ADDENDUM TO
"SINGULAR EQUIVARIANT ASYMPTOTICS AND WEYL'S LAW"

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ABSTRACT. Let M be a closed Riemannian manifold carrying an effective and isometric action of a compact connected Lie group G. We derive a refined remainder estimate in the stationary phase approximation of certain oscillatory integrals on $T^*M \times G$ with singular critical sets that were examined in [7] in order to determine the asymptotic distribution of eigenvalues of an invariant elliptic operator on M. As an immediate consequence, we deduce from this an asymptotic multiplicity formula for families of irreducible representations in $\mathrm{L}^2(M)$. In forthcoming papers, the improved remainder will be used to prove an equivariant semiclassical Weyl law [4] and a corresponding equivariant quantum ergodicity theorem [5].

CONTENTS

1. INTRODUCTION

Let M be a compact n-dimensional Riemannian manifold without boundary, carrying an isometric and effective action of a connected compact Lie group G with Lie algebra $\mathfrak{g}$. In the study of the spectral geometry of M one is led to an examination of the asymptotic behavior of oscillatory integrals of the form

$$(1.1) \qquad I(\mu) := \int_{T^*U} \int_G e^{i\mu\Phi(x,\xi,g)} a_\mu(x,\xi,g)\, dg\, d\,(T^*U)\,(x,\xi), \qquad \mu \to +\infty,$$

where (γ, U) denotes a chart on M, $a_\mu \in \mathrm{C}_c^\infty(T^*U \times G)$ an amplitude that might depend on the parameter $\mu > 0$, and the phase function is given by

$$\Phi(x,\xi,g) := \langle \gamma(x) - \gamma(g \cdot x), \xi \rangle, \qquad (x,\xi) \in T_x^*U,\ g \in G,$$

see [7, 6]. Here dg stands for the normalized Haar measure on G, and $d(T^*U)$ for the canonical symplectic volume form of the co-tangent bundle of U, which coincides with the Riemannian volume form given by the Sasaki metric on T^*U. It is assumed that $(x,\xi,g) \in \operatorname{supp} a_\mu$ implies $g \cdot x \in U$, where we wrote (x,ξ) for an element in $T^*U \simeq U \times \mathbb{R}^n$ with respect to the canonical trivialization of the co-tangent bundle over the chart domain. The phase function Φ represents a global analogue of the momentum map $\mathbb{J} : T^*M \to \mathfrak{g}^*$ of the Hamiltonian action of G on T^*M, and oscillatory integrals with phase function given by the latter have been examined in [8] in the context of equivariant cohomology. The critical set of Φ is given by

$$\operatorname{Crit}(\Phi) = \left\{ (x,\xi,g) \in T^*U \times G : (\Phi_*)_{(x,\xi,g)} = 0 \right\} = \mathcal{C} \cap (T^*U \times G),$$

where

$$\mathcal{C} := \{ (x,\xi,g) \in \Omega \times G :\ g \cdot (x,\xi) = (x,\xi) \}, \qquad \Omega := \mathbb{J}^{-1}(0).$$

Now, unless the G-action on T^*M is free, the momentum map $\mathbb{J}$ is not a submersion, so that the zero set Ω of the momentum map and the critical set of Φ are not smooth manifolds. The stationary phase theorem can therefore not immediately be applied to the integrals $I(\mu)$. Nevertheless, it was shown in [7] that by constructing a strong resolution of the set $\mathcal{N} := \{(x,g) \in M \times G : g \cdot x = x\}$ a partial desingularization $\mathcal{Z} : \widetilde{\mathbf{X}} \to \mathbf{X} := T^*M \times G$ of the critical set $\mathcal{C}$ can be achieved, and applying the stationary phase theorem in the resolution space $\widetilde{\mathbf{X}}$ an asymptotic description of $I(\mu)$ can be obtained. Indeed, the map $\mathcal{Z}$ yields a partial monomialization of the local ideal $I_\Phi = (\Phi)$ generated by the phase function Φ according to

$$\mathcal{Z}^*(I_\Phi) \cdot \mathcal{E}_{\tilde{x},\widetilde{\mathbf{X}}} = \prod_j \sigma_j^{l_j} \cdot \mathcal{Z}_*^{-1}(I_\Phi) \cdot \mathcal{E}_{\tilde{x},\widetilde{\mathbf{X}}},$$

where $\mathcal{E}_{\widetilde{\mathbf{X}}}$ denotes the structure sheaf of rings of $\widetilde{\mathbf{X}}$, $\mathcal{Z}^*(I_\Phi)$ the total transform, and $\mathcal{Z}_*^{-1}(I_\Phi)$ the weak transform of I_Φ, while the σ_j are local coordinate functions near each $\tilde{x} \in \widetilde{\mathbf{X}}$ and the l_j natural numbers. As a consequence, the phase function factorizes locally according to $\Phi \circ \mathcal{Z} \equiv \prod \sigma_j^{l_j} \cdot \tilde{\Phi}^{wk}$, and one shows that the weak transforms $\tilde{\Phi}^{wk}$ have clean critical sets. Asymptotics for the integrals $I(\mu)$ are then obtained by pulling them back to the resolution space $\widetilde{\mathbf{X}}$, and applying the stationary phase theorem to the $\tilde{\Phi}^{wk}$ with the variables σ_j as parameters. More precisely, let $\operatorname{Reg} \mathcal{C} \subset T^*M \times G$ denote the regular part of $\mathcal{C}$, and regard it as a Riemannian submanifold with Riemannian metric induced by the product metric of the Sasaki metric on T^*M and some left-invariant Riemannian metric on G. The corresponding induced Riemannian volume form will be denoted by $d(\operatorname{Reg} \mathcal{C})$. It was then shown in [7, Theorem 9.1] that for a μ-independent amplitude $a \in \mathrm{C}_{\mathrm{c}}^\infty(T^*U \times G)$ one has the asymptotic formula

$$I(\mu) = \left(\frac{2\pi}{\mu}\right)^\kappa \int_{\operatorname{Reg} \mathcal{C}} \frac{a(x,\xi,g)}{\left|\det \Phi''(x,\xi,g)_{|N_{(x,\xi,g)}\operatorname{Reg}\mathcal{C}}\right|^{1/2}} \, d(\operatorname{Reg}\mathcal{C})(x,\xi,g) + \mathfrak{R}(\mu),$$

where κ stands for the dimension of a G-orbit of principal type in M, $\Phi''(x,\xi,g)_{|N_{(x,\xi,g)}\operatorname{Reg}\mathcal{C}}$ denotes the restriction of the Hessian of Φ to the normal space of $\operatorname{Reg}\mathcal{C}$ inside $T^*U \times G$ at the point (x,ξ,g), and the remainder satisfies the estimate

$$(1.2) \qquad\qquad\qquad \mathfrak{R}(\mu) = O(\mu^{-\kappa-1}(\log\mu)^{\Lambda-1}),$$

Λ being the maximal number of elements of a totally ordered subset of the set of isotropy types of the G-action on M.

