THE EQUIVARIANT SPECTRAL FUNCTION OF AN INVARIANT ELLIPTIC OPERATOR. L^p-BOUNDS, CAUSTICS AND CONCENTRATION OF EIGENFUNCTIONS

PABLO RAMACHER

ABSTRACT. Let M be a compact boundaryless manifold, carrying an effective and isometric action of a compact connected Lie group G, and P_0 an invariant elliptic classical pseudodifferential operator on M. Using Fourier integral operator techniques, we prove a local Weyl law with remainder estimate for the equivariant (or reduced) spectral function of P_0 for each isotpyic component in the Peter-Weyl decomposition of $L^2(M)$. From this we deduce a generalized Kuznecov sum formula for periods of G-orbits, and recover the local Weyl law for orbifolds. Relying on recent results on singular equivariant asymptotics of oscillatory integrals, we further characterize the caustic behavior of the reduced spectral function near singular orbits, which allows us to give corresponding point-wise bounds for clusters of eigenfunctions in specific isotypic components that are sharp. In case that G acts on M without singular orbits, we are able to deduce refined L^p-bounds for $2 \leq p \leq \infty$ that improve on the classical estimates for generic eigenfunctions.

CONTENTS

1. INTRODUCTION

In this paper, we derive an asymptotic formula with remainder estimate for the equivariant (or reduced) spectral function of an invariant elliptic operator on a compact manifold with an effective and isometric action of a compact connected Lie group, generalizing previous work of Avacumovič [1], Levitan[19], Hörmander [13], and, more recently, Stanhope and Uribe [29]. If G acts on M with orbits of the same dimension, we obtain L^p-bounds for eigenfunctions belonging to specific isotypic components that improve on the classical estimates for generic eigenfunctions proved by Sogge [26], but cannot hold when singular orbits are present. In the latter case, we are able to describe the caustic behavior of the reduced spectral function as one approaches orbits of singular type, relying on recent results on singular equivariant asymptotics of oscillatory integrals obtained in the work [21] via desingularization techniques. In some sense, the present paper could be seen as culmination of the investigation initiated in that work. As an application, we are able to prove point-wise bounds for isotypic clusters of eigenfunctions, showing that they tend to concentrate on singular orbits. In

particular, this gives a new interpretation of the classical bounds for spherical harmonics in terms of caustics of the equivariant spectral function, generalizing them to eigenfunctions on arbitrary compact manifolds with symmetries. Our results can be viewed as part of the more general problem of studying eigenfunctions of a commuting family of differential operators on a general compact manifold that are independent in some sense, compare [20].

To explain our results, consider a closed[1] connected Riemannian manifold M of dimension n, together with an elliptic classical pseudodifferential operator

$$P_0 : \mathrm{C}^\infty(M) \longrightarrow \mathrm{L}^2(M)$$

of degree m, where $\mathrm{C}^\infty(M)$ denotes the space of smooth functions on M and $\mathrm{L}^2(M)$ the Hilbert space of square integrable functions with respect to the Riemannian volume density dM on M. We assume that P_0 is positive and symmetric, so that it has a unique self-adjoint extension P. Furthermore, the compactness of M implies that P has discrete spectrum. Let $\{E_\lambda\}$ be a spectral resolution of P, and denote by $e(x, y, \lambda)$ the Schwartz kernel of E_λ, which is called the spectral function of P. Within the theory of Fourier integral operators one can then show the following *local Weyl formula* [1, 19, 13]

$$(1.1) \qquad \left| e(x, x, \lambda) - \frac{\lambda^{\frac{n}{m}}}{(2\pi)^n} \int_{p(x,\xi)<1} d\xi \right| \leq C\lambda^{\frac{n-1}{m}}, \qquad x \in M, \ \lambda \to \infty,$$

for some constant $C > 0$ independent of x and λ, p being the principal symbol of P_0. By integrating over M one deduces from this for the spectral counting function $N(\lambda) := \sum_{t \leq \lambda} \dim \mathcal{E}_t = \int_M e(x, x, \lambda) \, dM(x)$ the *global Weyl formula*

$$N(\lambda) = \frac{\mathrm{vol}\, S^* M}{n(2\pi)^n} \lambda^{\frac{n}{m}} + O(\lambda^{\frac{n-1}{m}}),$$

where $\mathcal{E}_t$ denotes the eigenspace of P belonging to the eigenvalue t and $S^* M$ the co-sphere bundle $\{(x, \xi) \in T^* M : p(x, \xi) = 1\}$. In order to show (1.1) one first proves the estimate

$$(1.2) \qquad |e(x, x, \lambda + 1) - e(x, x, \lambda)| \leq C \cdot \lambda^{\frac{n-1}{m}}, \qquad x \in M,$$

which describes the order of magnitude of the discontinuities of $N(\lambda)$ or, more generally, the amount of eigenvalues in the interval $(\lambda, \lambda+1]$ as $\lambda \to +\infty$, yielding the asymptotics $N(\lambda+1) - N(\lambda) = O(\lambda^{\frac{n-1}{m}})$. The bound (1.2) is equivalent to

$$(1.3) \qquad \sum_{\lambda_j \in (\lambda, \lambda+1]} |e_j(x)|^2 \leq C \cdot \lambda^{\frac{n-1}{m}}, \qquad x \in M,$$

where $\{e_j\}$ denotes an arbitrary orthonormal basis of eigenfunctions $\{e_j\}$ of P in $\mathrm{L}^2(M)$ with eigenvalues $\{\lambda_j\}$, and actually implies the bound

$$(1.4) \qquad \|\chi_\lambda u\|_{\mathrm{L}^\infty(M)} \leq C(1 + \lambda)^{\frac{n-1}{2m}} \|u\|_{\mathrm{L}^2(M)}, \qquad u \in \mathrm{L}^2(M),$$

where χ_λ denotes the spectral projection onto the sum of eigenspaces with eigenvalues in the interval $(\lambda, \lambda+1]$ with Schwartz kernel $\chi_\lambda(x, y) = e(x, y, \lambda+1) - e(x, y, \lambda)$, since $\|\chi_\lambda\|^2_{\mathrm{L}^2 \to \mathrm{L}^\infty} \equiv \sup_{x \in M} \chi_\lambda(x, x)$. From this the estimate for $N(\lambda+1) - N(\lambda)$ immediately follows by taking the trace of χ_λ. In particular, one deduces from (1.4) the bound for eigenfunctions

$$(1.5) \qquad \|u\|_{\mathrm{L}^\infty(M)} \leq C \lambda^{\frac{n-1}{2m}}, \qquad u \in \mathcal{E}_\lambda, \ \|u\|_{\mathrm{L}^2} = 1.$$

Under the additional assumption that the co-spheres $S_x^* M$ are strictly convex, Seeger and Sogge [23] were also able to prove upper bounds for L^p-norms of eigenfunctions via analytic interpolation techniques, generalizing previous work of Sogge for second order elliptic differential operators [26]. More precisely, let

$$\delta_n(p) := \max\left(n \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{2}, 0 \right).$$

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[1] By a closed manifold we will understand a compact manifold without boundary.

Then, for $u \in \mathcal{E}_\lambda$, $\|u\|_{\mathrm{L}^2} = 1$ one has

$$(1.6) \qquad \|u\|_{\mathrm{L}^p(M)} \leq \begin{cases} C\lambda^{\frac{\delta(p)}{m}}, & \frac{2(n+1)}{n-1} \leq p \leq \infty, \\ C\lambda^{\frac{(n-1)(2-p')}{4mp'}}, & 2 \leq p \leq \frac{2(n+1)}{n-1}, \end{cases}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

In this paper, we shall prove bounds similar to those of (1.1)-(1.6) in the presence of symmetries. To explain our results, assume that M carries an effective and isometric action of a compact connected Lie group G with Lie algebra $\mathfrak{g}$, and that P commutes with the *left-regular representation* $(\pi, \mathrm{L}^2(M))$ of G in $\mathrm{L}^2(M)$ given by

$$\pi(g)u(x) = u(g^{-1} \cdot x), \qquad u \in \mathrm{L}^2(M),$$

so that each eigenspace of P becomes a unitary G-module. If $\widehat{G}$ denotes the set of equivalence classes of irreducible unitary representations of G, which we shall identify with the set of characters of G, the Peter-Weyl theorem asserts that

$$(1.7) \qquad \mathrm{L}^2(M) = \bigoplus_{\gamma \in \widehat{G}} \mathrm{L}_\gamma^2(M),$$

a Hilbert sum decomposition, where $\mathrm{L}_\gamma^2(M) = \Pi_\gamma \mathrm{L}^2(M)$ denotes the γ-isotypic component, and Π_γ the corresponding projection. Let $e_\gamma(x, y, \lambda)$ be the spectral function of the operator $P_\gamma := \Pi_\gamma \circ P \circ \Pi_\gamma = P \circ \Pi_\gamma = \Pi_\gamma \circ P$. Further, let $\mathbb{J} : T^*M \to \mathfrak{g}^*$ denote the momentum map of the Hamiltonian G-action on T^*M, induced by the action of G on M, and write $\Omega := \mathbb{J}^{-1}(\{0\})$. As our first result, we show in Theorem 4.3 the *equivariant local Wey law*

$$(1.8) \qquad \left| e_\gamma(x, x, \lambda) - \lambda^{\frac{n-\kappa_x}{m}} \frac{d_\gamma [\pi_{\gamma|G_x} : \mathbf{1}]}{(2\pi)^{n-\kappa_x}} \int_{(x,\xi) \in \Omega, \, p(x,\xi) < 1} \frac{d\xi}{\mathrm{vol}\, \mathcal{O}_{(x,\xi)}} \right| \leq C_x \, d_\gamma \, \lambda^{\frac{n-\kappa_x-1}{m}}, \quad x \in M,$$

as $\lambda \to \infty$, where $\kappa_x := \dim \mathcal{O}_x$ is the dimension of the orbit through x, d_γ denotes the dimension of an irreducible G-representation π_γ belonging to γ and $[\pi_{\gamma|G_x} : \mathbf{1}]$ the multiplicity of the trivial representation in the restriction of π_γ to the isotropy group G_x of x, while $C_x > 0$ is a constant depending on x but not on λ. It should be emphasized that κ_x, and therefore also the leading term and the constant C_x, which are independent of λ, will in general depend in a highly non-uniform way on $x \in M$. In fact, the description of $e_\gamma(x, y, \lambda)$ reduces in essence to the study of oscillatory integrals of the form

$$(1.9) \qquad I_{x,y}(\mu) := \int_G \int_{S_x^*Y} e^{i\mu \Phi_{x,y}(\omega,g)} a_\mu(x, y, \omega, g) \, d(S_x^*Y)(\omega) \, dg, \qquad \mu > 0,$$

with phase function

$$\Phi_{x,y}(\omega, g) := \langle \kappa(x) - \kappa(g \cdot y), \omega \rangle,$$

where (Y, κ) is a local chart on M and $a_\mu \in \mathrm{C}_c^\infty$ an amplitude that might depend on μ and is such that $(x, y, \omega, g) \in \mathrm{supp}\, a_\mu$ implies $x, g \cdot y \in Y$, while $d(S^*Y)$ and dg denote Liouville and Haar measure, respectively. Now, when trying to describe the asymptotic behavior of $I_{x,x}(\mu)$ as $\mu \to \infty$ uniformly in x via the stationary phase principle, one encounters the phenomenon that the critical set $\mathcal{C}_x$ of $\Phi_{x,x}$ changes abruptly its dimension when x passes through points of singular orbits, leading to a drastic change in the asymptotics of $I_{x,x}(\mu)$. Such points are called *caustics* [31], and are ultimately responsible for the qualitatively very different asymptotic behavior of the reduced spectral function as x approaches such points.

Though the leading coefficient in the asymptotic formula (1.8) for $e_\gamma(x, x, \lambda)$ is explicit, and has a clear geometric meaning, it does not unveil the caustic nature of $e_\gamma(x, x, \lambda)$, and blows up in an unknown way as x approaches singular orbits. To obtain a precise description of this caustic behavior it is necessary to examine the integrals (1.9) more carefully. For this, we shall rely on recent results on singular equivariant asymptotics obtained in [21] via resolution of singularities from which we will be able to deduce a uniform description of the integrals $I_{x,x}(\mu)$ and the behavior of $e_\gamma(x, x, \lambda)$ near singular orbits. More precisely, consider the stratification $M = M(H_1) \dot\cup \ldots \dot\cup M(H_L)$ of M into orbit

types, arranged in such a way that $(H_i) \leq (H_j)$ implies $i \geq j$, and let Λ be the maximal length that a maximal totally ordered subset of isotropy types can have. Write $M_{\mathrm{prin}} := M(H_L)$, M_{except}, and M_{sing} for the union of all orbits of principal, exceptional, and singular type, respectively, so that

$$M = M_{\mathrm{prin}} \,\dot\cup\, M_{\mathrm{except}} \,\dot\cup\, M_{\mathrm{sing}},$$

and denote by $\kappa := \dim G/H_L$ the dimension of an orbit of principal type. Then, by Theorem 7.7 one has for $x \in M_{\mathrm{prin}} \cup M_{\mathrm{except}}$ and $\lambda \to \infty$ the *singular equivariant local Weyl law*

(1.10)

$$\left| e_\gamma(x,x,\lambda) - \frac{d_\gamma \lambda^{\frac{n-\kappa}{m}}}{(2\pi)^{n-\kappa}} \sum_{N=1}^{\Lambda-1} \sum_{i_1 < \cdots < i_{N-1}} \prod_{l=1}^{N-1} |\tau_{i_l}|^{\dim G - \dim H_{i_l} - \kappa} \left[\mathcal{L}^{0,0}_{i_1 \ldots i_{N-1}}(x) + \right. \right.$$

$$\left. \left. \sum_{i_{N-1} < i_N} \mathcal{M}^{0,0}_{i_1 \ldots i_N}(x) |\tau_{i_N}|^{\dim G - \dim H_{i_N} - \kappa} \right] \right| \leq C d_\gamma \lambda^{\frac{n-\kappa-1}{m}} \sum_{N=1}^{\Lambda-1} \sum_{i_1 < \cdots < i_N} \prod_{l=1}^{N} |\tau_{i_l}|^{\dim G - \dim H_{i_l} - \kappa - 1},$$

where the multiple sums run over maximal, totally ordered subsets $\{(H_{i_1}), \ldots, (H_{i_N})\}$ of singular isotropy types, the coefficients $\mathcal{L}^{0,0}_{i_1 \ldots i_{N-1}}$ and $\mathcal{M}^{0,0}_{i_1 \ldots i_N}$ are explicitly given and bounded functions in x, and $\tau_{i_j} \in (-1, 1)$ are exceptional parameters that arise in the resolution process satisfying $|\tau_{i_j}| \approx \mathrm{dist}\,(x, M(H_{i_j}))$, while $C > 0$ is a constant independent of x. Thus, the combinatorial structure of the underlying group action is reflected in the shape of the equivariant spectral function. By integrating the asymptotic formulae (1.8) and (1.10) over $x \in M$, one obtains for the equivariant counting function $N_\gamma(\lambda) := \int_M e_\gamma(x,x,\lambda)\,dM(x)$ the *equivariant Weyl law*

(1.11) $$N_\gamma(\lambda) = \frac{d_\gamma [\pi_{\chi|H_L} : \mathbf{1}]}{(n-\kappa)(2\pi)^{n-\kappa}} \mathrm{vol}\,[(\Omega \cap S^*M)/G]\, \lambda^{\frac{n-\kappa}{m}} + O\big(\lambda^{(n-\kappa-1)/m} (\log \lambda)^\Lambda\big),$$

provided that $n - \kappa \geq 1$. This was the main result of [21]. Notice that in spite of the fact that the desingularization techniques developed in [21] are necessary to establish the remainder estimate in (1.11), singular and exceptional orbits, being of measure zero, do not contribute to (1.11), and remain hidden. It is only in the local Weyl laws (1.8) and (1.10) for the reduced spectral function that the whole orbit structure of the underlying group action becomes manifest.

As a major consequence, Theorems 4.3 and 7.7 lead to refined bounds for eigenfunctions. In the non-singular case, that is, when only principal and exceptional orbits are present, and consequently all G-orbits have the same dimension κ, the obtained bounds are still uniform in $x \in M$, while in the singular case, they show that eigenfunctions tend to concentrate along lower dimensional orbits. Indeed, as in the non-equivariant case, the crucial bound for obtaining (1.8) is a bound for $e_\gamma(x,x,\lambda+1) - e_\gamma(x,x,\lambda)$, which is equivalent to the non-uniform bound

(1.12) $$\sum_{\substack{\lambda_j \in (\lambda, \lambda+1], \\ e_j \in \mathrm{L}^2_\gamma(M)}} |e_j(x)|^2 \leq C_x \, d_\gamma \, \lambda^{\frac{n-\kappa_x-1}{m}}, \quad x \in M,$$

see Corollary 4.5. From this one immediately deduces in the non-singular case by Proposition 5.1 the L^∞-estimate

$$\|(\chi_\lambda \circ \Pi_\gamma) u\|_{\mathrm{L}^\infty(M)} \leq C \, d_\gamma \, (1+\lambda)^{\frac{n-\kappa-1}{2m}} \|u\|_{\mathrm{L}^2(M)}, \qquad u \in \mathrm{L}^2(M),$$

where $C > 0$ is a constant independent of λ. In particular, we obtain in this situation for arbitrary $\gamma \in \widehat{G}$ and any eigenfunction of P in the isotypic component $\mathrm{L}^2_\gamma(M)$ the bound

$$\|u\|_{\mathrm{L}^\infty(M)} \leq C \, d_\gamma \, \lambda^{\frac{n-\kappa-1}{2m}}, \qquad u \in \mathrm{L}^2_\gamma(M) \cap \mathcal{E}_\lambda, \|u\|_{\mathrm{L}^2} = 1.$$

Note that if $n = \kappa + 1$, this bound reads $\|u\|_\infty \leq C \, d_\gamma$. The proof of L^p-bounds is considerably more envolved, since it no longer suffices to study the integrals $I_{x,y}(\mu)$ restricted to the diagonal. Instead, it is necessary to estimate their growth as $\mu \to \infty$ in a neighborhood of the latter, for which we have

to assume that the co-spheres $S_x^* M$ are strictly convex. Using complex interpolation techniques, we are then able to prove in Theorem 5.3 the bounds

$$\|(\chi_\lambda \circ \Pi_\gamma)u\|_{L^q(M)} \leq \begin{cases} C\, d_\gamma\, \lambda^{\frac{\delta_{n-\kappa}(q)}{m}} \|u\|_{L^2(M)}, & \frac{2(n-\kappa+1)}{n-\kappa-1} \leq q \leq \infty, \\ C\, d_\gamma\, \lambda^{\frac{(n-\kappa-1)(2-q')}{4mq'}} \|u\|_{L^2(M)}, & 2 \leq q \leq \frac{2(n-\kappa+1)}{n-\kappa-1}, \end{cases}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$, $C > 0$ is a constant independent of λ, and

$$\delta_{n-\kappa}(q) := \max\left((n-\kappa)\left|\frac{1}{2} - \frac{1}{q}\right| - \frac{1}{2}, 0 \right).$$

In particular,

$$\|u\|_{L^q(M)} \leq \begin{cases} C\, d_\gamma\, \lambda^{\frac{\delta_{n-\kappa}(q)}{m}}, & \frac{2(n-\kappa+1)}{n-\kappa-1} \leq q \leq \infty, \\ C\, d_\gamma\, \lambda^{\frac{(n-\kappa-1)(2-q')}{4mq'}}, & 2 \leq q \leq \frac{2(n-\kappa+1)}{n-\kappa-1}, \end{cases}$$

for any eigenfunction of P with eigenvalue λ belonging to the isotypic component $L_\gamma^2(M)$, provided that G acts on M with orbits of the same dimension κ. For a comparison of our results and methods with the one of Seeger and Sogge [23], see Remark 5.4. Nevertheless, the L^p-bounds above cannot hold when singular orbits are present, and the situation in this case is described by Corollary 7.8 by which one has the uniform bound

$$(1.13) \qquad \sum_{\substack{\lambda_j \in (\lambda, \lambda+1], \\ e_j \in L_\gamma^2(M)}} |e_j(x)|^2 \leq \begin{cases} C\, d_\gamma\, \lambda^{\frac{n-1}{m}}, & x \in M_{\mathrm{sing}}, \\ C\, d_\gamma\, \lambda^{\frac{n-\kappa-1}{m}} \sum_{N=1}^{\Lambda-1} \sum_{i_1 < \cdots < i_N} \prod_{l=1}^{N} |\tau_{i_l}|^{\dim G - \dim H_{i_l} - \kappa - 1}, & x \in M - M_{\mathrm{sing}}, \end{cases}$$

for a constant $C > 0$ independent of x and λ. In comparison with the bound (1.12), where the dependency of the constant C_x on x remains unspecified, the bound (1.13) gives a rather precise description of the growth of eigenfunctions near singular orbits.

To illustrate our results, consider the classical case where $M = S^2$, and $G = \mathrm{SO}(2)$ acts on M by rotations around the symmetry axis through the poles. The eigenfunctions of the Laplace-Beltrami operator on $M = S^2$ are given by the spherical functions

$$Y_{k,m}(\phi, \theta) = \sqrt{\frac{2k+1}{4\pi} \frac{(k-m)!}{(k+m)!}} P_{k,m}(\cos\theta)e^{im\phi}, \qquad 0 \leq \phi < 2\pi, \, 0 \leq \theta < \pi,$$

with corresponding eigenvalues $k(k+1)$, where $k \in \mathbb{N}$, $|m| \leq k$, and $P_{k,m}$ are the associated Legendre polynomials. Furthermore, the Legendre polynomials $P_k(\cos\theta) := P_{k,0}(\cos\theta)$ satisfy the classical asymptotics

$$P_k(\cos\theta) = \sqrt{\frac{2}{\pi k \sin\theta}} \cos\left(\left(k+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right) + O\left(\frac{1}{(k\sin\theta)^{3/2}}\right), \qquad \theta \in (0, \pi),$$

where the remainder is uniform in θ on any interval $[\varepsilon, \pi - \varepsilon]$ with $0 < \varepsilon \ll 1$, see [12, Page 303]. From this one concludes in the limit $k \to \infty$ that

$$(1.14) \qquad |Y_{k,0}(\phi, \theta)|^2 = \frac{2k+1}{4\pi}|P_k(\cos\theta)|^2 \approx \begin{cases} k, & \theta = 0, \pi, \\ \frac{1}{\sin\theta}, & \theta \in (0, \pi). \end{cases}$$

Thus, as $k \to \infty$ the eigenfuntions $Y_{k,0}$ concentrate on the poles, which are precisely the fixed points of the $\mathrm{SO}(2)$-action on S^2, and maximize the bound (1.5). The bounds (1.13) are precisely of the type (1.14), and provide an interpretation of the latter ones in terms of the caustic behavior of the equivariant spectral function, compare also Example 7.9. On the other hand, as discussed in Section 8, the bounds (1.14) show that the point-wise bounds (1.13) are sharp in the spectral parameter.

Collecting everything, the main conclusions to be drawn from this work are that

- *symmetries lead to refined* L^p*-estimates for eigenfunctions* of invariant elliptic operators, provided that all orbits of the underlying group action have the same dimension;

- *lower dimensional orbits are responsible for concentration of eigenfunctions*, and this concentration is due to the caustic behavior of the equivariant spectral function. In other words, *the orbit structure is reflected in the shape of eigenfunctions.*

We would like to close this introduction by making some final comments. In the particular case that $\gamma = \gamma_{\mathrm{triv}}$ is the trivial representation, (1.8) actually implies in passing a generalized Kuznecov sum formula for periods of G-orbits, see Corollary 4.6, which generalizes previous results of Zelditch [32] on periods of closed geodesics. In case that G acts with finite isotropy groups on M, that is, when $\widetilde{M} := M/G$ is an orbifold, an asymptotic formula for the spectral function of an elliptic operator on $\widetilde{M}$ was given by Stanhope and Uribe in [29], and we recover their result in Corollary 4.7. Finally, let us mention that one can deduce also bounds for the spectral function $e(x, y, \lambda)$ of an elliptic operator of the form

$$|e(x, y, \lambda)| \leq C \cdot \lambda^{n/m}, \qquad x, y \in M,$$

by using heat-equation-methods or, equivalently, zeta-function-methods. Nevertheless, bounds of the form (1.2), which are necessary for proving the local Weyl law (1.1), are not accessible via these techniques, and can only be obtained within the theory of Fourier integral operators, see [13] and [24, Sections 15 and 21, in particular Problem 15.1 and Lemma 21.4]. In the equivariant case, bounds of the form

$$|e_\gamma(x, y, \lambda)| \leq C \cdot \lambda^{\frac{n-\kappa}{m}}, \qquad x, y \in M,$$

could in principle be deduced from work of Donnelly [7] and Brüning-Heintze [3], at least when G acts on M with orbits of the same dimension κ. But they would not be sufficient to imply our results, and the desingularization techniques developed in [21] are necessary in order to describe the precise nature of the reduced spectral function of an invariant elliptic operator.

L^p-bounds for spectral clusters for elliptic second-order differential operators on 2-dimensional compact manifolds with boundary and either Dirichlet or Neumann conditions were shown in [25], while manifolds with maximal eigenfunction growth were studied in [28]. For locally symmetric spaces of higher rank, improved L^p-bounds have been shown by Sarnak and Marshall in [22, 20]. For a general overview on eigenfunctions on Riemannian manifolds, we refer to the survey articles [34, 33].

Through the whole document, the notation $O(\mu^k), k \in \mathbb{R} \cup \{\pm\infty\}$, will mean an estimate that is uniform in all relevant variables, while $O_\aleph(\mu^k)$ will denote an upper bound of the form $C_\aleph \, \mu^k$ with a constant $C_\aleph > 0$ that depends on the indicated variable $\aleph$.

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2. The reduced spectral function of an invariant elliptic operator

Let M be a closed connected Riemannian manifold M of dimension n with Riemannian volume density dM, and P_0 an elliptic classical pseudodifferential operator on M of degree m which is positive and symmetric. Its principal symbol $p(x, \xi)$ is homogeneous in ξ of degree m, and strictly positive on $T^*M \setminus \{0\}$. Denote its unique self-adjoint extension by P with domain the m-th Sobolev space $H^m(M)$, and let $\{e_j\}_{j \geq 0}$ be an orthonormal basis of L$^2(M)$ consisting of eigenfunctions of P with eigenvalues $\{\lambda_j\}_{j \geq 0}$ repeated according to their multiplicity. Next, consider the m-th root $Q := \sqrt[m]{P}$ of P given by the spectral theorem. It is well known that Q is a classical pseudodifferential operator of order 1 with principal symbol $q(x, \xi) := \sqrt[m]{p(x, \xi)}$ and domain $H^1(M)$. Again, Q has discrete spectrum, and

its eigenvalues are given by $\mu_j := \sqrt[m]{\lambda_j}$. The spectral function $e(x, y, \lambda)$ of P can then be described by studying the spectral function of Q, which in terms of the basis $\{e_j\}$ is given by

$$e(x, y, \mu) := \sum_{\mu_j \leq \mu} e_j(x)\overline{e_j(y)},$$

and belongs to $C^\infty(M \times M)$ as a function of x and y for any $\mu \in \mathbb{R}$. Let χ_μ be the spectral projection onto the sum of eigenspaces of Q with eigenvalues in the interval $(\mu, \mu + 1]$, and denote its Schwartz kernel by $\chi_\mu(x, y) := e(x, y, \mu + 1) - e(x, y, \mu)$. To obtain an asymptotic description of the spectral function of Q, one first derives a description of $\chi_\mu(x, y)$ by approximating χ_μ by Fourier integral operators. To do so, let $\varrho \in \mathcal{S}(\mathbb{R}, \mathbb{R}_+)$ be such that $\varrho(0) = 1$ and supp $\hat{\varrho} \in (-\delta/2, \delta/2)$ for a given $\delta > 0$, and define the approximate spectral projection operator

$$(2.1) \qquad \widetilde{\chi}_\mu u := \sum_{j=0}^\infty \varrho(\mu - \mu_j)E_j u, \qquad u \in L^2(M),$$

where E_j denotes the orthogonal projection onto the subspace spanned by e_j. Clearly, $K_{\widetilde{\chi}_\mu}(x, y) := \sum_{j=0}^\infty \varrho(\mu - \mu_j)e_j(x)\overline{e_j(y)} \in C^\infty(M \times M)$ constitutes the kernel of $\widetilde{\chi}_\mu$. Now, notice that for $\mu, \tau \in \mathbb{R}$ one has

$$\varrho(\mu - \tau) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\varrho}(t)e^{-it\tau}e^{it\mu} \, dt,$$

where $\hat{\varrho}(t)$ denotes the Fourier transform of ϱ, so that for $u \in L^2(M)$ we obtain

$$\widetilde{\chi}_\mu u = \frac{1}{2\pi} \sum_{j=0}^\infty \int_{\mathbb{R}} \hat{\varrho}(t)e^{it\mu}e^{-it\mu_j} \, dt \, E_j u = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\varrho}(t)e^{it\mu}U(t)u \, dt,$$

where $U(t)$ denotes the one-parameter group of unitary operators in $L^2(M)$

$$U(t) = \int e^{-it\mu}dE_\mu^Q = e^{-itQ}, \qquad t \in \mathbb{R},$$

given by the Fourier transform of the spectral measure, $\{E_\mu^Q\}$ being a spectral resolution of Q. The central result of Hörmander [13] then says that $U(t) = e^{-itQ} : L^2(M) \to L^2(M)$ can be approximated by Fourier integral operators, yielding an asymptotic formula for the kernels of $\widetilde{\chi}_\mu$ and χ_μ, and finally for the spectral function of Q.

