function ${}^{(i_1 \dots i_N)}\tilde{\psi}^{wk}_\sigma$ has a clean critical set, meaning that

- the critical set $\mathrm{Crit}({}^{(i_1 \dots i_N)}\tilde{\psi}^{wk}_\sigma)$ is a C^∞-submanifold of codimension 2κ for arbitrary σ;
- the transversal Hessian

  $$\mathrm{Hess}\, {}^{(i_1 \dots i_N)}\tilde{\psi}^{wk}_\sigma(x^{(i_j)}, \tilde{v}^{(i_N)}, \alpha^{(i_j)}, \beta^{(i_N)}, p)_{|N_{(x^{(i_j)}, \tilde{v}^{(i_N)}, \alpha^{(i_j)}, \beta^{(i_N)}, p)} \mathrm{Crit}\left({}^{(i_1 \cdots i_N)}\tilde{\psi}^{wk}_\sigma\right)}$$

  defines a non-degenerate symmetric bilinear form for arbitrary σ at every point of the critical set of ${}^{(i_1 \dots i_N)}\tilde{\psi}^{wk}_\sigma$.

Thus, the necessary conditions for applying the principle of the stationary phase to the integral $J_{\sigma_{i_1}, \dots, \sigma_{i_N}}(\nu)$ are fulfilled, and we obtain the desired asymptotic expansion by Theorem C. To see the existence of the uniform bounds, note that as an examination of the proof of Theorem A shows, the constants $C_{N,\psi}$ in Theorem C are bounded from above by

$$\sup_{m\in\mathcal{C}\cap\operatorname{supp} a}\left\|\left(\psi''(m)_{|N_m\mathcal{C}}\right)^{-1}\right\|$$

see also [36, Remark 1]. We therefore have

$$C_{\tilde N,{}^{(i_1\dots i_N)}\tilde\psi^{wk}_\sigma}\leq C'_{\tilde N}\sup_{x^{(i_j)},\tilde v^{(i_N)},\alpha^{(i_j)},\beta^{(i_N)},p}\left\|\left(\text{Hess}\,{}^{(i_1\dots i_N)}\tilde\psi^{wk}_\sigma{}_{|N\text{Crit}({}^{(i_1\dots i_N)}\tilde\psi^{wk}_\sigma)}\right)^{-1}\right\|.$$

But since by Lemma 5 the transversal Hessian

$$\text{Hess}\,{}^{(i_1\dots i_N)}\tilde\psi^{wk}_\sigma{}_{|N_{(x^{(i_j)},\tilde v^{(i_N)},\alpha^{(i_j)},\beta^{(i_N)},p)}\text{Crit}({}^{(i_1\dots i_N)}\tilde\psi^{wk}_\sigma)}$$

is given by

$$\text{Hess}\,{}^{(i_1\dots i_N)}\tilde\psi^{wk}{}_{|N_{(\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)},\alpha^{(i_j)},\beta^{(i_N)},p)}\text{Crit}({}^{(i_1\dots i_N)}\tilde\psi^{wk})},$$

we finally obtain the estimate

$$C_{\tilde N,{}^{(i_1\dots i_N)}\tilde\psi^{wk}_\sigma}\leq C'_{\tilde N}\sup_{\sigma_{i_j},x^{(i_j)},\tilde v^{(i_N)},\alpha^{(i_j)},\beta^{(i_N)},p}\left\|\left(\text{Hess}\,{}^{(i_1\dots i_N)}\tilde\psi^{wk}{}_{|N\text{Crit}({}^{(i_1\dots i_N)}\tilde\psi^{wk})}\right)^{-1}\right\|\leq C_{\tilde N,i_1\dots i_N}$$

by a constant independent of σ. Similarly, one can show the existence of bounds of the form

$$|Q_j({}^{(i_1\dots i_N)}\tilde\psi^{wk}_\sigma;a_{i_1\dots i_N}\Phi_{i_1\dots i_N})|\leq\tilde C_{j,i_1\dots i_N},$$

with constants $\tilde C_{j,i_1\dots i_N}$ independent of σ. □

Remark 4. Before going on, let us remark that for the computation of the integrals ${}^1I^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\dots i_N}(\mu)$ it is only necessary to have an asymptotic expansion for the integrals $J^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\dots i_N}(\nu)$ in the case that $\sigma_{i_1}\cdots\sigma_{i_N}\neq 0$, which can also be obtained without Theorems 2 and 3 using only the factorization of the phase function ψ given by the resolution process, together with Lemma 6. Nevertheless, the main consequence to be drawn from Theorems 2 and 3 is that the constants $C_{\tilde N,{}^{(i_1\cdots i_N)}\tilde\psi^{wk}_\sigma}$ and the coefficients Q_j in Theorem 4 have uniform bounds in σ.

As a consequence of Theorem 4, we obtain for arbitrary $\tilde N\in\mathbb{N}$

$$\begin{aligned}&|J^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\dots i_N}(\nu)-(2\pi|\nu|)^\kappa Q_0({}^{(i_1\dots i_N)}\tilde\psi^{wk}_\sigma;a_{i_1\dots i_N}\Phi_{i_1\dots i_N})|\\&\qquad\leq\left|J^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\dots i_N}(\nu)-(2\pi|\nu|)^\kappa\sum_{l=0}^{\tilde N-1}|\nu|^lQ_l({}^{(i_1\dots i_N)}\tilde\psi^{wk}_\sigma;a_{i_1\dots i_N}\Phi_{i_1\dots i_N})\right|\\&\qquad+(2\pi|\nu|)^\kappa\sum_{l=1}^{\tilde N-1}|\nu|^l|Q_l({}^{(i_1\dots i_N)}\tilde\psi^{wk}_\sigma;a_{i_1\dots i_N}\Phi_{i_1\dots i_N})|\leq c_1|\nu|^{\tilde N}+c_2|\nu|^\kappa\sum_{l=1}^{\tilde N-1}|\nu|^l\end{aligned}$$

with constants $c_i > 0$ independent of both σ and ν. From this we deduce

$$\Big| {}^1 I^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\dots i_N}(\mu) - (2\pi\mu)^\kappa \int_{\varepsilon<|\tau_{i_j}(\sigma)|<1} Q_0 \prod_{j=1}^N |\tau_{i_j}(\sigma)|^{c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-1-\kappa} |\det D\delta_{i_1\dots i_N}(\sigma)|\, d\sigma \Big|$$
$$\leq c_3 \mu^{\tilde N} \int_{\varepsilon<|\tau_{i_j}(\sigma)|<1} \prod_{j=1}^N |\tau_{i_j}(\sigma)|^{c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-1-\tilde N} |\det D\delta_{i_1\dots i_N}(\sigma)|\, d\sigma$$
$$+ c_4 \mu^\kappa \sum_{l=1}^{\tilde N-1} \mu^l \int_{\varepsilon<|\tau_{i_j}(\sigma)|<1} \prod_{j=1}^N |\tau_{i_j}(\sigma)|^{c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-1-\kappa-l} |\det D\delta_{i_1\dots i_N}(\sigma)|\, d\sigma$$
$$\leq c_5 \mu^{\tilde N} \prod_{j=1}^N (-\log\varepsilon)^{i_j} \max\Big\{1, \prod_{j=1}^N \varepsilon^{c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-\tilde N}\Big\}$$
$$+ c_6 \sum_{l=1}^{\tilde N-1} \mu^{\kappa+l} \prod_{j=1}^N (-\log\varepsilon)^{i_{lj}} \max\Big\{1, \prod_{j=1}^N \varepsilon^{c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-\kappa-l}\Big\},$$

where the exponents i_j and i_{lj} can take the values 0 or 1. We now set $\varepsilon = \mu^{1/N}$. Taking into account Lemma 7, one infers that the right hand side of the last inequality can be estimated by

$$\mu^{k+1}(\log\mu)^N.$$

so that for sufficiently large $\tilde N \in \mathbb{N}$ we finally obtain an asymptotic expansion for $I^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\dots i_N}(\mu)$ by taking into account (52), and the fact that

$$(2\pi\mu)^\kappa \int_{0<|\tau_{i_j}|<\mu^{1/N}} Q_0 \prod_{j=1}^N |\tau_{i_j}|^{c^{(i_j)}+\sum_{r=1}^j d^{(i_r)}-1-\kappa}\, d\tau_{i_N}\dots d\tau_{i_1} = O(\mu^{\kappa+1}).$$

Theorem 5. *Let the assumptions of Theorem 2 be fulfilled. Then*

$$I^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\dots i_N}(\mu) = (2\pi\mu)^\kappa L^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\dots i_N} + O(\mu^{\kappa+1}(\log\mu)^N),$$

where the leading coefficient $L^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\dots i_N}$ is given by

$$L^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\dots i_N} = \int_{\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde\psi^{wk})} \frac{a^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\dots i_N}\, \Phi^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\dots i_N}\, d\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde\psi^{wk})}{|\mathrm{Hess}({}^{(i_1\dots i_N)}\tilde\psi^{wk})_{N\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde\psi^{wk})}|^{1/2}}, \tag{53}$$

where $d\mathrm{Crit}({}^{(i_1\dots i_N)}\tilde\psi^{wk})$ denotes the induced measure.