The goal of this note is to extend the asymptotic formula above to amplitudes $a_\mu \in \mathrm{C}_{\mathrm{c}}^\infty(T^*U \times G)$ that depend on μ, and derive a refined remainder estimate of the form

$$(1.3) \qquad\qquad\qquad |\mathfrak{R}(\mu)| \leq C \sup_{l \leq 2\kappa+3} \left\| D^l a_\mu \right\|_\infty \mu^{-\kappa-1}(\log\mu)^{\Lambda-1}$$

with a constant $C > 0$ and differential operators D^l of order l independent of μ and a_μ, where $\|\cdot\|_\infty$ denotes the supremum norm. This is accomplished in Theorem 2.1.

Integrals of the form $I(\mu)$ with μ-dependent amplitudes occur in several contexts, and the bound (1.3) allows a precise control of the contributions of such amplitudes to the remainder $\mathfrak{R}(\mu)$. As an immediate consequence of Theorem 2.1 we are able to prove an asymptotic multiplicity formula for families of unitary irreducible G-representations in the Hilbert space $\mathrm{L}^2(M)$ of square integrable functions on M, see Theorem 3.2, generalizing the Weyl law for the reduced spectral counting function of an invariant elliptic operator proven in [7] to families of representations whose cardinality is allowed to grow in a moderate way as the energy goes to infinity. Further applications will be given in [4] and [5], where the refinement (1.3) will be crucial to prove a sharpened equivariant semiclassical Weyl law and a corresponding equivariant quantum ergodicity theorem, which otherwise could not be obtained.

2. Refined singular equivariant asymptotics

Let the notation and assumptions be as in the introduction. The main purpose of this note is to prove the following

Theorem 2.1. *Let $\mathcal{K} \subset T^*U \times G$ be a compactum and $a_\mu \in C_c^\infty(T^*U \times G)$ a family of amplitudes such that for each $\mu > 0$ the support of a_μ is contained in $\mathcal{K}$. Then, as $\mu \to +\infty$ one has the asymptotic formula*

$$
(2.1) \quad
\left| I(\mu) - \left(\frac{2\pi}{\mu}\right)^\kappa \int_{Reg\,\mathcal{C}} \frac{a_\mu(x,\xi,g)}{\left| \det \Phi''(x,\xi,g)_{|N_{(x,\xi,g)}Reg\,\mathcal{C}} \right|^{1/2}} \, d(Reg\mathcal{C})(x,\xi,g) \right|
$$
$$
\leq C \sup_{l \leq 2\kappa+3} \left\| D^l a_\mu \right\|_\infty \mu^{-\kappa-1} (\log \mu)^{\Lambda-1},
$$

where the differential operators D^l and the constant $C > 0$ do not depend on μ nor on a_μ.

Remark 2.2. Note that both the derivatives and the supports of the amplitudes a_μ have to satisfy suitable assumptions about the way they depend on μ in order to obtain meaningful asymptotics.

Proof. For μ-independent amplitudes the statement of the theorem is essentially the content of [7, Theorem 9.1]. In case of a μ-dependent amplitude a_μ, the precise dependence of the remainder estimate on the amplitudes a_μ has to be taken into account. For this, let us recall the main arguments and results from [7], and consider the decomposition of M into orbit types

$$
(2.2) \quad M = M(H_1) \,\dot\cup\, \cdots \,\dot\cup\, M(H_L),
$$

where we suppose that the isotropy types $(H_1), \ldots, (H_L)$ are numbered in such a way that $(H_i) \geq (H_j)$ implies $i \leq j$, compare Figure 2.1. One then constructs a partial desingularization

$$
(2.3) \quad \mathcal{Z} : \widetilde{\mathbf{X}} \to \mathbf{X} := T^*M \times G
$$

of the critical set $\mathcal{C}$ as follows. For each $1 \leq N \leq \Lambda - 1$ and each maximal, totally ordered subset $\{(H_{i_1}), \ldots, (H_{i_N})\}$ of non-principal isotropy types one constructs sequences of consecutive local blow-ups $\mathcal{Z}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}$ whose respective centers are given by disjoint unions over maximal singular isotropy bundles labeled by the types $\{(H_{i_1}), \ldots, (H_{i_N})\}$, and are realized in a set of local charts labeled by
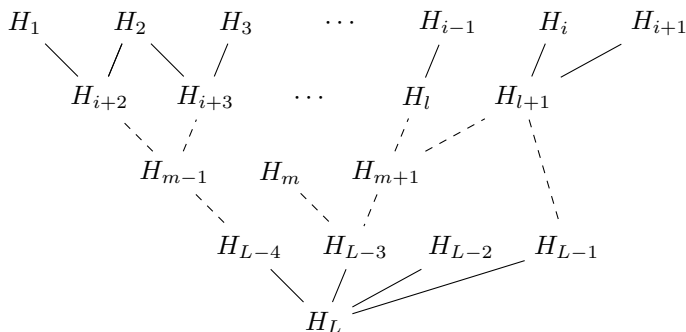

FIGURE 2.1. An isotropy tree corresponding to the decomposition (2.2). A line between two subgroups indicates partial ordering.

the indices $\varrho_{i_1}, \ldots, \varrho_{i_N}$, see [7, Eq. (6.7)]. The global morphism induced by the local transformations $\mathcal{Z}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}$ is then denoted by (2.3). For the precise construction, the reader is referred to [7, Beginning of Section 9 and Section 5]. Now, if we pull the phase function Φ back along the maps $\mathcal{Z}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}$, it factorizes locally according to

$$
\Phi \circ \mathcal{Z}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} = {}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{tot} = \tau_{i_1}(\sigma) \ldots \tau_{i_N}(\sigma) \, {}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk},
$$

where the τ_{i_j} are monomials in the exceptional parameters $\sigma_{i_1}, \ldots, \sigma_{i_N}$ given by the sequence of local quadratic transformations

$$\delta_{i_1 \ldots i_N} : (\sigma_{i_1}, \ldots \sigma_{i_N}) \mapsto \sigma_{i_1}(1, \sigma_{i_2}, \ldots, \sigma_{i_N}) = (\sigma'_{i_1}, \ldots, \sigma'_{i_N}) \mapsto \sigma'_{i_2}(\sigma'_{i_1}, 1, \ldots, \sigma'_{i_N}) = (\sigma''_{i_1}, \ldots, \sigma''_{i_N})$$

$$\mapsto \sigma''_{i_3}(\sigma''_{i_1}, \sigma''_{i_2}, 1, \ldots, \sigma''_{i_N}) = \cdots \mapsto \cdots = (\tau_{i_1}, \ldots, \tau_{i_N}).$$