Let us now come back to our initial problem, and assume that M carries an effective and isometric action of a compact connected Lie group G. Let P commute with the left-regular representation $(\pi, L^2(M))$ of G. Consider the Peter-Weyl decomposition (1.7) of $L^2(M)$, and let Π_γ be the projection onto the isotypic component belonging to $\gamma \in \widehat{G}$ which is given by the Bochner integral

$$\Pi_\gamma = d_\gamma \int_G \overline{\gamma(g)}\pi(g) \, d_G(g),$$

where d_γ is the dimension of an unitary irreducible representation of class γ, and $d_G(g) \equiv dg$ Haar measure on G which we assume to be normalized such that vol $G = 1$. In order to describe the spectral function of the operator $Q_\gamma := \Pi_\gamma \circ Q \circ \Pi_\gamma = Q \circ \Pi_\gamma = \Pi_\gamma \circ Q$ given by

$$(2.2) \qquad e_\gamma(x, y, \mu) := \sum_{\mu_j \leq \mu, \, e_j \in L^2_\gamma(M)} e_j(x)\overline{e_j(y)},$$

we consider the composition

$$(\chi_\mu \circ \Pi_\gamma)u = \sum_{\mu_j \in (\mu, \mu+1]} (E_j \circ \Pi_\gamma)u = \sum_{\mu_j \in (\mu, \mu+1], \, e_j \in L^2_\gamma(M)} E_j u, \qquad u \in L^2(M),$$

with kernel $K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x,y) = e_\gamma(x,y,\lambda+1) - e_\gamma(x,y,\lambda)$, together with the corresponding equivariant approximate spectral projection

$$(2.3) \qquad (\widetilde{\chi}_\mu \circ \Pi_\gamma)u = \sum_{j\geq 0,\, e_j \in \mathrm{L}^2_\gamma(M)} \varrho(\mu - \mu_j)E_j u = \frac{d_\gamma}{2\pi}\int_G \int_{\mathbb{R}} \hat{\varrho}(t)e^{it\mu}\overline{\gamma(g)}\big(U(t)\circ\pi(g)\big)u\, dt\, dg.$$

Its kernel can be written as

$$K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x,y) := \sum_{j\geq 0,\, e_j \in \mathrm{L}^2_\gamma(M)} \varrho(\mu - \mu_j)e_j(x)\overline{e_j(y)} \in \mathrm{C}^\infty(M\times M).$$

Put $m_\gamma(\mu_j) := d_\gamma \mathrm{mult}_\gamma(\mu_j)/\dim \mathcal{E}_{\mu_j}$, where $\mathrm{mult}_\gamma(\mu_j)$ denotes the multiplicity of an unitary irreducible representation of class γ in the eigenspace $\mathcal{E}_{\mu_j}$. In [21], an asymptotic formula for

$$\mathrm{tr}\,(\widetilde{\chi}_\mu \circ \Pi_\gamma) = \int_M K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x,x)\, dM(x) = \sum_{j=0}^\infty m_\gamma(\mu_j)\varrho(\mu - \mu_j)$$

was given in order to describe the behavior of the equivariant counting function as the eigenvalues become large, while now we are interested in the spectral function itself, which makes it necessary to derive asymptotics for the restriction of $K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}$ to the diagonal, or even to a neighborhood of it, and is more subtle than computing the trace.

As mentioned before, one can make use of the theory of Fourier integral operators to give an approximation of $U(t)$ in terms of oscillatory integrals. More precisely, let $\{(\kappa_\iota, Y_\iota)\}_{\iota\in I}$, $\kappa_\iota : Y_\iota \xrightarrow{\sim} \widetilde{Y}_\iota \subset \mathbb{R}^n$, be an atlas for M, $\{f_\iota\}$ a corresponding partition of unity and $\hat{v}(\eta) := \mathcal{F}(v)(\eta) := \int_{\mathbb{R}^n} e^{-i\langle \tilde{y}, \eta\rangle}v(\tilde{y})\, d\tilde{y}$ the Fourier transform of $v \in \mathrm{C}^\infty_c(\widetilde{Y}_\iota)$. Write $đ\eta := d\eta/(2\pi)^n$, and introduce $\widetilde{U}_\iota(t)$ the operator

$$[\widetilde{U}_\iota(t)v](\tilde{x}) := \int_{\mathbb{R}^n} e^{i\psi_\iota(t,\tilde{x},\eta)}a_\iota(t,\tilde{x},\eta)\hat{v}(\eta)đ\eta$$

on $\widetilde{Y}_\iota$, where $a_\iota \in S^0_{\mathrm{phg}}$ is a classical polyhomogeneous symbol satisfying $a_\iota(0,\tilde{x},\eta) = 1$ and ψ_ι the defining phase function given as the solution of the Hamilton-Jacobi equation

$$\frac{\partial\psi_\iota}{\partial t} + q\Big(x, \frac{\partial\psi_\iota}{\partial\tilde{x}}\Big) = 0, \qquad \psi_\iota(0,\tilde{x},\eta) = \langle\tilde{x},\eta\rangle,$$

see [15, Page 254]. Let us remark that ψ_ι is homogeneous in η of degree 1, so that Taylor expansion for small t gives

$$(2.4) \qquad \psi_\iota(t,\tilde{x},\eta) = \psi_\iota(0,\tilde{x},\eta) + t\frac{\partial\psi_\iota}{\partial t}(0,\tilde{x},\eta) + O(t^2|\eta|) = \langle\tilde{x},\eta\rangle - tq_\iota(\tilde{x},\eta) + O(t^2|\eta|),$$

where we wrote $q_\iota(\tilde{x},\eta) := q(\kappa_\iota^{-1}(\tilde{x}),\eta)$. In other words, there exists a smooth function ζ_ι which is homogeneous in η of degree 1 and satisfies

$$(2.5) \qquad \begin{aligned} \psi_\iota(t,\tilde{x},\eta) &= \langle\tilde{x},\eta\rangle - t\zeta_\iota(t,\tilde{x},\eta), & \zeta_\iota(0,\tilde{x},\eta) &= q_\iota(\tilde{x},\eta), \\ -2\,\partial_t\zeta_\iota(0,\tilde{x},\eta) &= \langle\partial_\eta q_\iota(\tilde{x},\eta), \partial_{\tilde{x}} q_\iota(\tilde{x},\eta)\rangle. & & \end{aligned}$$

Let now $\bar{U}_\iota(t)u := [\widetilde{U}_\iota(t)(u\circ\kappa_\iota^{-1})]\circ\kappa_\iota$, $u \in \mathrm{C}^\infty_c(Y_\iota)$. Consider further test functions $\bar{f}_\iota \in \mathrm{C}^\infty_c(Y_\iota)$ satisfying $\bar{f}_\iota \equiv 1$ on $\mathrm{supp}\, f_\iota$, and define

$$\bar{U}(t) := \sum_\iota F_\iota\, \bar{U}_\iota(t)\, \bar{F}_\iota,$$

where F_ι, $\bar{F}_\iota$ denote the multiplication operators corresponding to f_ι and $\bar{f}_\iota$, respectively. Then Hörmander showed that for small $|t|$

$$(2.6) \qquad\qquad R(t) := U(t) - \bar{U}(t)\ \text{is an operator with smooth kernel,}$$

compare [10, Page 134] and [24, Theorem 20.1]; in particular, the kernel $R_t(x,y)$ of $R(t)$ is smooth in t. In what follows we shall regard (x,η) as an element in $T^*Y \simeq Y \times \mathbb{R}^n$ with respect to the canonical trivialization of the co-tangent bundle over the chart domain. We now have the following

Proposition 2.1. *Let $\delta > 0$ be sufficiently small and $x, y \in M$. Then, as $\mu \to +\infty$,*

$$K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, y) = \frac{\mu^n d_\gamma}{(2\pi)^{n+1}} \sum_\iota \int_{-\infty}^{+\infty} \int_G \int_{\mathbb{R}^n} e^{i\mu t[1-\zeta_\iota(t, \kappa_\iota(x), \eta)]} e^{i\mu \langle \kappa_\iota(x) - \kappa_\iota(g \cdot y), \eta \rangle} \hat{\varrho}(t) \overline{\gamma(g)} f_\iota(x)$$

$$\cdot a_\iota(t, \kappa_\iota(x), \mu\eta) \bar{f}_\iota(g \cdot y) \alpha(q(x, \eta)) J_\iota(g, y) d\eta \, dg \, dt,$$

up to terms of order $O(\mu^{-\infty})$ which are uniform in x and y, where $0 \leq \alpha \in \mathrm{C}_c^\infty(1/2, 3/2)$ is a test function such that $\alpha \equiv 1$ in a neighborhood of 1, $J_\iota(g, y)$ is a Jacobian, and $d\eta$ denotes Lebesgue measure on $\mathbb{R}^n$. On the other hand, $K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, y)$ is rapidly decaying as $\mu \to -\infty$.

Proof. To obtain an explicit espression for the kernel of $\widetilde{\chi}_\mu \circ \Pi_\gamma$ let $u \in \mathrm{C}^\infty(M)$, and notice that

$$F_\iota \bar{U}_\iota(t) \bar{F}_\iota u(x) = f_\iota(x)[\widetilde{U}_\iota(t)(\bar{f}_\iota u \circ \kappa_\iota^{-1})] \circ \kappa_\iota(x)$$

$$= f_\iota(x) \int_{\mathbb{R}^n} e^{i\psi_\iota(t, \kappa_\iota(x), \eta)} a_\iota(t, \kappa_\iota(x), \eta) (\widehat{\bar{f}_\iota u \circ \kappa_\iota^{-1}})(\eta) d\eta$$

$$= \int_{\widetilde{Y}_\iota} \int_{\mathbb{R}^n} f_\iota(x) e^{i[\psi_\iota(t, \kappa_\iota(x), \eta) - \langle \tilde{y}, \eta \rangle]} a_\iota(t, \kappa_\iota(x), \eta) (\bar{f}_\iota u)(\kappa_\iota^{-1}(\tilde{y})) d\tilde{y} \, d\eta$$

$$= \int_{Y_\iota} \left[\int_{\mathbb{R}^n} e^{i[\psi_\iota(t, \kappa_\iota(x), \eta) - \langle \kappa_\iota(y), \eta \rangle]} a_\iota(t, \kappa_\iota(x), \eta) \, d\eta \, f_\iota(x) \, \bar{f}_\iota(y) (\beta_\iota^{-1} \circ \kappa_\iota)(y) \right] u(y) \, dM(y),$$

where we wrote $(\kappa_\iota^{-1})^*(dM) = \beta_\iota d\tilde{y}$. The last two expressions are oscillatory integrals with suitable regularizations. With (2.3) and (2.6) we therefore obtain for $K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, y)$ the expression

$$\frac{d_\gamma}{(2\pi)^{n+1}} \sum_\iota \int_{-\infty}^{+\infty} \int_G \int_{\mathbb{R}^n} \hat{\varrho}(t) e^{it\mu} \overline{\gamma(g)} f_\iota(x) e^{i[\psi_\iota(t, \kappa_\iota(x), \eta) - \langle \kappa_\iota(g \cdot y), \eta \rangle]} a_\iota(t, \kappa_\iota(x), \eta)$$

$$\cdot \bar{f}_\iota(g \cdot y) J_\iota(g, y) d\eta \, dg \, dt + O(|\mu|^{-\infty}),$$

since

$$\frac{1}{2\pi} \int_G \int_{-\infty}^{+\infty} \hat{\varrho}(t) e^{it\mu} R_t(x, g \cdot y) \, dt \, \overline{\gamma(g)} J_\iota(g, y) \, dg = \int_G \mathcal{F}^{-1}\big(\hat{\varrho}(\bullet) R_\bullet(x, g \cdot y)\big)(\mu) \, \overline{\gamma(g)} J_\iota(g, y) \, dg,$$

$\mathcal{F}^{-1}\big(\hat{\varrho}(\bullet) R_\bullet(x, g \cdot y)\big)$ being rapidly falling in μ; in particular, $O(|\mu|^{-\infty})$ is uniform in x, y. We are interested in the asymptotic behavior of $K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, y)$ as $\mu \to \pm\infty$. In order to study it by means of the stationary phase theorem, we define

$$\mathcal{G}(\tau, \tilde{x}, \eta) := \int_{-\infty}^{+\infty} e^{it\tau} \hat{\varrho}(t) a_\iota(t, \tilde{x}, \eta) e^{iO(t^2|\eta|)} dt,$$

where $O(t^2|\eta|)$ denotes the remainder in (2.4). Clearly, $\mathcal{G}(\tau, \tilde{x}, \eta)$ is rapidly decaying as a function in τ. On the other hand, there must exist a constant $C > 0$ such that

$$C|\eta| \geq q_\iota(\tilde{x}, \eta) \geq \frac{1}{C}|\eta| \qquad \forall \tilde{x} \in \widetilde{Y}_\iota, \, \eta \in \mathbb{R}^n,$$

which implies that for fixed μ, $\mathcal{G}(\mu - q_\iota(\tilde{x}, \eta), \tilde{x}, \eta)$ is rapidly decaying in η. This yields a regularization of the oscillatory integral above, and we obtain

$$K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, y) = \frac{d_\gamma}{(2\pi)^{n+1}} \sum_\iota \int_G \int_{\mathbb{R}^n} e^{i\langle \kappa_\iota(x) - \kappa_\iota(g \cdot y), \eta \rangle} \overline{\gamma(g)} f_\iota(x)$$

$$\cdot \mathcal{G}(\mu - q(x, \eta), \kappa_\iota(x), \eta) \bar{f}_\iota(g \cdot y) J_\iota(g, y) d\eta \, dg + O(|\mu|^{-\infty}).$$

But even more is true. $K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, y)$ is rapidly decreasing as $\mu \to -\infty$, reflecting the positivity of the spectrum. Furthermore, assume that $|1 - q_\iota(\tilde{x}, \eta/\mu)| \geq \mathrm{const} > 0$. Then

$$|\mathcal{G}(\mu - q_\iota(\tilde{x}, \eta), \tilde{x}, \eta)| \leq C_{N+M} \frac{1}{|\mu|^N} \frac{1}{|1 - q_\iota(\tilde{x}, \eta/\mu)|^N} \frac{1}{|\mu - q_\iota(\tilde{x}, \eta)|^M}$$

$$\leq C'_{N+M} \frac{1}{|\mu|^N} \frac{1}{|\mu - q_\iota(\tilde{x}, \eta)|^M}$$

for arbitrary $N, M \in \mathbb{N}$ and suitable constants. Let therefore $\alpha \in \mathrm{C}_c^\infty(1/2, 3/2)$ be as indicated, so that

$$1 - \alpha(q_\iota(\tilde{x}, \eta/\mu)) \neq 0 \quad \Longrightarrow \quad |1 - q_\iota(\tilde{x}, \eta/\mu)| \geq C > 0$$

for a constant depending only on α. Substituting $\eta = \mu\eta'$, we can re-write $K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, y)$ as

$$K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, y) = \frac{|\mu|^n d_\gamma}{(2\pi)^{n+1}} \sum_\iota \int_{-\delta/2}^{+\delta/2} \int_G \int_{\mathbb{R}^n} e^{i\mu\left[\psi_\iota(t, \kappa_\iota(x), \eta) - \langle \kappa_\iota(g \cdot y), \eta \rangle + t\right]} \hat{\varrho}(t) \overline{\gamma(g)} f_\iota(x)$$
$$\cdot a_\iota(t, \kappa_\iota(x), \mu\eta) \bar{f}_\iota(g \cdot y) \alpha(q(x, \eta)) J_\iota(g, y) \, d\eta \, dg \, dt + O(|\mu|^{-\infty}),$$

where all integrals are absolutely convergent, and the remainder is uniform in x, y. The proposition now follows with (2.5). $\qquad\square$

Since $\zeta_\iota(0, \tilde{x}, \omega) = q_\iota(\tilde{x}, \omega)$, there exists a constant $C > 0$ such that for sufficiently small $t \in (-\delta/2, \delta/2)$

$$C|\eta| \geq \zeta_\iota(t, \tilde{x}, \eta) \geq \frac{1}{C}|\eta| \qquad \forall \tilde{x} \in \widetilde{Y}_\iota, \, \eta \in \mathbb{R}^n.$$

We can therefore introduce in $\mathbb{R}^n \setminus \{0\}$ the coordinates

$$\eta = R\omega_1, \qquad R > 0, \qquad \zeta_\iota(t, \kappa_\iota(x), \omega_1) = 1.$$

Indeed, since $\zeta_\iota(t, \kappa_\iota(x), \eta)$ is homogeneous of degree 1 in η, its derivative in radial direction reads

$$\nabla_{\omega_1} \zeta_\iota(t, \kappa_\iota(x), \eta) = \lim_{s \to 0} s^{-1}(R + s - R)\zeta_\iota(t, \kappa_\iota(x), \omega_1) = 1,$$

so that for all $\eta = R\omega_1$ we have

$$(2.7) \qquad\qquad \langle \mathrm{grad}_\eta \zeta_\iota(t, \tilde{x}, \eta), \eta \rangle = R > 0.$$

Consequently, the Jacobian of the coordinate change $\eta = R\omega_1$ does not vanish. Re-writing the expression for the kernel of $\widetilde{\chi}_\mu \circ \Pi_\gamma$ in Proposition 2.1 in terms of the coordinates $\eta = R\omega_1$ we obtain

$$(2.8) \qquad K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, y) = \frac{\mu^n d_\gamma}{(2\pi)^{n+1}} \sum_\iota \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\mu[t - Rt]} \int_G \int_{\Sigma_{\iota,x}^{R,t}} e^{i\mu\langle \kappa_\iota(x) - \kappa_\iota(g \cdot y), \omega \rangle} \hat{\varrho}(t) \overline{\gamma(g)} f_\iota(x)$$
$$\cdot a_\iota(t, \kappa_\iota(x), \mu\omega) \bar{f}_\iota(g \cdot y) \alpha(q(x, \omega)) J_\iota(g, y) \, d\Sigma_{\iota,x}^{R,t}(\omega) \, dg \, dR \, dt$$

up to terms of order $O(\mu^{-\infty})$ which are uniform in x and y, where we set

$$(2.9) \qquad\qquad \Sigma_{\iota,x}^{R,t} := \{\omega \in \mathbb{R}^n : \zeta_\iota(t, \kappa_\iota(x), \omega) = R\}.$$

Here $d\Sigma_{\iota,x}^{R,t}(\omega)$ denotes the quotient of Lebesgue measure in $\mathbb{R}^n$ by Lebesgue measure in $\mathbb{R}$ with respect to $\zeta_\iota(t, \tilde{x}, \omega)$. Note that for sufficiently small $\delta > 0$ we can assume that the R-integration is over a compact set. Furthermore, R and t are close to 1 and 0, respectively. To describe the asymptotic behavior of $K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, y)$ as $\mu \to +\infty$, we shall now first apply the stationary phase theorem to the integral over R and t, and then to the integral over $G \times \Sigma_{\iota,x}^{R,t}$.

Corollary 2.2. *Let $\mu \geq 1$, $x, y \in M$, and with the notation of Proposition 2.1 put*

$$(2.10) \qquad I_\iota(\mu, R, t, x, y) := \int_G \int_{\Sigma_{\iota,x}^{R,t}} e^{i\mu\Phi_{\iota,x,y}(\omega, g)} \hat{\varrho}(t) \overline{\gamma(g)} f_\iota(x)$$
$$\cdot a_\iota(t, \kappa_\iota(x), \mu\omega) \bar{f}_\iota(g \cdot y) \alpha(q(x, \omega)) J_\iota(g, y) \, d\Sigma_{\iota,x}^{R,t}(\omega) \, dg,$$

where $\Phi_{\iota,x,y}(\omega, g) := \langle \kappa_\iota(x) - \kappa_\iota(g \cdot y), \omega \rangle$. Then, for each $\tilde{N} \in \mathbb{N}$

$$(2.11) \qquad K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, y) = \sum_\iota \left[(\mu/2\pi)^{n-1} \frac{d_\gamma}{2\pi} \sum_{j=0}^{\tilde{N}-1} D_{R,t}^{2j} I_\iota(\mu, R, t, x, y)_{|(R,t)=(1,0)} \, \mu^{-j} + d_\gamma \mathcal{R}_\iota(\mu, x, y) \right]$$

up to terms of order $O(\mu^{-\infty})$ which are uniform in x,y, where $D_{R,t}^{2j}$ are known differential operators of order $2j$ in R, t, and

$$|\mathcal{R}_\iota(\mu, x, y)| \leq C\mu^{n-\tilde{N}-1} \sum_{|\beta| \leq 2\tilde{N}+3} \sup_{R,t} \left| \partial_{R,t}^\beta I_\iota(\mu, R, t, x, y) \right|$$

for some constant $C > 0$.

Proof. Since $(R, t) = (1, 0)$ is the only critical point of $t - Rt$, the assertion follows immediately from (2.8) and the classical stationary phase theorem [10, Proposition 2.3]. $\qquad\square$

Thus, we are left with the task of describing the asymptotics of the oscillatory integrals $I_\iota(\mu, R, t, x, y)$ as $\mu \to +\infty$, which will occupy us in the next sections.

3. Equivariant asymptotics of oscillatory integrals

Let the notation be as in the previous section. As we have seen there, the question of describing the spectral function in the equivariant setting reduces to the study of oscillatory integrals of the form

$$(3.1) \qquad I_{x,y}(\mu) := \int_G \int_{\Sigma_x^{R,t}} e^{i\mu\Phi_{x,y}(\omega,g)} a_\mu(x, y, \omega, g) \, d\Sigma_x^{R,t}(\omega) \, dg, \qquad \mu \to +\infty,$$

with $\Sigma_x^{R,t}$ as in (2.9) and phase function

$$\Phi_{x,y}(\omega, g) := \langle \kappa(x) - \kappa(g \cdot y), \omega \rangle \,,$$

where we have skipped the index ι for simplicity of notation, and $a_\mu \in \mathrm{C}_c^\infty$ is an amplitude that might depend on μ such that $(x, y, \xi, g) \in \mathrm{supp}\, a_\mu$ implies $x, g \cdot y \in Y$. The asymptotic behavior of these integrals is related to that of oscillatory integrals of the form

$$(3.2) \qquad I(\mu) = \int_G \int_{T^*Y} e^{i\mu\Phi(x,\eta,g)} a_\mu(x, \eta, g) \, d(T^*Y)(x, \eta) \, dg, \qquad \mu \to +\infty,$$

with phase function

$$(3.3) \qquad\qquad \Phi(x, \eta, g) := \langle \kappa(x) - \kappa(g \cdot x), \eta \rangle \,.$$

Asymptotics for the integrals (3.2) were given in [21] using the stationary phase principle in combination with resolution of singularities, and we will rely on these results in the following to perform a similar analysis for the integrals $I_{x,y}(\mu)$. Write $\kappa(x) = (\tilde{x}_1, \ldots, \tilde{x}_n)$ so that the canonical local trivialization of T^*Y reads

$$Y \times \mathbb{R}^n \ni (x, \eta) \quad \equiv \quad \sum_{k=1}^n \eta_k (d\tilde{x}_k)_x \in T_x^* Y.$$

With respect to this trivialization, we shall identify $\Sigma_{x'}^{R,t}$ with a subset in $T_x^* Y$ for eventually different x and x', if convenient. Now, one computes for any $X \in \mathfrak{g}$

$$\frac{d}{dt} \Phi(x, \eta, e^{tX})_{|t=0} = \frac{d}{dt} \left\langle \kappa(e^{-tX} \cdot x), \eta \right\rangle_{|t=0} = \sum \eta_i \widetilde{X}_x(\tilde{x}_i) = \sum \eta_i (d\tilde{x}_i)_x(\widetilde{X}_x).$$

Furthermore, one has

$$\partial_{\tilde{x}} \Phi(\kappa^{-1}(\tilde{x}), \eta, g) = [\mathbf{1} - {}^T(\kappa \circ g \circ \kappa^{-1})_{*,\tilde{x}}]\eta = (\mathbf{1} - g_{\tilde{x}}^*)\eta,$$

so that $\partial_x \Phi(x, \eta, g) = 0$ amounts to the condition $g^*\eta = \eta$. In the same way, $\partial_\eta \Phi(x, \eta, g) = 0$ if, and only if, $g \cdot x = x$. Let $\Omega := \mathbb{J}^{-1}(\{0\})$ be the zero level set of the momentum map $\mathbb{J} : T^*M \to \mathfrak{g}^*$ of the underlying Hamiltonian G-action on T^*M. Since

$$(3.4) \qquad\qquad (x, \eta) \in \Omega \cap T_x^* M \quad \Longleftrightarrow \quad (x, \eta) \in \mathrm{Ann}(T_x(G \cdot x)),$$

where $\mathrm{Ann}\,(V_x) \subset T_x^* M$ denotes the annihilator of a vector subspace $V_x \subset T_x M$, the critical set of Φ is consequently given by

$$(3.5) \quad \mathrm{Crit}\, \Phi = \left\{ (x, \eta, g) \in T^*Y \times G : (\Phi_*)_{(x,\eta,g)} = 0 \right\} = \left\{ (x, \eta, g) \in (\Omega \cap T^*Y) \times G : g \in G_{(x,\eta)} \right\}.$$

Now, unless the G-action on T^*M is free, Ω and $\mathrm{Crit}\,\Phi$ are not smooth manifolds in general. In particular, the regular part of the critical set of Φ is given by

$$\mathrm{Reg}\,\mathrm{Crit}\,\Phi = \left\{(x,\eta,g) \in (\mathrm{Reg}\,\Omega \cap T^*Y) \times G : g \in G_{(x,\eta)}\right\},$$

where $\mathrm{Reg}\,\Omega = \left\{(x,\eta) \in \Omega : G_{(x,\eta)} \sim H_L\right\}$ denotes the regular part of Ω, and $\mathrm{Reg}\,\Omega \equiv \mathrm{Reg}\,\Omega \cap T^*(M_{\mathrm{prin}})$ up to a set of measure zero, while $M_{\mathrm{prin}} := M(H_L) \subset M$ represents the union of orbits of principal type. Furthermore, $\mathrm{Reg}\,\mathrm{Crit}\,\Phi$ is a manifold of co-dimension 2κ, where κ is the dimension of an orbit of principal type in M. For details, the reader is referred to [21, Sections 3 and 4]. We come now to the description of the critical set of the phase function $\Phi_{x,y}$. Let $\mathcal{O}_x := G \cdot x$ denote the G-orbit, and $G_x := \{g \in G : g \cdot x = x\}$ the stabilizer or isotropy group of a point $x \in M$. We then have the following

Lemma 3.1. *Let $x \in Y$, $\mathcal{O}_y \cap Y \neq \emptyset$, and*

$$\mathrm{Crit}\,\Phi_{x,y} := \left\{(\omega,g) \in \Sigma_x^{R,t} \times \{g \in G : g \cdot y \in Y\} : \partial_{\omega,g}\,\Phi_{x,y}(\omega,g) = 0\right\}$$

be the critical set of $\Phi_{x,y}$.

(a) *If $y \in \mathcal{O}_x$, $\mathrm{Crit}\,\Phi_{x,y}$ is given by*

$$\mathcal{C}_{x,y} := \left\{(\omega,g) : (g \cdot y, \omega) \in \Omega,\ x = g \cdot y\right\}$$

and a smooth sub-manifold of co-dimension $2\dim\mathcal{O}_x$; furthermore, the Hessian $\mathrm{Hess}\,\Phi_{x,y}$ is non-degenerate on $N_{(\omega,g)}\mathcal{C}_{x,y}$ for all $(\omega,g) \in \mathcal{C}_{x,y}$. In other words, $\mathrm{Crit}\,\Phi_{x,y}$ is clean.

(b) *In case that $y \notin \mathcal{O}_x$,*

$$\mathrm{Crit}\,\Phi_{x,y} = \left\{(\omega,g) : (g \cdot y, \omega) \in \Omega,\ \kappa(x) - \kappa(g \cdot y) \in N_\omega \Sigma_x^{R,t}\right\};$$

furthermore, assume that G acts on M with orbits of the same dimension κ, that is, $M = M_{\mathrm{prin}} \cup M_{\mathrm{except}}$, and that the co-spheres S_x^*M are strictly convex. Then, choosing Y sufficiently small one has that locally

$$\mathrm{Crit}\,\Phi_{x,y} \simeq G_y,$$

and $\mathrm{Crit}\,\Phi_{x,y}$ is clean and of co-dimension $n - 1 + \kappa$.

(c) *In case that $x \in Y \cap M_{\mathrm{prin}}$ one has*

$$\mathcal{C}_{x,x} = \mathrm{Crit}\,\Phi \cap (\Sigma_x^{R,t} \times G),$$

a transversal intersection. In particular $\mathcal{C}_{x,x}$ is a smooth sub-manifold of co-dimension 2κ.