□

9. Statement of the main result

Let us now return to our departing point, that is, the asymptotic behavior of the integral (11) in case that $\varsigma = 0$ is a singular value of the momentum map. For this, we still have to examine the contributions to $I(\mu)$ coming from integrals of the form

$$\tilde I^{\varrho_{i_1}\cdots\varrho_{i_\Theta}}_{i_1\dots i_\Theta}(\mu)=$$

$$\int_{M_{i_1}(H_{i_1})\times(-1,1)}\Big[\int_{\gamma^{(i_1)}((S_{i_1})_{x^{(i_1)}})_{i_2}(H_{i_2})\times(-1,1)}\cdots\Big[\int_{\gamma^{(i_{\Theta-1})}((S_{i_1\dots i_{\Theta-1}})_{x^{(i_{\Theta-1})}})_{i_\Theta}(H_{i_\Theta})\times(-1,1)}$$

$$\Big[\int_{\gamma^{(i_\Theta)}((S_{i_1\dots i_\Theta})_{x^{(i_\Theta)}})\times\mathfrak{g}_{x^{(i_\Theta)}}\times\mathfrak{g}^\perp_{x^{(i_\Theta)}}\times\cdots\times\mathfrak{g}^\perp_{x^{(i_1)}}\times T^*_{m^{(i_1\dots i_\Theta)}}W_{i_1}} e^{i\frac{\tau_1\dots\tau_\Theta}{\mu}\,{}^{(i_1\dots i_\Theta)}\tilde\psi^{wk}}\, a^{\varrho_{i_1}\cdots\varrho_{i_\Theta}}_{i_1\dots i_\Theta}\,\tilde\Phi^{\varrho_{i_1}\cdots\varrho_{i_\Theta}}_{i_1\dots i_\Theta}$$

$$d(T^*_{m^{(i_1\dots i_\Theta)}}W_{i_1})(\eta)\,dA^{(i_1)}\dots dA^{(i_\Theta)}\,dB^{(i_\Theta)}\,d\tilde v^{(i_\Theta)}\Big]\,d\tau_{i_\Theta}\,dx^{(i_\Theta)}\dots\Big]\,d\tau_{i_2}\,dx^{(i_2)}\Big]\,d\tau_{i_1}\,dx^{(i_1)},$$

where $\{(H_{i_1}),\dots,(H_{i_\Theta})\}$ is an arbitrary totally ordered subset of non-principal isotropy types, while $a^{\varrho_{i_1}\cdots\varrho_{i_\Theta}}_{i_1\dots i_\Theta}$ is a smooth amplitude which is supposed to have compact support in a system of $(\theta^{(i_1)},\dots,\theta^{(i_{N-1})},\alpha^{(i_N)})$-charts labeled by the indices $\varrho_{i_1}\dots\varrho_{i_\Theta}$, and

$$\tilde\Phi^{\varrho_{i_1}\cdots\varrho_{i_\Theta}}_{i_1\dots i_\Theta}=\prod_{j=1}^{\Theta}|\tau_{i_j}|^{c^{(i_j)}+\sum_r^j d^{(i_r)}-1}\Phi^{\varrho_{i_1}\cdots\varrho_{i_\Theta}}_{i_1\dots i_\Theta},$$

$\Phi_{i_1\dots i_\Theta}$ being a smooth function which does not depend on the variables τ_{i_j}. Now, a computation of the p-derivatives of ${}^{(i_1\dots i_\Theta)}\tilde\psi^{wk}$ in any of the $\alpha^{(i_\Theta)}$-charts shows that ${}^{(i_1\dots i_\Theta)}\tilde\psi^{wk}$ has no critical points there. By the non-stationary phase theorem, see Hörmander [24, Theorem 7.7.1], one then computes for arbitrary $\tilde N\in\mathbb{N}$

$$|\tilde I^{\varrho_{i_1}\cdots\varrho_{i_\Theta}}_{i_1\dots i_\Theta}(\mu)|\le c_7\mu^{\tilde N}\int_{\varepsilon<|\tau_{i_j}|<1}\prod_{j=1}^{\Theta}|\tau_{i_j}|^{c^{(i_j)}+\sum_r^j d^{(i_r)}-1-\tilde N}d\tau+c_8\varepsilon^{\Theta(\kappa+1)}\le c_9\max\left\{\mu^{\tilde N},\mu^{\kappa+1}\right\},$$

where we took $\varepsilon=\mu^{1/\Theta}$. Choosing $\tilde N$ large enough, we conclude that

$$|\tilde I^{\varrho_{i_1}\cdots\varrho_{i_\Theta}}_{i_1\dots i_\Theta}(\mu)|=O(\mu^{\kappa+1}).$$

As a consequence of this we see that, up to terms of order $O(\mu^{\kappa+1})$, $I(\mu)$ can be written as a sum

$$I(\mu)=\sum_{N=1}^{\Lambda-1}\sum_{\substack{i_1<\dots<i_N\\ \varrho_{i_1},\dots,\varrho_{i_N}}}I^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\dots i_N}(\mu)+\sum_{N=1}^{\Lambda-1}\sum_{\substack{i_1<\dots<i_{N-1}<L\\ \varrho_{i_1},\dots,\varrho_{i_{N-1}}}}I^{\varrho_{i_1}\cdots\varrho_{i_{N-1}}}_{i_1\dots i_{N-1}L}(\mu),\tag{54}$$

where the first term is a sum over maximal, totally ordered subsets of non-principal isotropy types, while the second term is a sum over totally ordered subsets of non-principal isotropy types. The asymptotic behavior of the integrals $I^{\varrho_{i_1}\cdots\varrho_{i_N}}_{i_1\dots i_N}(\mu)$ has been determined in the previous section, and using Lemma 6 it is not difficult to see that the integrals $I^{\varrho_{i_1}\cdots\varrho_{i_{N-1}}}_{i_1\dots i_{N-1}L}(\mu)$ have analogous asymptotic descriptions. We can now state the main result of this paper.

Theorem 6. *Let M be a connected Riemannian manifold, and G a compact, connected Lie group G with Lie algebra $\mathfrak{g}$ acting isometrically and effectively on M. Consider the oscillatory integral*

$$I(\mu)=\int_{T^*M}\int_{\mathfrak{g}}e^{i\psi(\eta,X)/\mu}a(\eta,X)\,dX\,d\eta,\qquad\mu>0,$$

where the phase function

$$\psi(\eta,X)=\mathbb{J}(\eta)(X)$$

is given by the momentum map $\mathbb{J}:T^*M\to\mathfrak{g}^*$ corresponding to the Hamiltonian action on T^*M, $d\eta$ is the Liouville measure on T^*M, and dX an Euclidean measure given by an $\mathrm{Ad}\,(G)$-invariant inner product on $\mathfrak{g}$, while $a\in\mathrm{C^\infty_c}(T^*M\times\mathfrak{g})$. Then $I(\mu)$ has the asymptotic expansion

$$I(\mu)=(2\pi\mu)^\kappa L_0+O(\mu^{\kappa+1}(\log\mu)^{\Lambda-1}),\qquad\mu\to0^+.$$

Here κ is the dimension of an orbit of principal type in M, Λ the maximal number of elements of a totally ordered subset of the set of isotropy types, and the leading coefficient is given by [4]

$$L_0 = \int_{\operatorname{Reg}\mathcal{C}} \frac{a(\eta, X)}{|\operatorname{Hess}\psi(\eta, X)_{|N_{(\eta,X)}\operatorname{Reg}\mathcal{C}}|^{1/2}} \, d(\operatorname{Reg}\mathcal{C})(\eta, X), \tag{55}$$

where $\operatorname{Reg}\mathcal{C}$ *denotes the regular part of the critical set* $\mathcal{C} = \operatorname{Crit}(\psi)$ *of* ψ*, and* $d(\operatorname{Reg}\mathcal{C})$ *the measure induced by* $d\eta\, dX$*. In particular, the integral over* $\operatorname{Reg}\mathcal{C}$ *exists.*

Remark 5. Note that equation (55) in particular means that the obtained asymptotic expansion for $I(\mu)$ is independent of the explicit partial resolution we used.