On the other hand, if we transform the oscillatory integral $I(\mu)$ under the global morphism $\mathcal{Z}$ using suitable partitions of unity we obtain the decomposition

$$(2.4) \qquad I(\mu) = \sum_{N=1}^{\Lambda-1} \sum_{\substack{i_1 < \cdots < i_N \\ \varrho_{i_1}, \ldots, \varrho_{i_N}}} I_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}(\mu) + \sum_{N=1}^{\Lambda-1} \sum_{\substack{j_1 < \cdots < j_{N-1} < L \\ \varrho_{j_1}, \ldots, \varrho_{j_{N-1}}}} I_{j_1 \ldots j_{N-1}L}^{\varrho_{j_1} \cdots \varrho_{j_{N-1}}}(\mu) + R(\mu),$$

where the first term is a sum over maximal, totally ordered subsets of non-principal isotropy types, while the second term is given by a sum over arbitrary, totally ordered subsets of non-principal isotropy types, and $R(\mu)$ denotes certain non-stationary contributions, see [7, Eq. (9.1)]. Here

$$I_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}(\mu) := \int_{(-1,1)^N} J_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} \left(\mu \cdot \tau_{i_1}(\sigma) \cdots \tau_{i_N}(\sigma) \right) \prod_{j=1}^N |\tau_{i_j}(\sigma)|^{e^{(i_j)}-1} |\det D\delta_{i_1 \ldots i_N}(\sigma)| \, d\sigma,$$

and

$$J_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}(\nu) := \int_{\widetilde{\mathbf{X}}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}} e^{i\nu \, {}^{(i_1 \ldots i_N)}\widetilde{\Phi}_\sigma^{wk}} \, a_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} \, \mathcal{J}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}},$$

where the $\widetilde{\mathbf{X}}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}$ are suitable submanifolds in the resolution space $\widetilde{\mathbf{X}}$ of codimension N, the amplitudes $a_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}$ are compactly supported and given by pullbacks of a_μ multiplied by elements of partitions of unity, the $\mathcal{J}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}$ are Jacobians independent of μ, and the $e^{(i_j)}$ are natural numbers satisfying $e^{(i_j)} - 1 \geq \kappa$, see [7, Section 8, in particular Lemma 8.1]. Besides, we introduced the new parameter

$$\nu := \mu \cdot \tau_{i_1}(\sigma) \cdots \tau_{i_N}(\sigma),$$

and wrote ${}^{(i_1 \ldots i_N)}\widetilde{\Phi}_\sigma^{wk}$ for the weak transform ${}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk}$ regarded as a function on $\widetilde{\mathbf{X}}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}$, while the variables $\sigma = (\sigma_{i_1}, \ldots, \sigma_{i_N})$ are regarded as parameters. It can then be shown that the weak transforms ${}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk}$ have clean critical sets on $(-1,1)^N \times \widetilde{\mathbf{X}}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}$ of codimension 2κ [7, Theorems 6.1 and 7.1]. By transversality, this implies that the phase functions ${}^{(i_1 \ldots i_N)}\widetilde{\Phi}_\sigma^{wk}$ have clean critical sets, too. Similarly,

$$I_{j_1 \ldots j_{N-1}j_N}^{\varrho_{j_1} \cdots \varrho_{j_{N-1}}}(\mu) := \int_{(-1,1)^{N-1}} J_{j_1 \ldots j_{N-1}j_N}^{\varrho_{j_1} \cdots \varrho_{j_{N-1}}} \left(\mu \cdot \tau_{j_1}(\sigma) \cdots \tau_{j_{N-1}}(\sigma) \right) \prod_{k=1}^{N-1} |\tau_{j_k}(\sigma)|^{e^{(j_k)}-1} |\det D\delta_{j_1 \ldots j_{N-1}}(\sigma)| \, d\sigma,$$

where now

$$J_{j_1 \ldots j_{N-1}j_N}^{\varrho_{j_1} \cdots \varrho_{j_{N-1}}}(\nu) := \int_{\widetilde{\mathbf{X}}_{j_1 \ldots j_{N-1}}^{\varrho_{j_1} \cdots \varrho_{j_{N-1}}}(H_{j_N})} e^{i\nu \, {}^{(j_1 \ldots j_{N-1})}\widetilde{\Phi}_\sigma^{wk}} \, a_{j_1 \ldots j_{N-1}}^{\varrho_{j_1} \cdots \varrho_{j_{N-1}}} \, \mathcal{J}_{j_1 \ldots j_{N-1}}^{\varrho_{j_1} \cdots \varrho_{j_{N-1}}} \, \chi_{j_1 \ldots j_{N-1}j_N},$$

$\chi_{j_1 \ldots j_{N-1}j_N}$ being a cut-off-function with compact support in $\widetilde{\mathbf{X}}_{j_1 \ldots j_{N-1}}^{\varrho_{j_1} \cdots \varrho_{j_{N-1}}}(H_{j_N})$. In case that $j_N = L$, ${}^{(j_1 \ldots j_{N-1})}\widetilde{\Phi}^{wk}$ has a clean critical set on $(-1,1)^{N-1} \times \widetilde{\mathbf{X}}_{j_1 \ldots j_{N-1}}^{\varrho_{j_1} \cdots \varrho_{j_{N-1}}}(H_L)$ of codimension 2κ, compare [7, Lemma 7.3].

