WONDERFUL VARIETIES. REGULARIZED TRACES AND CHARACTERS

STEPHANIE CUPIT-FOUTOU, APRAMEYAN PARTHASARATHY, PABLO RAMACHER

ABSTRACT. Let $\mathbf{G}$ be a connected reductive complex algebraic group with split real form G. In this paper, we introduce a distribution character for the regular representation of G on the real locus X of a strict wonderful $\mathbf{G}$-variety $\mathbf{X}$, showing that on a certain open subset of G of transversal elements it is locally integrable, and given by a sum over fixed points.

CONTENTS

1. INTRODUCTION

Let G be a real reductive group. In classical harmonic analysis a crucial role is played by the global character of an admissible representation (σ, H) of G on a Hilbert space H. It is a distribution $\Theta_\sigma : f \to \operatorname{tr} \sigma(f)$ on the group given in terms of the trace of the convolution operators

$$\sigma(f) = \int_G f(g)\sigma(g)\, d_G(g),$$

where f is a rapidly falling function on G, and d_G a Haar measure on G. By Harish-Chandra's regularity theorem, Θ_σ is known to be locally integrable, and is the natural generalization of the character of a finite-dimensional representation. The regularity theorem allowed Harish-Chandra to characterize tempered representations in terms of the growth properties of their global characters, and fully determine the irreducible L^2-integrable representations of G.

Let $\mathbf{G}$ be a connected reductive complex algebraic group with split real form G. In this paper, we introduce a similar character Θ_π for the regular representation $(\pi, C(X))$ of G on the Banach space $C(X)$ of continuous functions on the real locus X of a strict

Date: March 7, 2014.

1991 *Mathematics Subject Classification.* 22E46, 53C35, 32J05, 58J40, 58J37, 58J35, 47A10, 58C30, 20C15.

This first-named author was funded by the SFB/TR 12 of the German Research Foundation (DFG).

wonderful $\mathbf{G}$-variety $\mathbf{X}$. Since the G-action on X is no longer transitive, the corresponding convolution operators will no longer be smooth, and a regularized trace $\mathrm{Tr}_{reg}\,\pi(f)$ has to be considered. We then show that on a certain open set of transversal elements $G(X)$ the distribution Θ_π is locally integrable, and given by

$$\mathrm{Tr}_{reg}\,\pi(f) = \int_{G(X)} f(g)\,\mathrm{Tr}^\flat\,\pi(g) d_G(g), \qquad f \in \mathrm{C_c^\infty}(G(X)),$$

where the flat trace of $\pi(g)$

$$\mathrm{Tr}^\flat\,\pi(g) = \sum_{x\in \mathrm{Fix}(X,g)} \frac{1}{|\det(1 - d\Phi_g(x))|}$$

is given by a sum over the fixed points of an element $g \in G$.

This paper is based on the local structure theorem for strict wonderful $\mathbf{G}$-varieties recently proved by Akhiezer and Cupit-Foutou [ACF12], and generalizes results already obtained by Parthasarathy and Ramacher [PR12] for the Oshima compactification of a Riemannian symmetric space.

2. Wonderful varieties

Throughout this article we shall adopt the convention of writing complex objects with boldface letters and the corresponding real objects with the ordinary ones. Let G be the split real form of a connected reductive complex algebraic group $\mathbf{G}$ of rank n, and let $\sigma : \mathbf{G} \to \mathbf{G}$ be the involution defining the split real form G, so that $G = \mathbf{G}^\sigma = \{g \in \mathbf{G} : \sigma(g) = g\}$. In particular, G is a real reductive group. Since G is not necessarily connected, denote by G_0 the identity component of G. Fix a maximal algebraic torus $\mathbf{T}$ of $\mathbf{G}$ and a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ containing it. Denote the corresponding set of positive and negative roots by Σ^+ and Σ^-, respectively. We recall the definition of a wonderful variety.

Definition 1. ([Lun96]) *An algebraic $\mathbf{G}$-variety $\mathbf{X}$ is called* wonderful of rank r *if*

1. *$\mathbf{X}$ is projective and smooth;*
2. *$\mathbf{X}$ admits an open $\mathbf{G}$-orbit whose complement consists of a finite union of smooth prime divisors $\mathbf{X}_1 \dots, \mathbf{X}_r$ with normal crossings;*
3. *the $\mathbf{G}$-orbit closures of $\mathbf{X}$ are given by the partial intersections of the $\mathbf{X}_i$.*

In particular notice that $\mathbf{X}$ has a unique closed, hence projective $\mathbf{G}$-orbit. Further, recall that a real structure on $\mathbf{X}$ is an involutive anti-holomorphic map $\mu : \mathbf{X} \to \mathbf{X}$; it is said to be *$\sigma$-equivariant* if $\mu(g \cdot x) = \sigma(g) \cdot \mu(x)$ for all $(g, x) \in \mathbf{G} \times \mathbf{X}$. Crucial for the ensuing analysis is the existence of a unique σ-equivariant real structure on wonderful varieties. More precisely, one has the following

Theorem 1. [ACF12, Theorem 4.13] *Let $\mathbf{X}$ be a wonderful $\mathbf{G}$-variety of rank r whose points have a self-normalizing stabilizer. Then*

1. *there exists a unique σ-equivariant real structure μ on $\mathbf{X}$;*
2. *the real locus X of $(\mathbf{X}, \mu)$ is not empty, and constitutes a smooth, compact, analytic G-space with finitely many G-orbits and a unique projective G-orbit.*

□

Wonderful varieties whose points have self-normalizing stabilizers are called *strict*. In what follows, let $\mathbf{X}$ be a strict, wonderful $\mathbf{G}$-variety or rank r. From the classification results of [BCF10] and [Res10] one immediately infers

Proposition 1. *Let $\mathbf{X}$ be a wonderful $\mathbf{G}$-variety such that its $\mathbf{T}$-fixed-points are located on its closed $\mathbf{G}$-orbit. Then every point of $\mathbf{X}$ has a self-normalizing stabilizer.*

□

Examples of real loci of strict wonderful $\mathbf{G}$-varieties include the Oshima-Sekiguchi compactification of a Riemannian symmetric space, which is the real locus of the De-Concini-Procesi wonderful compactification of its complexification $\mathbf{X}$ up to a finite quotient, see [BJ06, Chapter 8, Section II.14].

