

SUBCONVEX BOUNDS FOR HECKE–MAASS FORMS ON COMPACT ARITHMETIC QUOTIENTS OF SEMISIMPLE LIE GROUPS

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ABSTRACT. Let H be a semisimple algebraic group, K a maximal compact subgroup of $G := H(\mathbb{R})$, and $\Gamma \subset H(\mathbb{Q})$ a congruence arithmetic subgroup. In this paper, we generalize existing subconvex bounds for Hecke–Maass forms on the locally symmetric space $\Gamma \backslash G/K$ to corresponding bounds on the arithmetic quotient $\Gamma \backslash G$ for cocompact lattices using the spectral function of an elliptic operator. The bounds obtained extend known subconvex bounds for automorphic forms to non-trivial K -types, yielding subconvex bounds for new classes of automorphic representations, and constitute subconvex bounds for eigenfunctions on compact manifolds with both positive and negative sectional curvature. We also obtain new subconvex bounds for holomorphic modular forms in the weight aspect.

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1. INTRODUCTION

Let M be a closed¹ Riemannian manifold M of dimension d and $P_0 : C^\infty(M) \rightarrow L^2(M)$ an elliptic classical pseudodifferential operator on M of degree m , where $C^\infty(M)$ denotes the space of smooth functions on M and $L^2(M)$ the space of square-integrable functions on M . Assume that P_0 is positive and symmetric. Denote its unique self-adjoint extension by P with the m -th Sobolev space as domain, and let $\{\phi_j\}_{j \geq 0}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of P with eigenvalues $\{\lambda_j\}_{j \geq 0}$ repeated according to their multiplicity. By a classical result of Avacumovic, Levitan, and Hörmander [1, 30, 22] one has for any $j \in \mathbb{N}$ the *convex bound*²

$$(1.1) \quad \|\phi_j\|_\infty \ll \lambda_j^{\frac{d-1}{2m}}.$$

If the ϕ_j are eigenfunctions of a larger family of commuting differential operators on M containing P_0 , this bound can be improved. Thus, assume that M carries an isometric action of a compact Lie group K such that all orbits have the same dimension $\kappa \leq d - 1$. Denote by \hat{K} the set of equivalence classes of irreducible unitary representations of K , which can be identified with the set of irreducible characters of K . Suppose further that P commutes with the family of differential operators generated

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¹ By a closed manifold we shall understand a compact boundaryless manifold.

² Here and in what follows we shall write $a \ll_\gamma b$ for two real numbers a and b and a variable γ , if there exists a constant $C_\gamma > 0$ depending only on γ such that $|a| \leq C_\gamma b$. If there are no relevant variables involved, we shall simply write $a \ll b$.

by the action of K , so that the eigenfunctions ϕ_j can be chosen to be compatible with the Peter–Weyl decomposition of $L^2(M)$ into σ -isotypic components $L^2_\sigma(M)$, where $\sigma \in \widehat{K}$. It was then shown in [40, 41] that the *equivariant convex bound*

$$(1.2) \quad \|\phi_j\|_\infty \ll \left(d_\sigma \sup_{u \leq \lfloor \kappa/2 + 1 \rfloor} \|D^u \sigma\|_\infty \right)^{1/2} \lambda_j^{\frac{d-\kappa-1}{2m}}, \quad \phi_j \in L^2_\sigma(M),$$

holds, where d_σ denotes the dimension of a representation of class σ , and D^u is a differential operator of order u on K . If $K = T$ is a torus, one actually has the almost sharp estimate

$$(1.3) \quad \|\phi_j\|_\infty \ll \lambda_j^{\frac{d-\kappa-1}{2m}}, \quad \phi_j \in L^2_\sigma(M), \quad \sigma \in \mathcal{W}_{\lambda_j},$$

where \mathcal{W}_λ denotes the subset of K -types occurring in the Peter–Weyl decomposition of $L^2(M)$ that grow at most with rate $\lambda^{1/m}/\log \lambda$.³