Now, let us proceed to the proof of (2.1), for which we intend to apply the stationary phase principle to the partially factorized phase functions. In order to deal with the competing asymptotics $\mu \to \infty$

and $\tau_{i_j} \to 0$, we define for a sufficiently small $\varepsilon > 0$ to be chosen later the integrals

$$^1 I_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}(\mu) := \int_{\varepsilon < |\tau_{i_j}(\sigma)| < 1} J_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}\left(\mu \cdot \tau_{i_1}(\sigma) \cdots \tau_{i_N}(\sigma)\right) \prod_{j=1}^{N} |\tau_{i_j}(\sigma)|^{e^{(i_j)}-1} |\det D\delta_{i_1 \ldots i_N}(\sigma)| \, d\sigma,$$

$$^2 I_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}(\mu) := \int_{0 < |\tau_{i_j}(\sigma)| < \varepsilon} J_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}\left(\mu \cdot \tau_{i_1}(\sigma) \cdots \tau_{i_N}(\sigma)\right) \prod_{j=1}^{N} |\tau_{i_j}(\sigma)|^{e^{(i_j)}-1} |\det D\delta_{i_1 \ldots i_N}(\sigma)| \, d\sigma.$$

Now, since the desingularization (2.3) of the critical set $\mathcal{C}$ is based on a strong resolution of the set $\mathcal{N} = \{(x, g) \in M \times G : g \cdot x = x\}$, the variable ξ is not affected by the resolution process, so that $\widetilde{\mathbf{X}}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} = \widetilde{\mathbf{Y}}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} \times \mathbb{R}_\xi^n$, where the $\widetilde{\mathbf{Y}}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}$ are sub-manifolds with compact closure, and consequently of finite volume, compare [7, Eq. (8.1)]. It is then immediate that

(2.5)

$$\left| {}^2 I_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}(\mu) \right| \leq \int_{(-\varepsilon, \varepsilon)^N} \left\| a_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} \, \mathcal{J}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} \right\|_\infty \mathrm{vol}_{\widetilde{\mathbf{X}}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}}(\mathrm{supp} \, a_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}) \prod_{j=1}^{N} |\tau_{i_j}|^{e^{(i_j)}-1} \, d\tau_{i_N} \ldots d\tau_{i_1}$$

$$\leq C \, \|a_\mu\|_\infty \int_{(-\varepsilon, \varepsilon)^N} \prod_{j=1}^{N} |\tau_{i_j}|^\kappa \, d\tau_{i_N} \ldots d\tau_{i_1} = \frac{2C}{\kappa + 1} \|a_\mu\|_\infty \, \varepsilon^{N(\kappa + 1)}$$

for some constant $C > 0$ independent of μ and a_μ, since we assumed that $\mathrm{supp} \, a_\mu \subset \mathcal{K}$ for all $\mu > 0$. Let us now turn to the integrals $^1 I_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}(\mu)$. As in [7, Theorem 8.2], the generalized stationary phase theorem [7, Theorem 4.1] yields for fixed $\sigma = (\sigma_{i_1}, \ldots, \sigma_{i_N})$ and arbitrary $\widetilde{N} \in \mathbb{N}$ the asymptotic expansion

(2.6)
$$J_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}(\nu) = (2\pi |\nu|^{-1})^\kappa \sum_{j=0}^{\widetilde{N}-1} Q_{j,\sigma} |\nu|^{-j} + R_{\widetilde{N}, \sigma}(\nu),$$

together with the explicit estimates

$$|Q_{j,\sigma}| \leq \widetilde{C}_{j,(i_1 \ldots i_N) \widetilde{\Phi}_\sigma^{wk}} \mathrm{vol}_{\widetilde{\mathcal{C}}_\sigma}(\mathrm{supp} \, a_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} \cap \widetilde{\mathcal{C}}_\sigma) \sup_{l \leq 2j} \left\| D^l (a_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} \, \mathcal{J}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}) \right\|_\infty,$$

$$|R_{\widetilde{N}, \sigma}(\nu)| \leq C_{\widetilde{N}, (i_1 \ldots i_N) \widetilde{\Phi}_\sigma^{wk}} \mathrm{vol}_{\widetilde{\mathbf{X}}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}}(\mathrm{supp} \, a_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}) \sup_{l \leq 2\kappa + 2\widetilde{N}+1} \left\| D^l (a_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} \, \mathcal{J}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}) \right\|_\infty \mu^{-\kappa - \widetilde{N}},$$

where we wrote $\widetilde{\mathcal{C}}_\sigma = \mathrm{Crit} \, {}^{(i_1 \ldots i_N)} \widetilde{\Phi}_\sigma^{wk}$. Moreover, the constants $\widetilde{C}_{j,(i_1 \ldots i_N) \widetilde{\Phi}_\sigma^{wk}}$ and $C_{\widetilde{N}, (i_1 \ldots i_N) \widetilde{\Phi}_\sigma^{wk}}$ are essentially bounded from above by

(2.7)
$$\sup_{\mathrm{supp} \, a_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} \cap \widetilde{\mathcal{C}}_\sigma} |\det \mathrm{Hess} \, {}^{(i_1 \ldots i_N)} \widetilde{\Phi}_\sigma^{wk}{}_{|N\widetilde{\mathcal{C}}}|^{-1/2} \cdot \left\| (\mathrm{Hess} \, {}^{(i_1 \ldots i_N)} \widetilde{\Phi}_\sigma^{wk}{}_{|N\widetilde{\mathcal{C}}})^{-1} \right\|^r$$

with $r = j, \widetilde{N}$, respectively, compare [7, Remark 4.2]. Now, the main consequence to be drawn from [7, Theorems 6.1 and 7.1] is that these bounds are uniform in σ, see [7, Remark 8.3 and proof of Theorem 8.2]. On the other hand, the amplitude $a_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}$ is given in terms of a_μ, so that the bounds (2.7) still depend on μ. But since the desingularization (2.3) does not affect the variable ξ, the weak transform $^{(i_1 \ldots i_N)} \widetilde{\Phi}_\sigma^{wk}$ depends linearly on ξ, and its tranversal Hessian will depend linearly on ξ as well, see [7, Page 54]. From this one concludes that (2.7) is uniformly bounded for all σ *and* μ, since $\xi \in \mathbb{R}^n$ is the only variable in the resolution space with a non-pre-compact domain of definition. Taking into

account that supp $a_\mu \subset \mathcal{K}$ for all $\mu > 0$, integration of (2.6) now yields for $\widetilde{N} = 1$

$$\left| {}^1 I_{i_1 \dots i_N}^{\varrho_{i_1} \dots \varrho_{i_N}}(\mu) - (2\pi/\mu)^\kappa \int_{\varepsilon < |\tau_{i_j}(\sigma)| < 1} Q_{0,\sigma} \prod_{j=1}^N |\tau_{i_j}(\sigma)|^{e^{(i_j)} - 1 - \kappa} |\det D\delta_{i_1 \dots i_N}(\sigma)| \, d\sigma \right|$$