Proof. We shall show (a) first by a transversality argument. Let $x \in Y$ and $y \in \mathcal{O}_x$ be fixed, and consider for $(x,\eta) \in T_x^*Y$ and $g \in \{g \in G : g \cdot y \in Y\}$ the function $\check{\Phi}_{x,y}(\eta,g) := \langle \kappa(x) - \kappa(g \cdot y), \eta \rangle$. The derivatives of $\check{\Phi}_{x,y}$ with respect to g read $\sum_{k=1}^n \eta_k (d\tilde{x}_k)_{g \cdot y}(\widetilde{X}_j)$, where $\{X_1, \ldots, X_d\}$ denotes a basis of $\mathfrak{g}$, and $\{\widetilde{X}_1, \ldots, \widetilde{X}_d\}$ the corresponding fundamental vector fields on M. Setting them equal zero yields, as in (3.5), $(g \cdot y, \eta) \in \Omega$. On the other hand, differentiation with respect to η gives $g \cdot y = x$, so that

(3.6) $$\mathrm{Crit}\,\check{\Phi}_{x,y} = (\Omega \cap T_x^*Y) \times \{g \in G : x = g \cdot y\}.$$

Next, introduce for fixed R and t in T_x^*Y the coordinates $\eta = s\omega$, $s > 0$, $\zeta(t, \kappa(x), \omega) = R$. Clearly,

(3.7) $$\partial_{s,\omega}[\check{\Phi}_{x,y}(s\omega,g)] = 0 \qquad \Longleftrightarrow \qquad [\partial_\eta \check{\Phi}_{x,y}](s\omega,g) = 0,$$

so that $\mathrm{Crit}\,\Phi_{x,y} \supset \mathcal{C}_{x,y}$. Further, note that $(\omega,g) \in \mathrm{Crit}\,\Phi_{x,y}$ iff $\partial_{\omega,g}[\check{\Phi}_{x,y}(s\omega,g)] = 0$ for arbitrary $s \neq 0$, since $\partial_{\omega,g}[\check{\Phi}_{x,y}(s\omega,g)] = s\,\partial_{\omega,g}[\langle \kappa(x) - \kappa(g \cdot y), \omega \rangle]$. We now assert that

(3.8) $$\partial_{\omega,g}[\check{\Phi}_{x,y}(s\omega,g)] = 0 \qquad \Longrightarrow \qquad \partial_s[\check{\Phi}_{x,y}(s\omega,g)] = 0.$$

Indeed, by the above $\partial_s[\check{\Phi}_{x,y}(s\omega,g)] = \langle \kappa(x) - \kappa(g \cdot y), \omega \rangle \equiv \Phi_{x,y}(\omega,g)$ is constant if $\partial_{\omega,g}[\check{\Phi}_{x,y}(s\omega,g)] = 0$. But $\partial_s[\check{\Phi}_{x,y}(s\omega,g)] = 0$ if $x = g \cdot y$, yielding that $\partial_s[\check{\Phi}_{x,y}(s\omega,g)]$ vanishes for $(\omega,g) \in \mathrm{Crit}\,\Phi_{x,y}$ since G is assumed to be connected, and $\Omega \cap T_x^*Y$ is connected in view of (3.4). Hence, (3.8) is proven. For $(\omega,g) \in \mathrm{Crit}\,\Phi_{x,y}$ the implications (3.7) and (3.8) yield $[\partial_{\eta,g}\,\check{\Phi}_{x,y}](s\omega,g) = 0$ for any $s \neq 0$, and with

(3.6) we obtain the first part of (a). In particular, $\Omega \cap \Sigma_x^{R,t} \subset T_x^* Y$ is a smooth sub-manifold in view of (3.4), and $\{g \in G : x = g \cdot y\} \simeq G_x$, too. To see the second, note that with (3.6) we have

$$\mathcal{C}_{x,y} = \text{Crit } \check{\Phi}_{x,y} \cap (\Sigma_x^{R,t} \times G),$$

a transversal intersection. As in [21, Proof of Lemma 7.3] one easily sees that $\check{\Phi}_{x,y}$ has a non-degenerate transversal Hessian. But then [21, Lemma 7.1] implies that $\Phi_{x,y}$ must have a non-degenerate transversal Hessian as well, and we obtain (a).

Alternatively, one can show (a) explicitly by considering a local parametrization

$$(3.9) \qquad F : \mathbb{R}^{n-1} \supset U \longrightarrow \Sigma_x^{R,t} \subset \mathbb{R}^n, \quad \alpha \longmapsto F(\alpha) = \omega,$$

of the hyerpsurface $\Sigma_x^{R,t}$, U being an open subset. While differentiating $\Phi_{x,y}$ with respect to g and setting the derivatives to zero yields $(g \cdot y, \omega) \in T_{g \cdot y}^* Y \cap \Omega \simeq N_{g \cdot y} \mathcal{O}_y$, differentiating $\Phi_{x,y}$ with respect to α gives the conditions $\langle \kappa(x) - \kappa(g \cdot y), \partial F / \partial \alpha_i \rangle = 0$ for $i = 1, \dots, n-1$, implying that $\kappa(x) - \kappa(g \cdot y)$ must be normal to $\Sigma_x^{R,t}$ at ω. Consequently,

$$(3.10) \qquad \text{Crit } \Phi_{x,y} = \left\{ (\omega, g) : (g \cdot y, \omega) \in \Omega, \; \kappa(x) - \kappa(g \cdot y) \in N_\omega \Sigma_x^{R,t} \right\}.$$

The second condition means that $\kappa(x) - \kappa(g \cdot y)$ is co-linear to $\text{grad}_\eta \zeta(t, \kappa_\iota(x), \omega)$. But in view of (2.7) we have the equality

$$(3.11) \qquad \left\langle \text{grad}_\eta \zeta(t, \tilde{x}, \omega), \omega \right\rangle = R > 0, \qquad \omega \in \Sigma_x^{R,t},$$

so that if $x \neq g \cdot y$ and $\kappa(x) - \kappa(g \cdot y) \in N_\omega \Sigma_x^{R,t}$, we deduce the lower bound

$$(3.12) \qquad \left| \left\langle \frac{\kappa(x) - \kappa(g \cdot y)}{\|\kappa(x) - \kappa(g \cdot y)\|}, \omega \right\rangle \right| \geq C > 0$$

for a uniform constant $C > 0$. Since the G-action on M is smooth, and hence, locally smooth, there is a linear tube around each G-orbit in M, and we may assume that the chart (κ, Y) is given in terms of such a tube around $\mathcal{O}_x$. Thus, let $\mathcal{O}_x \simeq G/H$, V be an Euclidean vector space with orthogonal H-action, and

$$\tau : G \times_H V \longrightarrow M$$

a linear tube around $\mathcal{O}_x$, that is, a G-equivariant embedding onto an open neighborhood of $\mathcal{O}_x$. If $H = G_x$, $S_x := \tau([e, V])$ is a slice at x, and

$$S_{g \cdot x} := \tau([g, V]) = \tau(g[e, V]) = g \cdot S_x$$

a slice at $g \cdot x$. Let $Y \subset \tau(G \times_H V)$, and identify $\kappa(S_{g \cdot x} \cap Y)$ with a subset of a linear subspace in $\mathbb{R}^n$, which in turn can be identified with $N_{g \cdot x} \mathcal{O}_x$, compare [2, Corollary VI.2.4]. Now, take $(\omega, g) \in \text{Crit } \Phi_{x,y}$, and assume that $y \in \mathcal{O}_x$, which means that the vector $\kappa(x) - \kappa(g \cdot y)$ must be approximately tangential to $\kappa(\mathcal{O}_x \cap Y)$, hence normal to $\kappa(S_{g \cdot y} \cap Y)$ at $\kappa(g \cdot y)$ for $d(x, g \cdot y) \ll 1$. For sufficiently small Y, the lower bound (3.12) then implies that $x = g \cdot y$, since ω is normal to $\mathcal{O}_x$ at $g \cdot y$. Thus, we conclude that $\text{Crit } \Phi_{x,y} = \mathcal{C}_{x,y}$. In order to see that $\text{Crit } \Phi_{x,y}$ is clean, note that with respect to the parametrization (3.9) of $\Sigma_x^{R,t}$ and canonical coordinates on G the Hessian of $\Phi_{x,y}$ at a critical point $(\omega, g) \in \mathcal{C}_{x,y}$ is given by the matrix

$$\text{Hess } \Phi_{x,y}(\omega, g) \equiv \begin{pmatrix} 0 & \sum_{k=1}^n \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_i} (d\tilde{x}_k)_x(\widetilde{X}_j) \\ \sum_{k=1}^n \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_j} (d\tilde{x}_k)_x(\widetilde{X}_i) & -\left\langle \widetilde{X}_{i,x}(\widetilde{X}_j(\kappa)), \omega \right\rangle \end{pmatrix},$$

where $\{X_1, \dots, X_d\}$ denotes a basis of $\mathfrak{g}$. The kernel of the corresponding linear transformation is given by those $(\tilde{\alpha}, \tilde{s}) \in \mathbb{R}^{n-1} \times \mathbb{R}^d$ satisfying the conditions

$$(3.13) \qquad \sum_k \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_i} (d\tilde{x}_k)_x \left(\sum_j \tilde{s}_j \widetilde{X}_j \right) = 0 \qquad \text{for all } i = 1, \dots, n-1,$$

$$(3.14) \qquad \sum_{j,k} \tilde{\alpha}_j \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_j} (d\tilde{x}_k)_x(\widetilde{X}_i) = 0 \qquad \text{for all } i = 1, \dots, d.$$

Indeed, (3.13) implies that $\left((d\tilde{x}_1)_x (\sum_j \tilde{s}_j \widetilde{X}_j), \ldots, (d\tilde{x}_n)_x (\sum_j \tilde{s}_j \widetilde{X}_j) \right)$ is co-linear to $\mathrm{grad}_\eta\, \zeta(t, \kappa_\iota(x), \omega)$. In view of (3.11) and the fact that $(\widetilde{X}_j)_x \in T_x \mathcal{O}_x$, $(x, \omega) \in N_x \mathcal{O}_x$, we conclude that $\sum_j \tilde{s}_j (\widetilde{X}_j)_x = 0$; in particular, the terms $\left\langle \widetilde{X}_{i,x}(\widetilde{X}_j(\kappa)), \omega \right\rangle$ do not contribute to the equations (3.14). Thus, the kernel in question is given by

$$\left\{ (\tilde{\alpha}, \tilde{s}) \in \mathbb{R}^{n-1} \times \mathbb{R}^d : \sum_j \tilde{s}_j (\widetilde{X}_j)_x = 0, \ \sum_{j,k} \tilde{\alpha}_j \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_i} (d\tilde{x}_k)_x \in \mathrm{Ann}\ (T_x \mathcal{O}_x) \right\} \simeq T_{(\omega,g)} \mathcal{C}_{x,y},$$

which means that Hess $\Phi_{x,y}$ is transversally non-degenerate on $\mathcal{C}_{x,y}$, yielding again (a).

To show (b), assume that $y \notin \mathcal{O}_x$. Note that without loss of generality we can assume that $y \in S_x \cap Y$. The first part of (b) is clear from (3.10). Now, assume that the co-spheres $S_x^* M$ are strictly convex. For small $|t| \ll 1$, the hypersurfaces $\Sigma_x^{R,t}$ will be strictly convex, too. In particular, $\Sigma_x^{R,t}$ is orientable, and the Gauss map

$$\mathcal{N} : \Sigma_x^{R,t} \ni \omega \longmapsto \mathcal{N}(\omega) \in N_\omega \Sigma_x^{R,t},$$

which assigns to each point of $\Sigma_x^{R,t}$ the *outer* normal unit vector to $\Sigma_x^{R,t}$ at that point, is a global diffeomorphism. Therefore, for each $x \neq \tilde{y} \in Y$ there is a unique $\omega_{\tilde{y}} \in \Sigma_x^{R,t}$ such that

$$\frac{\kappa(\tilde{y}) - \kappa(x)}{\|\kappa(\tilde{y}) - \kappa(x)\|} = \mathcal{N}(\omega_{\tilde{y}}).$$

Consequently, if $(\omega, g) \in \mathrm{Crit}\, \Phi_{x,y}$, ω is locally uniquely determined by the condition $\mathcal{N}(\omega) = \pm \mathcal{N}(\omega_{g \cdot y})$.

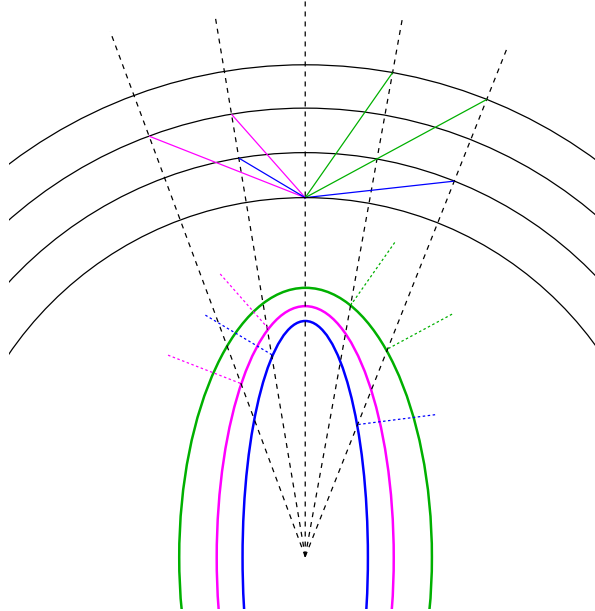FIGURE 3.1. Concerning the critical set of $\Phi_{x,y}$ in case that $y \notin \mathcal{O}_x$. The black circle segments represent G-orbits in $Y \equiv \kappa(Y) \subset \mathbb{R}^n$, the inner one through x and the outer ones through different points y; the black dotted lines represent normal spaces to the orbits. The three coloured ellipse segments depict different hypersurfaces $\Sigma_x^{R,t} \subset \mathbb{R}^n$ whose normal at $\omega \in N_{g \cdot y} \mathcal{O}_y \cap \Sigma_x^{R,t}$, depicted by a colored dotted line, is given by the corresponding colored line segments $\kappa(x) - \kappa(g \cdot y)$.

Now, introduce the sets

$$U_n := \tau(G \times_H V_{1/n}), \qquad V_{1/n} := \{v \in V : \|v\| < 1/n\}, \qquad n \in \mathbb{N},$$

and assume that for each $n \in \mathbb{N}$ there is a $y_n \in U_n \cap Y \cap S_x$ such that $\operatorname{Crit} \Phi_{x,y_n}$ is not empty, but $\operatorname{Crit} \Phi_{x,y_n} \not\simeq G_{y_n}$. In other words, assume that for each $n \in \mathbb{N}$ there is a smooth curve

$$\gamma_n : (-\varepsilon_n, \varepsilon_n) \ni t \longmapsto (\omega_n(t), g_n(t)) \in \operatorname{Crit} \Phi_{x,y_n}, \qquad \varepsilon_n > 0,$$

parametrized such that $\|\dot{\omega}_n(t)\| = 1$. In this way, we obtain for each $n \in \mathbb{N}$ a curve $\omega_n(t)$ in $\Sigma_x^{R,t}$ along which the unit normal vector field to $\Sigma_x^{R,t}$ is determined by the direction of $\kappa(x) - \kappa(g_n(t) \cdot y_n)$, so that $\mathcal{N}(\omega_n(t)) = \pm \mathcal{N}(\omega_{g_n(t) \cdot y_n})$. In view of (3.12), the curves

$$\{g_n(t) \cdot y_n : t \in (-\varepsilon_n, \varepsilon_n)\} \subset Y$$

converge to x as $n \to \infty$. Similarly, due to the compactness of $\Sigma_x^{R,t}$ the curves

$$\{\omega_n(t) : t \in (-\varepsilon_n, \varepsilon_n)\} \subset \Sigma_x^{R,t}$$

converge to at least one $\omega_\infty \in \Sigma_x^{R,t} \cap N_x \mathcal{O}_x$ after passing to a suitable convergent sub-sequence $\omega_{n_k}(t)$. Now, assume that G acts on M with orbits of the same dimension κ. If $\mathcal{O}_{\mathrm{prin}}$ is a principal orbit and $\mathcal{O}$ a principal or exceptional orbit, there is an equivariant covering map $\mathcal{O}_{\mathrm{prin}} \to \mathcal{O}$, so that $\mathcal{O}_{\mathrm{prin}}$ and $\mathcal{O}$ are locally diffeomorphic, compare [2, Page 181]. Therefore, we can assume that all orbits in Y are diffeomorphic, which implies that the more y_n approaches x, the faster the direction of $\kappa(x) - \kappa(g_n(t) \cdot y_n)$ changes as $t \in (-\varepsilon_n, \varepsilon_n)$ varies, and the faster $\mathcal{N}(\omega_n(t))$ changes as $t \in (-\varepsilon_n, \varepsilon_n)$ varies. Consequently, the Gaussian curvature of $\Sigma_x^{R,t}$ at ω_∞, which is given by the product of the principal curvatures, cannot stay bounded, compare Figure 3.1. Thus, we have shown that for sufficiently small Y we locally have

$$\operatorname{Crit} \Phi_{x,y} \simeq G_y, \qquad x, y \in Y,$$

which implies that $\operatorname{Crit} \Phi_{x,y}$ is a smooth sub-manifold of co-dimension $n - 1 + \dim \mathcal{O}_y$. We are left with the task of showing that $\operatorname{Hess} \Phi_{x,y}$ is transversally non-degenerate. For this, we are going to show that for each fixed $(\omega, g) \in \operatorname{Crit} \Phi_{x,y}$ one has $\operatorname{Ker} \operatorname{Hess} \Phi_{x,y}(\omega, g) \simeq \mathrm{T}_{(\omega,g)} \operatorname{Crit} \Phi_{x,y}$. To do so, note that with respect to the coordinates introduced in (a), the Hessian of $\Phi_{x,y}$ at a critical point (ω, g) is given by the matrix

$$(3.15) \quad \operatorname{Hess} \Phi_{x,y}(\omega, g) \equiv \begin{pmatrix} \left\langle \kappa(x) - \kappa(g \cdot y), \frac{\partial^2 F}{\partial \alpha_i \partial \alpha_j}(\alpha^{-1}(\omega)) \right\rangle & \sum_{k=1}^n \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_i}(d\tilde{x}_k)_{g \cdot y}(\widetilde{X}_j) \\ \sum_{k=1}^n \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_j}(d\tilde{x}_k)_{g \cdot y}(\widetilde{X}_i) & -\left\langle \widetilde{X}_{i,g \cdot y}(\widetilde{X}_j(\kappa)), \omega \right\rangle \end{pmatrix}.$$

Since $\kappa(g \cdot y) - \kappa(x) \in N_\omega \Sigma_x^{R,t}$, the sub-matrix in the first quadrant corresponds to a multiple of the second fundamental of $\Sigma_x^{R,t}$

$$\mathrm{II} : T\Sigma_x^{R,t} \times T\Sigma_x^{R,t} \longrightarrow \mathrm{C}^\infty(\Sigma_x^{R,t}), \quad \mathrm{II}(\mathcal{X}, \mathcal{Y}) := \langle \nabla_\mathcal{X} \mathcal{Y}, \mathcal{N} \rangle = \langle \mathcal{X}, A\mathcal{Y} \rangle,$$

where $\nabla_\mathcal{X} \mathcal{Y} \equiv \mathcal{X}(\mathcal{Y})$ denotes the covariant derivative in Euclidean space $\mathbb{R}^n$, and $A : T\Sigma_x^{R,t} \to T\Sigma_x^{R,t}$ the symmetric endomorphism induced by II [16, Chapter VII, Section 3]. Indeed, assume that $\kappa(x) - \kappa(g \cdot y)$ points in the direction of $-\mathcal{N}(\omega)$, and let $\partial / \partial \alpha_{i|\omega} := \partial F(\alpha^{-1}(\omega)) / \partial \alpha_i$, $1 \le i \le n-1$, be the coordinate frame given by the parametrization (3.9). Then, the sub-matrix in the first quadrant of (3.15) reads

$$(3.16) \qquad -\|\kappa(x) - \kappa(g \cdot y)\| \, \mathrm{II}\left(\frac{\partial}{\partial \alpha_{i|\omega}}, \frac{\partial}{\partial \alpha_{j|\omega}}\right) = -\|\kappa(x) - \kappa(g \cdot y)\| \left\langle \frac{\partial}{\partial \alpha_{i|\omega}}, A \frac{\partial}{\partial \alpha_{j|\omega}} \right\rangle.$$

To compute the kernel of the matrix (3.15), assume that the $X_1, \dots, X_d \in \mathfrak{g}$ are such that the vector fields $\widetilde{X}_1, \dots, \widetilde{X}_\kappa$ constitute an orthonormal basis of $T_{g \cdot y} \mathcal{O}_y$, and consider for $(\tilde{\alpha}, \tilde{s}) \in \mathbb{R}^{n-1} \times \mathbb{R}^d$ the system of equations

$$(3.17) \quad \sum_{j=1}^{n-1} \left\langle \kappa(x) - \kappa(g \cdot y), \frac{\partial^2 F}{\partial \alpha_i \partial \alpha_j}(\alpha^{-1}(\omega)) \right\rangle \tilde{\alpha}_j + \sum_{k=1}^n \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_i}(d\tilde{x}_k)_{g \cdot y}\left(\sum_{j=1}^\kappa \tilde{s}_j \widetilde{X}_j\right) = 0$$

with $i = 1, \ldots, n-1$, as well as

$$(3.18) \qquad \sum_{k=1}^{n} \sum_{j=1}^{n-1} \tilde\alpha_j \frac{\partial F_k(\alpha^{-1}(\omega))}{\partial \alpha_j} (d\tilde x_k)_{g\cdot y}(\widetilde X_i) - \left\langle \widetilde X_{i,g\cdot y}\Big(\sum_{j=1}^{\kappa} \tilde s_j \widetilde X_j(\kappa)\Big), \omega \right\rangle = 0$$

with $i = 1, \ldots, \kappa$. We have to show that the equations (3.17)–(3.18) only admit the solution $\tilde\alpha = 0$, $\tilde s_1 = \cdots = \tilde s_\kappa = 0$. Writing $\mathcal{W}_\omega(\tilde\alpha) := \sum_{j=1}^{n-1} \tilde\alpha_j \, \partial / \partial \alpha_{j|\omega}$ and identifying Y with $\kappa(Y)$, the system of equations (3.17) reads

$$-\|\kappa(x) - \kappa(g\cdot y)\| \left\langle \frac{\partial}{\partial \alpha_{i|\omega}}, A\,\mathcal{W}_\omega(\tilde\alpha) \right\rangle + \left\langle \frac{\partial}{\partial \alpha_{i|\omega}}, \widetilde X(\tilde s)_{g\cdot y} \right\rangle = 0, \qquad i = 1, \ldots, n-1,$$

and implies

$$(3.19) \qquad \mathcal{W}_\omega(\tilde\alpha) = \|x - g\cdot y\|^{-1} A^{-1}\big(\mathrm{proj}_{|T_\omega \Sigma_x^{R,t}}(\widetilde X(\tilde s)_{g\cdot y})\big),$$

where we wrote $X(\tilde s) := \sum_{j=1}^{\kappa} \tilde s_j X_j$ for short. Note that A is invertible, since the Gaussian curvature of $\Sigma_x^{R,t}$ does not vanish. Furthermore, the projection from $T_{g\cdot y}\mathcal{O}_y$ to $T_\omega \Sigma_x^{R,t}$ has a trivial kernel, since ω is normal to $\mathcal{O}_y$ at $g\cdot y$, and cannot be tangential to $\Sigma_x^{R,t}$ in view of (3.11). On the other hand, (3.18) amounts to the equations

$$(3.20) \qquad \langle \mathcal{W}_\omega(\tilde\alpha), \widetilde X_{i,g\cdot y}\rangle = \widetilde X_{i,g\cdot y}\big(\langle \widetilde X(\tilde s), \omega\rangle\big), \qquad i = 1, \ldots, \kappa.$$
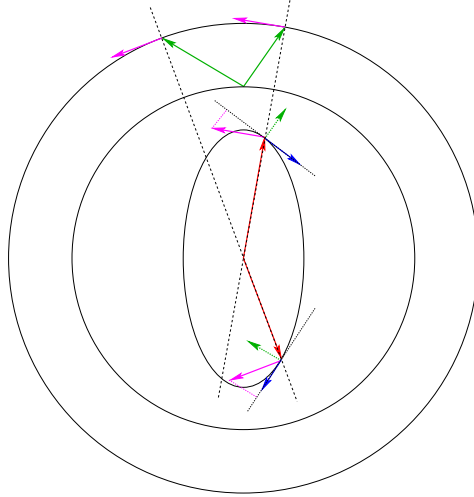

FIGURE 3.2. Concerning the cleanness of the critical set of $\Phi_{x,y}$ in case that $y \notin \mathcal{O}_x$. Black circles represent G-orbits in $Y \equiv \kappa(Y) \subset \mathbb{R}^n$ through x and y, respectively; the black dotted lines represent normal spaces to the orbits and tangent spaces to the hypersurface $\Sigma_x^{R,t}$, respectively, the latter being depicted by an ellipse. The red arrows represent different points $\omega \in \Sigma_x^{R,t}$, the green arrows different segments $\kappa(g\cdot y) - \kappa(x)$. The magenta arrows depict different vectors $\widetilde X(\tilde s)_{g\cdot y}$ and the blue arrows the corresponding vectors $\mathcal{W}_\omega(\tilde\alpha)$.

Since $\Sigma_x^{R,t}$ is strictly convex, the eigenvalues of A, which are given by the principal curvatures of $\Sigma_x^{R,t}$ with respect to the *outer* unit normal vector field, are strictly *negative*[2]. Hence A defines a non-positive operator on $T_\omega \Sigma_x^{R,t}$. In addition, the G-orbit through $g\cdot y$ must be convex with respect to x due to

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[2]Note that the sign convention used here is such that if $\Sigma_x^{R,t}$ equals the standard $(n-1)$-sphere $S^{n-1}(R)$ of radius R, then $A = -\mathbf{1}/R$, where $\mathbf{1}$ represents the identity transformation on $T_\omega S^{n-1}(R)$, see [16, Chapter VII, Example 4.2].

the condition $x - g \cdot y \in N_\omega \Sigma_x^{R,t}$ and the convexity of $\Sigma_x^{R,t}$. Consequently, if we assume as we may that the $\{X_1, \ldots, X_\kappa\}$ are such that

$$\left\langle \frac{\partial}{\partial \alpha_i \, |_\omega}, \widetilde{X}_{j, g \cdot y} \right\rangle \geq 0, \qquad 1 \leq i \leq n - 1, \quad 1 \leq j \leq \kappa,$$

$\langle \mathcal{W}_\omega(\tilde{\alpha}), \widetilde{X}_{i, g \cdot y} \rangle$ and $\widetilde{X}_{i, g \cdot y}(\langle X(\tilde{s}), \omega \rangle)$ must have opposite sign for all $i = 1, \ldots, \kappa$, so that Equations (3.19)–(3.20) only admit the solution $\tilde{s}_1 = \cdots = \tilde{s}_\kappa = 0$ and $\tilde{\alpha}_1 = \cdots = \tilde{\alpha}_{n-1} = 0$. If, on the contrary, $\kappa(x) - \kappa(g \cdot y)$ pointed in the direction of $+\mathcal{N}(\omega)$, $\mathcal{W}_\omega(\tilde{\alpha})$ would have opposite sign, which nevertheless would be compensated by a sign change of ω, the origin of $\mathbb{R}^n$ being contained in the interior of $\Sigma_x^{R,t}$, so that, again, Equations (3.19)–(3.20) would have only the trivial solution, compare Figure 3.2. Thus, we have shown

$$\mathrm{Ker\,Hess}\,\Phi_{x,y}(\omega, g) \simeq \{0\} \times \mathbb{R}^{d-\kappa} \simeq \mathrm{T}_{(\omega, g)} \mathrm{Crit}\,\Phi_{x,y},$$

and we obtain (b).

In order to show (c), let $x \in Y \cap M_{\mathrm{prin}}$ and $(\omega, g) \in \mathcal{C}_{x,x}$. If x is of principal isotropy type, G_x acts trivially on $N_x(G \cdot x)$ [2, pp. 308 and 181] and, via the identification $T^*M \simeq TM$, also on $\mathrm{Ann}(T_x(G \cdot x))$. But in view of (3.4) and Assertion (a) we have $\omega \in \mathrm{Ann}(T_x(G \cdot x))$, so that $g \cdot \omega = \omega$ in this case, and with (3.5) we obtain the desired inclusion and therefore (c). In particular, since $\mathrm{Crit}\,\Phi$ has co-dimension 2κ, $\mathcal{C}_{x,x}$ has co-dimension 2κ as well. $\qquad\square$

Remark 3.2.