The bounds (1.1) and (1.2) are known to be sharp in the eigenvalue aspect on the standard d -sphere, but if the considered eigenfunctions are joint eigenfunctions of an even larger family of commuting operators, they can be improved. Thus, let G be a semisimple real Lie group, K a maximal compact subgroup of G , $\Gamma \subset G$ a lattice, and $Y := \Gamma \backslash G/K$ the corresponding locally symmetric space of dimension d and rank r . If $\{\psi_j\}_{j \geq 0}$ constitutes an orthonormal basis in $L^2(Y)$ of simultaneous eigenfunctions of the full ring of invariant differential operators on Y , which is isomorphic to a finitely generated polynomial ring in r variables and contains the Beltrami–Laplace operator Δ , Sarnak [43] was able to show the *spherical convex bound*

$$(1.4) \quad \|\psi_j|_\Omega\|_\infty \ll_\Omega \lambda_j^{\frac{d-r}{4}}$$

for arbitrary compacta $\Omega \subset Y$, λ_j being the Beltrami–Laplace eigenvalue of ψ_j . From an arithmetic point of view, there is still an additional family of commuting operators on Y given by the Hecke operators, and in the case $G = \mathrm{SL}(2, \mathbb{R})$ and $K = \mathrm{SO}(2)$, Iwaniec and Sarnak [27] were able to strengthen the bound (1.4) for certain compact locally symmetric spaces $Y = \Gamma \backslash \mathbb{H}$ of rank $r = 1$, given as quotients of the complex upper half plane $\mathbb{H} \simeq G/K$ by suitable congruence arithmetic lattices Γ , and proved for any $\varepsilon > 0$ and $j \in \mathbb{N}$ the substantially stronger *spherical subconvex bound*

$$(1.5) \quad \|\psi_j\|_\infty \ll_\varepsilon \lambda_j^{\frac{5}{24} + \varepsilon},$$

provided that the ψ_j are also eigenfunctions of the ring of Hecke operators on $L^2(\Gamma \backslash \mathbb{H})$. More generally, if H is a semisimple algebraic group over \mathbb{Q} satisfying certain conditions, $\Gamma \subset H(\mathbb{Q})$ an arithmetic congruence lattice, and $G = H(\mathbb{R})$, Marshall [33] was able to strengthen the bound (1.4) and prove *spherical subconvex bounds* of the form

$$(1.6) \quad \|\psi_j|_\Omega\|_\infty \ll_\Omega \lambda_j^{\frac{d-r}{4} - \delta}$$

for some $\delta > 0$ and arbitrary compacta $\Omega \subset Y$, if the ψ_j are also eigenfunctions of the ring of Hecke operators on $L^2(Y)$, generalizing previous work of Blomer–Maga [3, 4] and Blomer–Pohl [7], among others. In fact, for negatively curved manifolds, much better bounds are expected to hold generically, the bound (1.5) being the strongest known bound up to now. The estimates (1.4)–(1.6) represent bounds for automorphic forms on G which are right K -invariant, and for this reason are called *spherical*.

In this paper, left Γ -invariant functions on G which are simultaneous eigenfunctions of an invariant elliptic differential operator and some module of Hecke operators will be called *Hecke–Maass forms of rank 1*. This class encompasses the usual concept of an automorphic form on G , and coincides with it in the rank 1 case, compare Section 5.3 and 7.4. Nevertheless, note that a Hecke–Maass form of rank 1 is not necessarily an eigenfunction of the full ring of invariant differential operators, since one can choose a very small submodule of the ring of Hecke operators, see Remark 7.3 for details. The goal of this paper is to extend the spherical subconvex bounds (1.5) and (1.6) to non-spherical situations, that

³The estimate is almost sharp in the sense that as a consequence of the equivariant Weyl law T -types in $L^2(M)$ can grow at most with rate $\lambda_j^{1/m}$, see [41].

is, to non-trivial K -types in the Peter-Weyl decomposition of $L^2(\Gamma \backslash G)$ for a large class of compact arithmetic quotients $\Gamma \backslash G$, sharpening the bounds (1.1) and (1.2) in case that the eigenfunctions ϕ_j are Hecke–Maass forms.