Let $\mathbf{Y}$ be the unique closed $\mathbf{G}$-orbit in $\mathbf{X}$, and consider a parabolic subgroup $\mathbf{B} \subset \mathbf{Q}$ of $\mathbf{G}$ such that $\mathbf{Y} \simeq \mathbf{G}/\mathbf{Q}$. Let $\mathbf{Q} = \mathbf{Q}^u\mathbf{L}$ be its Levi decomposition with $\mathbf{T} \subset \mathbf{L}$, and denote the parabolic subgroup of $\mathbf{G}$ opposite to $\mathbf{Q}$ relative to $\mathbf{L}$ by $\mathbf{P}$, so that $\mathbf{P} \cap \mathbf{Q} = \mathbf{L}$, and let $\mathbf{P} = \mathbf{P}^u\mathbf{L}$ be its Levi decomposition. Notice that both $\mathbf{P}^u$ and $(\mathbf{P}^u)^\sigma$ are connected, and following our convention, write P^u for $(\mathbf{P}^u)^\sigma$ and L for $\mathbf{L}^\sigma$.

The following local structure theorem describes the local structure of the real locus X, and will be essential for everything that follows.

Theorem 2. [ACF12, Theorem 1.22] *There exists a real algebraic L-subvariety Z of X such that*

(1) *The natural mapping*
$$P^u \times Z \to P^u \cdot Z$$
is a P^u-equivariant isomorphism;
(2) *each G_0-orbit in X contains points of the slice Z;*
(3) *the commutator (L, L) acts trivially on Z and the T-variety Z is isomorphic to $\mathbb{R}^r$ acted upon linearly by linearly independent characters of $\mathbf{T}$.*

□

Note that $P^u \cdot Z \simeq P^u \times Z$ is invariant under P, since $\mathbf{L}$ normalizes $\mathbf{P}^u$, so that
$$l \cdot (p, z) = (lpl^{-1}, lz) \in P^u \times Z \qquad \text{for } (p, z) \in P^u \times Z,\, l \in L.$$
By the first statement of Theorem 2, $P^u \cdot Z$ is an open subset of X isomorphic to $P^u \times \mathbb{R}^r$, and by the second, $G \cdot P^u \cdot Z = X$. We can therefore cover X by the G-translates
$$U_g := g \cdot U_e, \qquad U_e := P^u \cdot Z, \quad g \in G.$$
Consequently, there exists a real-analytic diffeomorphism
$$\varphi : \quad \mathbb{R}^d \supset \widetilde{U}_e \quad \longrightarrow \quad P^u \times Z \simeq P^u \cdot Z$$
and real-analytic diffeomorphisms φ_g
$$\varphi_g : \quad \mathbb{R}^d \supset \widetilde{U}_g \quad \xrightarrow{\varphi} \quad P^u \cdot Z \quad \xrightarrow{g} \quad g\, P^u \cdot Z, \qquad g \in G,$$
such that $\left\{(U_g, \varphi_g^{-1})\right\}_{g \in G}$ constitutes an atlas of X. More explicitly, if z_j denotes the j-th coordinate function on $Z \simeq \mathbb{R}^r$, and $p_1, \dots, p_k$ are coordinate functions on P^u, we write
$$\varphi_g^{-1} : U_g \ni x \longmapsto (p_1, \dots, p_k, z_1, \dots, z_r) = m \in \widetilde{U}_g := \varphi_g^{-1}(U_g).$$

Note that U_g is invariant under the subgroups gTg^{-1} and gP^ug^{-1}. Next, denote by

$$W = W(T) = N_G(T)/Z_G(T)$$

the Weyl group of G with respect to T, and write $(U_w, \varphi_w^{-1}) := (U_{n_w}, \varphi_{n_w}^{-1})$ for any element $w \in W$, $n_w \in N_G(T)$ being a representative of w. Note by definition of the Weyl group U_w is independent of n_w. Since $n_w T n_w^{-1} = T$, U_w carries a natural T-action. We shall now construct a more refined atlas for the class of wonderful $\mathbf{G}$-varieties introduced in Proposition 1. This atlas will be of crucial importance later.

Proposition 2. *Suppose that* $\mathbf{X}$ *is a wonderful* $\mathbf{G}$*-variety such that its* $\mathbf{T}$*-fixed-points are located on its closed* $\mathbf{G}$*-orbit. Then*

$$\left\{(U_w, \varphi_w^{-1})\right\}_{w\in W}$$

constitutes a finite atlas of X*.*

Proof. Let B^- denote the Borel subgroup of G such that $B \cap B^- = T$. The variety X has a unique projective G-orbit and, hence, a unique point fixed by B^- [ACF12]. This fixed point, denoted in the following by y_0, lies in the closed G-orbit by assumption. Next, let $\eta : s \mapsto (s^{a_1}, ..., s^{a_n})$, $a_i > 0$, be a morphism from $\mathbb{C}^*$ to the algebraic torus $\mathbf{T} \simeq (\mathbb{C}^*)^{\mathbf{r}}$, such that the set of $\mathbf{T}$-fixed-points in $\mathbf{X}$ coincides with the set of fixed points of $\{\eta(s)\}_{s\in\mathbb{C}^*}$ in $\mathbf{X}$. By [Bia73], there is a cell decomposition of $\mathbf{X}$ and, consequently, of X in terms of the sets

$$\{x \in X : \lim_{\mathbb{R}^*\ni s\to 0} \eta(s)\cdot x = y\},$$

where y runs over the set of T-fixed-points of X. Furthermore, the open subset $P^u \cdot Z \subset X$ is given by the cell

$$P^u \cdot Z = \{x \in X : \lim_{\mathbb{R}^*\ni s\to 0} \eta(s)\cdot x = y_0\},$$

see [ACF12] for details. By assumption, all T-fixed-points belong to the closed G-orbit of X. On the other side, it is well-known that the T-fixed-points of a projective G-orbit are indexed by the Weyl group W. More specifically, for each such y there exists a $w \in W$ such that $y = n_w y_0$ for any representative $n_w \in N_G(T)$ of w. Noticing that the aforementioned cells are just contained in the W-translates of $P^u \cdot Z$, one finally obtains the lemma. ☐

In what follows, we will always assume that the $\mathbf{T}$-fixed-points of $\mathbf{X}$ are located on its closed $\mathbf{G}$-orbit. Next, let $w \in W$ and $x \in U_w$, and denote by $V_{w,x} \subset G$ the set of $g \in G$ that leave U_w invariant. From the orbit structure and the analyticity of X one immediately deduces for $g \in V_{w,x}$

$$z_j(g\cdot x) = \chi_j(g,x) z_j(x), \tag{1}$$

where $\chi_j(g,x)$ is a function that is real-analytic in g and in x. Furthermore, one computes $1 = \chi_j(g^{-1}, g\cdot x)\cdot\chi_j(g,x)$, where $g^{-1} \in V_{w,gx}$. This implies

$$\chi_j(g,x) \neq 0 \qquad \forall\, x \in U_w, \quad g \in V_{w,x}, \tag{2}$$

since $\chi_j(g^{-1}, g\cdot x)$ is a finite complex number. We are interested in a more explicit description of the functions $\chi_j(g,x)$. For this, let $\gamma_1, \dots, \gamma_r$ be the characters of $\mathbf{T}$ mentioned in