Proof. By (54) and Theorem 5 one has

$$I(\mu) = (2\pi\mu)^\kappa L_0 + O(\mu^{\kappa+1}(\log\mu)^{\Lambda-1}), \qquad \mu \to 0^+,$$

where L_0 is given by a sum of integrals of the form (53). It therefore remains to show the equality (55). For this, we shall introduce certain cut-off functions for the singular part $\operatorname{Sing}\Omega$ of Ω. Choose a Riemmanian metric on T^*M, and denote the corresponding distance on T^*M by d. Let K be a compact subset in T^*M, $\delta > 0$, and consider the set

$$(\operatorname{Sing}\Omega \cap K)_\delta = \{\eta \in T^*M : d(\eta, \eta') < \delta \text{ for some } \eta' \in \operatorname{Sing}\Omega \cap K\}.$$

By using a partition of unity, one can show the existence of a test function $u_\delta \in \mathrm{C_c^\infty}((\operatorname{Sing}\Omega \cap K)_{3\delta})$ satisfying $u_\delta = 1$ on $(\operatorname{Sing}\Omega \cap K)_\delta$, see Hörmander [24, Theorem 1.4.1]. Now, let K be such that $\operatorname{supp}_\eta a \subset K$. We then assert that the limit

$$\lim_{\delta \to 0} \int_{\operatorname{Reg}\mathcal{C}} \frac{[a(1-u_\delta)](\eta, X)}{|\det\, \psi''(\eta, X)_{|N_{(\eta,X)}\operatorname{Reg}\mathcal{C}}|^{1/2}} d(\operatorname{Reg}\mathcal{C})(\eta, X) \tag{56}$$

exists and is equal to L_0, where $d(\operatorname{Reg}\mathcal{C})$ is the measure on $\operatorname{Reg}\mathcal{C}$ induced by $d\eta\, dX$. Indeed, define

$$I_\delta(\mu) = \int_{T^*M} \int_{\mathfrak{g}} e^{\frac{i}{\mu}\psi(\eta, X)} [a(1-u_\delta)](\eta, X)\, dX\, d\eta.$$

Since $(\eta, X) \in \operatorname{Sing}\mathcal{C}$ implies $\eta \in \operatorname{Sing}\Omega$, a direct application of Theorem C for fixed $\delta > 0$ gives

$$|I_\delta(\mu) - (2\pi\mu)^\kappa L_0(\delta)| \leq C_\delta \mu^{\kappa+1}, \tag{57}$$

where $C_\delta > 0$ is a constant depending only on δ, and

$$L_0(\delta) = \int_{\operatorname{Reg}\mathcal{C}} \frac{[a(1-u_\delta)](\eta, X)}{|\det\, \psi''(\eta, X)_{|N_{(\eta,X)}\operatorname{Reg}\mathcal{C}}|^{1/2}} d(\operatorname{Reg}\mathcal{C})(\eta, X).$$

On the other hand, applying our previous considerations to $I_\delta(\mu)$ instead of $I(\mu)$, we obtain again an asymptotic expansion of the form (57) for $I_\delta(\mu)$, where now the first coefficient is given by a sum of integrals of the form (53) with a replaced by $a(1-u_\delta)$. Since the first term in the asymptotic expansion (57) is uniquely determined, the two expressions for $L_0(\delta)$ must be identical. The existence of the limit (56) now follows by the Lebesgue theorem on bounded convergence, the corresponding limit being given by L_0. Let now $a^+ \in \mathrm{C_c^\infty}(T^*M \times \mathfrak{g}, \mathbb{R}^+)$. Since one can assume that $|u_\delta| \leq 1$, the lemma of Fatou implies that

$$\int_{\operatorname{Reg}\mathcal{C}} \lim_{\delta \to 0} \frac{[a^+(1-u_\delta)](\eta, X)}{|\det\, \psi''(\eta, X)_{|N_{(\eta,X)}\operatorname{Reg}\mathcal{C}}|^{1/2}} d(\operatorname{Reg}\mathcal{C})(\eta, X)$$

[4] A more explicit expression for L_0 will be given in Proposition 7.

is mayorized by the limit (56), with a replaced by a^+, and we obtain

$$\int_{\operatorname{Reg}\mathcal{C}} \frac{a^+(\eta,X)}{|\det\,\psi''(\eta,X)_{|N_{(\eta,X)}\operatorname{Reg}\mathcal{C}}|^{1/2}} |d(\operatorname{Reg}\mathcal{C})(\eta,X)| < \infty.$$

Choosing a^+ to be equal 1 on a neighborhood of the support of a, and applying the theorem of Lebesgue on bounded convergence to the limit (56), we obtain equation (55). $\square$

In what follows, we shall compute the leading term (55) in a more explicit way, and begin by computing the determinant of the transversal Hessian of the phase function $\psi(\eta,X)$, the notation being as in Theorem 6.

Lemma 8. *Let $(\eta,X)\in\operatorname{Reg}\mathcal{C}$ be fixed. Then*

$$\det\operatorname{Hess}\psi(\eta,X)_{|N_{(\eta,X)}\operatorname{Reg}\mathcal{C}} = \det\left(\Xi - L_X\circ L_X\right)_{|\mathfrak{g}\cdot\eta},$$

where $L_X:\mathfrak{g}\cdot\eta\to\mathfrak{g}\cdot\eta$ denotes the linear mapping (26) *given by the Lie derivative, and Ξ the linear transformation on $\mathfrak{g}\cdot\eta$ defined in* (17).

Proof. Let $(\eta,X)\in\operatorname{Reg}\mathcal{C}$ be fixed and $\{A_1,\dots,A_d\}$ an orthonormal basis of $\mathfrak{g}$ such that $\{A_1,\dots,A_\kappa\}$ is a basis of $\mathfrak{g}_\eta^\perp$ and $\{A_{\kappa+1},\dots,A_d\}$ a basis of $\mathfrak{g}\cdot\eta$. With respect to the basis

$$((\widetilde{\mathfrak{X}}_i)_\eta;0),\qquad (0;e_j),\qquad i=1,\dots,2n,\quad j=1,\dots,d,$$

of $T_{(\eta,X)}(T^*M\times\mathfrak{g}) = T_\eta(T^*M)\times\mathbb{R}^d$ introduced in the proof of Proposition 2, the Hessian

$$\operatorname{Hess}\psi: T_{(\eta,X)}(T^*M\times\mathfrak{g})\times T_{(\eta,X)}(T^*M\times\mathfrak{g})\to\mathbb{C},\qquad (v_1,v_2)\mapsto\tilde v_1(\tilde v_2(\psi))(\eta,X)$$

is given by the matrix

$$\mathcal{A} = \begin{pmatrix} \omega_\eta([\widetilde X,\widetilde{\mathfrak{X}}_i],\widetilde{\mathfrak{X}}_j) & -\omega_\eta(\widetilde A_j,\widetilde{\mathfrak{X}}_i) \\ -\omega_\eta(\widetilde A_i,\widetilde{\mathfrak{X}}_j) & 0\end{pmatrix}.$$

Indeed, $\widetilde{\mathfrak{X}}_i(J_X) = dJ_X(\widetilde{\mathfrak{X}}_i) = -\iota_{\widetilde X}\omega(\widetilde{\mathfrak{X}}_i)$, and by (6) we have $(\widetilde{\mathfrak{X}}_i)_\eta(\omega(\widetilde X,\widetilde{\mathfrak{X}}_j)) = -\omega_\eta([\widetilde X,\widetilde{\mathfrak{X}}_i],\widetilde{\mathfrak{X}}_j)$, since $\widetilde X_\eta=0$. If therefore $\mathcal{J}:T(T^*M)\to T(T^*M)$ denotes the bundle homomorphism introduced in Section 2, we obtain

$$\mathcal{A} = \begin{pmatrix} \mathcal{J}L_X & -g_\eta(\mathcal{J}\widetilde A_j,\widetilde{\mathfrak{X}}_i) \\ -g_\eta(\mathcal{J}\widetilde A_i,\widetilde{\mathfrak{X}}_j) & 0\end{pmatrix},$$

where $L_X: T_\eta(T^*M)\to T_\eta(T^*M),\mathfrak{X}\mapsto[\widetilde X,\widetilde{\mathfrak{X}}]_\eta$ denotes the linear transformation induced by the Lie derivative, and restricts to a map on $\mathfrak{g}\cdot\eta$ by Remark 3. Let $\{B_1,\dots,B_\kappa\}$ be another basis of $\mathfrak{g}_\eta^\perp$ such that $\{(\widetilde B_1)_\eta,\dots,(\widetilde B_\kappa)_\eta\}$ is an orthonormal basis of $\mathfrak{g}\cdot\eta$, and recall that by (15) we have $T_\eta\text{Reg }\Omega = (\mathfrak{g}\cdot\eta)^\omega$. Taking into account (23) and $\mathfrak{g}\cdot\eta\subset(\mathfrak{g}\cdot\eta)^\omega$ one sees that

$$\mathcal{B}_k = (\mathcal{J}(\widetilde B_k)_\eta;0),\qquad \mathcal{B}'_k = (L_X(\widetilde B_k)_\eta; g_\eta(\widetilde A_1,\widetilde B_k),\dots,g_\eta(\widetilde A_\kappa,\widetilde B_k),0,\dots,0),\qquad k=1,\dots,\kappa,$$