$$\leq c_1 \sup_{l \leq 2\kappa + 3} \left\| D^l a_\mu \right\|_\infty \mu^{-\kappa - 1} \int_{\varepsilon < |\tau_{i_j}| < 1} \prod_{j=1}^N |\tau_{i_j}|^{e^{(i_j)} - 2 - \kappa} \, d\tau$$

$$\leq c_2 \sup_{l \leq 2\kappa + 3} \left\| D^l a_\mu \right\|_\infty \mu^{-\kappa - 1} \prod_{j=1}^N \max\{1, (-\log \varepsilon)^{q_j}\},$$

where the exponents q_j can take the values 0 or 1, and the constants $c_i > 0$ are independent of σ, μ, and a_μ. Having in mind that we are interested in the case where $\mu \to +\infty$, we now set $\varepsilon = \mu^{-1/N}$. By taking into account (2.5) and the fact that

$$(2\pi/\mu)^\kappa \int_{0 < |\tau_{i_j}(\sigma)| < \mu^{-1/N}} Q_{0,\sigma} \prod_{j=1}^N |\tau_{i_j}(\sigma)|^{e^{(i_j)} - 1 - \kappa} |\det D\delta_{i_1 \dots i_N}(\sigma)| \, d\sigma \leq c_3 \left\| a_\mu \right\|_\infty \mu^{-\kappa - 1}$$

we finally obtain for each of the integrals $I_{i_1 \dots i_N}^{\varrho_{i_1} \dots \varrho_{i_N}}(\mu)$ the asymptotic expansion

$$I_{i_1 \dots i_N}^{\varrho_{i_1} \dots \varrho_{i_N}}(\mu) = (2\pi/\mu)^\kappa \mathcal{L}_{i_1 \dots i_N}^{\varrho_{i_1} \dots \varrho_{i_N}} + c_4 \sup_{l \leq 2\kappa + 3} \left\| D^l a_\mu \right\|_\infty \mu^{-\kappa - 1} (\log \mu)^N,$$

where the leading coefficient $\mathcal{L}_{i_1 \dots i_N}^{\varrho_{i_1} \dots \varrho_{i_N}}$ is given by

$$\mathcal{L}_{i_1 \dots i_N}^{\varrho_{i_1} \dots \varrho_{i_N}} := \int_{\mathrm{Crit}(^{(i_1 \dots i_N)} \widetilde{\Phi}^{wk})} \frac{a_{i_1 \dots i_N}^{\varrho_{i_1} \dots \varrho_{i_N}} \mathcal{J}_{i_1 \dots i_N}^{\varrho_{i_1} \dots \varrho_{i_N}} \, d\mathrm{Crit}(^{(i_1 \dots i_N)} \widetilde{\Phi}^{wk})}{|\det \mathrm{Hess}(^{(i_1 \dots i_N)} \widetilde{\Phi}^{wk})_{|N\mathrm{Crit}(^{(i_1 \dots i_N)} \widetilde{\Phi}^{wk})}|^{1/2}},$$

and $d\mathrm{Crit}(^{(i_1 \dots i_N)} \widetilde{\Phi}^{wk})$ denotes the induced Riemannian volume density. This is [7, Theorem 8.4] with a refined remainder estimate. Since the integrals $I_{j_1 \dots j_{N-1} L}^{\varrho_{j_1} \dots \varrho_{j_{N-1}}}(\mu)$ have analogous descriptions, we are left with the task of examining the non-stationary contributions $R(\mu)$ in (2.4). They are of two types: either they arise by localizing the integrals $J_{j_1 \dots j_{N-1} j_N}^{\varrho_{j_1} \dots \varrho_{j_{N-1}}}(\nu)$ to tubular neighborhoods of the relevant critical sets, or they correspond to integrals $J_{i_1 \dots i_N}^{\varrho_{i_1} \dots \varrho_{i_N}}(\nu)$ over charts of the resolution spaces where the weak transforms of the phase functions do not have critical points. In both cases, the considered non-stationary domains do have μ- and a_μ-independent distances strictly larger than zero to the relevant critical sets due to the particular resolution and cut off functions employed, see [7, Pages 27, 34, and 38]. Furthermore, the ξ-gradients of the weak transforms are given in terms of those distances, compare [7, Eq. (6.11)]. Therefore, the lengths of the gradients of the weak transforms of the non-stationary contributions are uniformly bounded from below on the pre-images of supp a_μ in any of the relevant charts, no matter how the support of a_μ varies as $\mu \to \infty$. The non-stationary phase principle [3, Page 19] then implies that they will contribute only terms of order

$$|R(\mu)| \leq c_5 \sup_{l \leq \kappa + 1} \left\| D^l a_\mu \right\|_\infty \mu^{-\kappa - 1},$$

compare [7, Page 62]. By taking into account (2.4) and the asymptotic descriptions of $I_{i_1 \dots i_N}^{\varrho_{i_1} \dots \varrho_{i_N}}(\mu)$ and $I_{j_1 \dots j_{N-1} L}^{\varrho_{j_1} \dots \varrho_{j_{N-1}}}(\mu)$, the assertion of the theorem follows, the computation of the leading term being already accomplished in [7, Theorem 9.1]. $\qquad \square$

3. An asymptotic multiplicity formula in $\mathrm{L}^2(M)$

3.1. **Asymptotic behavior of families of irreducible representations.** Let the notation be as in the introduction and $\widehat{G}$ the set of all equivalence classes of unitary irreducible representations of G. If $\chi \in \widehat{G}$ and $(\pi_\chi, H_\chi) \in \chi$, H_χ has finite dimension, and the *character of* χ is given by

$$\chi(g) := \mathrm{tr}\, \pi_\chi(g), \qquad g \in G.$$

It is denoted by the same letter. Let $d_\chi := \chi(e)$ be the dimension of π_χ. Endow M with the Riemannian volume density dM, and consider the Peter-Weyl decomposition

$$(3.1) \qquad \mathrm{L}^2(M) = \bigoplus_{\chi \in \widehat{G}} \mathrm{L}^2_\chi(M)$$

of the left-regular representation of G in $\mathrm{L}^2(M)$ into isotypic components. As an immediate consequence of Theorem 2.1 we shall generalize the Weyl law for the reduced spectral counting function of an invariant elliptic operator on M proven in [7, Theorem 9.5] to sums of isotypic components of the form $\bigoplus_{\chi \in \mathcal{W}_\lambda} \mathrm{L}^2_\chi(M)$, where $\mathcal{W}_\lambda \subset \widehat{G}$ are appropriate finite subsets whose cardinality does not grow too fast as $\lambda \to +\infty$. Thus, let

$$P_0 : \mathrm{C}^\infty(M) \longrightarrow \mathrm{L}^2(M)$$

be an invariant elliptic classical pseudodifferential operator of order m on M with principal symbol $p(x,\xi)$, where $\mathrm{C}^\infty(M)$ denotes the space of smooth functions on M. Assume that P_0 is positive and symmetric, and denote by P its unique self-adjoint extension. Denote by $\widehat{G}' \subset \widehat{G}$ the subset of equivalence classes of representations occuring in (3.1). Since P commutes with the G-action, each of its eigenspaces constitutes an unitary G-module, and we write $\mathrm{mult}_\chi(t)$ for the multiplicity of the unitary irreducible representation (π_χ, H_χ) in the eigenspace E_t of P belonging to the eigenvalue t. The following families of irreducible G-representations were first considered in [4] within a semiclassical context.