Theorem 2. These weights are usually called the *spherical roots of* $\mathbf{X}$. The T-action on $Z \simeq \mathbb{R}^r$ is given explicitly by

$$t \cdot z = (\gamma_1(t)z_1, \dots, \gamma_r(t)z_r) \qquad \text{for all} \quad z = (z_1, \dots, z_r) \in Z \quad \text{and} \quad t \in T. \tag{3}$$

Corollary 1. *For $t \in T$, $j = 1, \dots, r$, and $x \in U_w$ we have*

$$z_j(t \cdot x) = \chi_j(t, x) z_j(x) = \gamma_j(n_w^{-1} t n_w) z_j(x)$$

where $n_w \in N_G(T)$ is a representative of w. Furthermore,

$$z_j(n_w u n_w^{-1} \cdot x) = z_j(x)$$

for any element $u \in P^u$.

Proof. The first assertion follows readily from (1) and the definition of the open sets U_w. Indeed, let $x = n_w p \cdot z \in U_w$ and $t \in T$. Then $t = n_w t_1 n_w^{-1}$ for some $t_1 \in T$ and

$$\varphi_{n_w}^{-1}(t \cdot x) = \varphi^{-1}(t_1 p \cdot z) = \varphi^{-1}(t_1 p t_1^{-1}, t_1 \cdot z),$$

so that the z_j-coordinate of $t \cdot x$ reads $\gamma_j(t_1) z_j(x)$. The second assertion follows directly from Theorem 2 -(1). □

For later reference, we still mention the following

Corollary 2. *Let $I \subset \{1, \dots, r\}$, and put*

$$z_I = (z_1, \dots, z_r) \in Z \quad \Longleftrightarrow \quad z_i \neq 0 \quad \textit{iff} \quad i \in I.$$

Then, for all $x \in X$ there exists a z_I such that

1. *$G \cdot x = G \cdot z_I$;*
2. *$P^u \times (T / \cap_{i \in I} \ker \gamma_i)$ acts locally transitively on $G \cdot z_I$.*

Proof. This is a direct consequence of Theorem 2. □

3. Microlocal analysis of integral operators on wonderful varieties

As in Section 2, let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbb{C}$ and (G, σ) a split real form of $\mathbf{G}$. Let $\mathbf{X}$ be a strict wonderful $\mathbf{G}$-variety of rank r, and X the real locus of $\mathbf{X}$ with respect to the canonical real structure on it. As before, let $\mathbf{Y} = \mathbf{G}/\mathbf{Q}$ be the unique closed $\mathbf{G}$-orbit of $\mathbf{X}$, and $\mathbf{P}$ the parabolic subgroup opposite to $\mathbf{Q}$. Let $\mathbf{P} = \mathbf{P}^u \mathbf{L}$ be its Levi decomposition, where $\mathbf{P}^u$ is the unipotent radical of $\mathbf{P}$ and $\mathbf{L}$ its Levi component. Furthermore, fix some maximal torus $\mathbf{T}$ of $\mathbf{G}$ contained in $\mathbf{Q}$, and assume that the $\mathbf{T}$-fixed-points of $\mathbf{X}$ are located on its closed $\mathbf{G}$-orbit. Consider now the Banach space $\mathrm{C}(X)$ of continuous, complex valued functions on X, equipped with the supremum norm, and let $(\pi, \mathrm{C}(X))$ be the corresponding continuous regular representation of G_0 given by

$$\pi(g)\varphi(x) = \varphi(g^{-1} \cdot x), \qquad \varphi \in \mathrm{C}(X).$$

The representation of the universal enveloping algebra $\mathfrak{U}$ of the Lie algebra $\mathfrak{g}$ of G on the space of smooth vectors $\mathrm{C}(X)_\infty$ will be denoted by $d\pi$. Also, we will consider the regular representation of G_0 on $\mathrm{C}^\infty(X)$ which, equipped with the topology of uniform convergence, becomes a Fréchet space. We will denote this representation by π as well. Let $(L, \mathrm{C}^\infty(G_0))$ be the left regular representation of G_0. Let θ be a Cartan involution on $\mathfrak{g}$. With respect to the left-invariant Riemannian metric on G_0 given by the modified Cartan-Killing form

$$\langle A, B\rangle_\theta = -\langle A, \theta B\rangle\,, \qquad A, B \in \mathfrak{g},$$

we denote by $d(g,h)$ the distance between two points $g, h \in G_0$, and set $|g| = d(g,e)$, where e is the identity element of G. A function f on G_0 is said to be of *at most of exponential growth*, if there exists a $\kappa > 0$ such that $|f(g)| \leq Ce^{\kappa|g|}$ for some constant $C > 0$. Let further d_{G_0} be a Haar measure on G_0. We introduce now a certain class of rapidly decreasing functions on G_0.

Definition 2. *A function $f \in \mathrm{C}^\infty(G_0)$ is called* rapidly decreasing *if it satisfies the following condition: For every $\kappa \geq 0$ and $H \in \mathfrak{U}$ there exists a constant $C > 0$ such that*

$$|dL(H)f(g)| \leq Ce^{-\kappa|g|}.$$

The space of rapidly decreasing functions on G_0 will be denoted by $\mathcal{S}(G_0)$.

Remark 1. 1) Note that $f \in \mathcal{S}(G_0)$ implies that for every $\kappa \geq 0$ and $H \in \mathfrak{U}$ one has

$$dL(H)f \in \mathrm{L}^1(G_0, e^{\kappa|g|}d_{G_0}).$$

Indeed, let $c > 0$ be such that $e^{-c|g|} \in \mathrm{L}^1(G_0, d_{G_0})$, and $\kappa \geq 0$ and $X \in \mathfrak{U}$ be given. Then $|e^{(\kappa+c)|g|}dL(X)f(g)| \leq C$ for all $g \in G_0$ and a suitable constant $C > 0$, so that

$$\left\| dL(X)fe^{\kappa|\cdot|}\right\|_{\mathrm{L}^1(G_0,d_{G_0})} \leq C\left\|e^{-c|\cdot|}\right\|_{\mathrm{L}^1(G_0,d_{G_0})} < \infty.$$

2) If $f \in \mathcal{S}(G_0)$, $dR(X)f \in S(G_0)$. Furthermore, if one compares the space $\mathcal{S}(G)$ with the Fréchet spaces $\mathcal{S}_{a,b}(G)$ defined in [Wal88, Section 7.7.1], where a and b are smooth, positive, K-bi-invariant functions on G satisfying certain properties, one easily sees that $a(g) = e^{|g|}$ and $b(g) = 1$ satisfy the selfsame properties, except for the smoothness at $g = e$ and the K-bi-invariance of a. The introduction of the space $\mathcal{S}(G)$ was motivated by the study of strongly elliptic operators and the decay properties of the semigroups generated by them [Ram06].