As our first main result, we extend the bound (1.5) to automorphic forms on G of arbitrary K -type and Nebentypus character. Thus, let \mathcal{R} be an Eichler order in an indefinite division quaternion algebra A over \mathbb{Q} . Denote by $N(x)$ the reduced norm of an element $x \in A$, and write $\mathcal{R}(m) := \{\alpha \in \mathcal{R} \mid N(\alpha) = m\}$ for any $m \in \mathbb{N}_*$. Choose an embedding $\theta : \sqcup_{m=1}^{\infty} \mathcal{R}(m) \rightarrow G$, and set $\Gamma := \theta(\mathcal{R}(1))$. Then Γ constitutes a congruence arithmetic subgroup, and $\Gamma \backslash \mathbb{H} \simeq \Gamma \backslash G / K$ becomes a compact hyperbolic surface. Now, let χ be a Nebentypus character on Γ , and denote by $L^2_{\chi}(\Gamma \backslash G)$ the Hilbert space of measurable functions on G such that

$$f(\gamma x) = \chi(\gamma) f(x), \quad \gamma \in \Gamma, x \in G, \quad \|f\| := \left(\int_{\Gamma \backslash G} |f(x)|^2 dx \right)^{1/2} < \infty.$$

The space $L^2_{\chi}(\Gamma \backslash G)$ can be regarded as a closed subspace in $L^2(\Gamma_{\chi} \backslash G)$, where $\Gamma_{\chi} := \ker \chi$. Identifying $\mathcal{R}(n)$ with its image $\theta(\mathcal{R}(n))$ for each n prime to a fixed natural number which depends only on \mathcal{R} , the finite cosets $\Gamma \backslash \mathcal{R}(n)$ give rise to Hecke operators on $L^2_{\chi}(\Gamma \backslash G)$. Now, with the identification $K \simeq S^1 \simeq [0, 2\pi)$, any K -type $\sigma_l \in \widehat{K}$ can be realized as the character $\sigma_l(\theta) = e^{il\theta}$, $\theta \in [0, 2\pi)$, $l \in \mathbb{Z}$, and we denote by $L^2_{\sigma_l, \chi}(\Gamma \backslash G)$ the σ_l -isotypic component of $L^2_{\chi}(\Gamma \backslash G)$. It is then shown in Theorem 5.5 that for any orthonormal basis $\{\phi_j\}_{j \geq 0}$ of $L^2(\Gamma_{\chi} \backslash G)$ consisting of Hecke–Maass forms (of rank 1) with Beltrami–Laplace eigenvalues $0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ and compatible with the Peter-Weyl decomposition one has the *hybrid subconvex bound*

$$(1.7) \quad \|\phi_j\|_{\infty} \ll_{\varepsilon} \lambda_j^{\frac{5}{24} + \varepsilon}, \quad \phi_j \in L^2_{\chi}(\Gamma \backslash G),$$

for arbitrary small $\varepsilon > 0$ in the eigenvalue and isotypic aspect. This bound is the first sharpening the bound (1.3) for arbitrary K -types. If σ_l and χ are trivial, one recovers the spherical subconvex bound (1.5). Note that (1.7) is a subconvex bounds on a manifold which does have both positive and negative sectional curvature. It is stated from the perspective of elliptic operator theory, which is the natural one in our approach, while in the theory of automorphic forms it is more common to work within a representation-theoretic framework, and use the Casimir operator \mathcal{C} of G instead of the Beltrami–Laplace operator Δ , the former being no longer elliptic. But since on $L^2_{\sigma_l, \chi}(\Gamma \backslash G)$ the operators in question are related according to $\Delta = -\mathcal{C} + \frac{l^2}{4} \text{id}$, the bound (1.7) can be rephrased accordingly. Thus, for any Hecke eigenform $\phi \in L^2_{\sigma_l, \chi}(\Gamma \backslash G)$ satisfying $\|\phi\|_{L^2} = 1$ and $\mathcal{C}\phi = \frac{s^2-1}{8}\phi$ one has the *hybrid subconvex bound*

$$\|\phi\|_{\infty} \ll_{\varepsilon} (1 - s^2 + 2l^2)^{\frac{5}{24} + \varepsilon},$$

see Theorem 5.8. In this way, we obtain subconvex bounds for new classes of automorphic representations, in particular for the discrete series D_s and their limits $D_{\pm, 0}$, as well as the principal series $H(1, s)$, compare Section 5.3. Let us note that for fixed s we obtain the bound $\|\phi_j\|_{\infty} \ll_{\varepsilon} (1 + |l|)^{\frac{5}{12} + \varepsilon}$ for any $\phi_j \in L^2_{\sigma_l, \chi}(\Gamma \backslash G)$. This agrees with results of Venkatesh [50, p. 993], though by work of Reznikov [42, Theorem 1.5] one has in this case the much better bound $\|\phi_j\|_{\infty} \ll_{\varepsilon} (1 + |l|)^{\frac{1}{3} + \varepsilon}$. Nevertheless, our results do imply new results for a classical automorphic form $f : \mathbb{H} \rightarrow \mathbb{C}$ of weight $l \in \mathbb{N}$ and arbitrary Nebentypus character, for which we show in (5.11) the subconvex bound

$$\|f\|_{\infty} \ll_{\varepsilon} l^{\frac{5}{12} + \varepsilon}$$

in the weight aspect. The best previously known subconvex bound, proved by Das and Sengupta [12], had the exponent $\frac{1}{2} - \frac{1}{33} = \frac{31}{66}$.