constitutes a basis of $N_{(\eta,X)}\mathrm{Reg}\,\mathcal{C}$ with $\langle \mathcal{B}_k, \mathcal{B}_l\rangle = \delta_{kl}$, $\mathcal{B}_k \perp \mathcal{B}'_l$, and $\langle \mathcal{B}'_k, \mathcal{B}'_l\rangle = (\Xi + L_X L_X)_{kl}$, where Ξ was defined in (17). One now computes

$$\begin{aligned}
\mathcal{A}(\mathcal{B}_k) =&\Big(\mathcal{J} L_X \mathcal{J}(\widetilde{B}_k)_\eta; -\sum_{j=1}^{2n} g_\eta(\mathcal{J}\widetilde{A}_1, \widetilde{\mathfrak{X}}_j) g_\eta(\mathcal{J}\widetilde{B}_k, \widetilde{\mathfrak{X}}_j), \dots\Big)\\
=&(-L_X(\widetilde{B}_k)_\eta; -g_\eta(\mathcal{J}\widetilde{A}_1, \mathcal{J}\widetilde{B}_k), \dots, -g_\eta(\mathcal{J}\widetilde{A}_\kappa, \mathcal{J}\widetilde{B}_k), 0, \dots, 0) = -\mathcal{B}'_k,\\
\mathcal{A}(\mathcal{B}'_k) =&\Big(\mathcal{J} L_X L_X(\widetilde{B}_k)_\eta - \Big(\sum_{j=1}^{\kappa} g_\eta(\mathcal{J}\widetilde{A}_j, \widetilde{\mathfrak{X}}_1) g_\eta(\widetilde{A}_j, \widetilde{B}_k), \dots\Big);\\
&-\sum_{j=1}^{2n} g_\eta(\mathcal{J}\widetilde{A}_1, \widetilde{\mathfrak{X}}_j) g_\eta(L_X(\widetilde{B}_k)_\eta, \widetilde{\mathfrak{X}}_j), \dots\Big) = (\mathcal{J} L_X L_X(\widetilde{B}_k)_\eta + (g_\eta(\Xi(\widetilde{B}_k)_\eta, \mathcal{J}\widetilde{\mathfrak{X}}_1), \dots);\\
&- g_\eta(\mathcal{J}\widetilde{A}_1, L_X(\widetilde{B}_k)_\eta), \dots).
\end{aligned}$$

Since L_X defines an endomorphism of $\mathfrak{g}\cdot\eta$ and $\mathfrak{g}\cdot\eta \subset (\mathfrak{g}\cdot\eta)^\omega$ we have $g_\eta(\mathcal{J}\widetilde{A}_1, L_X(\widetilde{B}_k)_\eta) = \omega_\eta(\widetilde{A}_1, L_X(\widetilde{B}_k)_\eta) = 0$. Furthermore, the $\{\mathcal{J}(\widetilde{B}_1)_\eta, \dots, \mathcal{J}(\widetilde{B}_\kappa)_\eta\}$ form an orthonormal basis of $\mathcal{J}(\mathfrak{g}\cdot\eta)$, and we obtain

$$\mathcal{A}(\mathcal{B}'_k) = (\mathcal{J}(L_X L_X - \Xi)(\widetilde{B}_k)_\eta; 0) = \sum_{j=1}^{\kappa} g_\eta(\mathcal{J}(L_X L_X - \Xi)(\widetilde{B}_k)_\eta, \mathcal{J}(\widetilde{B}_j)_\eta)\, \mathcal{B}_j.$$

Taking all together, one sees that the transversal Hessian $\mathrm{Hess}\,\psi(\eta, X)_{|N_{(\eta,X)}\mathrm{Reg}\,\mathcal{C}}$ is given by the matrix

$$\begin{pmatrix} 0 & -\mathbf{1}_\kappa \\ (L_X L_X - \Xi)_{|\mathfrak{g}\cdot\eta} & 0 \end{pmatrix},$$

and the assertion follows. $\square$

Proposition 7. *The leading term in* (55) *is given by*

$$L_0 = \frac{vol\, G}{vol\, H} \int_{\mathrm{Reg}\,\Omega} \left[\int_{\mathfrak{g}_\eta} a(\eta, X)\, dX\right] \frac{d(\mathrm{Reg}\,\Omega)(\eta)}{vol\ \mathcal{O}_\eta},$$

where H denotes a principal isotropy group, and vol $\mathcal{O}_\eta$ the volume of the G-orbit through η, while dX is the measure on $\mathfrak{g}_\eta$ induced by the invariant inner product on $\mathfrak{g}$.

Proof. The proof is based on the following integration formula, compare [12, Lemma 3.4]. Let $(\mathbf{X}, h_{\mathbf{X}})$ and $(\mathbf{Y}, h_{\mathbf{Y}})$ be two Riemannian manifolds and $F : \mathbf{X} \to \mathbf{Y}$ a smooth submersion. Then, for $b \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X})$ one has

$$\int_{\mathbf{X}} b(x)\, d\mathbf{X}(x) = \int_{\mathbf{Y}} \left[\int_{F^{-1}(y)} b(z) \frac{d(F^{-1}(y))(z)}{|\det d_z F \circ {}^t d_z F)|^{1/2}}\right] d\mathbf{Y}(y), \tag{58}$$

where $d(F^{-1}(y))$ denotes the Riemannian measure induced by the one of $\mathbf{X}$ on $F^{-1}(y)$, and the transposed operator of the differential $d_x F : T_x\mathbf{X} \to T_{F(x)}\mathbf{Y}$ is given by the operator ${}^t d_x F : T_{F(x)}\mathbf{Y} \to T_x\mathbf{X}$ which is uniquely determined by the condition

$$h_{\mathbf{X}}(\mathfrak{X}, {}^t d_x F(\mathfrak{Y})) = h_{\mathbf{Y}}(d_x F(\mathfrak{X}), \mathfrak{Y}), \qquad \mathfrak{X} \in T_x\mathbf{X}, \quad \mathfrak{Y} \in T_{F(x)}\mathbf{Y}.$$

Consider now the map $P : \mathrm{Reg}\,\mathcal{C} \to \mathrm{Reg}\,\Omega, (\eta, X) \to \eta$, which is a submersion by Proposition 5. In order to apply the previous integration formula, we have to compute the determinant of

$d_{(\eta,X)}P \circ {}^t d_{(\eta,X)}P$ at a point $(\eta, X) \in \mathrm{Reg}\,\mathcal{C}$. For this, let $\mathcal{G}$ denote the orthogonal complement of $\mathfrak{g} \cdot \eta$ in $T_\eta \mathrm{Reg}\,\Omega$. We then assert that

$$d_{(\eta,X)}P \circ {}^t d_{(\eta,X)}P_{|\mathcal{G}} = \mathrm{id}\,. \tag{59}$$

Indeed, let $\mathfrak{Y} \in \mathcal{G}$. As was shown in the proof of Proposition 5, $[\widetilde{\mathfrak{Y}}, \widetilde{X}]_\eta \in \mathfrak{g} \cdot \eta$. On the other hand, the fact that $\mathfrak{g} \cdot \eta$ and $\mathcal{G}$ are invariant under G_η, together with (25), imply that $[\widetilde{\mathfrak{Y}}, \widetilde{X}]_\eta \in \mathcal{G}$. Hence $[\widetilde{\mathfrak{Y}}, \widetilde{X}]_\eta = 0$. Taking into account (23) we infer from this that $(\mathfrak{Y}, 0) \in T_{(\eta,X)}\mathrm{Reg}\,\mathcal{C}$, and consequently ${}^t dP_{(\eta,X)}(\mathfrak{Y}) = (\mathfrak{Y}, 0)$. Thus, $d_{(\eta,X)}P \circ {}^t d_{(\eta,X)}P(\mathfrak{Y}) = \mathfrak{Y}$, and (59) follows. For the computation of the determinant of $d_{(\eta,X)}P \circ {}^t d_{(\eta,X)}P$ it therefore suffices to consider its restriction to $\mathfrak{g} \cdot \eta$, and with the notation as in Lemma 8 we shall show that

$$d_{(\eta,X)}P \circ {}^t d_{(\eta,X)}P_{|\mathfrak{g}\cdot\eta} = (\Xi - L_X \circ L_X)^{-1} \circ \Xi. \tag{60}$$