Consider next on $\mathrm{C}(X)$ for each $f \in \mathcal{S}(G_0)$ the continuous linear operator

$$\pi(f) = \int_{G_0} \pi(g)f(g)d_{G_0}.$$

Its restriction to $\mathrm{C}^\infty(X)$ induces a continuous linear operator

$$\pi(f) : \mathrm{C}^\infty(X) \longrightarrow \mathrm{C}^\infty(X) \subset \mathcal{D}'(X),$$

with Schwartz kernel given by the distribution section $\mathcal{K}_f \in \mathcal{D}'(X \times X, \mathbf{1} \boxtimes \Omega_X)$. Observe that the restriction of $\pi(f)\varphi$ to any of the G_0-orbits depends only on the restriction of $\varphi \in \mathrm{C}(X)$ to that orbit. Let X_0 be an open orbit in X. The main goal of this section is to disclose the microlocal structure of the operators $\pi(f)$, and characterize them as totally characteristic pseudodifferential operators on the manifold with corners $\overline{X_0}$. Recall that according to Melrose [Mel82] a continuous linear map

$$A : \mathrm{C}^\infty_c(M) \quad\longrightarrow\quad \mathrm{C}^\infty(M)$$

on a smooth manifold with corners M is called a *totally characteristic pseudodifferential operator or order* $l \in \mathbb{R}$ if it can be written locally as an oscillatory integral

$$Au(m) = \int e^{im\cdot\xi} a(m,\xi)\hat{u}(\xi)\, đ\xi, \qquad u \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^{n,k}),$$

where $\hat{u}$ denotes the Fourier transform of u and $\mathbb{R}^{n,k} = [0,\infty)^k \times \mathbb{R}^{n-k}$ the standard manifold with corners with $0 \le k \le n$ and coordinates $m = (m_1, \dots, m_k, m')$, while $đ\xi = (2\pi)^{-n}\, d\xi$. The amplitude a is supposed to be of the form $a(m,\xi) = \widetilde{a}(m, m_1\xi_1, \dots, m_k\xi_k, \xi')$, where $\widetilde{a}(m,\xi)$ is a symbol of order l satisfying the lacunary condition

$$\int e^{i(1-t)\xi_j} a(m,\xi)\, d\xi_j = 0 \qquad \text{for } t < 0 \text{ and } 1 \le j \le k.$$

For a more detailed exposition on totally characteristic pseudodifferential operators, the reader is referred to [PR12].

To begin with our analysis, choose for each $x \in X$ open neighbourhoods $U_x \subset U'_x$ of x contained in U_w for some $w \in W$ depending on x. Since X is compact, we can take a finite sub-cover of the open cover $\{U_x\}_{x\in X}$ to obtain a finite atlas $\left\{(U_\varrho, \varphi_\varrho^{-1})\right\}_{\varrho\in R}$ on X, where $\varphi_\varrho = \varphi_{w(\varrho)}$ for a suitable $w(\varrho) \in W$. Let $\{\alpha_\varrho\}_{\varrho\in R}$ be a partition of unity subordinate to this atlas, and let $\{\bar{\alpha}_\varrho\}_{\varrho\in R}$ be another set of functions satisfying $\bar{\alpha}_\varrho \in \mathrm{C}^\infty_{\mathrm{c}}(U'_\varrho)$ and $\bar{\alpha}_{\varrho|U_\varrho} \equiv 1$. Write $\widetilde{U}_\varrho := \varphi_\varrho^{-1}(U_\varrho) \subset \mathbb{R}^{k+r}$, and consider the localization of $\pi(f)$ with respect to the atlas above given by

$$A^\varrho_f u = [\pi(f)_{|U_\varrho}(u \circ \varphi_\varrho^{-1})] \circ \varphi_\varrho, \qquad u \in \mathrm{C}^\infty_{\mathrm{c}}(\widetilde{U}_\varrho).$$

Writing $m = (m_1, \dots, m_{k+r}) = (p,z) \in \widetilde{U}_\varrho$ we obtain

$$A^\varrho_f u(m) = \int_{G_0} f(g)[(u\circ\varphi_\varrho^{-1})\bar{\alpha}_\varrho](g^{-1}\cdot\varphi_\varrho(m))\, d_{G_0}(g) = \int_{G_0} f(g)c_\varrho(m,g)(u\circ\varphi^g_{w(\varrho)})(m) d_{G_0}(g),$$

where we put $c_\varrho(m,g) = \bar{\alpha}_\varrho(g^{-1}\cdot\varphi_\varrho(m))$ and $\varphi^g_w = \varphi_w^{-1} \circ g^{-1} \circ \varphi_w$. Note that with the notation of (1) we have

$$\varphi^g_w(m) = (p_1(g^{-1}\cdot x), \dots, p_k(g^{-1}\cdot x), z_1(x)\chi_1(g^{-1},x), \dots, z_r(x)\chi_r(g^{-1},x))$$

for $x = \varphi_w(p,z) \in U_w$, $g^{-1} \in V_{w,x}$. Next, define the functions

$$\hat{f}_\varrho(m,\xi) = \int_{G_0} e^{i\varphi^g_{w(\varrho)}(m)\cdot\xi} c_\varrho(m,g) f(g) d_{G_0}(g), \qquad a^\varrho_f(m,\xi) = e^{-ix\cdot\xi}\hat{f}_\varrho(m,\xi),$$

which are seen to belong to $\mathrm{C}^\infty(U_\varrho \times \mathbb{R}^{k+r})$ by differentiating under the integral. Let now T_m be the diagonal $(r \times r)$-matrix with entries $m_{k+1}, \dots, m_{k+r}$, and introduce the auxiliary symbol