In an analogous way, we are able to derive equivariant and non-equivariant subconvex bounds for $G = \text{SU}(2)$, $K = \text{SO}(2)$, and $\Gamma := \{\pm 1\}$ in the setting of [31, 32] by identifying G with the group of units in the quaternion algebra over \mathbb{R} , and defining corresponding Hecke operators T_n on $L^2(\Gamma \backslash G)$. Thus, we obtain again in Theorem 6.1 the equivariant subconvex bound (1.7) for any simultaneous

eigenfunction $\phi_j \in L^2_{\sigma_j}(\Gamma \backslash G)$ of the Beltrami–Laplace operator Δ on G with eigenvalue λ_j and the T_n , where now $\Gamma \backslash G \cong \text{SO}(3)$. This generalizes a result of VanderKam [49, Theorem 1.1], where the case $l = 0$ with $L^2_{\sigma_0}(\Gamma \backslash G) \cong L^2(\Gamma \backslash G/K) \cong L^2(S^2)$ is treated, S^2 being the 2-sphere.

Our second main result concerns bounds of the form (1.6). As before, let H be a semisimple algebraic group over \mathbb{Q} which is assumed to be connected in the sense of Zariski. Write \mathbb{A}_{fin} for the finite adèle ring of \mathbb{Q} and $\mathbb{A} := \mathbb{R} \times \mathbb{A}_{\text{fin}}$ for the adèle ring. Choosing an open compact subgroup K_0 in $H(\mathbb{A}_{\text{fin}})$, we obtain an arithmetic subgroup $\Gamma := H(\mathbb{Q}) \cap (H(\mathbb{R})K_0)$ in the semisimple Lie group $G = H(\mathbb{R})$. Assume that $H(\mathbb{A}) = H(\mathbb{Q})(H(\mathbb{R})K_0)$ and that $H(\mathbb{Q}) \backslash H(\mathbb{A})$ is compact, so that $\Gamma \backslash G$ is also compact.⁴ From the point of view of automorphic representations, one has a suitable family of Hecke operators on $L^2(\Gamma \backslash G)$, which is given by unramified Hecke algebras over \mathbb{Q}_p for infinitely many primes p [33]. Now, let K be a maximal compact subgroup of G and $\{\phi_j\}_{j \geq 0}$ an orthonormal basis of $L^2(\Gamma \backslash G)$ consisting of Hecke–Maass forms of rank 1 with respect to an elliptic left-invariant differential operator P_0 on $\Gamma \backslash G$ of order m which commutes with the right regular representation of K . Assume that P_0 is positive and symmetric, and that the cosphere bundle defined by its principal symbol is strictly convex. Then, assuming the condition (WS) made in [33], we show in Theorem 7.4 that there exists a constant $\delta > 0$ independent of σ such that one has the *equivariant subconvex bound*

$$(1.8) \quad \|\phi_j\|_{\infty} \ll \sqrt{d_{\sigma}} \sup_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_{\infty} \lambda_j^{\frac{\dim G/K - 1}{2m} - \delta}, \quad \phi_j \in L^2_{\sigma}(\Gamma \backslash G),$$

where λ_j denotes the spectral eigenvalue of ϕ_j with respect to P_0 ; if $K = T$ is a torus, one has the stronger estimate

$$\|\phi_j\|_{\infty} \ll \lambda_j^{\frac{\dim G/K - 1}{2m} - \delta}, \quad \phi_j \in L^2(\Gamma \backslash G).$$