(1) Let $y \notin \mathcal{O}_x$. As an example where $\mathrm{Crit}\,\Phi_{x,y}$ is not isomorphic to G_y, and does not have co-dimension $n - 1 + \kappa$, consider the singular action of $G = \mathrm{SO}(2)$ on the standard 2-sphere $M = S^2 \subset \mathbb{R}^3$ by rotations around the poles x_N, x_S, and assume that $\Sigma_x^{R,t} = S^1$. Let (Y, κ) be an invariant tubular neighborhood around the fixed point x_N. Then, for any $y \in Y - \{x_N\}$ one has

$$\mathrm{Crit}\,\Phi_{x_N, y} = \{(\omega, g) : (g \cdot y, \omega) \in N_{g \cdot y}(G \cdot y), \ \kappa(x_N) - \kappa(g \cdot y) \parallel \omega\} \simeq \mathrm{SO}(2) \times \mathbb{Z}_2 \not\simeq G_y = \{e\},$$

which has co-dimension $\kappa = 1$ instead of 2.

(2) Note that Assertion (c) of Lemma 3.1 cannot hold in general for arbitrary $x \in Y \cap (M_{\mathrm{except}} \cup M_{\mathrm{sing}})$. In particular, if x were a fixed point we would have $\Phi_{x,x} \equiv 0$, so that $\mathrm{Crit}\,\Phi_{x,x} = \Sigma_x^{R,t} \times G$ in this case. Furthermore, Assertion (c) means that $\Phi_{x,x}$ does not have secondary critical points for $x \in Y \cap M_{\mathrm{prin}}$, that is, critical points which do not arise from critical points of Φ.

From the previous lemma one immediately deduces

Theorem 3.3. *For an arbitrary chart (κ, Y), consider the oscillatory integrals $I_{x,y}(\mu)$ defined in (3.1), and for fixed $x \in Y$ write*

$$I_x(\mu) := I_{x,x}(\mu), \qquad \Phi_x := \Phi_{x,x}, \qquad \mathcal{C}_x := \mathcal{C}_{x,x}.$$

(1) *For every $\tilde{N}$ one has the asymptotic formula*

$$I_x(\mu) = (2\pi/\mu)^{\dim \mathcal{O}_x} \sum_{k=0}^{\tilde{N}-1} Q_k(x)\mu^{-k} + \mathcal{R}_{\tilde{N}}(x, \mu), \qquad \mu \to +\infty,$$

with explicitly known coefficients and remainder. In particular,

$$Q_0(x) = \int_{\mathcal{C}_x} \frac{a_\mu(x, x, \omega, g)}{|\det \Phi_x''(\omega, g)_{N_{(\omega, g)} \mathcal{C}_x}|^{1/2}} \, d\mathcal{C}_x(\omega, g),$$

where $d\mathcal{C}_x$ denotes the induced volume density. Furthermore, $Q_k(x)$ and $\mathcal{R}_{\tilde{N}}(x,\mu)$ depend smoothly on R and t, and satisfy the bounds

$$|Q_k(x)| \le \widetilde{C}_{k,\Phi_x}\, vol\,(\mathrm{supp}\ a_\mu \cap \mathcal{C}_x) \max_{l \le 2k} \left\| D^l a_\mu \right\|_{\infty,\mathcal{C}_x},$$

$$|\mathcal{R}_{\tilde{N}}(x,\mu)| \le C_{\tilde{N},\Phi_x}\, vol\,(\mathrm{supp}\ a_\mu) \max_{l \le 2\dim \mathcal{O}_x + 2\tilde{N}+1} \left\| D^l a_\mu \right\|_{\infty,\Sigma_x^{R,t}Y\times G}\, \mu^{-\dim \mathcal{O}_x - \tilde{N}},$$

for suitable constants $\widetilde{C}_{k,\Phi_x} > 0$ and $C_{\tilde{N},\Phi_x} > 0$, where D^l denote differential operators of order l on $G \times \Sigma_x^{R,t}$. Moreover, as functions in x, $Q_k(x)$ and $\mathcal{R}_{\tilde{N}}(x,\mu)$ are smooth on $Y \cap M_{\mathrm{prin}}$, and the constants $\widetilde{C}_{k,\Phi_x}$ and $C_{\tilde{N},\Phi_x}$ are uniformly bounded in x if $M = M_{\mathrm{prin}} \cup M_{\mathrm{except}}$.

(2) Assume that $M = M_{\mathrm{prin}} \cup M_{\mathrm{except}}$, and that the co-spheres S_x^*M are strictly convex. Then, for sufficiently small Y and every $\tilde{N}$ one has the asymptotic formula

$$I_{x,y}(\mu) = (2\pi/\mu)^{\frac{\mathrm{codim\ Crit}\ \Phi_{x,y}}{2}} e^{i\mu\Phi_{x,y}^0} \sum_{k=0}^{\tilde{N}-1} Q_k(x,y)\mu^{-k} + \mathcal{R}_{\tilde{N}}(x,y,\mu), \qquad \mu \to +\infty,$$

with explicitly known coefficients and remainder of order $O\big(\mu^{\frac{\mathrm{codim\ Crit}\ \Phi_{x,y}}{2} - \tilde{N}}\big)$, where

$$\mathrm{codim\ Crit}\ \Phi_{x,y} = \begin{cases} 2\kappa, & y \in \mathcal{O}_x, \\ n - 1 + \kappa, & y \notin \mathcal{O}_x, \end{cases}$$

and $\kappa = \dim M/G$. The coefficients $Q_k(x,y)$ and the remainder term $\mathcal{R}_{\tilde{N}}(x,y,\mu)$ are uniformly bounded in x and y, and given by distributions depending smoothly on R,t with support in $\mathrm{Crit}\ \Phi_{x,y}$ and $\Sigma_x^{R,t} \times G$, respectively. $\Phi_{x,y}^0$ stands for the constant values of $\Phi_{x,y}$ on the connected components of its critical set, and is given by

$$\Phi_{x,y}^0(R,t) = R\,c_{x,g\cdot y}(t), \qquad c_{x,g\cdot y}(t) := \pm \frac{\|\kappa(x) - \kappa(g\cdot y)\|}{\|\mathrm{grad}_\eta\, \zeta(t,\kappa(x),\omega)\|}.$$

If $y \in \mathcal{O}_x$ one has $\Phi_{x,y}^0 = 0$.

Proof. By Lemma 3.1 (a), Φ_x has a clean critical set, so that the asymptotic formula for $I_x(\mu)$ follows directly from Theorem A.1. In particular, the smooth dependence of the coefficients $Q_k(x)$ and the remainder $\mathcal{R}_{\tilde{N}}(x,\mu)$ on the parameters R,t, and $x \in Y \cap M_{\mathrm{prin}}$ is seen by using a local trivialization $T^*Y \simeq Y \times \mathbb{R}^n$ and taking into account [14, Theorem 7.7.6]. Similarly, the asymptotic expansion for $I_{x,y}(\mu)$, together with the smoothness of the coefficients $Q_k(x,y)$ and the remainder $\mathcal{R}_{\tilde{N}}(x,y,\mu)$ in the parameters R,t is a direct consequence of Lemma 3.1 (b) together with [14, Theorem 7.7.6].

On the other hand, principal and exceptional orbits are locally diffeomorphic, and principal and exceptional isotropy groups are infinitesimally isomorphic. Therefore, if $M = M_{\mathrm{prin}} \cup M_{\mathrm{except}}$ the inverse of the transversal Hessian of $\Phi_{x,y}$ depends smoothly on x,y, as can be seen when computing it from (3.15). From the explicit form of the coefficients and the remainder in the proof of Theorem A.1 in [21, Theorem 4.1], which involves the inverse of the transversal Hessian, it then follows that the constants $\widetilde{C}_{k,\Phi_x}$, $C_{\tilde{N},\Phi_x}$, together with the coefficients $Q_k(x,y)$ and the remainder term $\mathcal{R}_{\tilde{N}}(x,y,\mu)$ are uniformly bounded in x and y if no singular orbits are present. Regarding the values of $\Phi_{x,y}$ on its critical set, note that for $(\omega,g) \in \mathrm{Crit}\ \Phi_{x,y}$ one computes with (2.7)

$$\Phi_{x,y}^0(R,t) = \langle \kappa(x) - \kappa(g\cdot y), \omega\rangle = \pm c_{x,g\cdot y}(t) \underbrace{\langle \mathrm{grad}_\eta\, \zeta(t,\kappa(x),\omega), \omega\rangle}_{=R} = R\,c_{x,g\cdot y}(t),$$

since $\kappa(x) - \kappa(g\cdot y)$ must be co-linear to $\mathrm{grad}_\eta\, \zeta(t,\kappa(x),\omega)$. In particular notice that $c_{x,g\cdot y}(t)$ is independent of R due to the fact that $\zeta(t,\kappa(x),\eta)$ is homogeneous of degree 1 in η, so that $\mathrm{grad}_\eta\, \zeta(t,\kappa(x),\omega)$ only depends on the direction of ω. $\qquad\square$

4. The equivariant local Weyl law

Let us now come back to our initial question of finding an asymptotic description of the equivariant spectral function. From the results in the previous section we deduce

Proposition 4.1. *Let the notation be as in Corollary 2.2, and $R, t \in \mathbb{R}$, $x \in Y_\iota$ be fixed. Then, for any $\tilde{N} \in \mathbb{N}$ one has*

$$\partial^\beta_{R,t} I_\iota(\mu, R, t, x, x) = (2\pi/\mu)^{\dim \mathcal{O}_x} \sum_{k=0}^{\tilde{N}-1} \mathcal{L}^k_{\iota,\beta}(R, t, x) \mu^{-k} + O_{R,t,x}(\mu^{-\dim \mathcal{O}_x - \tilde{N}}),$$

where the coefficients $\mathcal{L}^k_{\iota,\beta}(R, t, x)$ and the remainder term are given by distributions depending smoothly on $R, t,$ and $x \in Y \cap M_{\mathrm{prin}}$ with support in

$$\mathrm{Crit}_{R,t} \Phi_{\iota,x} := (\Omega \cap \Sigma^{R,t}_{\iota,x}) \times G_x$$

and $\Sigma^{R,t}_{\iota,x} \times G$, respectively. Furthermore, both the coefficients and the remainder are uniformly bounded in x if $M = M_{\mathrm{prin}} \cup M_{\mathrm{except}}$.

Proof. This is a direct consequence of Theorem 3.3 (1). Note that $\Phi_{\iota,x}$ vanishes on its critical set $\mathrm{Crit}_{R,t} \Phi_{\iota,x}$ no matter what values R and t take. Otherwise differentiation with respect to R and t of the factor $e^{i\mu\psi_0}$ in (A.2) with $\psi_0 \equiv \Phi_{\iota,x|\mathrm{Crit}_{R,t} \Phi_{\iota,x}}$ would yield additional positive powers of μ. Furthermore, $a_\iota \in \mathrm{S}^0_{phg}$ is a classical symbol of order 0, so that

$$\left| \partial^\alpha_\omega a_\iota(t, \kappa_\iota(x), \mu\omega) \right| = |\mu|^{|\alpha|} \left| (\partial^\alpha_\omega a_\iota)(t, \kappa_\iota(x), \mu\omega) \right| \leq C|\omega|^{-|\alpha|}.$$

Consequently, the dependence of the amplitude on μ in (2.10) does not interfer with the asymptotics, compare [8, Proposition 1.2.4]. $\square$

We now arrive at

Proposition 4.2 (Point-wise asymptotics for the kernel of the equivariant approximate projection). *For any fixed $x \in M$, $\gamma \in \widehat{G}$, and $\tilde{N} \in \mathbb{N}$ one has for $\mu \to +\infty$*

$$K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, x) = \sum_{j \geq 0, \, e_j \in \mathrm{L}^2_\gamma(M)} \varrho(\mu - \mu_j)|e_j(x)|^2$$

(4.1)

$$= (\mu/2\pi)^{n - \dim \mathcal{O}_x - 1} \sum_{k=0}^{\tilde{N}-1} \mathcal{L}_k(x) \mu^{-k} + O_x(d_\gamma \mu^{n - \dim \mathcal{O}_x - 1 - \tilde{N}})$$

with known coefficients $\mathcal{L}_k(x)$ and a remainder estimate that depend smoothly on $x \in M_{\mathrm{prin}}$; furthermore, they are uniformly bounded in x if $M = M_{\mathrm{prin}} \cup M_{\mathrm{except}}$. In particular,

$$\mathcal{L}_0(x) = \frac{d_\gamma}{2\pi} \hat{\varrho}(0)[\pi_{\gamma|G_x} : \mathbf{1}] \, vol[(\Omega \cap S^*_x M)/G],$$

where $S^*M := \{(x, \xi) \in T^*M : p(x, \xi) = 1\}$. For $\mu \to -\infty$, the function $K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, x)$ is rapidly decreasing in μ.

Proof. Corollary 2.2 and Proposition 4.1 immediately imply the asymptotic expansion (4.1) with

$$\mathcal{L}_0(x) = \sum_\iota \frac{f_\iota(x)\hat{\varrho}(0) \, d_\gamma}{2\pi} \int_{\mathrm{Crit}_{1,0} \Phi_{\iota,x}} \frac{\overline{\gamma(g)}}{|\det \Phi''_{\iota,x}(\omega, g)_{N_{(\omega,g)}\mathrm{Crit}_{1,0} \Phi_{\iota,x}}|^{1/2}} \, d(\mathrm{Crit}_{1,0} \Phi_{\iota,x})(\omega, g),$$

since $\alpha(q(x, \omega)) = 1$ on $\Sigma^{1,0}_{\iota,x}$ and $J_\iota(g, x) = 1$ for $g \in G_x$. In order to compute $\mathcal{L}_0(x)$, let us note that for any $x \in Y_\iota$ and smooth, compactly supported function f on $\Omega \cap \Sigma^{R,t}_{\iota,x}$ one has the formula

$$\int_{\mathrm{Crit}_{R,t} \Phi_{\iota,x}} \frac{\overline{\gamma(g)} f(x, \omega)}{|\det \Phi''_{\iota,x}(\omega, g)_{|N_{(\omega,g)}\mathrm{Crit}_{R,t} \Phi_{\iota,x}}|^{1/2}} d(\mathrm{Crit}_{R,t} \Phi_{\iota,x})(\omega, g)$$

$$= [\pi_{\gamma|G_x} : \mathbf{1}] \int_{\Omega \cap \Sigma^{R,t}_{\iota,x}} f(x, \omega) \frac{d(\Omega \cap \Sigma^{R,t}_{\iota,x})(\omega)}{\mathrm{vol} \, \mathcal{O}_{(x,\omega)}},$$

where we took into account that $\int_{G_x} \overline{\gamma(g)} dG_x(g) = [\pi_{\gamma|G_x} : \mathbf{1}]$, compare [5, Lemma 7], [21, Proof of Theorem 9.5], and [4, Section 3.3.2],

$$\mathrm{Crit}_{R,t}\, \Phi_{\iota,x} \to \Omega \cap \Sigma_{\iota,x}^{R,t}$$

being a submersion. As a consequence of this, we obtain for $\mathcal{L}_0(x)$ the expression

$$\mathcal{L}_0(x) = \frac{d_\gamma}{2\pi}\, \hat{\varrho}(0)[\pi_{\gamma|G_x} : \mathbf{1}] \sum_\iota f_\iota(x) \int_{\Omega \cap \Sigma_{\iota,x}^{1,0}} \frac{d(\Omega \cap \Sigma_{\iota,x}^{1,0})(\omega)}{\mathrm{vol}\, \mathcal{O}_{(x,\omega)}} = \frac{d_\gamma}{2\pi}\, \hat{\varrho}(0)[\pi_{\gamma|G_x} : \mathbf{1}] \,\mathrm{vol}\,[(\Omega \cap S_x^* M)/G].$$

$\square$

Using a standard Tauberian argument, we can now deduce from Proposition 4.2 our first main result.

Theorem 4.3 (**Equivariant local Weyl law**). *Let M be a closed connected Riemannian manifold M of dimension n carrying an isometric and effective action of a compact connected Lie group G, and P_0 a G-invariant elliptic classical pseudodifferential operator on M of degree m. Let $p(x,\xi)$ be its principal symbol, and assume that P_0 is positive and symmetric. Denote its unique self-adjoint extension by P, and for a given $\gamma \in \widehat{G}$ let $e_\gamma(x,y,\lambda)$ be its reduced spectral counting function. Further, let $\mathbb{J} : T^* M \to \mathfrak{g}^*$ be the momentum map of the G-action on M, and put $\Omega := \mathbb{J}^{-1}(\{0\})$. Then, for fixed $x \in M$ one has*

$$(4.2) \qquad \left| e_\gamma(x,x,\lambda) - \frac{d_\gamma[\pi_{\gamma|G_x} : \mathbf{1}]}{(2\pi)^{n-\kappa_x}} \lambda^{\frac{n-\kappa_x}{m}} \int_{\{\xi : (x,\xi) \in \Omega,\, p(x,\xi) < 1\}} \frac{d\xi}{\mathrm{vol}\, \mathcal{O}_{(x,\xi)}} \right| \leq C_x\, d_\gamma\, \lambda^{\frac{n-\kappa_x-1}{m}}$$

as $\lambda \to +\infty$, where $\kappa_x := \dim \mathcal{O}_x$, d_γ denotes the dimension of an irreducible G-representation π_γ belonging to γ and $[\pi_{\gamma|G_x} : \mathbf{1}]$ the multiplicity of the trivial representation in the restriction of π_γ to the isotropy group G_x of x, while $C_x > 0$ is a constant that depends smoothly on $x \in M_{\mathrm{prin}}$, and is uniformly bounded if $M = M_{\mathrm{prin}} \cup M_{\mathrm{except}}$.

Proof. This follows directly by integrating (4.1) with respect to μ from $-\infty$ to $\sqrt[m]{\lambda}$ with the arguments given in [9, Proof of Corollary 2.5 and the following Remarks]. $\square$

Remark 4.4.

  (1) Note that in view of (3.4) the integral in the leading term can be written as

$$\lambda^{n-\kappa_x} \int_{\{\xi : (x,\xi) \in \Omega,\, p(x,\xi) < 1\}} \frac{d\xi}{\mathrm{vol}\, \mathcal{O}_{(x,\xi)}} = \int_{\{\xi : (x,\xi) \in \Omega,\, p(x,\xi) < \lambda\}} \frac{d\xi}{\mathrm{vol}\, \mathcal{O}_{(x,\xi)}}.$$

  (2) The equivariant local Weyl law (4.2) implies the estimate

$$(4.3) \qquad\qquad |e_\gamma(x,x,\lambda+1) - e_\gamma(x,x,\lambda)| \leq C_x\, d_\gamma\, \lambda^{\frac{n-\kappa_x-1}{m}}, \qquad x \in M.$$

  But since $e_\gamma(x,y,\lambda+1) - e_\gamma(x,y,\lambda)$ is the kernel of a positive operator, one immediately infers from this with the Cauchy-Schwarz inequality the bound

$$|e_\gamma(x,y,\lambda+1) - e_\gamma(x,y,\lambda)| \leq d_\gamma \sqrt{C_x \lambda^{\frac{n-\kappa_x-1}{m}}} \sqrt{C_y \lambda^{\frac{n-\kappa_y-1}{m}}}, \qquad x,y \in M.$$

  From this, it is not difficult to deduce a corresponding equivariant local Weyl law for $e_\gamma(x,y,\lambda)$ in a neighborhood of the diagonal, see [13, pp. 68] or [24, Section 21].

As a first consequence of Theorem 4.3, let us note that the estimate (4.3) is equivalent to the following bound for spectral clusters.

Corollary 4.5 (**Point-wise bounds for isotypic spectral clusters**). *In the situation of Theorem 4.3 we have*

$$\sum_{\substack{\lambda_j \in (\lambda, \lambda+1], \\ e_j \in \mathrm{L}_\gamma^2(M)}} |e_j(x)|^2 \leq C_x\, d_\gamma\, \lambda^{\frac{n-\kappa_x-1}{m}}, \qquad x \in M,$$

where $\{e_j\}$ denotes an orthonormal basis of eigenfunctions of P with eigenvalues $\{\lambda_j\}$.

$\square$

A further implication of Theorem 4.3 is the following Kuznecov sum formula for periods of G-orbits, which generalizes the classical Kuznecov formula for periods of closed geodesics [32].

Corollary 4.6 (**Generalized Kuznecov sum formula for periods of G-orbits**). *In the setting of Theorem 4.3 we have*

$$\left| \sum_{\lambda_j \leq \lambda} \left| \int_G e_j(g^{-1} \cdot x) \, dg \right|^2 - \frac{vol \, G_x}{(2\pi)^{n-\kappa_x}} \lambda^{\frac{n-\kappa_x}{m}} \int_{\{\xi : (x,\xi) \in \Omega, \, p(x,\xi) < 1\}} \frac{d\xi}{vol \, \mathcal{O}_{(x,\xi)}} \right| \leq C_x \, d_\gamma \, \lambda^{\frac{n-\kappa_x-1}{m}}.$$

Proof. Let $\gamma = \gamma_{\mathrm{triv}}$ correspond to the trivial representation. Then

$$e_{\gamma_{\mathrm{triv}}}(x, x, \lambda) = \sum_{\lambda_j \leq \lambda, \, e_j \in \mathrm{L}^2_{\gamma_{\mathrm{triv}}}(M)} |e_j(x)|^2 = \sum_{\lambda_j \leq \lambda} \left| \int_G e_j(g^{-1} \cdot x) \, dg \right|^2,$$

and the assertion follows from the previous theorem with

$$[\pi_{\gamma_{\mathrm{triv}}|G_x} : \mathbf{1}] = \int_{G_x} \overline{\gamma_{\mathrm{triv}}(g)} \, dG_x(g) = \mathrm{vol} \, G_x.$$

$\square$

In case that $\widetilde{M} := M/G$ is an orbifold we essentially recover the description of the spectral function of a Riemannian orbifold given by Stanhope and Uribe in [29]. More precisely, we immediately infer

Corollary 4.7 (**Local Weyl law for Riemannian orbifolds**). *In the situation of Theorem 4.3, assume that G acts on M with finite isotropy groups. Then, for fixed $x \in M$ and $\gamma \in \widehat{G}$ the asymptotic formula (4.2) holds with $n - \kappa_x = n - \kappa$ being equal to the dimension of $\widetilde{M}$. Moreover, let γ_{triv} be the trivial representation. Then $e_{\gamma_{triv}}(x, x, \lambda)$ is G-invariant, and descends to a function on $\widetilde{M} \times \widetilde{M}$ satisfying*

$$\left| e_{\gamma_{triv}}(\tilde{x}, \tilde{x}, \lambda) - \frac{|G_{\tilde{x}}|}{(2\pi)^{\dim \widetilde{M}}} \lambda^{\frac{\dim \widetilde{M}}{m}} vol\big(S^*_{\tilde{p}, \tilde{x}}(\widetilde{M})\big) \right| \leq C_{\tilde{x}} \lambda^{\frac{\dim \widetilde{M} - 1}{m}}, \qquad \tilde{x} \in \widetilde{M},$$

where $(G_{\tilde{x}})$ denotes the isotropy type of $\tilde{x} := G \cdot x$, $|G_{\tilde{x}}|$ its cardinality, while $S^*_{\tilde{p}, \tilde{x}}(\widetilde{M})$ equals the fiber over $\tilde{x}$ of the orbifold bundle $S^*_{\tilde{p}}(\widetilde{M}) := \left\{ (\tilde{x}, \xi) \in T^*\widetilde{M} : \tilde{p}(\tilde{x}, \xi) = 1 \right\}$, $\tilde{p}$ being the function on $\widetilde{M}$ induced by p.

Proof. The first assertion is clear, since all G-orbits on M have the same dimension $\kappa = \dim \widetilde{M}$, so that no singular orbits are present. To see the second note that since G_x is finite, $d_{\gamma_{\mathrm{triv}}} = 1$, and $\gamma_{\mathrm{triv}}(g) = 1$ for all $g \in G$ one computes

$$[\pi_{\gamma|G_x} : \mathbf{1}] = \int_{G_x} \overline{\gamma(g)} dG_x(g) = \sum_{l=1}^{|G_x|} 1 = |G_x|,$$

dG_x being the counting measure. For the volume factor, see [17, Remark 4.2]. $\square$

Example 4.8. Let us consider the case where $M = T^2 \subset \mathbb{R}^3$ is the standard 2-torus on which $G = \mathrm{SO}(2)$ acts by rotations. Then all orbits are 1-dimensional and of principal type, and Theorem 4.3 yields for the reduced spectral function of the Laplace-Beltrami operator

$$\left| e_\gamma(x, x, \lambda) - \frac{1}{2\pi} \sqrt{\lambda} \int_{\{\xi : (x,\xi) \in \Omega, \, p(x,\xi) < 1\}} \frac{d\xi}{\mathrm{vol} \, \mathcal{O}_{(x,\xi)}} \right| \leq C_x, \qquad x \in T^2, \qquad \gamma \in \widehat{\mathrm{SO}(2)}.$$

Example 4.9. Consider a connected semisimple Lie group G with finite center and Lie algebra $\mathfrak{g}$, together with a discrete co-compact subgroup Γ. In particular, Γ might have torsion, meaning that there are non-trivial elements of Γ conjugate in G to an element of K. Let K be a maximal compact subgroup of G, and choose a left-invariant metric on G given by an $\mathrm{Ad}\,(K)$-invariant bilinear form on $\mathfrak{g}$. The quotient $M := \Gamma\backslash G$ is a compact manifold without boundary, and has a Riemannian structure induced by the one of G. Furthermore, K acts on $\Gamma\backslash G$ from the right in an isometric and effective way, and the isotropy group of a point Γg is conjugate to the finite group $gKg^{-1}\cap\Gamma$. Hence, all K-orbits in $\Gamma\backslash G$ are either principal or exceptional, $\Gamma\backslash G/K$ is an orbifold, and Corollary 4.7 applies.

Example 4.10. Let us now consider a case where singular orbits are present, and $M = S^2 \subset \mathbb{R}^3$ be the standard 2-sphere and $G = \mathrm{SO}(2) \subset \mathrm{SO}(3)$ the group of rotations around the x_3-axis with fixed points $x_N = (0,0,1)$ and $x_S = (0,0,-1)$. In this case the phase function of $I_x(\mu)$ reads $\Phi_x(\omega,g) = \langle x - g\cdot x,\omega\rangle$. with respect to standard coordinates in $\mathbb{R}^3$. For $x = x_N, x_S$ it simply vanishes, so that $I_x(\mu)$ is independent of μ in this case, which is consistent with the asymptotics

$$I_x(\mu) = \begin{cases} O(\mu^0), & x = x_N, x_S, \\ O(\mu^{-1}), & \text{otherwise,} \end{cases}$$

implied by Theorem 3.3. Let us now apply Theorem 4.3 to the Laplace-Beltrami operator $-\Delta$ on S^2, and notice for this that the orbit volume $\mathrm{vol}\,\mathcal{O}_{(x,\xi)}$ is of order $\sqrt{\xi_1^2 + \xi_2^2} + \sqrt{x_1^2 + x_2^2}$ for arbitrary x and ξ. By Theorem 4.3 and with the identification $\widehat{\mathrm{SO}(2)} \simeq \mathbb{Z}$ the reduced spectral function satisfies on $S^2_{\mathrm{prin}} = S^2 - \{x_N, x_S\}$ the estimate

$$(4.4) \qquad \left| e_m(x,x,\lambda) - \frac{\sqrt{\lambda}}{2\pi}\int_{\left\{\xi:\,(x,\xi)\in\Omega,\,\|\xi\|_x<1\right\}} \frac{d\xi}{\mathrm{vol}\,\mathcal{O}_{(x,\xi)}} \right| \leq C_x, \qquad x \in S^2_{\mathrm{prin}}, \, m \in \mathbb{Z}.$$

In this case, $\Omega\cap T_x^*(S^2)$ is 1-dimensional; the integral in (4.4) is finite, but as $S^2_{\mathrm{prin}} \ni x \to x_N$ or x_S the orbit volume becomes of order $\sqrt{\xi_1^2 + \xi_2^2}$, so that the mentioned integral goes to infinity. On the other hand, for the fixed points $x = x_N, x_S$ the space $\Omega\cap T_x^*S^2 = T_x^*S^2$ is 2-dimensional and Theorem 4.3 yields

$$(4.5) \qquad \left| e_m(x,x,\lambda) - \frac{[\pi_{m|G}:\mathbf{1}]}{(2\pi)^2}\lambda\int_{\left\{\xi:\,\|\xi\|_x<1\right\}} \frac{d\xi}{\mathrm{vol}\,\mathcal{O}_{(x,\xi)}} \right| \leq C_x\sqrt{\lambda}, \qquad x = x_N, x_S, \, m \in \mathbb{Z},$$

where

$$[\pi_{m|G}:\mathbf{1}] = \begin{cases} 1, & m = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, at the fixed points only the trivial representation contributes to the main term in the asymptotic formula for the spectral function given by the local Weyl law (1.1). Further note that, though for $x = x_N, x_S$ the orbit volume is proportional to $\sqrt{\xi_1^2 + \xi_2^2}$, its inverse is still locally integrable on $T_x^*S^2$, and the integral in (4.5) certainly exists. Ultimately, the leading coefficient in (4.4) must blow up as x approaches the fixed points in order to compensate for the fact that the leading power changes abruptly from $\sqrt{\lambda}$ to λ at the fixed points. Note that the remainder estimates in (4.4) and (4.5) are consistent with the asymptotics (1.14) for the spherical function $Y_{k,0}$.