(4)
$$\tilde{a}^\varrho_f(m,\xi) = a^\varrho_f(m, (\mathbf{1}_k \otimes T_m^{-1})\xi) = e^{-i(m_1,\dots,m_k,1,\dots,1)\cdot\xi} \int_{G_0} \psi^{w(\varrho)}_{\xi,m}(g^{-1}) c_\varrho(m,g) f(g) d_{G_0}(g)$$

where we put

$$\psi^w_{\xi,m}(g) = e^{i(p_1(g\cdot x),\dots,p_k(g\cdot x),\chi_1(g,x),\dots,\chi_r(g,x))\cdot\xi}.$$

Clearly, $\tilde{a}_f^\varrho(m,\xi) \in \mathrm{C}^\infty(U_\varrho \times \mathbb{R}^{k+r})$. Our next goal is to show that $\tilde{a}_f^\varrho(m,\xi)$ is a *lacunary* symbol of order $-\infty$. To key argument is contained in the following

Proposition 3. *Let $w \in W$ and (U_w, φ_w) be an arbitrary chart of X. Let further $\{\mathcal{P}_1, \dots, \mathcal{P}_k\}$ and $\{\mathcal{T}_1, \dots, \mathcal{T}_r\}$ be bases for* $\mathrm{Lie}\,(n_w P^u n_w^{-1})$ *and* $\mathrm{Lie}\,(T)$*, respectively, n_w being a representative of w. With $m = (p,z) \in \widetilde{U}_w$, $x = \varphi_w(m) \in U_w$, and $g \in V_{w,x}$ one has*

$$\begin{pmatrix} dL(\mathcal{P}_1)\psi^w_{\xi,m}(g) \\ \vdots \\ dL(\mathcal{T}_r)\psi^w_{\xi,m}(g) \end{pmatrix} = i\psi^w_{\xi,m}(g)\Gamma(m,g)\xi, \tag{5}$$

where

$$\Gamma(m,g) = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{pmatrix} = \left(\begin{array}{c|c} dL(\mathcal{P}_i)p_{j,x}(g) & dL(\mathcal{P}_i)\chi_j(g,x) \\ \hline dL(\mathcal{T}_i)p_{j,x}(g) & dL(\mathcal{T}_i)\chi_j(g,x) \end{array}\right) \tag{6}$$

belongs to $\mathrm{GL}(r+k,\mathbb{R})$*, and $p_{j,x}(g) = p_j(g\cdot x)$.*

Proof. Let m, x, g be as above. For $\mathcal{G} \in \mathfrak{g}$, one computes

$$\begin{aligned} dL(\mathcal{G})\psi^w_{\xi,m}(g) = \frac{d}{ds} e^{i(\mathbf{1}_k \otimes T_x^{-1})\varphi_w^{\mathrm{e}^{-s\mathcal{G}}\, g}(m)\cdot\xi}{}_{|s=0} = i\psi^w_{\xi,m}(g)\Big[\sum_{i=1}^k \xi_i dL(\mathcal{G})p_{i,x}(g) \\ + \sum_{j=1}^l \xi_{k+j} dL(\mathcal{G})\chi_j(g,x)\Big], \end{aligned}$$

showing the first equality. To see the invertibility of the matrix $\Gamma(m,g)$, note that for small $s \in \mathbb{R}$

$$\chi_j(\mathrm{e}^{-s\mathcal{G}}\, g, x) = \chi_j(g,x)\chi_j(\mathrm{e}^{-s\mathcal{G}}, g\cdot x).$$

Corollary 1 then yields that Γ_4 is non-singular. In the same way, the matrix Γ_1 is non-singular. Its $(ij)^{th}$ entry reads

$$dL(\mathcal{P}_i)p_{j,x}(g) = \frac{d}{ds} p_{j,x}(\mathrm{e}^{-s\mathcal{P}_i}\cdot g)_{|s=0} = (-\mathcal{P}_{i|X})_{g\cdot x}(p_j),$$

and the assertion follows from Corollary 2. On the other hand, Corollary 1 implies

$$dL(\mathcal{P}_i)\chi_j(g,x) = \chi_j(g,x)\frac{d}{ds}\Big(\chi_j(e^{-s\mathcal{P}_i}, g\cdot x)\Big)_{|s=0} = 0,$$

showing that Γ_2 is identically zero. Geometrically, this amounts to the fact that the fundamental vector field corresponding to $\mathcal{T}_j$ is transversal to the hypersurface defined by $z_j = \text{const} \in \mathbb{R}\setminus\{0\}$, while the vector fields corresponding to the Lie algebra elements $\mathcal{P}_r, \mathcal{T}_i$, $i \neq j$, are tangential. We therefore conclude that $\Gamma(m,g)$ is non-singular. ∎

We can now state the main result of this section.

Theorem 3. *Let X be the real locus of a strict wonderful variety $\boldsymbol{X}$. For $f \in \mathcal{S}(G_0)$, the operators $\pi(f)$ are locally of the form*

$$A^{\varrho}_f u(m) = \int e^{im\cdot\xi} a^{\varrho}_f(m,\xi)\hat{u}(\xi)d\xi, \qquad u \in \mathrm{C}^\infty_{\mathrm{c}}(\widetilde{U}_\varrho), \tag{7}$$

where $a^{\varrho}_f(m,\xi) = \tilde{a}^{\varrho}_f(m,\xi_1,\dots,\xi_k,m_{k+1}\xi_{k+1},\dots,\xi_{k+r}m_{k+r})$, and $\tilde{a}^{\varrho}_f(m,\xi) \in \mathrm{S}^{-\infty}_{\mathrm{la}}(\widetilde{U}_\varrho \times \mathbb{R}^{k+r}_\xi)$ is given by (4). In particular, the kernel of the operator A^{ϱ}_f is determined by its restrictions to $\widetilde{U}^*_\varrho \times \widetilde{U}^*_\varrho$, where $\widetilde{U}^*_\varrho = \{m = (p,t) \in \widetilde{U}_\varrho : t_1 \cdots t_r \neq 0\}$, and given by the oscillatory integral

$$K_{A^{\varrho}_f}(m,y) = \int e^{i(m-y)\cdot\xi} a^{\varrho}_f(m,\xi)d\xi. \tag{8}$$