Definition 3.1. *Let $\{\mathcal{W}_\lambda\}_{\lambda \in (0,\infty)}$ be a family of finite sets $\mathcal{W}_\lambda \subset \widehat{G}'$ such that there is a $\vartheta \geq 0$ so that for each $l \in \mathbb{N}$ and each differential operator D^l on G of order l there is a constant $C > 0$ independent of λ with*

$$\max_{\chi \in \mathcal{W}_\lambda} \frac{\left\| D^l \, \overline{\chi} \right\|_\infty}{[\pi_{\chi|H} : \mathbb{1}]} \leq C \, \lambda^{\vartheta l} \qquad \forall \, \lambda \in (0,\infty),$$

where $[\pi_{\chi|H} : \mathbb{1}]$ denotes the multiplicity of the trivial representation in the restriction of π_χ to a principal isotropy group H. The smallest possible ϑ is called the growth rate *of the family $\mathcal{W}_\lambda$.*

Remark 3.1. Note that $[\pi_{\chi|H} : \mathbb{1}] = \int_H \overline{\chi(h)} \, dh$, see [1, Eq. (3.33)], and by the Frobenius reciprocity theorem one has $[\pi_{\chi|H} : \mathbb{1}] = [\mathrm{L}^2(G/H) : \pi_\chi]$. Furthermore, the irreducible G-representations appearing in the Peter-Weyl decomposition of $\mathrm{L}^2(M)$ are precisely those G-representations appearing in $\mathrm{L}^2(G/H)$, so that $[\pi_{\chi|H} : \mathbb{1}] \geq 1$ for $\chi \in \widehat{G}'$. If the orbit space M/G has dimension greater than one then each irreducible G-representation appears an infinite number of times, compare [2, Section 2].

The following result is a generalization of [7, Theorem 9.5] to growing families of isotypic components.

Theorem 3.2. *With the notation as above assume that $n - \kappa \geq 1$, set*

$$\mathcal{M}_\chi(\lambda) := \sum_{t \leq \lambda} \mathrm{mult}_\chi(t),$$

and write $S^*M := \{(x,\xi) \in T^*M : p(x,\xi) = 1\}$. If $\mathcal{W}_\lambda \subset \widehat{G}'$ is a family of growth rate $\vartheta \in \left[0, \frac{1}{(2\kappa+3)m}\right)$, then as $\lambda \to +\infty$

$$\frac{1}{|\mathcal{W}_\lambda|} \sum_{\chi \in \mathcal{W}_\lambda} \frac{\mathcal{M}_\chi(\lambda)}{[\pi_{\chi|H} : \mathbb{1}]} = \frac{\mathrm{vol} \left[(\Omega \cap S^*M)/G \right]}{(n-\kappa)(2\pi)^{n-\kappa}} \lambda^{\frac{n-\kappa}{m}} + O\big(\lambda^{\frac{n-\kappa-1}{m} + \vartheta(2\kappa+3)}(\log \lambda)^\Lambda\big).$$

Proof. Let $Q = (P)^{1/m}$ be the m-th root of P given by the spectral theorem. It is a classical pseudo-differential operator of order 1 with principal symbol $q(x,\xi) = p(x,\xi)^{1/m}$, and if $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ are the eigenvalues of P repeated according to their multiplicity, the eigenvalues of Q are $\mu_j := (\lambda_j)^{1/m}$. The operator Q is invariant, and we write $\mathrm{mult}^Q_\chi(\mu_j)$ for the multiplicity of the representation π_χ in the eigenspace $E^Q_{\mu_j}$ of Q belonging to the eigenvalue μ_j. Put $\mathcal{M}^Q_\chi(\mu) := \sum_{t \leq \mu} \mathrm{mult}^Q_\chi(t)$. As explained in [7, Section 2], an asymptotic description of the reduced spectral counting function

$N_\chi(\lambda) := d_\chi \mathcal{M}_\chi(\lambda) = d_\chi \mathcal{M}_\chi^Q(\lambda^{1/m})$ can be obtained by studying the singularities of the tempered distribution

$$\sum_{j=1}^\infty m_\chi^Q(\mu_j)\, e^{-it\mu_j} \in \mathcal{S}'(\mathbb{R}), \qquad m_\chi^Q(\mu_j) := d_\chi \mathrm{mult}_\chi^Q(\mu_j)/\dim E_{\mu_j}^Q.$$

It corresponds to the distribution trace of $P_\chi \circ U(t)$, where $U(t) := e^{-itQ}$ denotes the Fourier transform of the spectral measure of Q and P_χ the projector onto the isotypic component $\mathrm{L}_\chi^2(M)$. This distribution trace is the Fourier transform of the distribution

$$\sigma_\chi(\mu) := \sum_{j=1}^\infty m_\chi^Q(\mu_j)\, \delta(\mu - \mu_j),$$

and the main singularity of $\hat\sigma_\chi$ at $t = 0$ can be described by approximating $U(t)$ via Fourier integral operators, yielding for sufficiently small $\delta > 0$ and $\mu \geq 1$ a description of

$$\hat\sigma_\chi(\check\varrho e^{i(\cdot)\mu}) = \sum_{j=1}^\infty m_\chi^Q(\mu_j)\hat\varrho(\mu - \mu_j), \qquad \varrho \in \mathrm{C}_\mathrm{c}^\infty(-\delta,\delta),$$