5. Equivariant L^p-bounds of eigenfunctions for non-singular group actions

Let the notation be as in the previous sections. From the asymptotic formula for the equivariant spectral function proven Theorem 4.3 we already deduced in Corollary 4.5 point-wise bounds for isotypic spectral clusters. Similarly, one immediately shows in the non-singular case the following equivariant L^∞-bounds for eigenfunctions.

Proposition 5.1 (L$^\infty$-**bounds for isotypic spectral clusters**)**.** *Assume that G acts on M with orbits of the same dimension κ, and denote by χ_λ the spectral projection onto the sum of eigenspaces of P with eigenvalues in the interval $(\lambda, \lambda + 1]$. Then,*

$$(5.1) \qquad \|(\chi_\lambda \circ \Pi_\gamma) u\|_{\mathrm{L}^\infty(M)} \leq C(1+\lambda)^{\frac{n-\kappa-1}{2m}} \|u\|_{\mathrm{L}^2(M)}, \qquad u \in \mathrm{L}^2(M),\, \gamma \in \widehat{G},$$

where $C > 0$ is a constant independent of λ proportional to d_γ. In particular,

$$\|u\|_{\mathrm{L}^\infty(M)} \leq C\,\lambda^{\frac{n-\kappa-1}{2m}}, \qquad u \in \mathrm{L}_\gamma^2(M),\, \|u\|_{\mathrm{L}^2} = 1,$$

for any eigenfunction of P with eigenvalue λ in the isotypic component $\mathrm{L}_\gamma^2(M)$.

Proof. The assertion is a direct consequence of Theorem 4.3. In fact, standard arguments [27, Eq. (3.2.6)] imply that

$$\|\chi_\lambda \circ \Pi_\gamma\|_{\mathrm{L}^2 \to \mathrm{L}^\infty}^2 = \left[\sup_x \left(\int_M |K_{\chi_\lambda \circ \Pi_\gamma}(x,y)|^2 dM(y) \right)^{1/2} \right]^2$$

$$= \sup_x K_{\chi_\lambda \circ \Pi_\gamma}(x,x) = \sup_x \left[e_\gamma(x,x,\lambda+1) - e_\gamma(x,x,\lambda) \right].$$

Since $M = M_{\mathrm{prin}} \cup M_{\mathrm{except}}$, the assertion follows from (4.2). $\qquad\qquad\square$

It is instructive to see how Proposition 5.1 can be deduced directly from Proposition 4.2 by transfering the arguments given in [27, Page 50] to the equivariant setting. By duality, the estimate (5.1) is equivalent to

$$(5.2) \qquad \|(\chi_\lambda \circ \Pi_\gamma) u\|_{\mathrm{L}^2(M)} \leq C(1+\lambda)^{\frac{n-\kappa-1}{2m}} \|u\|_{\mathrm{L}^1(M)}.$$

In order to show the latter estimate, one considers again a Schwartz function $\varrho \in \mathcal{S}(\mathbb{R}, \mathbb{R}_+)$ satisfying $\varrho(0) = 1$ and supp $\hat{\varrho} \in (-\delta/2, \delta/2)$ for a given $\delta > 0$. If $\widetilde{\chi}_\lambda$ denotes the corresponding approximate spectral projection, one then shows that (5.2) is implied by

$$(5.3) \qquad \|(\widetilde{\chi}_\lambda \circ \Pi_\gamma) u\|_{\mathrm{L}^2(M)} \leq C(1+\lambda)^{\frac{n-\kappa-1}{2m}} \|u\|_{\mathrm{L}^1(M)}.$$

Thus, one is left with the task of proving (5.3). Now, the $\mathrm{L}^1 \to \mathrm{L}^2$ operator norm can be estimated according to

$$(5.4) \qquad \begin{aligned} \|\widetilde{\chi}_\lambda \circ \Pi_\gamma\|_{\mathrm{L}^1 \to \mathrm{L}^2}^2 &= \sup_{y \in M} \int_M |K_{\widetilde{\chi}_\lambda \circ \Pi_\gamma}(x,y)|^2 \, dM(x) \\ &= \sup_{y \in M} \sum_{j \geq 0,\, e_j \in \mathrm{L}_\gamma^2(M)} [\varrho(\lambda - \lambda_j)]^2 |e_j(y)|^2 \leq \|\varrho\|_{\mathrm{L}^\infty(\mathbb{R})} \sup_{y \in M} K_{\widetilde{\chi}_\lambda \circ \Pi_\gamma}(y,y). \end{aligned}$$

Hence, everything is shown, since by Proposition 4.2 we have the uniform bound

$$(5.5) \qquad |K_{\widetilde{\chi}_\lambda \circ \Pi_\gamma}(y,y)| \leq C(1+\lambda)^{\frac{n-\kappa-1}{m}}, \qquad y \in M = M_{\mathrm{prin}} \cup M_{\mathrm{except}},$$

and we obtain again (5.1).

Example 5.2. In the situation of Example 4.8, where $M = T^2 \subset \mathbb{R}^3$ is the standard 2-torus on which $G = \mathrm{SO}(2)$ acts by rotations, Proposition 5.1 implies the bounds

$$\|u\|_{\mathrm{L}^\infty(T^2)} \leq C, \qquad u \in \mathrm{L}_\gamma^2(T^2),\, \|u\|_{\mathrm{L}^2} = 1,$$

for any eigenfunction of P in a specific isotypic component, which in case of the Laplace-Beltrami operator Δ are well-known. Indeed, via the identification

$$\mathbb{R}^2/\mathbb{Z}^2 \xrightarrow{\simeq} T^2 \simeq S^1 \times S^1, (x_1, x_2) \longmapsto (e^{2\pi i x_1}, e^{2\pi i x_2}),$$

the standard orthonormal basis of Δ is given by $\left\{ e^{2\pi i k_1 x_1} e^{2\pi i k_2 x_1} : (k_1, k_2) \in \mathbb{Z}^2 \right\}$.

In what follows, we shall derive refined L^p-bounds for isotypic spectral clusters using complex interpolation techniques. For this, we shall need the additional assumption that the co-spheres S_x^*M are strictly convex. In essence, the proof is an elaboration of arguments from [23] applied to the equivariant setting. Nevertheless, while for the proof of the L^∞-bounds in the previous proposition it was sufficient to consider the asymptotic behavior of the integrals $I_{x,y}(\mu)$ in case that $x = y$, the proof of L^p-estimates actually requires estimates for the integrals $I_{x,y}(\mu)$ in a neighborhood of the diagonal, making things significantly more involved. This leads us to our second main result.

Theorem 5.3 (L^p-**bounds for isotypic spectral clusters**)**.** *Let M be a closed connected Riemannian manifold M of dimension n on which a compact connected Lie group G acts effectively and isometrically with orbits of the same dimension κ. Further, let P be the unique self-adjoint extension of a G-invariant elliptic positive symmetric classical pseudodifferential operator on M of degree m, and assume that its principal symbol $p(x, \xi)$ is such that the co-spheres $S_x^*M := \{(x, \xi) \in T^*M : p(x, \xi) = 1\}$ are strictly convex. Denote by χ_λ the spectral projection onto the sum of eigenspaces of P with eigenvalues in the interval $(\lambda, \lambda + 1]$, and by Π_γ the projection onto the isotypic component $\mathrm{L}_\gamma^2(M)$, where $\gamma \in \widehat{G}$. Then, for $u \in \mathrm{L}^2(M)$*

$$(5.6) \qquad \|(\chi_\lambda \circ \Pi_\gamma)u\|_{\mathrm{L}^q(M)} \leq \begin{cases} C\,\lambda^{\frac{\delta_{n-\kappa}(q)}{m}} \|u\|_{\mathrm{L}^2(M)}, & \frac{2(n-\kappa+1)}{n-\kappa-1} \leq q \leq \infty, \\ C\,\lambda^{\frac{(n-\kappa-1)(2-q')}{4mq'}} \|u\|_{\mathrm{L}^2(M)}, & 2 \leq q \leq \frac{2(n-\kappa+1)}{n-\kappa-1}, \end{cases}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$,

$$\delta_{n-\kappa}(q) := \max\left((n-\kappa)\left|\frac{1}{2} - \frac{1}{q}\right| - \frac{1}{2}, 0\right),$$

and $C > 0$ is a constant independent of λ proportional to d_γ. In particular,

$$\|u\|_{\mathrm{L}^q(M)} \leq \begin{cases} C\,\lambda^{\frac{\delta_{n-\kappa}(q)}{m}}, & \frac{2(n-\kappa+1)}{n-\kappa-1} \leq q \leq \infty, \\ C\,\lambda^{\frac{(n-\kappa-1)(2-q')}{4mq'}}, & 2 \leq q \leq \frac{2(n-\kappa+1)}{n-\kappa-1}, \end{cases}$$

for any eigenfunction of P with eigenvalue λ and L^2-norm 1 in the isotypic component $\mathrm{L}_\gamma^2(M)$.

Proof. By duality, (5.6) is equivalent to

$$(5.7) \qquad \|(\chi_\mu \circ \Pi_\gamma)u\|_{\mathrm{L}^2(M)} \leq \begin{cases} C\,\mu^{\delta_{n-\kappa}(p)} \|u\|_{\mathrm{L}^p(M)}, & 1 \leq p \leq \frac{2(n-\kappa+1)}{n-\kappa+3}, \\ C\,\mu^{\frac{(n-\kappa-1)(2-p)}{4p}} \|u\|_{\mathrm{L}^p(M)}, & \frac{2(n-\kappa+1)}{n-\kappa+3} \leq p \leq 2, \end{cases}$$

where χ_μ denotes the spectral projection onto the sum of eigenspaces of $Q := \sqrt[m]{P}$ with eigenvalues in the interval $(\mu, \mu + 1]$. The case $p = 1$ follows from the equivariant local Weyl law, and has already been dealt with in (5.2). On the other hand, orthogonality arguments immediately imply

$$\|(\chi_\mu \circ \Pi_\gamma)u\|_{\mathrm{L}^2(M)} \leq \|u\|_{\mathrm{L}^2(M)}.$$

By the Riesz interpolation theorem [30, Chapter V, Theorem 1.3] it therefore suffices to prove (5.7) in case that $p = \frac{2(n-\kappa+1)}{n-\kappa+3}$, which can be inferred from the corresponding bound

$$(5.8) \qquad \|(\widetilde{\chi}_\mu \circ \Pi_\gamma)u\|_{\mathrm{L}^2(M)} \leq C\mu^{\delta_{n-\kappa}(p)} \|u\|_{\mathrm{L}^p(M)}, \qquad p = \frac{2(n-\kappa+1)}{n-\kappa+3},$$

for the approximate spectral projection $\widetilde{\chi}_\mu$ defined in (2.1). Now, by Hölder's inequality one computes

$$\|(\widetilde{\chi}_\mu \circ \Pi_\gamma)u\|_{\mathrm{L}^2(M)}^2 = \int_M \left| \sum_{j \geq 0,\, e_j \in \mathrm{L}_\gamma^2(M)} \varrho(\mu - \mu_j) E_j u(x) \right|^2 dM(x)$$

$$= \int_M \sum_{j \geq 0,\, e_j \in \mathrm{L}_\gamma^2(M)} \varrho^2(\mu - \mu_j) E_j u(x)\overline{u(x)}\, dM(x) \leq \|(\check{\chi}_\mu \circ \Pi_\gamma)u\|_{\mathrm{L}^{p'}(M)} \|u\|_{\mathrm{L}^p(M)},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, and we put $\check{\chi}_\mu u := \sum_{j=0}^\infty \varrho^2(\mu - \mu_j) E_j u$ for $u \in \mathrm{L}^2(M)$. In order to see (5.8) it is therefore sufficient to prove

$$(5.9) \qquad \left\|(\check{\chi}_\mu \circ \Pi_\gamma) u\right\|_{\mathrm{L}^{p'}(M)} \leq C\mu^{2\delta_{n-\kappa}(p)} \left\|u\right\|_{\mathrm{L}^p(M)}, \qquad p = \frac{2(n - \kappa + 1)}{n - \kappa + 3}.$$

In order to show the latter, we shall use analytic interpolation [30, Chapter V, Theorem 4.1], and consider the analytic family of operators

$$\check{\chi}_\mu^z := \frac{e^{z^2}}{2\pi} \int_\mathbb{R} \widehat{\varrho^2}(t)\, e^{it\mu}\, (t - i\,0)^z\, U(t)\, dt, \qquad z \in \mathbb{C},$$

where $(t - i0)^z$ denotes the distribution $\lim_{\varepsilon \to 0+}(t - i\varepsilon)^z$. Clearly, $\check{\chi}_\mu^z = \check{\chi}_\mu$ if $z = 0$, and since $2\delta_{n-\kappa}(2(n - \kappa + 1)/(n - \kappa + 3)) = (n - \kappa - 1)/(n - \kappa + 1)$, analytic interpolation theory implies that (5.9) would follow if we were able to show that

$$(5.10) \qquad \left\|(\check{\chi}_\mu^z \circ \Pi_\gamma) u\right\|_{\mathrm{L}^2(M)} \leq C \left\|u\right\|_{\mathrm{L}^2(M)}, \qquad\qquad\qquad \mathrm{Re}\, z = -1,$$

$$(5.11) \qquad \left\|(\check{\chi}_\mu^z \circ \Pi_\gamma) u\right\|_{\mathrm{L}^\infty(M)} \leq C\, \mu^{\frac{n-\kappa-1}{2}} \left\|u\right\|_{\mathrm{L}^1(M)}, \qquad \mathrm{Re}\, z = \frac{n - \kappa - 1}{2}.$$

The crucial observation for the following estimates is that the Fourier transform of the distribution $\tau_+^z / \Gamma(z + 1)$ is given by the formula

$$(5.12) \qquad \int_\mathbb{R} e^{-it\tau} \frac{\tau_+^z}{\Gamma(z + 1)}\, d\tau = e^{-i\pi(z+1)/2}(t - i\,0)^{-z-1}, \qquad z \in \mathbb{C},$$

where Γ denotes the Gamma function, see [14, Example 7.1.17]; in particular, the singularity of $\tau_+^z / \Gamma(z + 1)$ at $\tau = 0$ determines the asymptotic behavior of $(t - i0)^{-z-1}$ as $t \to \infty$, and viceversa. From this (5.10) immediately follows. The non-trivial bound to be proven is (5.11), which would follow if we were able to show that the Schwartz kernel of $\check{\chi}_\mu^z \circ \Pi_\gamma$ satisfies

$$(5.13) \qquad |K_{\check{\chi}_\mu^z \circ \Pi_\gamma}(x, y)| \leq C\mu^{\frac{n-\kappa-1}{2}}, \qquad \mathrm{Re}\, z = \frac{n - \kappa - 1}{2},$$

uniformly in $x, y \in M$. Note that, in contrast, $|K_{\check{\chi}_\mu \circ \Pi_\gamma}(x, x)| \leq C\mu^{n-\kappa-1}$, compare Proposition 4.2. Now, it is clear from (2.8) that

$$K_{\check{\chi}_\mu^z \circ \Pi_\gamma}(x, y) = \frac{\mu^n d_\gamma\, e^{z^2}}{(2\pi)^{n+1}} \sum_\iota \int_\mathbb{R} \int_\mathbb{R} e^{i\mu[t - Rt]}(t - i\,0)^z\, I_\iota(\mu, R, t, x, y)\, dt\, dR$$

where $I_\iota(\mu, R, t, x, y)$ is as in (2.10) with ϱ replaced by ϱ^2. Due to the presence of the distribution $(t - i\,0)^z$ we cannot apply the stationary phase theorem to the (R, t)-integral. Instead, we shall apply the stationary phase principle to the integrals $I_\iota(\mu, R, t, x, y)$ first, and then use (5.12) to deal with the (R, t)-integral. If $x \notin Y_\iota$ or $\mathcal{O}_y \cap Y_\iota = \emptyset$, $I_\iota(\mu, R, t, x, y) = 0$. Otherwise, one deduces from Theorem 3.3 (2) for fixed $R, t \in \mathbb{R}$, and any $\tilde{N} \in \mathbb{N}$ the asymptotic expansion

$$I_\iota(\mu, R, t, x, y) = (2\pi/\mu)^{\frac{\mathrm{codim}\,\mathrm{Crit}\,\Phi_{\iota,x,y}}{2}} e^{i\mu\Phi_{\iota,x,y}^0} \sum_{k=0}^{\tilde{N}-1} \mathcal{L}_\iota^k(R, t, x, y)\mu^{-k} + O_{R,t,x,y}(\mu^{\frac{-\dim\,\mathrm{Crit}\,\Phi_{\iota,x,y}}{2} - \tilde{N}}),$$

where

$$\mathrm{codim}\,\mathrm{Crit}\,\Phi_{\iota,x,y} = \begin{cases} 2\kappa, & y \in \mathcal{O}_x, \\ n - 1 + \kappa, & y \notin \mathcal{O}_x. \end{cases}$$

The coefficients $\mathcal{L}_\iota^k(R, t, x, y)$ and the remainder term are given by distributions depending smoothly on R, t with support in $\mathrm{Crit}\Phi_{\iota,x,y}$ and $\Sigma_{\iota,x}^{R,t} \times G$, respectively. Furthermore, they and their derivatives with respect to R, t are uniformly bounded in x and y, while $\Phi_{\iota,x,y}^0(R, t) = R\, c_{x,g\cdot y}(t)$ denotes the

constant value of $\Phi_{\iota,x,y}$ on its critical set. If $y \in \mathcal{O}_x$ one has $\Phi^0_{\iota,x,y} = 0$, so that up to terms of order $O_{R,t,x,y}(\mu^{-\kappa-\tilde{N}})$ the kernel $K_{\tilde{\chi}^z_\mu \circ \Pi_\gamma}(x,y)$ is given by a sum of terms of the form

$$(5.14) \qquad \frac{\mu^{n-\kappa-k} d_\gamma\, e^{z^2}}{(2\pi)^{n-\kappa+1}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\mu(t-Rt)}(t-i\,0)^z\, \mathcal{L}^k_\iota(R,t,x,y)\, dt\, dR,$$

and if $y \notin \mathcal{O}_x$, by a sum of terms of the form

$$(5.15) \qquad \frac{\mu^{n-(n-1+\kappa)/2-k} d_\gamma\, e^{z^2}}{(2\pi)^{n-(n-1+\kappa)/2+1}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\mu(t-Rt)}(t-i\,0)^z\, e^{i\mu\Phi^0_{\iota,x,y}(R,t)} \mathcal{L}^k_\iota(R,t,x,y)\, dt\, dR,$$

where $k = 0, \dots, \tilde{N}-1$. Now, as a consequence of (5.12), one has for any $f \in C^\infty_c(\mathbb{R} \times \mathbb{R})$, that might depend on μ as a parameter, and $z \in \mathbb{C}$

$$\left\langle (t-i0)^z, e^{i\mu(1-R)t} f(R,t) \right\rangle = \frac{e^{-i\pi z/2}}{\Gamma(-z)} \left\langle \tau^{-z-1}_+, \widehat{f(R,\cdot)}(\tau - \mu(1-R)) \right\rangle.$$

Let us consider first the case when $z = 0, 1, 2, 3, \dots$, and write $-l := -z - 1$. Since $\tau^{-l}_+ / \Gamma(-l+1) = \delta^{(l-1)}_0$, compare [14, (3.2.17)'], partial integration yields

$$\int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\mu(1-R)t}(t-i0)^z f(R,t)\, dt\, dR = e^{-i\pi z/2}(-1)^{l-1} \int_{\mathbb{R}} \int_{\mathbb{R}} (-it)^{l-1} e^{it\mu(1-R)} f(R,t)\, dt\, dR$$

$$= e^{-i\pi z/2} \mu^{-l+1} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it\mu(1-R)} (\partial^{l-1}_R f)(R,t)\, dt\, dR.$$

The relevant integrals in (5.14) and (5.15) therefore read

$$(5.16) \qquad e^{-i\pi z/2} \mu^{-l+1} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it\mu(1-R)} \partial^{l-1}_R [e^{i\mu\Phi^0_{\iota,x,y}(R,t)} \mathcal{L}^k_\iota(R,t,x,y)]\, dt\, dR,$$

and an application of the stationary phase theorem [10, Proposition 2.3] to the (R,t)-integral allows us to deduce for $z = 0, 1, 2, 3, \dots$ the bounds

$$(5.17) \qquad K_{\tilde{\chi}^z_\mu \circ \Pi_\gamma}(x,y) = \begin{cases} O(\mu^{n-\kappa-z-1}), & y \in \mathcal{O}_x, \\ O(\mu^{n-(n-1+\kappa)/2-1}), & y \notin \mathcal{O}_x, \end{cases}$$

yielding (5.13) in this case. Indeed, if $y \in \mathcal{O}_x$ the phase function in (5.16) simply reads $t(1-R)$, and the only critical point is $(R_0, t_0) = (1, 0)$, which is non-degenerate, the determinant of the Hessian being -1. If $y \notin \mathcal{O}_x$, the phase function is given by $t(1-R) + \Phi^0_{\iota,x,y}(R,t)$, and a computation shows that the determinant of the matrix of its second derivatives is given by

$$(5.18) \qquad -\left(1 - c'_{x,g\cdot y}(t)\right)^2 \approx -(1 \pm O(\|\kappa_\iota(x) - \kappa_\iota(g\cdot y)\|))^2$$

since $c_{x,g\cdot y}(t) = \pm \|\kappa_\iota(x) - \kappa_\iota(g\cdot y)\| / \|\text{grad}_\eta\, \zeta_\iota(t, \kappa(x), \omega)\|$. By choosing the charts Y_ι sufficiently small so that $|\kappa_\iota(x) - \kappa_\iota(g\cdot y)| \ll 1$, we can therefore achieve that in a sufficiently small neighborhood of $(R,t) = (1, 0)$, which is where $\mathcal{L}^k_\iota(R,t,x,y)$ is supported, the phase function $t(1-R) + \Phi^0_{\iota,x,y}(R,t)$ has, if at all, only non-degenerate, hence isolated, critical points. If we now apply the stationary phase theorem to the integral (5.16) with respect to the phase function $t(1-R)$ and $t(1-R) + \Phi^0_{\iota,x,y}(R,t)$, respectively, we obtain (5.17) for $z = 0, 1, 2, 3, \dots$. Next, let us turn to the case where $z \neq 0, 1, 2, 3, \dots$, and note that by homogeneity of τ^z_+ one has

$$\left\langle (t-i0)^z, e^{i\mu(1-R)t} f(R,t) \right\rangle = \frac{e^{-i\pi z/2}}{\Gamma(-z)} \left\langle \tau^{-z-1}_+, \mu^{-1} \widehat{f(R,\cdot/\mu)} \left(\tau/\mu - 1 + R\right) \right\rangle$$

$$= \frac{e^{-i\pi z/2}}{\Gamma(-z)} \mu^{-z-1} \left\langle \tau^{-z-1}_+, \widehat{f(R,\cdot/\mu)}(\tau - 1 + R) \right\rangle,$$

compare [14, (3.2.7)]. By definition of τ_+^{-z-1} and partial integration one computes

$$-z(-z+1)\ldots(-z-1+l)(-1)^l \int_{\mathbb{R}}\int_{\mathbb{R}} \tau_+^{-z-1} f\widehat{(R,\cdot/\mu)}(\tau-1+R)\,d\tau\,dR$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \tau_+^{-z-1+l} \partial_\tau^l \left[f\widehat{(R,\cdot/\mu)}(\tau-1+R)\right] d\tau\,dR$$

$$= (-1)^l \mu \int_{\mathbb{R}} \tau_+^{-z-1+l} \left[\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-it\mu(\tau-1+R)}(\partial_R^l f)(R,t)\,dt\,dR\right] d\tau,$$

where $l > \operatorname{Re} z$ is any sufficiently large positive integer, so that τ_+^{-z-1+l} becomes locally integrable. Note that we have, as we may, interchanged the integrals over τ and R, while the integrals over τ and t cannot be interchanged. As a consequence, the relevant integrals in (5.14) and (5.15) are given by linear combinations of terms of the form

$$(5.19) \qquad \mu^{-z} \int_{\mathbb{R}} \tau_+^{-z-1+l} \left[\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-it\mu(\tau-1+R)} \partial_R^l [e^{i\mu\Phi_{\iota,x,y}^0(R,t)}\mathcal{L}_\iota^k(R,t,x,y)]\,dt\,dR\right] d\tau.$$

Again, let us examine the (R,t)-integral by means of the stationary phase. If $y \in \mathcal{O}_x$, the phase function is given by $t(\tau-1+R)$, the only critical point is $(R_0,t_0) = (1-\tau,0)$, and we obtain for (5.19) the estimate

$$2\pi\mu^{-z-1} \int_{\mathbb{R}} \tau_+^{-z-1+l} \left[(\partial_R^l \mathcal{L}_\iota^k)(1-\tau,0,x,y) + O_\tau(\mu^{-1})\right] d\tau = O(\mu^{-z-1}),$$

the remainder $O_\tau(\mu^{-1})$ being rapidly falling in τ, since $\mathcal{L}_\iota^k$ has compact (R,t)-support. Now, if $y \notin \mathcal{O}_x$, the phase function reads $t(1-R) + \Phi_{\iota,x,y}^0(R,t) - t\tau$, and the determinant of the matrix of its second derivatives is again given by (5.18). By the arguments above, we can therefore assume that in a sufficiently small neighborhood of $(R,t) = (1,0)$ the phase function $t(1-R) + \Phi_{\iota,x,y}^0(R,t) - t\tau$ has only one non-degenerate critical point (R_0,t_0). It satisfies the relations

$$t_0 = c_{x,g\cdot y}(t_0) \approx 0, \qquad R_0 = \frac{1-\tau}{1-c'_{x,g\cdot y}(t)} \approx 1-\tau,$$

and at this point, the phase function takes the value $t_0(1-R_0) + \Phi_{\iota,x,y}^0(R_0,t_0) - t_0\tau = t_0(1-\tau)$. Thus, we obtain for (5.19) the bound

$$2\pi\mu^{-z-1} \int_{\mathbb{R}} \tau_+^{-z-1+l} \left[e^{i\mu t_0(1-\tau)} \sum_{j=0}^{N-1} \sum_{l_1'+l_2'+l''=l} c_{l_1',l_2',l''}\mu^{-j}\right.$$

$$\left. D_{R,t}^{2j}\left[\partial_R^{l_2'}[(i\mu\Phi_{\iota,x,y}^0(R,t))^{l_1'}]\partial_R^{l''}[\mathcal{L}_\iota^k(R,t,x,y)]\right]_{(R,t)=(R_0,t_0)} + O_\tau(\mu^{-N+l})\right] d\tau = O(\mu^{-1}),$$

where $N \in \mathbb{N}$ is sufficiently large, the $D_{R,t}^{2j}$ are differential operators of order $2j$ in the variables R,t, the $c_{l_1',l_2',l''}$ certain coefficients, and the remainder is rapidly falling in τ. Here we took into account that for any $w \in \mathbb{C}$ with $\operatorname{Re} w > -1$ and $g \in \mathcal{S}(\mathbb{R})$ one has

$$(5.20) \qquad \int_{\mathbb{R}} e^{-i\mu\tau}\tau_+^w g(\tau)\,d\tau \approx \frac{\Gamma(w+1)}{(-i\mu)^{w+1}} = O(\mu^{-\operatorname{Re} w-1}),$$

compare (5.12). Consequently, we have shown (5.17) for $z \neq 0,1,2,3,\ldots$ as well, and we obtain (5.13), all estimates being uniform in x and y. This completes the proof of Theorem 5.3. $\qquad\square$

Remark 5.4. It might be instructive to illustrate our arguments by showing how they imply the L^p-bounds (1.6) proved by Seeger and Sogge. Assume that the co-spheres S_x^*M are strictly convex. In their notation, the crucial bound to be shown is[3]

$$I_{x,y}^z(\lambda) := \int_{\mathbb{R}^n} e^{i\langle x-y,\eta\rangle}\beta\left(\frac{\tilde{p}(\eta)}{\lambda}\right) q_z(\lambda-\tilde{p}(\eta),\eta)\,d\eta = O(\lambda^{\frac{n-1}{2}}), \qquad x,y \in \mathbb{R}^n, \quad \lambda \to +\infty,$$

---

[3]Note that in [23, Eq. (2.24)] the factor $\beta\left(1 - \frac{\tilde{p}(\eta)}{\lambda}\right)$ should read $\beta\left(\frac{\tilde{p}(\eta)}{\lambda}\right)$.