Proof. The proof follows essentially the proof of [PR12, Theorem 2]. Indeed, as a consequence of Proposition 3 one computes that $\psi^w_{\xi,m}(g)$ can be written for arbitrary $N \in \mathbb{N}$ as

$$\psi^w_{\xi,m}(g) = (1+|\xi|^2)^{-N} \sum_{j=0}^{2N} \sum_{|\alpha|=j} b^N_\alpha(m,g) dL(\mathcal{G}^\alpha)\psi^w_{\xi,m}(g)$$

with suitable $\mathcal{G}^\alpha \in \mathfrak{U}$ and coefficients $b^N_\alpha(m,g)$ that are at most of exponential growth in g. Since $(\partial^\alpha_\xi \partial^\beta_m \tilde{a}^{\varrho}_f)(m,\xi)$ is given by a finite sum of terms of the form

$$\xi^{\beta'} e^{-i(m_1,\dots,m_k,1,\dots,1)\cdot\xi} \int_G f(g) d^\alpha_{\beta'\beta''}(m,g)\psi^w_{\xi,m}(g^{-1})(\partial^{\beta''}_m c_\varrho)(m,g) d_{G_0}(g),$$

the functions $d^\alpha_{\beta'\beta''}(m,g)$ being at most of exponential growth in g, we finally obtain for arbitrary α,β, and $N \in \mathbb{N}$ the estimate

$$|(\partial^\alpha_\xi \partial^\beta_m \tilde{a}^{\varrho}_f)(m,\xi)| \leq \frac{1}{(1+\xi^2)^N} C_{\alpha,\beta,\mathcal{K}} \qquad m \in \mathcal{K},$$

where $\mathcal{K}$ denotes an arbitrary compact set in U_ϱ. This proves that $\tilde{a}^{\varrho}_f(m,\xi)$ is a symbol of order $-\infty$. Since equation (7) follows immediately from the Fourier inversion formula, and the lacunarity of $\tilde{a}^{\varrho}_f(m,\xi)$ is a direct consequence of the orbit structure of X, the assertion of Theorem 3 follows. For further details we refer the reader to the proof of [PR12, Theorem 2]. □

As a consequence of the above theorem, one obtains the following

Corollary 3. *Let X_0 be an open G_0-orbit in X. Then the continuous linear operators*

$$\pi(f)_{|\overline{X_0}} : \mathrm{C}^\infty_{\mathrm{c}}(\overline{X_0}) \longrightarrow \mathrm{C}^\infty(\overline{X_0}),$$

are totally characteristic pseudodifferential operators of class $\mathrm{L}^{-\infty}_b$ on the manifold with corner $\overline{X_0}$.

□

Remark 2. Note that if in the previous corollary X_0 is a Riemannian symmetric space, then its closure $\overline{X_0}$ in X is the maximal Satake compactification of X_0, see Remark II.14.10, [BJ06].

As the most important consequence, Theorem 3 enables us to write the kernel of $\pi(f)$ locally in the form

$$
\begin{aligned}
K_{A_f^\varrho}(m,m') &= \int e^{i(m-m')\cdot\xi} a_f^\varrho(m,\xi) đ\xi = \int e^{i(m-m')\cdot(\mathbf{1}_k\otimes T_m^{-1})\xi} \tilde{a}_f^\varrho(m,\xi) \\
&\quad \cdot |\det(\mathbf{1}_k \otimes T_m^{-1})'(\xi)| đ\xi \\
&= \frac{1}{|m_{k+1}\cdots m_{k+r}|} \tilde{A}_f^\varrho(m, m_1-m_1', \dots, 1-\frac{m'_{k+1}}{m_{k+1}}, \dots),
\end{aligned} \tag{9}
$$

where $\tilde{A}_f^\varrho(m,y)$ denotes the inverse Fourier transform of the lacunary symbol $\tilde{a}_f^\varrho(m,\xi)$, and $m_{k+1}\cdots m_{k+r} \neq 0$. The restriction of the kernel of A_f^ϱ to the diagonal is given by

$$
K_{A_f^\varrho}(m,m) = \frac{1}{|m_{k+1}\cdots m_{k+r}|} \tilde{A}_f^\varrho(m,0), \qquad m_{k+1}\cdots m_{k+r} \neq 0.
$$

These restrictions yield a family of smooth functions $k_f^\varrho(x) = K_{A_f^\varrho}(\varphi_\varrho^{-1}(x), \varphi_\varrho^{-1}(x))$, which define a density k_f on the union of the open G_0-orbits on X. Nevertheless, the functions $k_f^\varrho(x)$ are not locally integrable on all of X, so that we cannot define a trace of $\pi(f)$ by integrating the density k_f over the diagonal $\Delta_{X\times X} \simeq X$. Instead, the explicit form of the local kernels (9) suggests a natural regularization of the integral operators $\pi(f)$, based on a classical result of Bernstein-Gelfand on the meromorphic continuation of complex powers.

Proposition 4. *Let $\{\alpha_\varrho\}$ be the partition of unity subordinate to the atlas $\{(U_\varrho, \varphi_\varrho^{-1})\}_{\varrho\in R}$. Let $f \in \mathcal{S}(G_0)$, $s \in \mathbb{C}$, and define for* $\operatorname{Re} s > 0$

$$
\begin{aligned}
\operatorname{Tr}_s \pi(f) &= \sum_\varrho \int_{\widetilde{U}_\varrho} (\alpha_\varrho \circ \varphi_\varrho)(m) |m_{k+1}\cdots m_{k+r}|^s \widehat{A}_f^\varrho(m,0) dm \\
&= \left\langle |m_{k+1}\cdots m_{k+r}|^s, \sum_\varrho (\alpha_\varrho\circ\varphi_\varrho) \widehat{A}_f^\varrho(\cdot,0) \right\rangle.
\end{aligned}
$$

Then $\operatorname{Tr}_s \pi(f)$ can be continued analytically to a meromorphic function in s with at most poles at $-1,-3,\dots$. Furthermore, for $s \in \mathbb{C} - \{-1,-3,\dots\}$,

$$
\Theta_\pi^s : \mathrm{C_c^\infty}(G) \ni f \mapsto \operatorname{Tr}_s \pi(f) \in \mathbb{C}
$$

defines a distribution density on G.