by oscillatory integrals of the form (1.1). Indeed, consider an atlas $\{(\kappa_\gamma, U_\gamma)\}$ for M with subordinated partition of unity $\{f_\gamma\}$, and additional test functions $\bar f_\gamma \in \mathrm{C}_\mathrm{c}^\infty(U_\gamma)$ satisfying $\bar f_\gamma \equiv 1$ on supp f_γ. Let further $\varrho \in \mathrm{C}_\mathrm{c}^\infty(-\delta,\delta)$ be such that $1 = 2\pi\varrho(0)$, and $0 \leq \alpha \in \mathrm{C}_\mathrm{c}^\infty(1/2,3/2)$ satisfy $\alpha \equiv 1$ in a neighborhood of 1, while $\Delta_{\varepsilon,r} \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R})$ denotes an approximation of the δ-distribution at $r \in \mathbb{R}$ as $\varepsilon \to 0$. Then, by [7, Theorem 2.4] one has

$$\hat\sigma_\chi(\varrho e^{i(\cdot)\mu}) = \lim_{\varepsilon \to 0} \sum_\gamma \left[\frac{\mu^{n-1}d_\chi}{(2\pi)^{n-1}} I_{\gamma,\varepsilon}^0(\mu,1,0) + \mathfrak{R}_{\gamma,\varepsilon}(\mu) \right], \quad \mathfrak{R}_{\gamma,\varepsilon}(\mu) = O\Big(d_\chi\, \mu^{n-2} \sum_{|\beta| \leq 5} \sup_{R,t} |I_{\gamma,\varepsilon}^\beta(\mu,R,t)|\Big),$$

where

$$I_{\gamma,\varepsilon}^\beta(\mu,R,t) := \int_{T^*U_\gamma} \int_G e^{i\mu\Phi_\gamma(x,\eta,g)} \overline{\chi(g)} f_\gamma(x) \bar f_\gamma(gx) J_\gamma(g,x)\alpha(q(x,\eta))$$
$$\cdot \partial_{R,t}^\beta \left[\varrho(t)a_\gamma(t,\kappa_\gamma(x),\mu\eta)\Delta_{\varepsilon,R}(\zeta_\gamma(t,\kappa_\gamma(x),\eta)) \right]\, dg\, d(T^*U_\gamma)(x,\eta),$$

and $\Phi_\gamma(x,\eta,g) = \langle \kappa_\gamma(x) - \kappa_\gamma(g \cdot x), \eta \rangle$. Here $a_\gamma \in S_{\mathrm{phg}}^0$ are classical symbols with $a_\gamma(0,\kappa_\gamma(x),\eta) = 1$, and ζ_γ certain smooth functions homogeneous in η of degree 1 satisfying $\zeta_\gamma(0,\kappa_\gamma(x),\eta) = q(x,\eta)$, while $J_\gamma(x,g)$ is a Jacobian. Based on the remainder estimate (1.2) it was then shown in [7, Proposition 9.6] that

$$(3.2) \qquad \lim_{\varepsilon \to 0} I_{\gamma,\varepsilon}^\beta(\mu,R,t) = \mathcal{L}_\gamma^\beta(R,t)(2\pi/\mu)^\kappa + O_\chi(\mu^{-\kappa-1}(\log\mu)^{\Lambda-1}),$$

where the coefficients $\mathcal{L}_\gamma^\beta(R,t)$ are given in terms of distributions supported on the regular part of the critical set $\mathcal{C} \cap T^*U_\gamma$ of Φ_γ intersected with $G \times S_{t,R}^*U_\gamma$, and the remainder term by distributions supported on $G \times S_{t,R}^*U_\gamma$, where $S_{t,R}^*U_\gamma := \{(x,\omega) \in T^*U_\gamma : \zeta_\gamma(t,\kappa_\gamma(x),\omega) = R\}$. In particular, $\mathcal{L}_\gamma^0(1,0) = [\pi_{\chi|H} : \mathbf{1}]\, \varrho(0) \mathrm{vol}\, [(\mathrm{Reg}\,\Omega \cap S^*M)/G]$. This yielded the estimate

$$\hat\sigma_\chi(\varrho e^{i(\cdot)\mu}) = d_\chi[\pi_{\chi|H} : \mathbf{1}]\, \varrho(0)\, \mathrm{vol}\,[(\mathrm{Reg}\,\Omega \cap S^*M)/G]\, (\mu/2\pi)^{n-\kappa-1} + O_\chi\big(\mu^{n-\kappa-2}(\log\mu)^{\Lambda-1}\big)$$

from which an asymptotic description for $N_\chi(\lambda)$, and consequently $\mathcal{M}_\chi(\lambda)$, was obtained via a classical Tauberian argument, see [7, Proof of Theorem 9.5]. Now, in order to prove Theorem 3.2, note that by the improved bounds (1.3) of Theorem 2.1 the integrals $I_{\gamma,\varepsilon}^\beta(\mu,R,t)$ actually have asymptotic descriptions with leading terms of order $\mu^{-\kappa}$ and remainder terms bounded from above by

$$C_{\gamma,\varepsilon}^\beta(R,t) \sup_{l \leq 2\kappa+3} \left\| D^l \overline\chi \right\|_\infty \mu^{-\kappa-1}(\log\mu)^{\Lambda-1}$$

with constants $C_{\gamma,\varepsilon}^\beta(R,t) > 0$ independent of χ and μ. In analogy to (3.2) one therefore deduces for any $\chi \in \mathcal{W}_{\mu^m}$

$$\lim_{\varepsilon \to 0} I_{\gamma,\varepsilon}^\beta(\mu,R,t) = \mathcal{L}_\gamma^\beta(R,t)(2\pi/\mu)^\kappa + O([\pi_{\chi|H} : \mathbf{1}]\, \mu^{\vartheta(2\kappa+3)m}\mu^{-\kappa-1}(\log\mu)^{\Lambda-1}),$$

yielding with $d_\chi[\pi_{\chi|H} : \mathbf{1}] \geq 1$ the asymptotic formula

$$\frac{1}{|\mathcal{W}_{\mu^m}|} \sum_{\chi \in \mathcal{W}_{\mu^m}} \frac{\hat{\sigma}_\chi(\varrho e^{i(\cdot)\mu})}{d_\chi[\pi_{\chi|H} : \mathbf{1}]} = \varrho(0) \operatorname{vol}\left[(\operatorname{Reg}\Omega \cap S^*M)/G\right](\mu/2\pi)^{n-\kappa-1}$$

$$+ O(\mu^{\vartheta(2\kappa+3)m}\mu^{n-\kappa-2}(\log\mu)^{\Lambda-1}).$$

The assertion of the theorem now follows again from a classical Tauberian argument, compare [7, Proof of Theorem 9.5]. $\qquad\square$