compare [23, Eq. (2.24)], where $\tilde{p}(\eta) \equiv \tilde{p}_{x,y}(\eta)$ is a symbol homogeneus of degree 1 in η, $\beta \in C_c^\infty(\mathbb{R})$ a test function satisfying $\beta(s) = 1$ when $s \in [1/2, 1]$ and $\beta(s) = 0$ if $s \notin [1/4, 2]$, and

$$q_z(\cdot, \eta) = e^{z^2} \int_{\mathbb{R}} (t - i0)^z \hat{\chi}(t) q(t, x, y, \eta) e^{-it\cdot} \, dt, \qquad \mathrm{Re}\,(z) = \frac{n-1}{2},$$

q being a classical symbol in $S_{1,0}^0$ and $\chi \in \mathcal{S}(\mathbb{R})$ a Schwartz function with $\mathrm{supp}\,\hat{\chi} \subset (-\varepsilon, \varepsilon)$. Now, introducing the coordinates $\eta = R\omega_1$, $R > 0$ with $\omega_1 \in \widetilde{\Sigma} := \{\eta \in \mathbb{R}^n : \tilde{p}(\eta) = 1\}$ we obtain with (5.12) for arbitrary $z \in \mathbb{C}$

$$I_{x,y}^z(\lambda) = \lambda^n e^{z^2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-it\lambda(1-R)} (t - i0)^z \hat{\chi}(t) \beta(R) J_{x,y}^{R,t}(\lambda) \, dR \, dt$$

$$= \lambda^n e^{z^2} e^{-i\pi z/2} \int_{\mathbb{R}} \frac{\tau_+^{-z-1}}{\Gamma(-z)} \left[\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-it[\lambda(1-R)+\tau]} \hat{\chi}(t) \beta(R) J_{x,y}^{R,t}(\lambda) \, dR \, dt \right] d\tau,$$

where we wrote

$$J_{x,y}^{R,t}(\lambda) := \int_{\widetilde{\Sigma}_R} e^{i\lambda\langle x-y,\omega\rangle} q(t, x, y, \lambda\omega) \, d\omega, \qquad \widetilde{\Sigma}_R := \{\eta : \tilde{p}(\eta) = R\}.$$

Note that the λ-dependence of the integrand is unproblematic, since $q \in S_{1,0}^0$. Clearly, $J_{x,y}^{R,t}(\lambda) = O(1)$ if $x = y$. Let us therefore assume that $x \neq y$. The critical set of the phase function $\langle x - y, \omega \rangle$ as a function in ω is clean due to the strict convexity assumption, and given by the isolated points $\mathcal{C} := \{\omega \in \widetilde{\Sigma}_R : x - y \in N_\omega \widetilde{\Sigma}_R\}$, so the stationary phase theorem yields for any $N \in \mathbb{N}$ the asymptotic expansion

$$J_{x,y}^{R,t}(\lambda) = (2\pi/\lambda)^{\frac{n-1}{2}} \sum_{\omega \in \mathcal{C}} e^{i\lambda\langle x-y,\omega\rangle} e^{i\pi\sigma_\omega/4} \sum_{k=0}^{N-1} D_\omega^{2k}[q(t, x, y, \lambda\omega)](\omega)\lambda^{-k} + O_{R,t,x,y}(\lambda^{\frac{n-1}{2}-N}),$$

compare [14, Theorem 7.7.14], with certain differential operators D_ω^{2k} and $\sigma_\omega \in \mathbb{Z}$ being given by the principal curvatures at ω, and an explicit remainder depending smoothly on R, t, x, and y. Let us first consider the case when $z = 0, 1, 2, 3, \ldots$, and write $-l := -z - 1$. Then $\tau_+^{-l}/\Gamma(-l+1) = \delta_0^{(l-1)}$, compare [14, (3.2.17)'], and finding an upper bound for $I_{x,y}^z(\lambda)$ reduces to the question of examining the asymptotic behavior of

$$\left\langle \delta_{\tau=0}^{(l-1)}, \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-it[\lambda(1-R)+\tau]} e^{i\lambda R\langle x-y,\omega_1\rangle} f(R, t) \, dR \, dt \right\rangle$$

$$= \lambda^{-l+1} (-1)^{l-1} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it\lambda(R-1)} \partial_R^{l-1} \left[e^{i\lambda R\langle x-y,\omega_1\rangle} f(R, t) \right] dt \, dR,$$

where $f \in C_c^\infty(\mathbb{R} \times \mathbb{R})$ has support near $(R, t) = (1, 0)$, and $\omega \equiv R\omega_1$. To do so, we proceed as in (5.16), and note that the phase function $t(R - 1) + R\langle x - y, \omega_1 \rangle$ has the only critical point

$$(R_0, t_0) = (1, -\langle x - y, \omega_1 \rangle).$$

Applying the stationary phase principle to the last (R, t)-integral we obtain $I_{x,y}^z(\lambda) = O(\lambda^{\frac{n-1}{2}})$ in case that $z = (n-1)/2 = 0, 1, 2, 3, \ldots$. On the other hand, if $z \neq 0, 1, 2, 3, \ldots$, things reduce to a description of oscillatory integrals of the form

$$-z(-z+1)\ldots(-z-1+l)(-1)^l \left\langle \tau_+^{-z-1}, \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-it[\lambda(1-R)+\tau]} e^{i\lambda R\langle x-y,\omega_1\rangle} f(R, t) \, dR \, dt \right\rangle$$

$$= \lambda^{-l} \int_{\mathbb{R}} \tau_+^{-z-1+l} \left[\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-it[\lambda(1-R)+\tau]} \partial_R^l \left[e^{i\lambda R\langle x-y,\omega_1\rangle} f(R, t) \right] dt \, dR \right] d\tau$$

$$= \lambda^{-z} \int_{\mathbb{R}} \tau_+^{-z-1+l} \left[\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\lambda t[(1-R)+\tau]} \partial_R^l \left[e^{i\lambda R\langle x-y,\omega_1\rangle} f(R, t) \right] dt \, dR \right] d\tau,$$

where $l \in \mathbb{N}$ is such that $\operatorname{Re} - z - 1 + l > -1$. The only critical point of the phase function $t[(R-1) - \tau] + R \langle x - y, \omega_1 \rangle$ as a function of R and t is

$$(R_0, t_0) = (1 + \tau, \langle x - y, \omega_1 \rangle),$$

and its value at this point is $R_0 t_0$. By applying the stationary phase theorem to the last (R, t)-integral, the uniform bound $I_{x,y}^z(\lambda) = O(\lambda^{\frac{n-1}{2}})$ would follow for $z \neq 0, 1, 2, 3, \ldots$ and $\operatorname{Re} z = (n-1)/2$, if

$$\int \tau_+^{-z-1+l} e^{i\lambda(1+\tau)\langle x-y,\omega_1\rangle} g(\tau) \, d\tau = O(\lambda^{\operatorname{Re} z - l})$$

for any $g \in \mathcal{S}(\mathbb{R})$, which nevertheless is implied by (5.20).

Example 5.5. Let us resume Example 4.9 of a connected semisimple Lie group G with finite center, discrete co-compact subgroup Γ, and maximal compact subgroup K. The group K acts on $\Gamma \backslash G$ with orbits of principal and exceptional type, all orbits having the dimension $\dim K$, and we deduce from Proposition 5.1 for each $\gamma \in \widehat{K}$ the estimate

$$\|u\|_{\mathrm{L}^\infty(\Gamma\backslash G)} \leq C \lambda^{\frac{\dim G/K - 1}{2m}}, \qquad u \in \mathrm{L}_\gamma^2(\Gamma\backslash G), \|u\|_{\mathrm{L}^2} = 1,$$

for any eigenfunction u of a K-invariant elliptic positive symmetric classical pseudodifferential operator P on $\Gamma \backslash G$ of degree m with eigenvalue λ. More generally, with $\frac{1}{q} + \frac{1}{q'} = 1$ and

$$\delta(q) := \max \left(\dim G/K \left| \frac{1}{2} - \frac{1}{q} \right| - \frac{1}{2}, 0 \right)$$

we have by Theorem 5.3 the bound

$$\|u\|_{\mathrm{L}^q(\Gamma\backslash G)} \leq \begin{cases} C \lambda^{\frac{\delta(q)}{m}}, & \frac{2(\dim G/K + 1)}{\dim G/K - 1} \leq q \leq \infty, \\ C \lambda^{\frac{(\dim G/K - 1)(2 - q')}{4mq'}}, & 2 \leq q \leq \frac{2(\dim G/K + 1)}{\dim G/K - 1}, \end{cases}$$

provided that P satisfies the strict convexity assumption in Theorem 5.3. In case that Γ has no torsion and $p = \infty$, we recover for any eigenfunction of the Beltrami-Laplace operator on $\Gamma\backslash G/K$ the classical bound

$$\|u\|_{\mathrm{L}^\infty(\Gamma\backslash G/K)} \leq C \lambda^{\frac{\dim \Gamma\backslash G/K - 1}{4}}, \qquad u \in \mathrm{L}^2(\Gamma\backslash G/K), \|u\|_{\mathrm{L}^2} = 1,$$

$\mathrm{L}^2(\Gamma\backslash G/K) \simeq \mathrm{L}^2(\Gamma\backslash G)^K$ corresponding to the trivial isotypic component in the Peter-Weyl decomposition of $\mathrm{L}^2(\Gamma\backslash G)$. Thus, our results generalize the classical bounds for Maass forms on $\Gamma\backslash G$ to arbitrary K-types.

6. The desingularization process

As already noted, the asymptotic formula for the reduced spectral function $e_\gamma(x, x, \lambda)$ given in Theorem 4.3 depends in a highly non-smooth way on $x \in M$ if exceptional or even singular orbits are present, and does not give a precise description of the caustic behavior of $e_\gamma(x, x, \lambda)$. In particular, it remains unclear if the coefficients in the expansion of $e_\gamma(x, x, \lambda)$ are integrable in x, and how one could deduce from Theorem 4.3 asymptotics for the equivariant spectral counting function $N_\gamma(\lambda) := \int_M e_\gamma(x, x, \lambda) \, dM(x)$. In what follows, we would like to give a description of $e_\gamma(x, x, \lambda)$ that interpolates between the asymptotics for different values of x, and in particular to characterize the behavior of the leading coefficient and the remainder term in Theorem 4.3 as $x \in M_{\mathrm{prin}}$ approaches singular orbits. For this, we shall make use of resolution of singularities. As we shall see, the major difficulty resides in the fact that, unless the G-action on T^*M is free, so that the considered momentum map becomes a submersion, Ω and $\operatorname{Crit} \Phi$ are not smooth manifolds. Nevertheless, it was shown in [21] that by constructing a strong resolution of the set

$$(6.1) \qquad\qquad \mathcal{N} := \{(p, g) \in M \times G : g \cdot p = p\}$$

a partial desingularization

$$(6.2) \qquad\qquad \mathcal{Z} : \widetilde{\mathbf{X}} \to \mathbf{X} := T^*M \times G$$

of the critical sets $\operatorname{Crit}\Phi$ can be achieved, and after applying the stationary phase theorem in the resolution space $\widetilde{\mathbf{X}}$, an asymptotic description of the integrals $I(\mu)$ defined in (3.2) was obtained, leading to an asymptotic formula for $N_\gamma(\lambda)$. In the ensuing sections, we shall use the partial desingularization (6.2) to obtain an asymptotic formula for the integrals $I_x(\mu) := I_{x,x}(\mu)$ defined in (3.1) that describes the caustic behavior of the coefficients $Q_k(x)$ in Theorem 3.3 (1) as one approaches singular orbits. One can deduce from this the asymptotic description of the integrals $I(\mu)$ given in [21], but the converse implication is more subtle and not straight-forward. For this reason, a careful re-examination of the results of [21] is needed in order to obtain a precise description of the coefficients in the asymptotic formula for the integrals $I_x(\mu)$ and, ultimately, of the leading coefficient in the asymptotic formula for the equivariant spectral function.

Let M be a closed connected Riemannian manifold and G a connected, compact Lie group acting on M by isometries. In what follows, we shall recall the construction of the partial desingularization (6.2) of the critical set $\mathcal{C} := \big\{(x,\eta,g) \in (\Omega \cap T^*M) \times G : g \in G_{(x,\eta)}\big\}$ performed in [21]. The desingularization process presented here is exactly the same, only that we apply it now to the study of the integrals (3.1) instead of the integrals (3.2). For details, the reader is referred to [21]. Consider the decomposition of M into orbit types

(6.3) $$M = M(H_1) \,\dot\cup \cdots \dot\cup\, M(H_L),$$

where we suppose that the isotropy types are numbered in such a way that $(H_i) \geq (H_j)$ implies $i \leq j$, (H_L) being the principal isotropy type, see Figure 6.1.
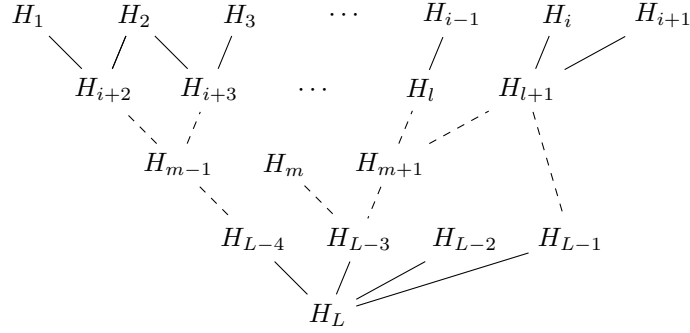

FIGURE 6.1. An isotropy tree corresponding to the decomposition (6.3). A line between two subgroups indicates partial ordering.

To construct (6.2), an iterative process along the strata of the G-action on M is set up, where the centers of the blow-ups are successively chosen as isotropy bundles over unions of maximally singular orbits. For simplicity, one assumes that at each step the union of maximally singular orbits is connected.

Beginning of iteration. Let $f_k : \nu_k \to M_k$ be an invariant tubular neighborhood of $M_k(H_k)$ in

$$M_k := M - \bigcup_{i=1}^{k-1} f_i(\overset{\circ}{D}_{1/2}(\nu_i)), \qquad k = 1, \ldots, L,$$

a manifold with corners on which G acts with the isotropy types $(H_k), (H_{k+1}), \ldots, (H_L)$. Here ν_k denotes the normal G-vector bundle of $M_k(H_k)$, $\overset{\circ}{D}_{1/2}(\nu_i) := \{v \in \nu_i : \|v\| < 1/2\}$,

$$f_k(p^{(k)}, v^{(k)}) := (\exp_{p^{(k)}} \circ \gamma^{(k)})(v^{(k)}), \qquad p^{(k)} \in M_k(H_k),\ v^{(k)} \in (\nu_k)_{p^{(k)}},$$

is an equivariant diffeomorphism given in terms of the exponential map, and

$$\gamma^{(k)}(v^{(k)}) := \frac{F_k(p^{(k)})}{(1 + \|v^{(k)}\|^2)^{1/2}} v^{(k)},$$

where $F_k : M_k(H_k) \to \mathbb{R}$ is a smooth, G-invariant, positive function, see [2, Page 306]. Let S_k be the unit sphere bundle over $M_k(H_k)$, and put $W_k := f_k(\overset{\circ}{D}_1(\nu_k))$, $W_L := \overset{\circ}{M}_L$, so that we obtain the open covering

$$(6.4) \qquad\qquad M = W_1 \cup \cdots \cup W_L.$$

Fix an inner product on $\mathfrak{g}$, which induces a Riemannian structure on G, and consider for each k and $p^{(k)} \in M_k(H_k)$ the decomposition

$$T_e G \simeq \mathfrak{g} = \mathfrak{g}_{p^{(k)}} \oplus \mathfrak{g}_{p^{(k)}}^\perp,$$

where $\mathfrak{g}_{p^{(k)}} \simeq T_e G_{p^{(k)}}$ denotes the Lie algebra of the stabilizer $G_{p^{(k)}}$ of $p^{(k)}$, and $\mathfrak{g}_{p^{(k)}}^\perp$ its orthogonal complement with respect to the above Riemannian structure. Now, introduce a partition of unity $\{\chi_k\}_{k=1,\ldots,L}$ subordinated to the covering (6.4), and define

$$I_k(x, \mu) := \chi_k(x) I_x(\mu)$$

with $I_x(\mu) = I_{x,x}(\mu)$ as in (3.1). By Theorem 3.3 (1) the asymptotic expansion for $I_L(x, \mu)$ depends smoothly on $x \in W_L \cap Y$. Let us therefore turn to the case when $1 \le k \le L - 1$ and $W_k \cap Y \ne \emptyset$. For fixed k and $x = f_k(p^{(k)}, v^{(k)}) \in W_k \cap Y$ Lemma 3.1 (a) implies that

$$\operatorname{Crit} \Phi_x = \left\{ (\omega, g) \in \Sigma_x^{R,t} \times G : (x, \omega) \in \Omega, \, g \cdot x = x \right\} \subset \Sigma_x^{R,t} \times G_{p^{(k)}},$$

where $\Phi_x := \Phi_{x,x}$. Up to non-stationary contributions, it will therefore suffice to evaluate the integrals $I_k(x, \mu)$ in a neighborhood of $G_{p^{(k)}}$. To this end, consider the isotropy bundle $\operatorname{Iso} M_k(H_k) \to M_k(H_k)$ over $M_k(H_k)$, as well as the canonical projection

$$\pi_k : W_k \to M_k(H_k), \qquad f_k(p^{(k)}, v^{(k)}) \mapsto p^{(k)}, \qquad p^{(k)} \in M_k(H_k), \, v^{(k)} \in (\nu_k)_{p^{(k)}}.$$

Further, let

$$\pi_k^* \operatorname{Iso} M_k(H_k) = \left\{ (f_k(p^{(k)}, v^{(k)}), h^{(k)}) \in W_k \times G : h^{(k)} \in G_{p^{(k)}} \right\}$$

be the induced bundle. Let U_k be a tubular neighborhood of $\pi_k^* \operatorname{Iso} M_k(H_k)$ in $W_k \times G$, and note that the fiber of the normal bundle $N \pi_k^* \operatorname{Iso} M_k(H_k)$ at a point $(f_k(p^{(k)}, v^{(k)}), h^{(k)})$ may be identified with the fiber of the normal bundle to $G_{p^{(k)}}$ at the point $h^{(k)}$. Consider further an orthonormal basis $\{A_1(p^{(k)}), \ldots, A_{d^{(k)}}(p^{(k)})\}$ of $\mathfrak{g}_{p^{(k)}}^\perp$, and introduce canonical coordinates of the second kind

$$(6.5) \qquad \mathbb{R}^{d^{(k)}} \times G_{p^{(k)}} \ni (\alpha_1^{(k)}, \ldots, \alpha_{d^{(k)}}^{(k)}, h^{(k)}) \longmapsto e^{\sum_i \alpha_i^{(k)} A_i(p^{(k)})} h^{(k)}$$

in a neighborhood of $G_{p^{(k)}}$, see [11, Page 146]. Denote by b_μ the amplitude a_μ multiplied by a smooth cut-off-function with support in U_k which is equal to 1 in a small neighborhood of $\pi_k^* \operatorname{Iso} M_k(H_k)$. Taking into account the non-stationary phase theorem [14, Theorem 7.7.1] one computes

$$(6.6) \qquad I_k(x, \mu) = \chi_k(x) \int_{G_{p^{(k)}} \times \mathfrak{g}_{p^{(k)}}^\perp \times \Sigma_x^{R,t}} e^{i\mu \Phi_x} b_\mu \, d(\Sigma_x^{R,t})(\omega) \, dA^{(k)} \, dh^{(k)} + O(\mu^{-\infty}),$$

where $dh^{(k)}, dA^{(k)}$ are suitable volume densities on the sets $G_{p^{(k)}}$ and $\mathfrak{g}_{p^{(k)}}^\perp \simeq N_{h^{(k)}} G_{p^{(k)}}$, respectively, such that $dg \equiv dA^{(k)} \, dh^{(k)}$, compare [21, (5.4)], and the remainder estimate is uniform in x.

We shall now sucessively resolve the singularities of (6.1) in order to obtain a factorization of Φ_x. Note that by [21, Eq. (5.1)]

$$\mathcal{N} = \mathcal{N}_L \cup \bigcup_{k=1}^{L-1} \mathcal{N}_k,$$

where $\mathcal{N}_k := \mathcal{N} \cap U_k$, $\mathcal{N}_L := \operatorname{Iso} W_L$, $\operatorname{Iso} W_L \to W_L$ being the isotropy bundle over W_L. While $\mathcal{N}_L$ is a smooth submanifold, $\mathcal{N}_k$ is in general singular. In particular, if $\dim H_k \ne \dim H_L$, $\mathcal{N}_k$ has a maximal singular locus given by $\operatorname{Iso} M_k(H_k)$. One then performs for each $k \in \{1, \ldots, L-1\}$ a blow-up

$$\zeta_k : B_{Z_k}(U_k) \longrightarrow U_k$$

with center $Z_k := \operatorname{Iso} M_k(H_k) \subset \mathcal{N}_k$, and by piecing these transformations together one obtains the global blow-up

$$\zeta^{(1)} : B_{Z^{(1)}}\mathcal{M} \longrightarrow \mathcal{M}, \qquad Z^{(1)} := \bigsqcup_{k=1}^{L-1} Z_k,$$

where we put $\mathcal{M} := M \times G$. To get a local description, fix k, and let $\{v_1^{(k)}, \ldots, v_{c^{(k)}}^{(k)}\}$ be an orthonormal frame in ν_k, and $(\theta_1^{(k)}, \ldots, \theta_{c^{(k)}}^{(k)})$ coordinates in $\gamma^{(k)}((\nu_k)_{p^{(k)}})$. Similarly, let $(\alpha_1^{(k)}, \ldots, \alpha_{d^{(k)}}^{(k)})$ be the coordinates introduced in (6.5). If one now covers $B_{Z_k}(U_k)$ with standard projective charts $\{(\phi_k^\varrho, \mathcal{O}_k^\varrho)\}$ one obtains in the so-called $\theta^{(k)}$-charts $\{\mathcal{O}_k^\varrho\}_{1 \leq \varrho \leq c^{(k)}}$, in which the $\theta_\varrho^{(k)}$-coordinate is non-zero, for ζ_k the local expressions

$$(6.7) \qquad \zeta_k^\varrho = \zeta_k \circ (\phi_k^\varrho)^{-1} : (p^{(k)}, \tau_k, \tilde{v}^{(k)}, A^{(k)}, h^{(k)}) \mapsto \left(\exp_{p^{(k)}} \tau_k \tilde{v}^{(k)}, \, e^{\tau_k A^{(k)}} h^{(k)} \right) = (x, g),$$

where

$$p^{(k)} \in M_k(H_k), \qquad A^{(k)} \in \mathfrak{g}_{p^{(k)}}^\perp, \qquad h^{(k)} \in G_{p^{(k)}}, \qquad \tilde{v}^{(k)} \in \gamma^{(k)}\big((S_k^+)_{p^{(k)}}\big),$$

and $S_k^+ := \left\{ v \in \nu_k : v := \sum s_i v_i^{(k)}, s_\varrho > 0, \|v\| = 1 \right\}$, while $\tau_k \in (-1, 1)$, see [21, Eq. (5.6)]. A similar description of ζ_k is given in the so-called $\alpha^{(k)}$-charts $\{\mathcal{O}_k^\varrho\}_{c^{(k)}+1 \leq \varrho \leq c^{(k)}+d^{(k)}}$, in which the $\alpha_\varrho^{(k)}$-coordinate does not vanish. By performing Taylor expansion at $\tau_k = 0$ one can then show that the phase function (3.3) factorizes according to

$$(6.8) \qquad \Phi \circ (\operatorname{id}_\eta \otimes \zeta_k^\varrho) = {}^{(k)}\widetilde{\Phi}^{tot} = \tau_k \cdot {}^{(k)}\widetilde{\Phi}^{wk},$$

${}^{(k)}\widetilde{\Phi}^{tot}$ and ${}^{(k)}\widetilde{\Phi}^{wk}$ being the *total* and *weak transform* of the phase function Φ, respectively, see [21, Eqs. (5.8) and (5.9)]. Since ζ_k is a real-analytic surjective proper map, which is a diffeomorphism on the complement of $\zeta_k^{-1}(Z_k)$, we can lift the integral $I_k(x, \mu)$ along the restriction of ζ_k to the fiber over $\{x\} \times G$ to the resolution space $B_{Z_k}(U_k)$. To obtain local expressions, introduce a compactly supported partition $\{u_k^\varrho\}$ of unity subordinate to the covering $\{\mathcal{O}_k^\varrho\}$, set $a_k^\varrho := (u_k^\varrho \circ (\phi_k^\varrho)^{-1}) \cdot [(b_\mu \chi_k) \circ (\operatorname{id}_\omega \otimes \zeta_k^\varrho)]$, and define for $x = \exp_{p^{(k)}} \tau_k \tilde{v}^{(k)} \in W_k \cap Y$ and $1 \leq \varrho \leq c^{(k)}$ the integrals

$$I_k^\varrho(x, \mu) := |\tau_k|^{d^{(k)}} \int_{G_{p^{(k)}} \times \mathfrak{g}_{p^{(k)}}^\perp \times \Sigma_x^{R,t}} e^{i\mu\tau_k \, {}^{(k)}\widetilde{\Phi}^{wk}_{\tau_k, p^{(k)}, \tilde{v}^{(k)}}} a_k^\varrho \, d(\Sigma_x^{R,t})(\omega) \, dA^{(k)} \, dh^{(k)},$$

and for $c^{(k)} + 1 \leq \varrho \leq c^{(k)} + d^{(k)}$ corresponding integrals $\widetilde{I}_k^\varrho(x, \mu)$. Here $\widetilde{\Phi}^{wk}_{\tau_k, p^{(k)}, \tilde{v}^{(k)}}$ denotes the weak transform regarded as a function of the variables $\omega, A^{(k)}, h^{(k)}$, while $\tau_k, p^{(k)}, \tilde{v}^{(k)}$ are considered as parameters. Let us emphasize that the amplitudes a_k^ϱ are compactly supported. In view of (6.6) we arrive for $x \in W_k$ at the decomposition

$$I_k(x, \mu) = \sum_{\varrho=1}^{c^{(k)}} I_k^\varrho(x, \mu) + \sum_{\varrho=c^{(k)}+1}^{d^{(k)}} \widetilde{I}_k^\varrho(x, \mu)$$

up to terms of order $O(\mu^{-\infty})$. As we shall see in Corollary 7.2, the weak transforms $\widetilde{\Phi}^{wk}_{\tau_k, p^{(k)}, \tilde{v}^{(k)}}$ have no critical points in the $\alpha^{(k)}$-charts, which will imply that the integrals $\widetilde{I}_k^\varrho(x, \mu)$ contribute to $I(x, \mu)$ with terms of order $O(\mu^{-\infty})$. If G acts on S_k only with isotropy type (H_L), we shall see in the next section that in each of the $\theta^{(k)}$-charts the critical sets of the weak transforms ${}^{(k)}\widetilde{\Phi}^{wk}$ are clean, so that one can apply the stationary phase theorem in order to obtain asymptotics for each of the $I_k^\varrho(x, \mu)$. But in general, G will act on S_k with singular orbit types, so that neither $\mathcal{N}_k$ is resolved, nor do the weak transforms ${}^{(k)}\widetilde{\Phi}^{wk}$ have clean critical sets, and we are forced to continue with the iteration.