Proof. The proof is analogous to the proof of [PR12, Proposition 4]. In particular, the fact that $\operatorname{Tr}_s \pi(f)$ can be continued meromorphically is a consequence of the analytic continuation of $|m_{k+1}\cdots m_{k+r}|^s$ as a distribution in $\mathbb{R}^{k+r}$. □

Consider next the Laurent expansion of $\Theta_\pi^s(f)$ at $s=-1$. For this, let $u \in \mathrm{C_c^\infty}(\mathbb{R}^{k+r})$ be a test function, and consider the expansion

$$
\langle |m_{k+1}\cdots m_{k+r}|^s, u\rangle = \sum_{j=-l}^{\infty} S_j(u)(s+1)^j,
$$

where $S_k \in \mathcal{D}'(\mathbb{R}^{k+r})$. Since $|m_{k+1}\cdots m_{k+r}|^{s+1}$ has no pole at $s=-1$, we necessarily must have

$$|m_{k+1}\cdots m_{k+r}|\cdot S_j = 0 \quad \text{for } j<0, \qquad |m_{k+1}\cdots m_{k+r}|\cdot S_0 = 1$$

as distributions. Thus $S_0 \in \mathcal{D}'(\mathbb{R}^{k+r})$ represents a distributional inverse of $|m_{k+1}\cdots m_{k+r}|$. By the same arguments that led to Proposition 4 we arrive at the following

Proposition 5. *For $f \in \mathcal{S}(G)$, let the regularized trace of the operator $\pi(f)$ be defined by*

$$\operatorname{Tr}_{reg}\pi(f) = \left\langle S_0, \sum_\varrho (\alpha_\varrho \circ \varphi_\varrho)\tilde{A}_f^\varrho(\cdot, 0)\right\rangle.$$

Then $\Theta_\pi : \mathrm{C_c^\infty}(G) \ni f \mapsto \operatorname{Tr}_{reg}\pi(f) \in \mathbb{C}$ constitutes a distribution density on G, which is called the character of the representation *π.*

□

Remark 3. Alternatively, a similar regularized trace can be defined using the calculus of b-pseudodifferential operators developed by Melrose. For a detailed description, the reader is referred to [Loy98], Section 6.

In what follows, we shall identify distributions with distribution densities on G via the Haar measure d_G. Our next aim is to understand the distributions Θ_π^s and Θ_π in terms of the G-action on X. We shall actually show that on a certain open set of transversal elements, they are represented by locally integrable functions given in terms of fixed points. Similar expressions where derived by Atiyah and Bott [AB68] for the global character of an induced representation of G.

4. Character formulae

In what follows, we shall prove similar formulae for the distributions Θ_π and Θ_π^s defined in the previous section. Let the notation be as before, and denote by $\Phi_g(x) = g^{-1}\cdot x$ the action of an element $g \in G$ on X. Recall that Φ_g is called *transversal*, if all its fixed points are *simple*, meaning that $\det(\mathbf{1} - (d\Phi_g)_{x_0}) \neq 0$ for a fixed point $x_0 \in X$. Further note that the set $G(X) \subset G$ of elements acting transversally on X is open. We then have the following

Theorem 4. *Let $f \in \mathrm{C_c^\infty}(G)$ have support in $G(X)$, and $s \in \mathbb{C}$ be such that $\operatorname{Re} s > -1$. Let further $\operatorname{Fix}(X,g)$ denote the set of fixed points of an element $g \in G$ on X. Then*

(10)
$$\operatorname{Tr}_s \pi(f) = \int_{G(X)} f(g)\left(\sum_{x\in \operatorname{Fix}(X,g)}\sum_\varrho \frac{\alpha_\varrho(x)|m_{k+1}(\kappa_\varrho^{-1}(x))\cdots m_{k+r}(\kappa_\varrho^{-1}(x))|^{s+1}}{|\det(\mathbf{1} - d\Phi_g(x))|}\right) d_G(g).$$

In particular, $\Theta_\pi^s : \mathrm{C_c^\infty}(G) \ni f \to \operatorname{Tr}_s \pi(f) \in \mathbb{C}$ is regular on $G(X)$.

Proof. The proof is analogous to the proof of Theorem 7 in [PR12]. By Proposition 3,

$$\operatorname{Tr}_s \pi(f) = \sum_\varrho \int_{\widetilde{U}_\varrho} (\alpha_\varrho \circ \varphi_\varrho)(m)|m_{k+1}\cdots m_{k+r}|^s \widetilde{A}_f^\varrho(m,0)dm$$

is a meromorphic function in s with possible poles at $-1, -3, \dots,$ and we assume that $\operatorname{Re} s > -1$. Since $\widetilde{A}^\varrho_f(m,0) = \int \tilde{a}^\varrho_f(m,\xi) đ\xi$, where $\tilde{a}^\varrho_f(m,\xi) \in \mathrm{S}^{-\infty}_{la}(\widetilde{U}_\varrho \times \mathbb{R}^{k+r})$ is rapidly decaying in ξ by Theorem 3, the order of integration can be interchanged, yielding

$$\operatorname{Tr}_s \pi(f) = \sum_\varrho \int \int_{\widetilde{U}_\varrho} (\alpha_\varrho \circ \varphi_\varrho)(m) |m_{k+1} \cdots m_{k+r}|^s \tilde{a}^\varrho_f(m,\xi) dm\, đ\xi.$$

Let $\chi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^{k+r}, \mathbb{R}^+)$ be equal 1 in a neighborhood of 0, and $\varepsilon > 0$. Then, by Lebesgue's theorem on bounded convergence,

$$\operatorname{Tr}_s \pi(f) = \lim_{\varepsilon \to 0} I_\varepsilon,$$

where we set

$$I_\varepsilon = \sum_\varrho \int \int_{\widetilde{U}_\varrho} (\alpha_\varrho \circ \varphi_\varrho)(m) |m_{k+1} \cdots m_{k+r}|^s \tilde{a}^\varrho_f(m,\xi) \chi(\varepsilon\xi)\, dm\, đ\xi.$$

Interchanging the order of integration once more, one obtains with (4)

$$I_\varepsilon = \int_G f(g) \sum_\varrho \int \int_{\widetilde{U}_\varrho} e^{i\Psi_{w(\varrho)}(g^{-1},m)\cdot\xi} c_\varrho(m,g) (\alpha_\varrho \circ \varphi_\varrho)(m) |m_{k+1} \cdots m_{k+r}|^s \chi(\varepsilon\xi) dm\, đ\xi\, d_G(g),$$

everything being absolutely convergent, where we wrote

$$\begin{aligned}\Psi_w(g,m) &= [(\mathbf{1}_k \otimes T_m^{-1})(\varphi^g_w(m) - m)] \\ &= (m_1(g\cdot x) - m_1(x), \dots, m_k(g \cdot x) - m_k(x), \chi_1(g,x) - 1, \dots, \chi_r(g,x) - 1).\end{aligned}$$