3.2. Families of irreducible representations and the Cartan-Weyl classification.

In what follows, we shall apply Theorem 3.2 to specific families of representations given in terms of the Cartan-Weyl classification of unitary irreducible representations of G. Thus [10], let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$ and $T \subset G$ a maximal torus with Lie algebra $\mathfrak{t}$. The exponential function $\exp$ is a covering homomorphism of $\mathfrak{t}$ onto T, and its kernel L a lattice in $\mathfrak{t}$. Let $\widehat{T}$ denote the *set of characters of T*, that is, of all continuous homomorphisms of T into the circle. The differential of a character $\mu : T \to S^1$, denoted by the same letter, is a linear form $\mu : \mathfrak{t} \to i\mathbb{R}$ which is *integral* in the sense that $\mu(L) \subset 2\pi i\mathbb{Z}$. On the other hand, if μ is an integral linear form, one defines

$$t^\mu := e^{\mu(X)}, \qquad t = \exp X \in T,$$

setting up an identification of $\widehat{T}$ with the integral linear forms on $\mathfrak{t}$. Let $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{t}_\mathbb{C}$ denote the complexifications of $\mathfrak{t}$ and $\mathfrak{g}$, respectively. Then $\mathfrak{t}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$, and we denote the corresponding system of roots by $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. Let Σ^+ denote a set of positive roots. Since roots define integral linear forms on $\mathfrak{t}$, one can regard them as characters of T.

As before, let $\widehat{G}$ be the set of all equivalence classes of irreducible unitary representations of G and $\chi \in \widehat{G}$. Due to the invariance of the trace under cyclic permutations the character of χ satisfies $\chi(t) = \chi(gtg^{-1})$ for all $t, g \in G$. Since any element in G is conjugated to an element of T, $\chi(g)$ is fully determined by its restriction to T. Now, as a consequence of the Cartan-Weyl classification of irreducible finite dimensional representations of reductive Lie algebras over $\mathbb{C}$ one has the identification

$$(3.3) \qquad \widehat{G} \simeq \{\Lambda \in \mathfrak{t}_\mathbb{C}^* : \ \Lambda \text{ is dominant integral and } T\text{-integral}\}.$$

Here an element $\Lambda \in \mathfrak{t}_\mathbb{C}^*$ is called *dominant integral* if $2(\Lambda, \alpha)/(\alpha, \alpha)$ is a non-negative integer for any $\alpha \in \Sigma^+$, $(\cdot, \cdot)$ being the symmetric non-degenerate form on $\mathfrak{t}_\mathbb{C}^*$ induced by an $\operatorname{Ad}(G)$-invariant inner product on $\mathfrak{g}$.

Next, assume that G is semi-simple, write $\varrho := \frac{1}{2}\sum_{\alpha \in \Sigma^+} \alpha$, and let $W := W(G, T)$ be the Weyl group. For an integral linear form $\mu \in \widehat{T}$, define the alternating sum $A(\mu)(t) := \sum_{w \in W} \det w \, t^{w\mu}$. If $\chi \in \widehat{G}$, let $\Lambda_\chi \in \mathfrak{t}_\mathbb{C}^*$ be the *highest weight* given by the isomorphism (3.3). Then, the Weyl character formula asserts that on T one has [9]

$$\chi_{|T}(t) = \frac{A(\Lambda_\chi + \varrho)(t)}{t^\varrho \prod_{\alpha \in \Sigma^+}(1 - t^{-\alpha})} = \frac{A(\Lambda_\chi + \varrho)(t)}{\prod_{\alpha \in \Sigma^+}(t^{\alpha/2} - t^{-\alpha/2})}.$$

If G is simply connected this can be written as $\chi_{|T}(t) = A(\Lambda_\chi + \varrho)(t)/A(\varrho)(t)$. Since $\mu(H) \in i\mathbb{R}$ for all $H \in \mathfrak{t}$ and any integral linear form μ on $\mathfrak{t}$, one immediately deduces for any $l \in \mathbb{N}$ as $|\Lambda_\chi| \to \infty$ the estimate

$$\frac{d^l}{ds^l}\chi_{|T}(\exp sH) = O(|\Lambda_\chi|^l).$$

Writing $g = h(g)t(g)h(g)^{-1}$ for an arbitrary element $g \in G$ with $t(g) \in T$ and $h(g) \in G$ we obtain with $\chi(g) = \chi_{|T}(t(g))$ the following simple consequence. If D^l is a differential operator on G of order l and $\chi \in \widehat{G}$ a class with highest weight Λ_χ, then

$$(3.4) \qquad \left\|D^l\chi\right\|_\infty = O(|\Lambda_\chi|^l), \qquad |\Lambda_\chi| \to \infty.$$

From this we deduce the following

Corollary 3.3. *In the setup of Section 3.1 define* $\mathcal{W}_\lambda := \left\{ \chi \in \widehat{G}' \,:\, |\Lambda_\chi| \leq C\,\lambda^\vartheta \right\}$ *for some* $\vartheta \in \left[0, \frac{1}{(2\kappa+3)m}\right)$ *and a constant* $C > 0$. *Then* $\{\mathcal{W}_\lambda\}_{\lambda \in (0,\infty)}$ *constitutes a family with growth rate* ϑ, *and Theorem 3.2 applies.*

Proof. By (3.4) we have for each $l \in \mathbb{N}$ and each differential operator D^l on G of order l

$$\max_{\chi \in \mathcal{W}_\lambda} \frac{\left\| D^l \overline{\chi} \right\|_\infty}{\left[\pi_{\chi|_H} : \mathbb{1} \right]} \leq \max_{\chi \in \mathcal{W}_\lambda} \left\| D^l \overline{\chi} \right\|_\infty \leq C\,\lambda^{\vartheta l} \qquad \forall\, \lambda \in (0,\infty)$$

with a constant $C > 0$ independent of λ.[1]

$\square$

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[1]Note that the Weyl character formula implies the dimension formula $d_\chi = \prod_{\alpha \in \Sigma^+} (\Lambda_\chi + \varrho, \alpha)/(\alpha, \alpha)$. Thus, for the set $\partial\mathcal{W}_\lambda := \left\{ \chi \in \widehat{G}' : |\Lambda_\chi| = \lambda^\vartheta \right\}$ one gets the stronger estimate

$$\max_{\chi \in \partial\mathcal{W}_\lambda} \frac{\left\| D^l \overline{\chi} \right\|_\infty}{d_\chi \left[\pi_{\chi|_H} : \mathbb{1} \right]} \leq C\,\lambda^{\vartheta(l - |\Sigma^+|)} \qquad \forall\, \lambda \in (0,\infty).$$