Consider thus an element $\mathfrak{X} \in \mathfrak{g} \cdot \eta$, and write ${}^t d_{(\eta,X)}P(\mathfrak{X}) = (\mathfrak{Y}, w)$. Denote the $\mathrm{Ad}\,(G)$-invariant inner product in $\mathfrak{g}$ by $\langle \cdot, \cdot \rangle$, and let again $\{A_1, \dots, A_d\}$ be an orthonormal basis of $\mathfrak{g}$ such that $\mathfrak{g}_\eta^\perp$ is spanned by the elements $\{A_1, \dots, A_\kappa\}$, and $\mathfrak{g}_\eta$ by $\{A_{\kappa+1}, \dots, A_d\}$. From (23) it is clear that for each $j = 1, \dots, \kappa$ we have $((\widetilde{A}_j)_\eta; \langle [X, A_j], A_1 \rangle, \dots, \langle [X, A_j], A_d \rangle) \in T_{(\eta,X)}\mathrm{Reg}\,\mathcal{C}$. By definition of the transposed we therefore have

$$g(\mathfrak{X}, (\widetilde{A}_j)_\eta) = g(\mathfrak{Y}, (\widetilde{A}_j)_\eta) + \sum_{k=1}^d w_k \langle [X, A_j], A_k \rangle \,.$$

Consequently, $g(\mathfrak{X} - \mathfrak{Y}, (\widetilde{A}_j)_\eta) = \sum_{k=1}^d w_k \langle [X, A_j], A_k \rangle$. If Ξ denotes the linear transformation introduced in (17), we obtain

$$\Xi(\mathfrak{X} - \mathfrak{Y}) = \sum_{j=1}^\kappa \sum_{k=1}^d w_k \langle [X, A_j], A_k \rangle\, (\widetilde{A}_j)_\eta = \sum_{j=1}^d \sum_{k=1}^d w_k \langle A_j, [A_k, X] \rangle\, (\widetilde{A}_j)_\eta = \sum_{k=1}^d w_k [\widetilde{A_k, X}]_\eta.$$

Let $f \in \mathrm{C}^\infty(T^*M)$. Due to $\widetilde{X}_\eta = 0$ we have $[\widetilde{A_k, X}]_\eta f = (\widetilde{A}_k)_\eta (\widetilde{X} f)$. Combined with the fact that $\sum_{k=1}^d w_k (\widetilde{A}_k)_\eta = -[\widetilde{\mathfrak{Y}}, \widetilde{X}]_\eta$ this implies

$$-\sum_{k=1}^d w_k [\widetilde{A_k, X}]_\eta f = [\widetilde{\mathfrak{Y}}, \widetilde{X}]_\eta (\widetilde{X} f) = [[\widetilde{\mathfrak{Y}}, \widetilde{X}], \widetilde{X}]_\eta f = [\widetilde{X}, [\widetilde{X}, \widetilde{\mathfrak{Y}}]]_\eta f,$$

and consequently

$$\Xi(\mathfrak{Y} - \mathfrak{X}) = [\widetilde{X}, [\widetilde{X}, \widetilde{\mathfrak{Y}}]]_\eta = L_X([\widetilde{X}, \widetilde{\mathfrak{Y}}]_\eta) = L_X \circ L_X(\mathfrak{Y}).$$

Thus, $\mathfrak{Y} = (\Xi - L_X \circ L_X)^{-1} \circ \Xi(\mathfrak{X})$, and (60) follows. Taking all together we have shown that

$$\det d_{(\eta,X)}P \circ {}^t d_{(\eta,X)}P = \det{}^{-1}(\Xi - L_X \circ L_X) \cdot \det \Xi,$$

and with Lemma 8 and the integration formula (58) we obtain

$$L_0 = \int_{\mathrm{Reg}\,\mathcal{C}} \frac{a(\eta, X)\, d(\mathrm{Reg}\,\mathcal{C})(\eta, X)}{|\mathrm{Hess}\,\psi(\eta, X)_{|N_{(\eta,X)}\mathrm{Reg}\,\mathcal{C}}|^{1/2}} = \int_{\mathrm{Reg}\,\Omega} \left[\int_{\mathfrak{g}_\eta} a(\eta, X)\, dX \right] \frac{d(\mathrm{Reg}\,\Omega)(\eta)}{|\det \Xi_{|\mathfrak{g}\cdot\eta}|^{1/2}},$$

where $d(\mathrm{Reg}\,\Omega)$ denotes the volume form induced by $d\eta\, dX$. The assertion of the proposition now follows by noting that $|\det \Xi_{|\mathfrak{g}\cdot\eta}|^{1/2} = \mathrm{vol}\,\mathcal{O}_\eta \cdot \mathrm{vol}\,G_\eta / \mathrm{vol}\,G$, compare [12, Lemma 3.6].

□

10. Residue formulae for $\mathbf{X} = T^*M$

We are now in position to derive residue formulae for the cotangent bundle of a G-manifold. Thus, let M be an n-dimensional, connected Riemannian manifold, and G a d-dimensional, compact, connected Lie group with maximal torus $T \subset G$ acting on M by isometries. Let Θ be the Liouville form on T^*M, $\omega = d\Theta$ the symplectic form, denote the corresponding momentum map by $\mathbb{J} : T^*M \to \mathfrak{g}^*$, $\mathbb{J}(\eta)(X) = J_X(\eta) = \Theta(\widetilde{X})(\eta)$, and write $\Omega = \mathbb{J}^{-1}(0)$. Let further $\pi : \operatorname{Reg}\Omega \to \operatorname{Reg}\mathbf{X}_{red} = \operatorname{Reg}\Omega/G$ be the canonical projection, and consider the map

$$\widetilde{\mathcal{K}} : H_G^{*+\kappa}(T^*M) \xrightarrow{r} H_G^*(\operatorname{Reg}\Omega) \xrightarrow{(\pi^*)^{-1}} H^*(\operatorname{Reg}\mathbf{X}_{red}),$$

where $r : \Lambda^*(T^*M) \to \Lambda^{*-\kappa}(\operatorname{Reg}\Omega)$ denotes the natural restriction map described in (64) and κ is the dimension of a principal G-orbit. As an application of Theorem 6, we are able to compute the limit (2) in case that κ equals $d = \dim\mathfrak{g}$. It corresponds to the leading term in the expansion.

Corollary 3. *Assume that the dimension κ of a principal G-orbit in M equals $d = \dim\mathfrak{g}$. Let $\alpha \in \Lambda_c(T^*M)$ and $\varphi \in \mathrm{C}_c^\infty(\mathfrak{g}^*)$ have total integral one. Then*

$$\lim_{\varepsilon\to 0} \langle \mathcal{F}_{\mathfrak{g}} L_\alpha, \varphi_\varepsilon\rangle = \frac{(2\pi)^d\, vol\, G}{|H|} \int_{\operatorname{Reg}\Omega} a(\eta) \frac{d(\operatorname{Reg}\Omega)(\eta)}{vol\, \mathcal{O}_\eta} = \frac{(2\pi)^d\, vol\, G}{|H|} \int_{\operatorname{Reg}\Omega} \frac{r(\alpha)}{vol\, \mathcal{O}_\eta},$$

where H denotes a principal isotropy group of the G-action, and we wrote $\alpha_{[2n]} = a(\eta)d\eta$, $d\eta$ being Liouville measure.

Proof. By (2), Theorem 6, and Proposition 7 one deduces

$$L_0 = \lim_{\varepsilon\to 0} \langle \mathcal{F}_{\mathfrak{g}} L_\alpha, \varphi_\varepsilon\rangle = \frac{(2\pi)^d \mathrm{vol}\, G}{\mathrm{vol}\, H} \int_{\operatorname{Reg}\Omega} \left[\int_{\mathfrak{g}_\eta} \hat{\varphi}(X)\, dX\right] a(\eta) \frac{d(\operatorname{Reg}\Omega)(\eta)}{\mathrm{vol}\, \mathcal{O}_\eta}.$$

Since $\kappa = \dim\mathfrak{g}$, we have $\mathfrak{g}_\eta = \{0\}$ for all $\eta \in \operatorname{Reg}\Omega$; in particular, $H \sim G_\eta$ is a finite group. Hence, $\mathrm{vol}\, H \equiv |H|$ and $\int_{\mathfrak{g}_\eta} \hat{\varphi}(X)\, dX = \hat{\varphi}(0) = 1$, and we obtain the first equality. To see the second, assume that α is supported in a neighborhood of $\mathcal{C}$. Let $K \subset T^*M$ be a compact subset such that $\operatorname{supp}\alpha \subset K$, and $u_\delta \in \mathrm{C}_c^\infty(\operatorname{Sing}\Omega \cap K)_{3\delta}$ a family of cut-off functions as in the proof of Theorem 6. Denote the normal bundle to $\operatorname{Reg}\mathcal{C} = \operatorname{Reg}\Omega \times \{0\} \equiv \operatorname{Reg}\Omega$ by $\nu : N\operatorname{Reg}\mathcal{C} \to \mathcal{C}$, and identify a tubular neighborhood of $\operatorname{Reg}\mathcal{C}$ with a neighborhood of the zero section in $N\operatorname{Reg}\mathcal{C}$. A direct application of Theorem A then yields with Lemma 8

$$L_0(\delta) = \lim_{\varepsilon\to 0} \int_{\mathfrak{g}} \int_{\mathbf{X}} e^{iJ_X/\varepsilon}(1-u_\delta)\, \alpha\, \hat{\varphi}(X) \frac{dX}{\varepsilon^d} = \frac{(2\pi)^d\, \mathrm{vol}\, G}{|H|} \int_{\operatorname{Reg}\Omega} \frac{r((1-u_\delta)\alpha)}{\mathrm{vol}\, \mathcal{O}_\eta},$$

where only the leading term (63) is relevant. Repeating the arguments in the proof of Theorem 6 then shows that

$$L_0 = \lim_{\delta\to 0} L_0(\delta) = \frac{(2\pi)^d\, \mathrm{vol}\, G}{|H|} \int_{\operatorname{Reg}\Omega} \frac{r(\alpha)}{\mathrm{vol}\, \mathcal{O}_\eta}.$$