Let us now define

$$I_\varepsilon(g) = f(g) \sum_\varrho \int \int_{\widetilde{U}_\varrho} e^{i\Psi_{w(\varrho)}(g^{-1},m)\cdot\xi} c_\varrho(m,g) (\alpha_\varrho \circ \varphi_\varrho)(m) |m_{k+1} \cdots m_{k+r}|^s \chi(\varepsilon\xi) dm\, đ\xi,$$

so that $I_\varepsilon = \int_G I_\varepsilon(g)\, d_G(g)$. In order to pass to the limit under the integral, we shall show that $\lim_{\varepsilon\to 0} I_\varepsilon(g)$ is an integrable function on G. Now, it is not difficult to see that, as $\varepsilon \to 0$, the main contributions to $I_\varepsilon(g)$ originate from the fixed points of g, which are also the fixed points of g^{-1}. To examine these contributions, note that due to the fact that all fixed points are simple, $m \mapsto \varphi^g_\varrho(m) - m$ defines a diffeomorphism near the fixed points. Performing the change of variables $y = m - \varphi^g_\varrho(m)$ one obtains

$$\lim_{\varepsilon\to 0} I_\varepsilon(g) = f(g) \sum_{x \in \operatorname{Fix}(X,g)} \sum_\varrho \frac{\alpha_\varrho(x) |m_{k+1}(\kappa_\varrho^{-1}(x)) \cdots m_{k+r}(\kappa_\varrho^{-1}(x))|^{s+1}}{|\det(\mathbf{1} - d\Phi_g(x))|}.$$

The limit function $\lim_{\varepsilon\to 0} I_\varepsilon(g)$ is therefore clearly integrable on G for $\operatorname{Re} s > -1$. Passing to the limit under the integral then yields

$$\begin{aligned}\operatorname{Tr}_s \pi(f) &= \lim_{\varepsilon\to 0} I_\varepsilon = \lim_{\varepsilon \to 0} \int_G I_\varepsilon(g)\, d_G(g) = \int_G \lim_{\varepsilon\to 0} \big(I^{(1)}_\varepsilon + I^{(2)}_\varepsilon\big)(g) d_G(g) \\ &= \int_G f(g) \sum_{x \in \operatorname{Fix}(X,g)} \sum_\varrho \frac{\alpha_\varrho(x) |m_{k+1}(\kappa_\varrho^{-1}(x)) \cdots m_{k+r}(\kappa_\varrho^{-1}(x))|^{s+1}}{|\det(1 - d\Phi_g(x))|}\, d_G(g).\end{aligned}$$

The assertion of the theorem now follows. $\square$

From the previous theorem it is now clear that if $f \in \mathrm{C}_c^\infty(G(X))$, $\mathrm{Tr}_s\,\pi(f)$ is not singular at $s=-1$. Consequently, we obtain

Corollary 4. *Let $f \in \mathrm{C}_c^\infty(G)$ have support in $G(X)$. Then*

$$\mathrm{Tr}_{reg}\,\pi(f) = \mathrm{Tr}_{-1}\,\pi(f) = \int_{G(X)} f(g) \sum_{x\in \mathrm{Fix}(X,g)} \frac{1}{|\det(1-d\Phi_g(x))|}\, d_G(g).$$

In particular, the distribution $\Theta_\pi : f \to \mathrm{Tr}_{reg}(f)$ is regular on $G(X)$.

Proof. By (10), $\mathrm{Tr}_s\,\pi(f)$ has no pole at $s=-1$. Therefore, the Laurent expansion of $\Theta^s_\pi(f)$ at $s=-1$ must read

$$\mathrm{Tr}_s\,\pi(f) = \left\langle |m_{k+1}\cdots m_{k+r}|^s, \sum_\varrho (\alpha_\varrho \circ \varphi_\varrho)\widehat{A}^\varrho_f(\cdot,0)\right\rangle = \sum_{j=0}^\infty S_j\Big(\sum_\varrho (\alpha_\varrho\circ\varphi_\varrho)\widehat{A}^\varrho_f(\cdot,0)\Big)(s+1)^j,$$

where $S_k \in \mathcal{D}'(\mathbb{R}^{k+r})$. Thus,

$$\mathrm{Tr}_{-1}\,\pi(f) = \left\langle S_0, \sum_\varrho (\alpha_\varrho \circ \varphi_\varrho)\widehat{A}^\varrho_f(\cdot,0)\right\rangle = \mathrm{Tr}_{reg}\,\pi(f),$$

and the assertion follows with the previous theorem. □

Corollary 4 implies that $\mathrm{Tr}_{reg}\,\pi(f)$ is invariantly defined. Furthermore, interpreting $\pi(g)$ as a geometric endomorphism on the trivial bundle $E = X\times\mathbb{C}$ over X, a flat trace $\mathrm{Tr}^\flat\,\pi(g)$ of $\pi(g)$ can be defined. As it turns out [AB67],

$$\mathrm{Tr}^\flat\,\pi(g) = \sum_{x\in\mathrm{Fix}(X,g)} \frac{1}{|\det(1-d\Phi_g(x))|},$$

so that we finally obtain

$$\mathrm{Tr}_{reg}\,\pi(f) = \int_{G(X)} f(g)\,\mathrm{Tr}^\flat\,\pi(g) d_G(g), \qquad f\in \mathrm{C}_c^\infty(G(X)).$$

References

[AB67] M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes: I*, Ann. of Math. **86** (1967), 374–407.

[AB68] M.F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes: II. Applications*, Ann. of Math. **88** (1968), 451–491.

[ACF12] D. Akhiezer and S. Cupit-Foutou, *On the canonical real structure on wonderful varieties*, arXiv:1202.6607, to be published in Crelle's Journal, 2012.

[BCF10] P. Bravi and S. Cupit-Foutou, *Classification of strict wonderful varieties*, Annales de l'institut Fourier **60** (2010), no. 2, 641–681.

[Bia73] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups.*, Ann. Math. (2) **98** (1973), 480–497.

[BJ06] A. Borel and L. Ji, *Compactifications of symmetric and locally symmetric spaces*, Birkhäuser Boston Inc., Boston, 2006.

[Loy98] P. Loya, *On the b-pseudodifferential calculus on manifolds with corners*, PhD thesis, 1998.

[Lun96] D. Luna, *Toute variété magnifique est sphérique*, Transform. Groups **1** (1996), no. 3, 249–258.

[Mel82] R. Melrose, *Transformation of boundary problems*, Acta Math. **147** (1982), 149–236.

[PR12] A. Parthasarathy and P. Ramacher, *Integral operators on the Oshima compactification of a Riemannian symmetric spaces of non-compact type*, arXiv: 1102.5069 and 1106.0482, 2012.

[Ram06] P. Ramacher, *Pseudodifferential operators on prehomogeneous vector spaces*, Comm. Partial Diff. Eqs. **31** (2006), 515–546.

[Res10] N. Ressayre, *Spherical homogeneous spaces of minimal rank*, Advances in Mathematics **224** (2010), no. 5, 1784–1800.

[Wal88] N. R. Wallach, *Real reductive groups*, vol. I, Academic Press, Inc., 1988.