Iteration step from $N-1$ to N. Denote by $\Lambda \leq L$ the maximal number of elements that a totally ordered subset of the set of isotropy types can have. Assume that $2 \leq N < \Lambda$, and let $\{(H_{i_1}), \ldots, (H_{i_N})\}$ be a totally ordered subset of the set of isotropy types such that $i_1 < \cdots < i_N < L$. Let f_{i_1}, S_{i_1}, as well as $p^{(i_1)} \in M_{i_1}(H_{i_1})$ be defined as at the beginning of the iteration, and assume that $f_{i_1 \ldots i_j}$, $S_{i_1 \ldots i_j}, p^{(i_j)}, \ldots$ have already been defined for $j < N$. For every fixed $p^{(i_{N-1})}$, denote by $\gamma^{(i_{N-1})}((S_{i_1 \ldots i_{N-1}})_{p^{(i_{N-1})}})_{i_N}$ the sub-manifold with corners of the closed $G_{p^{(i_{N-1})}}$-manifold $\gamma^{(i_{N-1})}((S_{i_1 \ldots i_{N-1}})_{p^{(i_{N-1})}})$ from which all orbit types less than G/H_{i_N} have been removed. Consider the invariant tubular neighborhood

$$f_{i_1 \ldots i_N} := \exp \circ \gamma^{(i_N)} : \nu_{i_1 \ldots i_N} \to \gamma^{(i_{N-1})}((S_{i_1 \ldots i_{N-1}})_{p^{(i_{N-1})}})_{i_N}$$

of the set $\gamma^{(i_{N-1})}((S_{i_1 \ldots i_{N-1}})_{p^{(i_{N-1})}})_{i_N}(H_{i_N})$, where $\nu_{i_1 \ldots i_N}$ denotes its normal $G_{p^{(i_{N-1})}}$-vector bundle, and $\exp \circ \gamma^{(i_N)}$ the corresponding equvariant diffeomorphism, and define $S_{i_1 \ldots i_N}$ as the sphere sub-bundle in $\nu_{i_1 \ldots i_N}$, while

$$S^+_{i_1 \ldots i_N} := \left\{ v \in S_{i_1 \ldots i_N} : v = \sum s_i v_i^{(i_1 \ldots i_N)}, \ s_{\varrho_{i_N}} > 0 \right\}$$

for some ϱ_{i_N}. Put $W_{i_1 \ldots i_N} := f_{i_1 \ldots i_N}(\overset{\circ}{D}_1(\nu_{i_1 \ldots i_N}))$, and denote the corresponding integral in the decomposition of $I^{\varrho_{i_1} \cdots \varrho_{i_{N-1}}}_{i_1 \ldots i_{N-1}}(x, \mu)$ by $I^{\varrho_{i_1} \cdots \varrho_{i_{N-1}}}_{i_1 \ldots i_N}(x, \mu)$. Here we can assume that, modulo terms of order $O(\mu^{-\infty})$, the $W_{i_1 \ldots i_N} \times G_{p^{(i_{N-1})}}$-support of the integrand in $I^{\varrho_{i_1} \cdots \varrho_{i_{N-1}}}_{i_1 \ldots i_N}(\mu)$ is contained in a compactum of a tubular neighborhood of the induced bundle $\pi^*_{i_1 \ldots i_N} \mathrm{Iso}\, \gamma^{(i_{N-1})}((S_{i_1 \ldots i_{N-1}})_{p^{(i_{N-1})}})_{i_N}(H_{i_N})$, where $\pi_{i_1 \ldots i_N} : W_{i_1 \ldots i_N} \to \gamma^{(i_{N-1})}((S_{i_1 \ldots i_{N-1}})_{p^{(i_{N-1})}})_{i_N}(H_{i_N})$ denotes the canonical projection. For a given point $p^{(i_N)} \in \gamma^{(i_{N-1})}((S^+_{i_1 \ldots i_{N-1}})_{p^{(i_{N-1})}})_{i_N}(H_{i_N})$, consider further the decomposition

$$\mathfrak{g}_{p^{(i_{N-1})}} = \mathfrak{g}_{p^{(i_N)}} \oplus \mathfrak{g}^\perp_{p^{(i_N)}},$$

and set $d^{(i_N)} := \dim \mathfrak{g}^\perp_{p^{(i_N)}}$, $e^{(i_N)} := \dim \mathfrak{g}_{p^{(i_N)}}$. This yields the decomposition

$$(6.9) \qquad \mathfrak{g} = \mathfrak{g}_{p^{(i_1)}} \oplus \mathfrak{g}^\perp_{p^{(i_1)}} = (\mathfrak{g}_{p^{(i_2)}} \oplus \mathfrak{g}^\perp_{p^{(i_2)}}) \oplus \mathfrak{g}^\perp_{p^{(i_1)}} = \cdots = \mathfrak{g}_{p^{(i_N)}} \oplus \mathfrak{g}^\perp_{p^{(i_N)}} \oplus \cdots \oplus \mathfrak{g}^\perp_{p^{(i_1)}}.$$

Denote by $\left\{ A_r^{(i_N)}(p^{(i_1)}, \ldots, p^{(i_N)}) \right\}$ a basis of $\mathfrak{g}^\perp_{p^{(i_N)}}$, and let $\left(\alpha_1^{(i_N)}, \ldots, \alpha_{d^{(i_N)}}^{(i_N)} \right)$ be corresponding coordinates. Further, let $\left\{ v_1^{(i_1 \ldots i_N)}, \ldots, v_{c^{(i_N)}}^{(i_1 \ldots i_N)} \right\}$ be an orthonormal frame in $\nu_{i_1 \ldots i_N}$, and $(\theta_1^{(i_N)}, \ldots, \theta_{c^{(i_N)}}^{(i_N)})$ corresponding coordinates. Now, let the blow-up $\zeta^{(1)}$ be defined as in the beginning of the iteration, and assume that the blow-ups $\zeta^{(j)}$ have already been defined for $j < N$. Put $\widetilde{\mathcal{M}}^{(j)} := B_{Z^{(j)}}(\widetilde{\mathcal{M}}^{(j-1)})$, $\widetilde{\mathcal{M}}^{(0)} := \mathcal{M} = M \times G$, and consider the blow-up

$$(6.10) \qquad \zeta^{(N)} : B_{Z^{(N)}}(\widetilde{\mathcal{M}}^{(N-1)}) \to \widetilde{\mathcal{M}}^{(N-1)}, \qquad Z^{(N)} := \bigcup_{i_1 < \cdots < i_N < L} Z_{i_1 \ldots i_N},$$

where the union is over all totally ordered subsets $\{(H_{i_1}), \ldots, (H_{i_N})\}$ of N elements with $i_1 < \cdots < i_N < L$, and

$$Z_{i_1 \ldots i_N} \simeq \bigcup_{p^{(i_1)}, \ldots, p^{(i_{N-1})}} (-1, 1)^{N-1} \times \mathrm{Iso}\, \gamma^{(i_{N-1})}((S_{i_1 \ldots i_{N-1}})_{p^{(i_{N-1})}})_{i_N}(H_{i_N})$$

are the possible maximal singular loci of $(\zeta^{(1)} \circ \cdots \circ \zeta^{(N-1)})^{-1}(\mathcal{N})$. Denote by $\zeta_{i_1}^{\varrho_{i_1}} \circ \cdots \circ \zeta_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}$ a local realization of the sequence of blow-ups $\zeta^{(1)} \circ \cdots \circ \zeta^{(N)}$ corresponding to the totally ordered subset $\{(H_{i_1}), \ldots, (H_{i_N})\}$ in a set of charts labeled by the indices $\varrho_{i_1}, \ldots, \varrho_{i_N}$. As a consequence, we obtain local factorizations of the phase function according to

$$\Phi \circ ((\zeta_{i_1}^{\varrho_{i_1}} \circ \cdots \circ \zeta_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}) \otimes \mathrm{id}_\eta) = {}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{tot} = \tau_{i_1} \cdots \tau_{i_N} \, {}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk},$$

see [21, Page 39]. Assume now that the indices $\varrho_{i_1}, \ldots, \varrho_{i_N}$ correspond to a set of $(\theta^{(i_1)}, \ldots, \theta^{(i_N)})$-charts. Then $\zeta_{i_1}^{\varrho_{i_1}} \circ \cdots \circ \zeta_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}$ is explicitly given by

$$(\tau_{i_1}, \ldots, \tau_{i_N}, p^{(i_1)}, \ldots, p^{(i_N)}, \tilde{v}^{(i_N)}, A^{(i_1)}, \ldots, A^{(i_N)}, h^{(i_N)}) \quad \longmapsto \quad (x_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}, g_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}) = (x, g),$$

where we set

$$x_{i_j \ldots i_N}^{\varrho_{i_j} \cdots \varrho_{i_N}} := \exp_{p^{(i_j)}}[\tau_{i_j} \exp_{p^{(i_{j+1})}}[\tau_{i_{j+1}} \exp_{p^{(i_{j+2})}}[\ldots [\tau_{i_{N-2}} \exp_{p^{(i_{N-1})}}[\tau_{i_{N-1}} \exp_{p^{(i_N)}}[\tau_{i_N} \tilde{v}^{(i_N)}]]] \ldots]]],$$

$$g_{i_j \ldots i_N}^{\varrho_{i_j} \cdots \varrho_{i_N}} := \mathrm{e}^{\tau_{i_j} \cdots \tau_{i_N} A^{(i_j)}} \, \mathrm{e}^{\tau_{i_{j+1}} \cdots \tau_{i_N} A^{(i_{j+1})}} \cdots \mathrm{e}^{\tau_{i_{N-1}} \tau_{i_N} A^{(i_{N-1})}} \, \mathrm{e}^{\tau_{i_N} A^{(i_N)}} \, h^{(i_N)}.$$

In this situation we define

$$(6.11) \qquad I_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}(x, \mu) := \prod_{j=1}^{N} |\tau_{i_j}|^{\sum_{r=1}^{j} d^{(i_r)}} \int_{\widetilde{\mathbf{X}}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} \times \Sigma_x^{R,t}}$$

$$\cdot \, \mathrm{e}^{i\mu \tau_1 \ldots \tau_N \, (i_1 \ldots i_N) \widetilde{\Phi}^{wk}_{\tau_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}} \, a_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} \, d\omega \, dA^{(i_1)} \ldots dA^{(i_N)} \, dh^{(i_N)},$$

where

- $\widetilde{\mathbf{X}}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} := G_{p^{(i_N)}} \times \mathfrak{g}_{p^{(i_N)}}^{\perp} \times \cdots \times \mathfrak{g}_{p^{(i_1)}}^{\perp}$,
- $\widetilde{\Phi}^{wk}_{\tau_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}$ denotes the weak transform regarded as a function on $\widetilde{\mathbf{X}}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} \times \Sigma_x^{R,t}$, while the $\tau_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}$ are regarded as parameters,
- the $a_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}$ are amplitudes with compact support in a system of $(\theta^{(i_1)}, \ldots, \theta^{(i_N)})$-charts labeled by the indices $\varrho_{i_1}, \ldots, \varrho_{i_N}$,
- $dA^{(i_1)}, \ldots, dA^{(i_N)}, dh^{(i_N)}$ are suitable measures on $\mathfrak{g}_{p^{(i_1)}}^{\perp}, \ldots, \mathfrak{g}_{p^{(i_N)}}^{\perp}$, and $G_{p^{(i_N)}}$, respectively.

Similarly, one defines analogous integrals $\widetilde{I}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}(x, \mu)$ in the $(\theta^{(i_1)}, \ldots, \theta^{(i_{N-1})}, \alpha^{(i_N)})$-charts. As we shall see in Section 7, $I_x(\mu)$ will be given by a sum involving both the integrals $I_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}(x, \mu)$ and $\widetilde{I}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}}(x, \mu)$.

Now, for each $p^{(i_{N-1})}$, the isotropy group $G_{p^{(i_{N-1})}}$ acts on $\gamma^{(i_{N-1})}((S_{i_1 \ldots i_{N-1}})_{p^{(i_{N-1})}})_{i_N}$ by the isotropy types $(H_{i_N}), \ldots, (H_L)$. The types occuring in $W_{i_1 \ldots i_N}$ constitute a subset of these, and $G_{p^{(i_{N-1})}}$ acts on the sphere bundle $S_{i_1 \ldots i_N}$ over the submanifold $\gamma^{(i_{N-1})}((S_{i_1 \ldots i_{N-1}})_{p^{(i_{N-1})}})_{i_N}(H_{i_N}) \subset W_{i_1 \ldots i_N}$ with one type less.

End of iteration. After $N = \Lambda - 1$ steps, the end of the iteration is reached, yielding a strong desingularization of $\mathcal{N}$, see [21, Theorem 5.1], and a factorization of the phase function Φ_x that will allow us to interpolate between the different asymptotics for the integrals $I_x(\mu)$ described in Theorem 3.3.

7. Asymptotics in the resolution space. Caustics and concentration of eigenfunctions

We are now ready to give an asymptotic formula for the integrals (6.11) that will result in a corresponding description of the integrals (3.1) on the diagonal. With the notation as before, consider for fixed $1 \leq N \leq \Lambda - 1$ a maximal, totally ordered subset $\{(H_{i_1}), \ldots, (H_{i_N})\}$ of non-principal isoptropy types in the sense that if there is an isotropy type $(H_{i_{N+1}})$ with $i_N < i_{N+1}$ such that $\{(H_{i_1}), \ldots, (H_{i_{N+1}})\}$ is a totally ordered subset, then $(H_{i_{N+1}}) = (H_L)$. Assign to each such subset the sequence of consecutive local blow-ups

$$\mathcal{Z}_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} := (\zeta_{i_1}^{\varrho_{i_1}} \circ \cdots \circ \zeta_{i_1 \ldots i_N}^{\varrho_{i_1} \cdots \varrho_{i_N}} \circ (\delta_{i_1 \ldots i_N} \otimes \mathrm{id})) \otimes \mathrm{id}_\eta$$

where $\delta_{i_1 \ldots i_N}$ denotes the sequence of local quadratic transformations

$$\delta_{i_1 \ldots i_N} : (\sigma_{i_1}, \ldots \sigma_{i_N}) \mapsto \sigma_{i_1}(1, \sigma_{i_2}, \ldots, \sigma_{i_N}) = (\sigma_{i_1}', \ldots, \sigma_{i_N}') \mapsto \sigma_{i_2}'(\sigma_{i_1}', 1, \ldots, \sigma_{i_N}') = (\sigma_{i_1}'', \ldots, \sigma_{i_N}'')$$

$$\mapsto \sigma_{i_3}''(\sigma_{i_1}'', \sigma_{i_2}'', 1, \ldots, \sigma_{i_N}'') = \cdots \mapsto \cdots = (\tau_{i_1}, \ldots, \tau_{i_N}).$$

The global morphism induced by the local transformations $\mathcal{Z}_{i_1 \ldots i_N}^{\varrho_{i_1} \ldots \varrho_{i_N}}$ is then denoted by

$$\mathcal{Z} : \widetilde{\mathbf{X}} \to \mathbf{X} := T^* M \times G,$$

and constitutes a partial desingularization of the critical set $\mathcal{C}$, see [21, Section 9]. Pulling the phase function Φ back along the maps $\mathcal{Z}_{i_1 \ldots i_N}^{\varrho_{i_1} \ldots \varrho_{i_N}}$ then yields the local factorization

$$\Phi \circ \mathcal{Z}_{i_1 \ldots i_N}^{\varrho_{i_1} \ldots \varrho_{i_N}} = {}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{tot} = \tau_{i_1}(\sigma) \ldots \tau_{i_N}(\sigma) \, {}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk},$$

where the τ_{i_j} are monomials in the exceptional parameters $\sigma_{i_1}, \ldots, \sigma_{i_N}$. The principal result in [21] is

Theorem 7.1. *In any of the $(\theta^{(i_1)}, \ldots, \theta^{(i_N)})$-charts, the critical sets of the weak transforms ${}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk}$ are smooth sub-manifolds in the resolution space of co-dimension 2κ, and the Hessians $\mathrm{Hess}\, {}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk}$ are transversally non-degenerate. In other words, the weak transforms ${}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk}$ have clean critical sets in the mentioned charts. On the other hand, the weak transforms ${}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk}$ have no critical points in any of the $(\theta^{(i_1)}, \ldots, \theta^{(i_{N-1})}, \alpha^{(i_N)})$-charts.*

Proof. See [21, Theorems 6.1 and 7.2, as well as pp. 61-62]. $\qquad\square$

In order to prove Theorem 7.1 for the $(\theta^{(i_1)}, \ldots, \theta^{(i_N)})$-charts one first shows that

$$\partial_{\eta, \alpha^{(i_1)}, \ldots, \alpha^{(i_N)}, h^{(i_N)}} \, {}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk} = 0 \quad \Longrightarrow \quad \partial_{\sigma_{i_1}, \ldots, \sigma_{i_N}, p^{(i_1)}, \ldots, p^{(i_N)}, \tilde{v}^{(i_N)}} \, {}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk} = 0,$$

see [21, Eq. (6.17)]. If therefore

$${}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}} \big(\alpha^{(i_j)}, h^{(i_N)}, \eta \big)$$

denotes the weak transform of Φ regarded as a function of the variables $(\alpha^{(i_1)}, \ldots, \alpha^{(i_N)}, h^{(i_N)}, \eta)$ alone, while the variables $(\sigma_{i_1}, \ldots, \sigma_{i_N}, p^{(i_1)}, \ldots, p^{(i_N)}, \tilde{v}^{(i_N)})$ are kept fixed at constant values, its critical set is given by the transversal intersection

$$\mathrm{Crit}\big({}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}} \big) = \mathrm{Crit}\big({}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk} \big) \cap \big\{ \sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)} = \text{ constant} \big\}.$$

In fact, $\mathrm{Crit}\big({}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk} \big)$ turns out to be a fibre bundle, and the critical set of ${}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}$ is equal to the fiber over $(\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)})$ of this bundle, in particular being a smooth sub-manifold. Furthermore, [21, Lemma 7.1] implies that the transversal Hessian of ${}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk}$ is non-degenerate iff the transversal Hessian of ${}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}$ is non-degenerate, the latter fact being proven in [21, Proposition 7.4]. Thus, we arrive at

Corollary 7.2. *In any of the $(\theta^{(i_1)}, \ldots, \theta^{(i_N)})$-charts, the weak transforms ${}^{(i_1 \ldots i_N)}\widetilde{\Phi}^{wk}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}$ have clean critical sets of co-dimension 2κ as functions on $\widetilde{\mathbf{X}}_{i_1 \ldots i_N}^{\varrho_{i_1} \ldots \varrho_{i_N}} \times \Sigma_x^{R,t}$. They do not have critical points in the $(\theta^{(i_1)}, \ldots, \theta^{(i_{N-1})}, \alpha^{(i_N)})$-charts.*

Proof. The assertion is a direct consequence of the foregoing explanations and transversality arguments like those given in the proof of Lemma 3.1 (a). $\qquad\square$

From this we immediately deduce

Proposition 7.3. *For every $\tilde{N} \in \mathbb{N}$, $\varepsilon > 0$, any $(\theta^{(i_1)}, \ldots, \theta^{(i_N)})$-chart labeled by the indices $\varrho_{i_1}, \ldots, \varrho_{i_N}$, and $x = x_{i_1 \ldots i_N}^{\varrho_{i_1} \ldots \varrho_{i_N}}$ (or $x \in Y \cap M_{\mathrm{prin}}$ and $\varepsilon \geq 0$) one has the asymptotic formula*

$$I_{i_1 \ldots i_N}^{\varrho_{i_1} \ldots \varrho_{i_N}}(x, \mu) = (2\pi)^\kappa \prod_{j=1}^N |\tau_{i_j}|^{\dim G - \dim H_{(i_j)}} \left(\sum_{k=0}^{\tilde{N}-1} \frac{{}^k \mathcal{Q}_{i_1 \ldots i_N}^{\varrho_{i_1} \ldots \varrho_{i_N}}(x)}{(\mu |\tau_{i_1} \cdots \tau_{i_N}| + \varepsilon)^{\kappa + k}} + \mathcal{R}_{\tilde{N}}(x, \mu) \right),$$

where the ${}^k \mathcal{Q}_{i_1 \ldots i_N}^{\varrho_{i_1} \ldots \varrho_{i_N}}(x)$ are explicitly known coefficients that are uniformly bounded in x, and

$$\mathcal{R}_{\tilde{N}}(x, \mu) = O\big((\mu |\tau_{i_1} \cdots \tau_{i_N}| + \varepsilon)^{-\kappa - \tilde{N}} \big).$$

In particular, with $\widetilde{\Phi}^{wk} := {}^{(i_1 \ldots i_N)} \widetilde{\Phi}^{wk}_{\sigma_{i_j}, p^{(i_j)}, \tilde{v}^{(i_N)}}$ *we have*

$$
{}^0\mathcal{Q}^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \ldots i_N}(x) = \int_{\operatorname{Crit} \widetilde{\Phi}^{wk}} \frac{a^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \ldots i_N}}{\left| \det \operatorname{Hess} \widetilde{\Phi}^{wk}_{|N\operatorname{Crit} \widetilde{\Phi}^{wk}} \right|^{1/2}}.
$$

Proof. By definition we have $d^{(i_r)} = \dim H_{i_{r-1}} - \dim H_{i_r}$ with $H_{i_0} := G$. Consequently, $\sum_{r=1}^{j} d^{(i_r)} = \dim G - \dim H_{i_j}$. By Corollary 7.2 we can apply Theorem A.1 and the remarks following it to the integral (6.11) with asymptotic parameter $\mu |\tau_{i_1}(\sigma) \cdots \tau_{i_N}(\sigma)| + \varepsilon$, yielding the assertion, since $e^{-i\varepsilon \widetilde{\Phi}^{wk}} = 1$ on $\operatorname{Crit} \widetilde{\Phi}^{wk}$. Regarding the uniform boundedness of the coefficients and the remainder, see [21, Proof of Theorem 8.2 and Remark 8.3]. $\square$

Proposition 7.4. *In any* $(\theta^{(i_1)}, \ldots, \theta^{(i_{N-1})}, \alpha^{(i_N)})$*-chart labeled by the indices* $\varrho_{i_1}, \ldots, \varrho_{i_N}$ *one has*

$$
\widetilde{I}^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \ldots i_N}(x, \mu) = O(\mu^{-\infty}).
$$

Proof. This is an immediate consequence of the previous corollary and the non-stationary phase principle [14, Theorem 7.7.1]. $\square$

Now, if we transform the oscillatory integral $I_x(\mu) = I_{x,x}(\mu)$ defined in (3.1) under the global morphism $\mathcal{Z}$ we obtain with our previous notation the decomposition

$$
(7.1) \qquad I_x(\mu) = \sum_{N=1}^{\Lambda-1} \sum_{\substack{i_1 < \cdots < i_{N-1} < L \\ \varrho_{i_1}, \ldots, \varrho_{i_{N-1}}}} \left(I^{\varrho_{i_1} \cdots \varrho_{i_{N-1}}}_{i_1 \ldots i_{N-1}L}(x, \mu) + \sum_{\substack{i_{N-1} < i_N \\ \varrho_{i_N}}} I^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \ldots i_N}(x, \mu) \right) + \mathcal{R}(x, \mu),
$$

where the first multiple sum is given by a sum over arbitrary, totally ordered subsets of non-principal isotropy types and corresponding charts, while the second multiple sum is a sum over non-principal isotropy types (H_{i_N}) and corresponding charts such that $\{(H_{i_1}), \ldots, (H_{i_N})\}$ forms a maximal, totally ordered subset, and $\mathcal{R}(\mu, x)$ denotes the non-stationary contributions of order $O(\mu^{-\infty})$ that arise by localizing the relevant integrals to tubular neighborhoods of the relevant critical sets, or correspond to integrals over charts of the resolution spaces where the weak transforms of the phase functions do not have critical points, compare [21, Eq. (9.1)]. Here $I^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \ldots i_N}(x, \mu) = 0$ unless $x = x^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \ldots i_N}$ lies in the corresponding chart, and similarly for $I^{\varrho_{i_1} \cdots \varrho_{i_{N-1}}}_{i_1 \ldots i_{N-1}L}(x, \mu)$ and the coefficients in the corresponding asymptotic expansions. Since the latter integrals have the same asymptotic description than the integrals $I^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \ldots i_N}(x, \mu)$ we arrive at

Theorem 7.5. *For every* $\tilde{N}$*,* $x \in Y$ *and* $\varepsilon > 0$ *(or* $x \in Y \cap M_{\operatorname{prin}}$ *and* $\varepsilon \geq 0$*) one has*

$$
I_x(\mu) = (2\pi)^{\kappa} \sum_{N=1}^{\Lambda-1} \sum_{\substack{i_1 < \cdots < i_{N-1} < L \\ \varrho_{i_1}, \ldots, \varrho_{i_{N-1}}}} \prod_{l=1}^{N-1} |\tau_{i_l}|^{\dim G - \dim H_{i_l}}
$$

$$
\cdot \left[\sum_{k=0}^{\tilde{N}-1} \frac{{}^k\mathcal{P}^{\varrho_{i_1} \cdots \varrho_{i_{N-1}}}_{i_1 \ldots i_{N-1}L}(x)}{(\mu |\tau_{i_1} \cdots \tau_{i_{N-1}}| + \varepsilon)^{\kappa+k}} + O\left((\mu |\tau_{i_1} \cdots \tau_{i_{N-1}}| + \varepsilon)^{-\kappa-\tilde{N}} \right) \right.
$$

$$
\left. + \sum_{\substack{i_{N-1} < i_N \\ \varrho_{i_N}}} |\tau_{i_N}|^{\dim G - \dim H_{i_N}} \left(\sum_{k=0}^{\tilde{N}-1} \frac{{}^k\mathcal{Q}^{\varrho_{i_1} \cdots \varrho_{i_N}}_{i_1 \ldots i_N}(x)}{(\mu |\tau_{i_1} \cdots \tau_{i_N}| + \varepsilon)^{\kappa+k}} + O\left((\mu |\tau_{i_1} \cdots \tau_{i_N}| + \varepsilon)^{-\kappa-\tilde{N}} \right) \right) \right]
$$

up to terms of order $O(\mu^{-\infty})$*, where the multiple sums run over maximal, totally ordered subsets* $\{(H_{i_1}), \ldots, (H_{i_N})\}$ *of non-principal isotropy types. Furthermore, all coefficients are given explicitly in terms of distributions on the resolution space, and are uniformly bounded in* x*.*

$\square$

Theorem 7.5 gives a simultaneous description of the competing asymptotics $\lambda \to \infty$ and $\tau_{i_j} \to 0$, and for $\varepsilon > 0$ interpolates between the different asymptotics in Theorem 3.3. For $\varepsilon = 0$, it yields a description of the singular behavior of the coefficients in the expansion of $I_x(\mu)$ in Theorem 3.3 (1) as $x \in M_{\mathrm{prin}}$ approaches singular orbits. Note that the factors $|\tau_{i_l}|^{\dim G - \dim H_{i_l}}$ in the expansion of Theorem 7.5 reflect the fact that the coefficients become more singular as the dimension of the stabilizer groups H_{i_l} become large, that is, as one approaches more and more singular orbits, answering for the different asymptotics in Theorem 3.3 (1) given by the exponents $\kappa_x = \dim \mathcal{O}_x$.[4] For an exceptional orbit of type (H_{i_l}) one has $\dim G - \dim H_{i_l} = \kappa$, so that the corresponding factors $|\tau_{i_l}|^\kappa$ cancel each other, in concordance with Theorem 3.3 (1), by which the summands in the expansion of $I_x(\mu)$ in Theorem 7.5 must stay bounded as one approaches exceptional orbits. Besides, note that the terms with $k \geq 1$ involve derivatives with respect to g that give rise to additional positive powers in the exceptional parameters. In the same way that Theorem 4.3 was deduced from Theorem 3.3 (1), the previous theorem allows us to derive the asymptotic formula for the reduced spectral function we were looking for. First, one deduces

Proposition 7.6 (Singular point-wise asymptotics for the kernel of the equivariant approximate projection). *For arbitrary $\tilde{N}_1, \tilde{N}_2 = 1, 2, 3, \ldots$, fixed $x \in M$ and $\varepsilon > 0$ (or $x \in M_{\mathrm{prin}} \cup M_{\mathrm{except}}$ and $\varepsilon \geq 0$) one has for $\mu \to \infty$ the asymptotic expansion*

$$K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x,x) = \frac{\mu^{n-1} d_\gamma}{(2\pi)^{n-\kappa}} \sum_{j=0}^{\tilde{N}_1 - 1} \mu^{-j} \sum_{N=1}^{\Lambda-1} \sum_{i_1 < \cdots < i_{N-1} < L} \prod_{l=1}^{N-1} |\tau_{i_l}|^{\dim G - \dim H_{i_l}}$$

$$\cdot \left[\sum_{k=0}^{\tilde{N}_2 - 1} \frac{\mathcal{L}_{i_1 \ldots i_{N-1} L}^{j,k}(x)}{(\mu |\tau_{i_1} \cdots \tau_{i_{N-1}}| + \varepsilon)^{\kappa + k}} + O\big((\mu |\tau_{i_1} \cdots \tau_{i_{N-1}}| + \varepsilon)^{-\kappa - \tilde{N}_2}\big) \right.$$

$$+ \sum_{i_{N-1} < i_N} |\tau_{i_N}|^{\dim G - \dim H_{i_N}} \left(\sum_{k=0}^{\tilde{N}_2 - 1} \frac{\mathcal{M}_{i_1 \ldots i_N}^{j,k}(x)}{(\mu |\tau_{i_1} \cdots \tau_{i_N}| + \varepsilon)^{\kappa + k}} + O\big((\mu |\tau_{i_1} \cdots \tau_{i_N}| + \varepsilon)^{-\kappa - \tilde{N}_2}\big) \right) \right]$$

up to terms of order $O(\mu^{n - \tilde{N}_1 - 1})$, where the multiple sums run over maximal, totally ordered subsets $\{(H_{i_1}), \ldots, (H_{i_N})\}$ of singular isotropy types, and all coefficients are explicitly given by distributions on the resolution space bounded uniformly in x. For $\mu \to -\infty$, the function $K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x,x)$ is rapidly decreasing in μ.