$\square$

With the notation as in Sections 2 and 4, we finally arrive at the following

Theorem 7. *Let $\varrho \in H_G^*(T^*M)$ be of the form $\varrho(X) = \alpha + D\nu(X)$, where α is a closed, basic differential form on T^*M of compact support, and ν an equivariant differential form of compact support. Assume that the dimension κ of a principal G-orbit equals $d = \dim\mathfrak{g}$. Then*

$$(2\pi)^d \int_{\operatorname{Reg}\mathbf{X}_{red}} \widetilde{\mathcal{K}}(e^{-i\omega}\alpha) = \frac{|H|}{|W|\, vol\, T} \operatorname{Res}\Big(\Phi^2 \sum_{F\in\mathcal{F}} u_F\Big).$$

Proof. Let α be a basic differential form on T^*M. By definition, α is G-invariant and satisfies $\iota_{\widetilde{X}}\alpha = 0$ for all $X \in \mathfrak{g}$. It is therefore a constant map from $\mathfrak{g}$ to $\Lambda(T^*M)$, and belongs to $(S(\mathfrak{g}^*) \otimes \Lambda(T^*M))^G$. Furthermore, $D\alpha = 0$ iff $d\alpha = 0$, so that $\alpha \in H^*_G(T^*M)$. The assertion is now a consequence of Corollaries 2 and 3, together with Lemma 2, by which

$$\frac{\operatorname{vol} G}{|W| \operatorname{vol} T} \operatorname{Res}\Big(\Phi^2 \sum_{F \in \mathcal{F}} u_F\Big) = \lim_{\varepsilon \to 0} \Big\langle \mathcal{F}_{\mathfrak{g}}\Big(L_{e^{-i\omega}\varrho(\cdot)}(\cdot)\Big), \varphi_\varepsilon \Big\rangle = \frac{(2\pi)^d \operatorname{vol} G}{|H|} \int_{\operatorname{Reg} \Omega} \frac{r(e^{-i\omega}\alpha)}{\operatorname{vol} \mathcal{O}_\eta}$$

$$= \frac{(2\pi)^d \operatorname{vol} G}{|H|} \int_{\operatorname{Reg} \mathbf{X}_{red}} \widetilde{\mathcal{K}}(e^{-i\omega}\alpha).$$

□

Remark 6. In order to fully describe the cohomology of the quotient $\operatorname{Reg} \mathbf{X}_{red}$, it would still be necessary to consider more general forms $\varrho \in H^*_G(T^*M)$ than the ones examined in Theorem 7. For this, one would need a full asymptotic expansion for the integrals studied in Theorem 6, and we intend to tackle this problem in a future paper. Nevertheless, the considered forms ϱ are already quite general in the following sense. Let G act *locally freely* on a symplectic manifold $\mathbf{X}$, which means that all stabilizer groups are finite, and assume that the action is Hamiltonian. As a consequence, 0 is a regular value of the momentum map and $\mathbf{X}/G$ is an orbifold. Furthermore, one has the isomorphism

$$H^*_G(\mathbf{X}) \simeq H^*(\mathbf{X}/G),$$

which implies that any equivariantly closed differential form ϱ can be written in the form

$$\varrho(X) = \alpha + D\nu(X),$$

where α is a closed, basic differential form on T^*M of compact support, and ν is an equivariant differential form of compact support [20].

Let $\mathbf{X}$ be a $2n$-dimensional symplectic manifold with a Hamiltonian G-action. For general, not necessarily equivariantly closed $\alpha \in \Lambda_c(\mathbf{X})$, no similar formulae can be expected, and non-local remainder terms will occur. To see this, let us first deduce an expansion for $L_\alpha(X)$ using the stationary phase principle. For this, recall that for fixed $X \in \mathfrak{g}$ the critical set of J_X is clean in the sense of Bott, and equal to F^T in case that $X \in \mathfrak{t}'$ is a regular element.

Lemma 9. *Let $X \in \mathfrak{g}$, and suppose that* $\operatorname{supp} \alpha \cap \operatorname{Crit} J_X = \emptyset$. *Then $L_\alpha \in \mathcal{S}(\mathfrak{g})$.*

Proof. Let $(\gamma, \mathcal{O})$ be a Darboux chart on $\mathbf{X}$, so that the symplectic form ω and the corresponding Liouville form read

$$\omega \equiv \sum_{i=1}^n dp_i \wedge dq_i, \qquad \frac{\omega^n}{n!} \equiv dp_1 \wedge dq_1 \wedge \cdots \wedge dp_n \wedge dq_n.$$

Assume that $\alpha_{[2n]} = f \cdot \frac{\omega^n}{n!} \in \Lambda_c(\mathbf{X})$ is supported in $\mathcal{O}$, so that

$$\int_{\mathbf{X}} e^{iJ_X}\alpha = \int_{\gamma(\mathcal{O})} e^{iJ_X \circ \gamma^{-1}(q,p)} (f \circ \gamma^{-1})(q,p)\, dq\, dp,$$

where $J_X \circ \gamma^{-1}(q,p)$ depends linearly on X. Let now $\operatorname{supp} \alpha \cap \operatorname{Crit} J_X = \emptyset$. Writing

$$e^{iJ_X \circ \gamma^{-1}} = \frac{1}{i|(J_X \circ \gamma^{-1})'|^2} \sum_{j=1}^n \left(\frac{\partial}{\partial q_j}(J_X \circ \gamma^{-1}) \frac{\partial}{\partial q_j} + \frac{\partial}{\partial p_j}(J_X \circ \gamma^{-1}) \frac{\partial}{\partial p_j} \right) e^{iJ_X \circ \gamma^{-1}},$$

and integrating by parts we obtain $L_\alpha(X) = O(|X|^{-\infty})$ on $\mathfrak{g}$. Similarly, if $\{X_1, \dots, X_d\}$ denotes a basis of $\mathfrak{g}$, and $X = \sum s_i X_i$, the same arguments yield for arbitrary multi-indices γ the estimate $\partial_s^\gamma L_\alpha(X) = O(|X|^{-\infty})$ on $\mathfrak{g}$, and the assertion follows. □

Next, let $Y \in \mathfrak{t}'$ be a regular element, so that $\mathrm{Crit} J_Y = F^T$, $F \in \mathcal{F}$ a connected component of F^T, and $\nu : NF \to F$ the normal bundle of F. As usual, we identify a neighborhood of the zero section of NF with a tubular neighborhood of F, and assume in the following that the support of α is contained in that neighborhood. Integration along the fiber yields

$$L_\alpha(Y) = \int_F \nu_*(e^{iJ_Y}\alpha).$$

To obtain a localization formula for $L_\alpha(Y)$ via the stationary phase principle, consider an oriented trivialization $\{(U_j, \varphi_j)\}_{j \in I}$ of $\nu : NF \to F$. Let $\{s_1, \dots, s_l\}$ be the fiber coordinates on $NF_{|U_j}$ given by φ_j, and Assume that α is given on $\nu^{-1}(U_j)$ by

$$\alpha_j = f_j(x,s)\,(\nu^*\beta_j) \wedge ds_1 \wedge \cdots \wedge ds_l, \qquad \beta_j \in \Lambda^{2n-l}(U_j), \quad x \in U_j,$$

where f_j is compactly supported. The cleanness of $\mathrm{Crit} J_Y$ implies that the function $s \mapsto J_Y(x,s) = J_Y \circ \varphi_j^{-1}(x,s)$ has a non-degenerate critical point at $s = 0$ for each $x \in U_j$, so that by choosing the support of f_j sufficiently small we can assume that there are no other critical points. Define now the function $H_Y(x,s) = J_Y(x,s) - \langle J_Y''(x,0)s, s\rangle/2$, which depends linearly on Y. As in the proof of Theorem A one computes for any $N \in \mathbb{N}$

$$\nu_*(e^{iJ_Y}\alpha_j) = \frac{1}{\det\left(J_Y''(x,0)/2\pi i\right)^{1/2}}$$
$$\cdot \left[\sum_{r-k \le N} \sum_{3k \le 2r} \frac{1}{r!k!} \left(\left\langle D_s, \frac{J_Y''(x,0)^{-1}}{2i} D_s \right\rangle^r (iH_Y(x,\cdot))^k f_j(x,\cdot) \right)(x,0) + R_{j,N+1}(Y) \right] \cdot \beta_j,$$

where $R_{j,N+1}$ is an explicitly given smooth function on $\mathfrak{t}'$ of order $O(|Y|^{-N-1})$ given by

$$R_{j,N+1}(Y) = \frac{\beta_j}{\det\left(J_Y''(x,0)/2\pi i\right)^{1/2}}$$
$$\cdot \sum_{k=0}^\infty \int_{\mathbb{R}^l} \sum_{r=3N+1}^\infty \frac{1}{(2\pi)^l k! r!} \left(\frac{\langle J_Y''(x,0)^{-1}\xi, \xi\rangle}{2i} \right)^r \mathcal{F}\big(H_Y(x,\cdot)^k f_j(x,\cdot)\big)(\xi)\, d\xi.$$

As a consequence, we obtain the desired localization formula.

Proposition 8. *Let $\alpha \in \Lambda_c(T^*M)$, and $Y \in \mathfrak{t}'$. Then, for arbitrary $N \in \mathbb{N}$,*

$$L_\alpha(Y) = \sum_{F \in \mathcal{F}} \sum_j \int_F \frac{1}{\det\left(J_Y''(x,0)/2\pi i\right)^{1/2}}$$
$$\cdot \left[\sum_{r-k \le N} \sum_{3k \le 2r} \frac{1}{r!k!} \left(\left\langle D_s, \frac{J_Y''(x,0)^{-1}}{2i} D_s \right\rangle^r (iH_Y(x,\cdot))^k f_j(x,\cdot) \right)(x,0) \right] \cdot \beta_j + R_{N+1}(Y),$$

where R_{N+1} is an explicitely given, smooth function on $\mathfrak{t}'$ of order $O(|Y|^{-N-1})$.

□

The limit (3) can now be studied taking into account (7) and Cauchy's integral theorem, together with the theorems of Paley-Wiener-Schwartz, leading to corresponding residue formulae with non-local terms.

Appendix A. The generalized stationary phase theorem

In this appendix, we include a proof of the generalized stationary phase theorem in the setting of vector bundles. It is a direct consequence of the projection formula and the stationary phase approximation, and implies the classical generalized stationary phase theorem for manifolds. Sketches of proofs for the latter can also be found in Combescure-Ralston-Robert [14, Theorem 3.3], as well as Varadarajan [39, pp. 199].

Theorem A (Stationary phase theorem for vector bundles)**.** *Let M be an n-dimensional, oriented manifold, and $\pi : E \to M$ an oriented vector bundle of rang l. Let further $\alpha \in \Lambda^q_{cv}(E)$ be a differential form on E with compact support along the fibers, $\tau \in \Lambda^{n+l-q}_c(M)$ a differential form on M of compact support, $\psi \in \mathrm{C}^\infty(E)$, and consider the integral*

$$I(\mu) = \int_E e^{i\psi/\mu}(\pi^*\tau) \wedge \alpha, \qquad \mu > 0.$$

Let $\iota : M \hookrightarrow E$ denote the zero section. Assume that the critical set of ψ coincides with $\iota(M)$, and that the transversal Hessian $\mathrm{Hess}_{trans}\,\psi$ of ψ is non-degenerate along $\iota(M)$. Then, for each $N \in \mathbb{N}$, $I(\mu)$ possesses an asymptotic expansion of the form

$$I(\mu) = e^{i\psi_0/\mu} e^{\frac{i\pi}{4}\sigma_\psi}(2\pi\mu)^{\frac{l}{2}} \sum_{j=0}^{N-1} \mu^j Q_j(\psi; \alpha, \tau) + R_N(\mu), \tag{61}$$

where ψ_0 and σ_ψ denote the value of ψ and the signature of the transversal Hessian along $\iota(M)$, respectively. The coefficients Q_j are given by measures supported on M, and can be computed explicitly, as well as the remainder term $R_N(\mu)$ which is of order $O(\mu^{l/2+N})$.

Proof. Let $\pi_* : \Lambda^*_{cv}(E) \to \Lambda^{*-l}(M)$ denote integration along the fiber in E, which lowers the degree by the fiber dimension. By the projection formula [8, Proposition 6.15] one has

$$\int_E e^{i\psi/\mu}(\pi^*\tau) \wedge \alpha = \int_M \tau \wedge \pi_*(e^{i\psi/\mu}\alpha).$$

Let $\{U_j\}_{j\in I}$ be an open covering of M and $\{(U_j, \varphi_j)\}_{j\in I}$, $\varphi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{R}^l$, an oriented trivialization of $\pi : E \to M$. Write $s_1, \dots, s_l$ for the fiber coordinates on $E_{|U_j}$ given by φ_j. Since $I(\mu)$ vanishes if $q < l$, we assume in the following that $q \geq l$ and that α is given on $\pi^{-1}(U_j)$ by

$$\alpha_j = f_j(x, s)\,(\pi^*\beta_j) \wedge ds_1 \wedge \cdots \wedge ds_l, \qquad \beta_j \in \Lambda^{q-l}(U_j), \quad x \in U_j,$$

where the function $f_j \in \mathrm{C}^\infty(U_j \times \mathbb{R}^l)$ is compactly supported along the fibers. By assumption, $s \mapsto \psi(x, s) = \psi \circ \varphi_j^{-1}(x, s)$ has a non-degenerate critical point at $s = 0$ for each $x \in U_j$, so that in view of the non-stationary phase theorem [24, Theorem 7.7.1] we can assume that there are no other critical points by choosing the support of f_j sufficiently small. Then, letting $\psi(x, 0) = 0$ and setting $H(x, s) = \psi(x, s) - \langle \psi''(x, 0)s, s\rangle /2$, one computes on $\pi^{-1}(U_j)$

$$\begin{aligned}\pi_*(e^{i\psi/\mu}\alpha_j) &= \int_{\mathbb{R}^l} e^{i\psi(x,s)/\mu} f_j(x, s) ds \cdot \beta_j = \int_{\mathbb{R}^l} e^{i\left\langle \psi''(x,0)s,s\right\rangle/2\mu} e^{iH(x,s)/\mu} f_j(x, s) ds \cdot \beta_j \\ &= \sum_{k=0}^{\infty} \frac{i^k}{\mu^k k!} \int_{\mathbb{R}^l} e^{i\left\langle \psi''(x,0)s,s\right\rangle/2\mu} H(x, s)^k f_j(x, s) ds \cdot \beta_j.\end{aligned}$$

Note that it is permissible to interchange the order of summation and integration, since $H(x, s) = O(|s|^3)$, so that under the hypothesis $\mathrm{supp}_s\, f_j(x, \cdot) \subset B(0, 1)$ one has for suitable $C > 0$ the estimate

$$\Big| f_j(x,\cdot) \sum_{k=0}^{\tilde N} \frac{H(x,\cdot)^k}{\mu^k k!} \Big| \leq C |f_j(x,\cdot)| \sum_{k=0}^{\tilde N} \frac{1}{\mu^k k!} \leq C e^{1/\mu} |f_j(x,\cdot)|, \qquad \tilde N \in \mathbb{N},$$

yielding an integrable majorand. Put $D_k = -i\,\partial_k$. Taking into account

$$\int_{\mathbb{R}^l} \langle \xi, \psi''(x,0)^{-1}\xi \rangle^r \mathcal{F}\big(H(x,\cdot)^k f_j(x,\cdot)\big)(\xi)\, d\xi = (2\pi)^l \Big(\big\langle D_s, \psi''(x,0)^{-1} D_s \big\rangle^r H(x,\cdot)^k f_j(x,\cdot) \Big)(0)$$

we obtain with Parseval's formula for arbitrary $\tilde N \in \mathbb{N}$

$$\begin{aligned} \pi_*(e^{i\psi/\mu}\alpha_j) &= \frac{\beta_j}{\det(\psi''(x,0)/2\pi\mu i)^{1/2}} \sum_{k=0}^\infty \frac{i^k}{(2\pi)^l \mu^k k!} \int_{\mathbb{R}^l} e^{-i\mu \langle \psi''(x,0)^{-1}\xi,\xi\rangle/2} \mathcal{F}\big(H(x,\cdot)^k f_j(x,\cdot)\big)(\xi)\, d\xi \\ &= \frac{\beta_j}{\det(\psi''(x,0)/2\pi\mu i)^{1/2}} \sum_{k=0}^\infty \frac{i^k}{\mu^k k!} \Bigg[\sum_{r=0}^{\tilde N -1} \frac{(-i\mu)^r}{2^r r!} \Big(\big\langle D_s, \psi''(x,0)^{-1} D_s \big\rangle^r H(x,\cdot)^k f_j(x,\cdot)\Big)(0) \\ &\qquad + \int_{\mathbb{R}^l} \sum_{r=\tilde N}^\infty \frac{(-i\mu)^r}{(2\pi)^l 2^r r!} \big(\langle \psi''(x,0)^{-1}\xi, \xi\rangle \big)^r \mathcal{F}\big(H(x,\cdot)^k f_j(x,\cdot)\big)(\xi)\, d\xi \Bigg]. \end{aligned}$$