Proof. According to Theorem 3.3 (1), the summands in the expansion of $I_x(\mu)$ in Theorem 7.5 must stay bounded as one approaches exceptional orbits. By gathering the contributions from exceptional and principal isotropy types, and collecting the terms from different charts corresponding to the same subset of isotropy types, the assertion follows from Corollary 2.2 by applying Theorem 7.5 to the integrals (2.10). $\square$

Using standard Tauberian arguments we obtain as our third main result

Theorem 7.7 (Singular equivariant local Weyl law). *Let M be a closed connected Riemannian manifold M of dimension n with an isometric and effective action of a compact connected Lie group G, and P_0 a G-invariant elliptic classical pseudodifferential operator on M of degree m. Let $p(x, \xi)$ be its principal symbol, and assume that P_0 is positive and symmetric. Denote its unique self-adjoint extension by P, and for a given $\gamma \in \widehat{G}$ let $e_\gamma(x, y, \lambda)$ be its reduced spectral counting function. Denote*

---

[4]Indeed, assume that $M_{\mathrm{prin}} \ni x_{i_1 \ldots i_N}^{\varrho_{i_1} \ldots \varrho_{i_N}} \to y \in M(H_{i_q})$ in such a way that among the indices $i_1 < \cdots < i_N$ the index τ_{i_q} goes to zero with rate $\tau_{i_q} \approx \mu^{-1} \to 0$. Then, if $\kappa = \dim G$,

$$\prod_{l=1}^{N} \frac{|\tau_{i_l}|^{\dim G - \dim H_{i_l}}}{(\mu |\tau_{i_1} \cdots \tau_{i_N}|)^\kappa} = \prod_{l=1}^{N} \frac{|\tau_{i_l}|^{-\dim H_{i_l}}}{\mu^\kappa} \approx O(\mu^{-\dim G + \dim H_{i_q}}) = O(\mu^{-\dim \mathcal{O}_y}).$$

by κ the dimension of an G-orbit in M of principal type and by d_γ the dimension of an irreducible G-representation π_γ of class γ. Then, for $x \in M_{\mathrm{prin}} \cup M_{\mathrm{except}}$ one has the asymptotic formula

$$\left| e_\gamma(x,x,\lambda) - \frac{d_\gamma \lambda^{\frac{n-\kappa}{m}}}{(2\pi)^{n-\kappa}} \sum_{N=1}^{\Lambda-1} \sum_{i_1 < \cdots < i_{N-1} < L} \prod_{l=1}^{N-1} |\tau_{i_l}|^{\dim G - \dim H_{i_l} - \kappa} \left[\mathcal{L}_{i_1 \ldots i_{N-1} L}^{0,0}(x) \right.\right.$$

$$\left.\left. + \sum_{i_{N-1} < i_N} \mathcal{M}_{i_1 \ldots i_N}^{0,0}(x) |\tau_{i_N}|^{\dim G - \dim H_{i_N} - \kappa} \right] \right| \le C\, d_\gamma\, \lambda^{\frac{n-\kappa-1}{m}} \sum_{N=1}^{\Lambda-1} \sum_{i_1 < \cdots < i_N} \prod_{l=1}^{N} |\tau_{i_l}|^{\dim G - \dim H_{i_l} - \kappa - 1}$$

as $\lambda \to +\infty$, where the multiple sums run over maximal, totally ordered subsets $\{(H_{i_1}), \ldots, (H_{i_N})\}$ of singular isotropy types, and all coefficients are bounded in x, while $C > 0$ is a constant independent of x, and the τ_{i_j} are parameters satisfying $|\tau_{i_j}| \approx \mathrm{dist}\,(x, M(H_{i_j}))$.

Proof. The assertion follows by integrating the expression for $K_{\Pi_\gamma \circ \widetilde{\chi}_\mu}(x,x)$ in Proposition 7.6 from $-\infty$ to ν for the values $\varepsilon = 0$, $\tilde{N}_1 = \kappa + 1$, $\tilde{N}_2 = 1$ with the arguments given in the proof of Theorem 4.3. $\qquad\square$

As an immediate consequence this yields

Corollary 7.8 (**Singular point-wise bounds for isotypic spectral clusters**). *In the setting of Theorem 7.7 we have*

$$\sum_{\substack{\lambda_j \in (\lambda, \lambda+1], \\ e_j \in \mathrm{L}_\gamma^2(M)}} |e_j(x)|^2 \le \begin{cases} C \lambda^{\frac{n-1}{m}}, & x \in M_{\mathrm{sing}}, \\[2ex] C \lambda^{\frac{n-\kappa-1}{m}} \displaystyle\sum_{N=1}^{\Lambda-1} \sum_{i_1 < \cdots < i_N} \prod_{l=1}^{N} |\tau_{i_l}|^{\dim G - \dim H_{i_l} - \kappa - 1}, & x \in M - M_{\mathrm{sing}}, \end{cases}$$

for a constant $C > 0$ independent of x and λ proportional to d_γ. In particular, the bound holds for each individual $e_j \in \mathrm{L}_\gamma^2(M)$ with $\lambda_j \in (\lambda, \lambda+1]$.

$\qquad\square$

We would like to remark that the expansion in Theorem 7.7 is only meaningful if μ is sufficiently large compared to the exceptional parameters τ_{i_j}, more precisely, if

$$\lambda^{1/m} \prod_l |\tau_{i_l}| > 1$$

for all possible combinations of exceptional parameters, since $-\dim H_{i_l} \le \dim G - \dim H_{i_l} - \kappa \le 0$ for all i_l. While (4.2) describes the asymptotics of the equivariant spectral function for arbitrary, but fixed $x \in M$, Theorem 7.7 gives a uniform description of the behavior of the coefficients as $x \in M_{\mathrm{prin}}$ approaches singular orbits. Nevertheless, an asymptotic formula for $e_\gamma(x,x,\lambda)$ that interpolates between the various asymptotic behaviors in Theorem 4.3, in the same way than Theorem 7.5 interpolates between the different asymptotics in Theorem 3.3 (1), seems not accessible via Tauberian techniques. Indeed, while $(\mu|\tau_{i_1} \cdots \tau_{i_N}| + \varepsilon)^{-1}$ depends continuously on the parameters τ_{i_j} for $\varepsilon > 0$, its anti-derivative

$$\int \frac{1}{\mu|\tau_{i_1} \cdots \tau_{i_N}| + \varepsilon}\, d\mu = \begin{cases} \frac{\log(\mu|\tau_{i_1} \cdots \tau_{i_N}| + \varepsilon)}{|\tau_{i_1} \cdots \tau_{i_N}|} + C, & \tau_{i_1} \cdots \tau_{i_N} \ne 0, \\[1ex] \mu/\varepsilon + C, & \text{otherwise,} \end{cases}$$

no longer does.

Example 7.9. To illustrate the desingularization process and our results, let us resume Example 4.10, where we considered the action of $G = \mathrm{SO}(2)$ on the standard 2-sphere $M = S^2 \subset \mathbb{R}^3$ by rotations around the x_3-axis. The isotropy types are $H_1 = \mathrm{SO}(2)$ and $H_2 = \{e\}$, and the set of maximally singular orbits $M_1(H_1) = \{x_N, x_S\}$ is disconnected in this case. Instead of working with the covering (6.4), we can cover S^2 with the two charts $Y_1 := S^2 - \{x_N\}$ and $Y_2 := S^2 - \{x_S\}$ by introducing geodesic polar coordinates $x = \exp_{x_S}(\tau_1 \tilde{v})$ and $x = \exp_{x_N}(\tau_2 \tilde{v})$ around the poles, respectively, where $\tilde{v} \in S^1$, and $\tau_i \ge 0$ equals the Riemannian distance of x to the corresponding pole. Note that $\mathfrak{g}_{x_N}^\perp = \mathfrak{g}_{x_S}^\perp = \{0\}$,

so that it is not necessary to perform a blow-up in the group variables, and no additional $O(\mu^{-\infty})$-terms arise. After one iteration, the action is desingularized, and one obtains in agreement with Theorem 7.5 for arbitrary $\tilde{N} \in \mathbb{N}$ and $\varepsilon \geq 0$ the asymptotic formula

$$I_x(\mu) = 2\pi \sum_{i=1,2} \left[\sum_{k=0}^{\tilde{N}-1} {}^k\mathcal{Q}^i(x)\,(\mu\tau_i + \varepsilon)^{-1-k} + O((\mu\tau_i + \varepsilon)^{-1-\tilde{N}}) \right],$$

all coefficients being bounded in x. In particular, setting $\varepsilon = 0$ one sees that the leading coefficient in Theorem 3.3 (1) is given by

$$Q_0(x) = \frac{1}{\tau_1} {}^0\mathcal{Q}^1(x) + \frac{1}{\tau_2} {}^0\mathcal{Q}^2(x), \qquad x \neq x_N,\, x_S,$$

which describes its singular behavior as one approaches the fixed points. This implies for the reduced spectral counting function of the Laplace-Beltrami operator $-\Delta$ on S^2 the asymptotics

$$\left| e_m(x, x, \lambda) - \frac{\sqrt{\lambda}}{2\pi} \frac{\mathcal{L}(x)}{\text{dist}(x, \{x_N, x_S\})} \right| \leq \frac{C}{\text{dist}^2(x, \{x_N, x_S\})}, \qquad m \in \mathbb{Z},\, x \neq x_N,\, x_S,$$

provided that $\sqrt{\lambda}\,\text{dist}(x, \{x_N, x_S\}) > 1$, all coefficients being bounded in x, in agreement with Theorem 7.7. From this, we immediately deduce the following pointwise bounds for spherical harmonics. Let $Y_{k,m}$ be the classical spherical functions with $k \in \mathbb{N}, m \in \mathbb{Z} \simeq \widehat{\text{SO}(2)}$, $|m| \leq l$ satisfying

$$-\Delta Y_{k,m} = \lambda_k\, Y_{k,m}, \qquad \lambda_k = k(k+1).$$

Then, from

$$e_m(x, x, \lambda + 1) - e_m(x, x, \lambda) = \sum_{\lambda_k \in (\lambda, \lambda+1]} \left| Y_{k,m}(x) \right|^2$$

one directly infers for fixed m the point-wise bounds

$$|Y_{k,m}(x)|^2 = \begin{cases} O(\sqrt{\lambda_k}), & x = x_N,\, x_S, \\ O\big([\text{dist}(x, \{x_N, x_S\})]^{-2}\big), & x \neq x_N,\, x_S, \end{cases}$$

as $k \to \infty$, where we took into account the bound (4.5). In particular, this is consistent with (1.14). Thus, spherical harmonics with fixed m concentrate on the poles as k becomes large. This fact is in accordance with the probability of finding a classical particle of zero angular momentum near singular orbits and the corresponding equivariant quantum limits, see [18, Section 6.2]. Furthermore, if $\mathbf{c}$ denotes a closed geodesic on S^2 we obtain for the restriction of $Y_{k,m}$ to $\mathbf{c}$ the L^∞-bounds

$$\left\| Y_{k,m|\mathbf{c}} \right\|_\infty = \begin{cases} O_m(\lambda_k^{1/4}), & \text{if } x_N, x_S \in \mathbf{c}, \\ O_{m,\mathbf{c}}(1), & \text{otherwise}, \end{cases}$$

as $k \to \infty$. The foregoing considerations can be immediately generalized to surfaces of revolution diffeomorphic to the 2-sphere.

8. Sharpness

To conclude, we show that the obtained bounds are sharp and that, as in the classical case, they are already attained on the 2-dimensional sphere. Let us begin by reviewing the non-equivariant case [27, Section 3.4]. Thus, denote by $M = S^n$ the standard sphere in $\mathbb{R}^{n+1}$ endowed with the induced metric, and let Δ be the Laplace-Beltrami operator on S^n. The eigenvalues of $-\Delta$ are given by the numbers $\lambda_k = k(k + n - 1)$, where $k = 0, 1, 2, 3, \ldots$ and the corresponding d_k-dimensional eigenspaces $\mathcal{H}_k$ are spanned by the classical spherical functions Y_{kl}, $1 \leq l \leq d_k$, so that

$$-\Delta Y_{kl} = \lambda_k\, Y_{kl}.$$

The Y_{kl} are orthonormal to each other, and by the spectral theorem we have the decomposition $L^2(M) = \bigoplus_{k=0}^\infty \mathcal{H}_k$. Furthermore, in accordance with Weyl's law one computes $d_k = \dim \mathcal{H}_k =$

$\frac{2k^{n-1}}{(n-1)!} + O(k^{n-2})$. Now, let $\mu := \mu_k - 1 = \sqrt{\lambda_k} - 1$. In this case, χ_μ corresponds to the orthogonal projection $\mathrm{L}^2(M) \to \mathcal{H}_k$ so that its kernel reads

$$\chi_\mu(x,y) := \sum_{l=1}^{d_k} Y_{kl}(x)\overline{Y_{kl}(y)}.$$

Since $\chi_\mu(x,y)$ is invariant under the group of rotations $\mathrm{SO}(n+1)$, which acts transitively on S^n, the kernel $\chi_\mu(x,y)$ must be constant on the diagonal. More precisely, $\chi_\mu(x,x) \equiv \frac{1}{\mathrm{vol}\, S^n} \int_{S^n} \chi_\mu(x,x) dS^n(x) = \frac{d_k}{\mathrm{vol}\, S^n}$ for all $x \in \mathcal{S}^n$, from which one concludes that

$$\|\chi_\mu\|_{\mathrm{L}^2 \to \mathrm{L}^\infty}^2 = \sup_x \chi_\mu(x,x) \approx \mu_k^{\,n-1}, \qquad \mu = \sqrt{\lambda_k} - 1.$$

Consequently, the bound (1.4) for the norm $\|\chi_\lambda\|_{\mathrm{L}^2 \to \mathrm{L}^\infty}^2$ is sharp. Next, for fixed $x \in S^n$ and $k \in \mathbb{N}_0$ consider the zonal eigenfunction

$$e_{\mu_k} : S^n \ni y \longmapsto \sum_{l=1}^{d_k} Y_{kl}(x)\overline{Y_{kl}(y)} \in \mathbb{C}.$$

By the previous considerations, e_{μ_k} is an eigenfunction of $\sqrt{-\Delta}$ with eigenvalue μ_k, and with $\mu = \sqrt{\lambda_k} - 1$ we have

$$|e_{\mu_k}(x)| = \chi_\mu(x,x) \approx \mu_k^{n-1}, \qquad \|e_{\mu_k}\|_{\mathrm{L}^2}^2 = \left(\sum_{l=1}^{d_k} |Y_{kl}(x)|^2 \right)^{1/2} = (\chi_\mu(x,x))^{1/2} \approx \mu_k^{\frac{n-1}{2}}.$$

From this one concludes that for a general eigenfunction f of $-\Delta$ with $\|f\|_{\mathrm{L}^2} = 1$ and eigenvalue λ even the estimate (1.5), which in this case reads $\|f\|_{\mathrm{L}^\infty(M)} \leq C \lambda^{\frac{n-1}{4}}$ for some $C > 0$, cannot be improved.

Now, let $G \subset \mathrm{SO}(n)$ be a subgroup of the isotropy group of a point in $S^n \simeq \mathrm{SO}(n+1)/\mathrm{SO}(n)$, and

$$\mathcal{H}_k = \bigoplus_{\gamma \in \widehat{G}} \mathcal{H}_k^\gamma$$

be the decomposition of the eigenspace $\mathcal{H}_k$ into its isotypic components. Clearly, $d_k = \sum_{\gamma \in \widehat{G}} m_\gamma(k) d_\gamma$, where $m_\gamma(k)$ denotes the multiplicity of π_γ in $\mathcal{H}_k$. Let $\{Z_{kj}^\gamma\} \subset \{Y_{kl}\}_{l=1}^{d_k}$ be an orthonormal basis of $\mathcal{H}_k^\gamma$ so that with $\mu = \mu_k - 1$

$$K_{\chi_\mu \circ \Pi_\gamma}(x,y) = \sum_{j=1}^{m_\gamma(k)d_\gamma} Z_{kj}^\gamma(x)\overline{Z_{kj}^\gamma(y)},$$

$\chi_\mu \circ \Pi_\gamma$ being the projection onto $\mathcal{H}_k^\gamma$. In contrast to the situation encountered before, $K_{\chi_\mu \circ \Pi_\gamma}(x,y)$ is no longer constant on the diagonal, but instead we have by Theorem 4.3 the bound

$$|K_{\chi_\mu \circ \Pi_\gamma}(x,x)| = |e_\gamma(x,x,\mu_k) - e_\gamma(x,x,\mu_k - 1)| \leq C_x \mu_k^{n-\kappa_x-1}, \qquad C_x > 0,\ x \in S^n,$$

while the behavior near singular orbits is described in Theorem 7.7. Following our previous considerations we now define for fixed $x \in S^n$ the isotypic zonal eigenfunction

$$e_{\mu_k}^\gamma : S^n \ni y \longmapsto \sum_{j=1}^{m_\gamma(k)d_\gamma} Z_{kj}^\gamma(x)\overline{Z_{kj}^\gamma(y)} \in \mathbb{C},$$

which is again an eigenfunction of $\sqrt{-\Delta}$ for the eigenvalue μ_k and satisfies

$$\|e_{\mu_k}^\gamma\|_{\mathrm{L}^2}^2 = \left(\sum_{j=0}^{m_\gamma(k)d_\gamma} |Z_{kj}^\gamma(x)|^2 \right)^{1/2} = (K_{\chi_\mu \circ \Pi_\gamma}(x,x))^{1/2}.$$

In order to examine the sharpness of the bounds obtained, we specialize to the case where $n = 2$ and $G = \mathrm{SO}(2)$ acts by rotations around the symmetry axis through the poles. In this case, $\mathcal{H}_k^\gamma$, $\gamma \equiv m \in \mathbb{Z}$, $|m| \leq k$ is spanned by the spherical function

$$Y_{k,m}(\phi, \theta) = \sqrt{\frac{2k+1}{4\pi} \frac{(k-m)!}{(k+m)!}} P_{k,m}(\cos \theta) e^{im\phi}, \qquad 0 \leq \phi < 2\pi, \, 0 \leq \theta < \pi,$$

where $P_{k,m}$ is the associated Legendre polynomial

$$P_{k,m}(\alpha) := (-1)^m \left(1 - \alpha^2\right)^{\frac{m}{2}} \frac{d^m}{d\alpha^m} P_k(\alpha) := \frac{(-1)^m}{2^k k!} \left(1 - \alpha^2\right)^{\frac{m}{2}} \frac{d^{k+m}}{d\alpha^{k+m}} \left(\alpha^2 - 1\right)^k.$$

Furthermore, for the Legendre polynomials $P_k(\cos \theta)$ one has the asymptotics

$$(8.1) \qquad P_k(\cos \theta) = \sqrt{\frac{2}{\pi k \sin \theta}} \cos \left(\left(k + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right) + O\left(\frac{1}{(k \sin \theta)^{3/2}}\right),$$

where the remainder is uniform in θ on any interval $[\varepsilon, \pi - \varepsilon]$ with $0 < \varepsilon \ll 1$, see e.g. [12, Page 303]. Thus, in the special case where $m = 0$ we see that with $\mu = \mu_k - 1$ one has in the limit $k \to \infty$

$$K_{\chi_\mu \circ \Pi_\gamma}(x, x) = |Y_{k,0}(x)|^2 = \frac{2k+1}{4\pi} |P_{k,0}(\cos \theta)|^2 \approx \begin{cases} \sqrt{\lambda_k}, & x = x_N, x_S, \\ \frac{1}{\sin \theta} \approx \frac{1}{\mathrm{dist}(x, \{x_N, x_S\})}, & x \in S^2 - \{x_N, x_S\}, \end{cases}$$

where x_N and x_S denote the poles. Consequently, we conclude that the remainder estimates in Theorems 4.3 and 7.7 are sharp in the spectral parameter λ, but not optimal in the exceptional parameters τ_{i_j}, since in the present case we have $\lambda \approx k^2$, $\sin \theta \approx \theta \approx \tau_{i_j}$, compare also Example 7.9. Nevertheless, the estimate given in Theorem 7.7 qualitatively reflects the singular behavior of $Y_{k,0}(x)$ as x approaches the poles, and suggests that the asymptotic formula (8.1) should have a structural explanation in terms of caustics of oscillatory integrals. On the other hand, the bound for $|Y_{k,0}(x)|$ implies similar bounds for $e_{\mu_k}^\gamma(y) = Y_{k,0}(x)\overline{Y_{k,0}(y)}$, and that for a general eigenfunction $f \in \mathrm{L}^2(S^2)$ of $-\Delta$ belonging to a specific isotypic component with $\|f\|_{\mathrm{L}^2} = 1$ and eigenvalue λ the estimate

$$|f(x)| \leq C_x \, \lambda^{\frac{n - \kappa_x - 1}{4}}, \qquad x \in S^2,$$

in Corollary 4.5 cannot be improved.

To close, let us mention that in the considered case $M = S^2$ and $G = \mathrm{SO}(2)$ the previous considerations imply for the equivariant counting function $N_\gamma(\lambda)$ of the Beltrami-Laplace operator with $\gamma \equiv m$ the estimate

$$N_\gamma(\lambda) = d_\gamma \sum_{\lambda_k \leq \lambda} \mathrm{mult}_\gamma(\lambda_k) = \sum_{k(k+1) \leq \lambda, \, |m| \leq k} 1 \approx \sum_{|m| \leq k \leq \sqrt{\lambda}} 1 \approx \sqrt{\lambda} - |m|,$$

as $\lambda \to \infty$, where $\mathrm{mult}_\gamma(\lambda_k)$ denotes the multiplicity of an unitary irreducible representation of class γ in the eigenspace $\mathcal{E}_{\lambda_k}$, showing that the equivariant Weyl law proved in [21, Theorem 9.5] is sharp up to a logarithmic factor in the remainder estimate. From this one recovers the classical Weyl law

$$N(\lambda) = \sum_{k(k+1) \leq \lambda} \dim \mathcal{E}_{\lambda_k} = \sum_{m \in \mathbb{Z}} N_m(\lambda) \approx \sum_{|m| \leq \sqrt{\lambda}} (\sqrt{\lambda} - |m|) \approx (2\sqrt{\lambda} + 1)\sqrt{\lambda} - 2\frac{\sqrt{\lambda}(\sqrt{\lambda} + 1)}{2} = \lambda.$$

APPENDIX A. STATIONARY PHASE ASYMPTOTICS AND CAUSTICS

Our analysis relies on the generalized stationary phase principle, which we state below. Sketches of proofs can be found in [6, Theorem 3.3] and [31, Theorem 2.12]. For a detailed proof, which includes explicit expressions for the coefficients in the stationary phase expansion, see [21, Theorem 4.1].

Theorem A.1. *Consider an n-dimensional Riemannian manifold $\mathcal{M}$ with volume density $d\mathcal{M}$, $\psi \in$ $C^\infty(\mathcal{M}, \mathbb{R})$, and set*

(A.1)
$$\mathcal{I}(\mu) = \int_{\mathcal{M}} e^{i\mu\psi(m)} a(m)\, d\mathcal{M}(m), \qquad \mu > 0,$$

where $a(m) \in C_c^\infty(\mathcal{M})$. In addition, assume that the critical set

$$\mathcal{C} := \mathrm{Crit}(\psi) = \big\{ m \in \mathcal{M} : \psi_* : T_m\mathcal{M} \to T_{\psi(m)}\mathbb{R} \text{ is zero} \big\}$$

of the phase function ψ is clean. Then, for all $\tilde{N} \in \mathbb{N}$,

(A.2)
$$\mathcal{I}(\mu) := e^{i\mu\psi_0} (2\pi/\mu)^{\frac{n-p}{2}} \sum_{r=0}^{\tilde{N}-1} \mu^{-r} Q_r(\psi, a) + \mathcal{R}_{\tilde{N}}(\psi, a; \mu),$$

where p denotes the dimension of $\mathcal{C}$, ψ_0 is the constant value of ψ on $\mathcal{C}$, and the expressions $Q_r(\psi, a)$ and $\mathcal{R}_{\tilde{N}}(\psi, a; \mu)$ can be computed explicitly. Furthermore, there exist constants $\tilde{C}_{r,\psi} > 0$ and $C_{\tilde{N},\psi} > 0$ such that

$$|Q_r(\psi; a)| \leq \tilde{C}_{r,\psi} vol\,(\mathrm{supp}\ a \cap \mathcal{C}) \sup_{l \leq 2r} \big\| D^l a \big\|_{\infty, \mathcal{C}},$$

$$|R_{\tilde{N}}(\mu)| \leq C_{\tilde{N},\psi} vol\,(\mathrm{supp}\ a) \sup_{l \leq n-p+2\tilde{N}+1} \big\| D^l a \big\|_{\infty, \mathcal{M}} \mu^{-(n-p)/2-\tilde{N}},$$

where D^l is a differential operator on $\mathcal{M}$ of order l. In particular,

$$Q_0(\psi, a) = \int_{\mathcal{C}} \frac{a(m)}{|\det \psi''(m)_{|N_m\mathcal{C}}|^{1/2}} d\sigma_{\mathcal{C}}(m) e^{i\frac{\pi}{4}\sigma_{\psi''}},$$

where $d\sigma_{\mathcal{C}}$ stands for the induced volume density on $\mathcal{C}$ and $\sigma_{\psi''}$ for the constant value of the signature of the transversal Hessian $\psi''(m)_{|N_m\mathcal{C}}$ on $\mathcal{C}$.

$\square$

Remark A.2. As stated, the expansion (A.2) is valid for arbitrary $\mu > 0$, though the case of interest is when $\mu \to \infty$, since then the error becomes smaller than the other terms. In essence, the point is that by Taylor's formula one has

$$\left| e^{it} - \sum_{k=0}^{N-1} \frac{(it)^k}{k!} \right| = O(|t|^N) \qquad \text{for arbitrary } t \in \mathbb{R},$$

no matter how large $|t|$ is, though the estimate is only meaningful for $|t| < 1$.

Our main concern in this paper consists in extrapolating between asymptotic expansions of different orders. More precisely, consider an integral of the form (A.1) with a clean critical set, let $\tau \geq 0$ be an additional parameter, and define the integral

$$\mathcal{I}(\mu, \tau) := \int_{\mathcal{M}} e^{i\mu\tau\psi(m)} a(m)\, d\mathcal{M}(m).$$

Depending on the value of τ, it will exhibit different asymptotic behaviors in μ. Indeed, for $\tau > 0$ the integral $\mathcal{I}(\mu, \tau)$ decreases with order $O(\mu^{-\frac{n-p}{2}})$, while for $\tau = 0$ it is actually independent of μ. This behavior is reflected in the fact that if we apply the previous theorem to the integral $\mathcal{I}(\mu, \tau)$, either with $\mu\tau$ as asymptotic parameter, or with $\tau\psi$ as phase function, we would arrive at an expansion of the form (A.2) in which the coefficients in the expansion blow up as $\tau \to 0$ due to the abrupt change of the critical set of the phase function $\tau\psi(m)$ when τ becomes zero. In general, if $\psi_\aleph \in C^\infty(\mathcal{M}, \mathbb{R})$ denotes a family of functions depending on a parameter $\aleph$ such that $\mathrm{Crit}(\psi_\aleph)$ is clean for generic values of $\aleph$, one understands by a *caustic point* for this family a parameter value $\aleph$ such that $\mathrm{Crit}(\psi_\aleph)$ is not clean [31]. With this terminology, in the situation above $\tau = 0$ constitutes a caustic point. Nevertheless, it is possible to derive an adequate asymptotic expansion for $\mathcal{I}(\mu, \tau)$ that smoothly interpolates between

the different asymptotics, and takes into account the competing asymptotics $\mu \to \infty$ and $\tau \to 0$, based on the following simple idea. Let $\varepsilon \geq 0$ be a fixed positive real number, and consider the integral

$$\mathcal{I}_\varepsilon(\mu) := \int_\mathcal{M} e^{i\mu\psi(m)} e^{-i\varepsilon\psi(m)} a(m) \ d\mathcal{M}(m).$$

Clearly, $\mathcal{I}(\mu) = \mathcal{I}_\varepsilon(\mu + \varepsilon)$. Since $e^{-i\varepsilon\psi}$ is independent of μ, we can apply the previous theorem with $\mu + \varepsilon$ as parameter, obtaining for each $\tilde{N} \in \mathbb{N}$ and each $\varepsilon \geq 0$ the asymptotic formula

$$(A.3) \qquad \mathcal{I}(\mu) = e^{i(\mu+\varepsilon)\psi_0} \Big(\frac{2\pi}{\mu+\varepsilon} \Big)^{\frac{n-p}{2}} \sum_{r=0}^{\tilde{N}-1} (\mu+\varepsilon)^{-r} Q_r(\psi, e^{-i\varepsilon\psi} a) + R_{\tilde{N}}(\psi, e^{-i\varepsilon\psi} a; \mu+\varepsilon).$$

Because

$$\frac{1}{\mu+\varepsilon} = \frac{1}{\mu} \cdot \frac{1}{1+\frac{\varepsilon}{\mu}} = \frac{1}{\mu} \sum_{k=0}^{\infty} \Big(\frac{-\varepsilon}{\mu} \Big)^k = \frac{1}{\mu} - \frac{\varepsilon}{\mu^2} + \frac{\varepsilon^2}{\mu^3} - \cdots, \qquad \varepsilon/\mu < 1,$$

the expansion (A.2) is consistent with the expansion (A.3), the corrections being of lower order. Now, if we apply the previous argument to $\mathcal{I}(\mu, \tau) = \mathcal{I}(\mu\tau)$ we obtain

$$\mathcal{I}(\mu, \tau) = e^{i(\mu\tau+\varepsilon)\psi_0} \Big(\frac{2\pi}{\mu\tau+\varepsilon} \Big)^{\frac{n-p}{2}} \sum_{r=0}^{\tilde{N}-1} (\mu\tau+\varepsilon)^{-r} Q_r(\psi, e^{-i\varepsilon\psi} a) + \mathcal{R}_{\tilde{N}}(\psi, e^{-i\varepsilon\psi} a; \mu\tau+\varepsilon)$$

as $\mu \to \infty$. The formula is only meaningful for $\tau\mu+\varepsilon > 1$, and simultaneously describes the asymptotic behavior of $\mathcal{I}(\mu, \tau)$ in the competing parameters τ and μ. For $\varepsilon > 0$, it interpolates between the asymptotics $O(\mu^{-\frac{n-p}{2}})$ and $O(\mu^0)$ in a smooth way; in fact, for $\tau = 0$ it simply collapses to $\int_\mathcal{M} a \ d\mathcal{M}$.

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