Note that interchanging integration and summation in the last term is in general not possible due to the lack of an integrable majorand. Since $H(x,s)$ vanishes of third order at $s=0$, the local terms are zero unless $3k \leq 2r$. Consequently, for general ψ and arbitrary $N \in \mathbb{N}$ we arrive at

$$\begin{aligned} \pi_*(e^{i\psi/\mu}\alpha_j) &= \frac{e^{i\psi(x,0)/\mu}\cdot \beta_j}{\det(\psi''(x,0)/2\pi\mu i)^{1/2}} \\ &\cdot \Bigg[\sum_{r-k\leq N} \mu^{r-k} \sum_{3k \leq 2r} \frac{1}{r!\,k!\,2^r\, i^{r-k}} \Big(\big\langle D_s, \psi''(x,0)^{-1} D_s\big\rangle^r H(x,\cdot)^k f_j(x,\cdot)\Big)(0) + R_{j,N+1} \Bigg], \end{aligned} \tag{62}$$

where $R_{j,N+1}$ is explicitly given by

$$\begin{aligned} R_{j,N+1} &= \frac{e^{i\psi(x,0)/\mu}\cdot\beta_j}{\det(\psi''(x,0)/2\pi\mu i)^{1/2}} \\ \cdot \sum_{k=0}^\infty \frac{i^k}{\mu^k k!} \int_{\mathbb{R}^l} &\sum_{r=3N+1}^\infty \frac{(-i\mu)^r}{(2\pi)^l 2^r r!} \big(\langle \psi''(x,0)^{-1}\xi,\xi\rangle\big)^r \mathcal{F}\big(H(x,\cdot)^k f_j(x,\cdot)\big)(\xi)\, d\xi. \end{aligned}$$

Moreover, by [24, Theorem 7.7.5] one has $R_{j,N+1} = O(\mu^{N+1})$. The assertion now follows by integrating over M, and by taking $\det(\psi''(x,0)/2\pi\mu i)^{1/2} = (2\pi\mu)^{-l/2}|\det \psi''(x,0)|^{1/2} e^{\frac{-i\pi}{4}\sigma_\psi}$ into account. In particular, the leading coefficient is given by

$$Q_0(\psi;\alpha,\tau) = \int_M \frac{\tau \wedge r(\alpha)}{|\det \mathrm{Hess}_{trans}\, \psi|^{1/2}}, \tag{63}$$

where the restriction map $r : \Lambda^q(E) \to \Lambda^{q-l}(M)$ is locally given by

$$h_j\,(\pi^*\gamma_j) \wedge ds_{\sigma(1)} \wedge \dots \wedge ds_{\sigma(p)} \quad \longmapsto \quad \begin{cases} (-1)^{\mathrm{sgn}\,\sigma} \iota^*(h_j)\,\gamma_j, & p = l, \\ 0, & p < l, \end{cases} \tag{64}$$

$\gamma_j \in \Lambda^{q-p}(U_j)$, $h_j \in \mathrm{C}^\infty(U_j \times \mathbb{R}^l)$, σ being a permutation in p variables.

$\square$

Remark B. (1) In the proof of the last theorem, one can also use the lemma of Morse. This simplifies the proof, but gives less explicit expressions for the coefficients Q_j, since the Morse diffeomorphism is not given explicitly. Indeed, by Morse's Lemma, we can choose the trivialization of $\pi : E \to M$ in such a way that

$$\psi(x,s) = \frac{1}{2}\langle s, S_x s\rangle, \qquad S_x \in \mathrm{Sym}(l,\mathbb{R}),\, \det S_x \neq 0,$$

where the symmetric matrix S_x depends smoothly on $x \in U_j$. Parseval's formula then yields

$$\begin{aligned}\pi_*(e^{i\psi/\mu}\alpha_j) &= \int_{\mathbb{R}^l} e^{i\psi(x,s)/\mu} f_j(x,s)ds \cdot \beta_j \\ &= \frac{e^{i\pi \operatorname{sgn} S_x/4}\mu^{l/2}}{(2\pi)^{l/2}|\det S_x|^{1/2}} \int_{\mathbb{R}^l} e^{-i\mu\langle S_x^{-1}\xi,\xi\rangle/2}\mathcal{F}\big(f_j(x,\cdot)\big)(\xi)\, d\xi \cdot \beta_j \\ &= \frac{e^{i\pi \operatorname{sgn} S_x/4}\mu^{l/2}}{(2\pi)^{l/2}|\det S_x|^{1/2}} \Bigg[(2\pi)^l \sum_{r=0}^{N-1} \frac{\mu^r}{r!}\left(\left\langle D_s, \frac{S_x^{-1}}{2i}D_s\right\rangle^r f_j(x,\cdot)\right)(x,0) \\ &\quad + \int_{\mathbb{R}^l}\sum_{r=N}^{\infty}\frac{\mu^r}{r!}\Big(\frac{\langle S_x^{-1}\xi,\xi\rangle}{2i}\Big)^r \mathcal{F}\big(f_j(x,\cdot)\big)(\xi)\, d\xi\Bigg] \cdot \beta_j.\end{aligned}$$

By integrating over M, the assertion of Theorem A follows.

(2) In general, it is not possible to say anything about the convergence of the sum in (61) as $N \to \infty$, and consequently, about the limit $\lim_{N\to\infty} R_N(\mu)$, due to the lack of control of the growth of the derivatives $\partial_s^\alpha f_j(x,0)$ as $|\alpha| \to \infty$.

From Theorem A we can now infer the classical generalized stationary phase theorem.

Theorem C (Generalized stationary phase theorem for manifolds)**.** *Let M be a n-dimensional, orientable Riemannian manifold with volume form dM, $\psi \in \mathrm{C}^\infty(M)$ a real valued phase function, $\mu > 0$, and set*

$$I(\mu) = \int_M e^{i\psi(m)/\mu} a(m)\, dM(m),$$

where $a(m) \in \mathrm{C}^\infty_c(M)$ denotes a compactly supported function on M. Let

$$\mathcal{C} = \big\{m \in M : \psi_* : T_m M \to T_{\psi(m)}\mathbb{R} \text{ is zero}\big\}$$

be the critical set of the phase function ψ, and assume that $\mathcal{C}$ is clean in the sense that

1. *$\mathcal{C}$ is a smooth submanifold of M of dimension p in a neighborhood of the support of a;*
2. *for all $m \in \mathcal{C}$, the restriction $\psi''(m)_{|N_m\mathcal{C}}$ of the Hessian of ψ at the point m to the normal space $N_m\mathcal{C}$ is a non-degenerate quadratic form.*

Then, for all $N \in \mathbb{N}$, there exists a constant $C_{N,\psi} > 0$ such that

$$|I(\mu) - e^{i\psi_0/\mu} e^{\frac{i\pi}{4}\sigma_\psi}(2\pi\mu)^{\frac{n-p}{2}} \sum_{j=0}^{N-1} \mu^j Q_j(\psi;a)| \leq C_{N,\psi}\mu^N \sup_{l\leq 2N}\left\|D^l a\right\|_{\infty,M},$$

where D^l is a differential operator on M of order l and ψ_0 the constant value of ψ on $\mathcal{C}$, while σ_ψ denotes the constant value of the signature of the transversal Hessian $\mathrm{Hess}\,\psi(m)_{|N_m\mathcal{C}}$ on $\mathcal{C}$. The coefficients Q_j can be computed explicitly, and for each j there exists a constant $\tilde{C}_{j,\psi} > 0$ such that

$$|Q_j(\psi;a)| \leq \tilde{C}_{j,\psi} \sup_{l \leq 2j}\left\|D^l a\right\|_{\infty,\mathcal{C}}.$$

In particular,

$$Q_0(\psi; a) = \int_{\mathcal{C}} \frac{a(m)}{|\det \operatorname{Hess} \psi(m)_{|N_m\mathcal{C}}|^{1/2}} d\sigma_{\mathcal{C}}(m),$$

where $d\sigma_{\mathcal{C}}$ is the induced volume form on $\mathcal{C}$.

Proof. Due to the non-stationary phase principle, we can assume that $a\, dM$ is supported in a tubular neighborhood of $\mathcal{C}$. Identifying the latter with the total space $N\mathcal{C}$ of the normal bundle of $\mathcal{C}$, the assertion follows with Theorem A. □

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