The bound (1.8) bound sharpens the bound (1.2) for a large class of examples. If σ is trivial, it is implied by (1.6). An example would be given by $H = \text{SL}(1, D)$, where D is any central division algebra of index n over \mathbb{Q} , and $G = \text{SL}(n, \mathbb{R})$. Furthermore, we show in Theorem 7.9 for some $\delta > 0$ the weaker *non-equivariant subconvex bound*

$$(1.9) \quad \|\phi_j\|_{\infty} \ll \lambda_j^{\frac{\dim G - 1}{2m} - \delta}, \quad \phi_j \in L^2(\Gamma \backslash G),$$

for an orthonormal basis of $L^2(\Gamma \backslash G)$ consisting of suitable Hecke–Maass forms, sharpening the bound (1.1), but without assuming the condition (WS) of [33]. An example is again $H = \text{SL}(1, D)$, where now D is any central division algebra over \mathbb{Q} , except when $G = \text{SL}(1, \mathbb{H})$. As before, (1.8) and (1.9) constitute first arithmetic subconvex bounds on a large class of manifolds which are both positively and negatively curved, and if P_0 is the Beltrami–Laplace operator, the bounds can be rephrased in terms of the eigenvalues of the Casimir operator of G . Indeed, by Theorem 7.12 we have for each $\phi_j \in L^2_{\sigma}(\Gamma \backslash G)$ with Casimir eigenvalue μ_j the bound

$$\|\phi_j\|_{\infty} \ll \sqrt{d_{\sigma}} \sup_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_{\infty} (-\mu_j + 2\mu_{\sigma})^{\frac{\dim G/K - 1}{4} - \delta}, \quad \phi_j \in L^2_{\sigma}(\Gamma \backslash G),$$

provided that $H = \text{Res}_{F/\mathbb{Q}} \underline{G}$ and (WS) is fulfilled, while in general

$$\|\phi_j\|_{\infty} \ll (-\mu_j + 2\mu_{\sigma})^{\frac{\dim G - 1}{4} - \delta},$$

μ_{σ} being the eigenvalue of the Casimir operator of K on σ . If $K = T$ is a torus,

$$\|\phi_j\|_{\infty} \ll (-\mu_j + 2\mu_{\sigma})^{\frac{\dim G/K - 1}{4} - \delta}.$$

Let us briefly say a few words about the methods employed. While in the theory of automorphic forms representation-theoretic tools prevail, our analysis is mainly based on the spectral theory of

⁴Note that $H(\mathbb{A}) = H(\mathbb{Q})(H(\mathbb{R})K_0)$ is satisfied for any K_0 if H has the strong approximation property. In this case, $G = H(\mathbb{R})$ can be any of the groups $\text{SL}(n, \mathbb{R})$, $\text{SL}(n, \mathbb{C})$, $\text{SL}(n, \mathbb{H})$, $\text{SU}(m, n, \mathbb{R})$, $n \geq 2$, $m \geq 1$, and their products, compare [38, Section 2.3 and Theorem 7.12].

elliptic operators, and uses Fourier integral operator methods. Thus, let P be an elliptic pseudodifferential operator on a closed Riemannian manifold M as above. Our main tool is the *spectral function* $e(x, y, \mu)$ of the m -th root $Q := \sqrt[m]{P}$ of P given by

$$e(x, y, \mu) := \sum_{\mu_j \leq \mu} \phi_j(x) \overline{\phi_j(y)} \in C^\infty(M \times M), \quad \mu \in \mathbb{R}, \quad \mu_j := \sqrt[m]{\lambda_j}.$$

In the spherical situations [27, 3, 4, 7, 33] examined before, a crucial role is played by asymptotics for spherical functions, see [27, Eq. (1.3)] and [33, Eq. (8)]. Since we cannot rely on them in our setting⁵, we consider instead the spectral expansion of $e(x, y, \mu)$ itself and the asymptotic behaviour of

$$s_\mu(x, y) := e(x, y, \mu + 1) - e(x, y, \mu),$$

which represents the Schwartz kernel of the spectral projection s_μ onto the sum of eigenspaces of Q with eigenvalues in the interval $(\mu, \mu + 1]$. More precisely, if M carries an effective and isometric action of a compact Lie group K and $\sigma \in \widehat{K}$, denote by Π_σ the projector onto the σ -isotypic component in the Peter-Weyl decomposition of $L^2(M)$. In order to show the L^∞ -bounds (1.2), and analogous equivariant convex L^p -bounds, an asymptotic formula for the Schwartz kernel of $s_\mu \circ \Pi_\sigma$, or rather of $\tilde{s}_\mu \circ \Pi_\sigma$, where \tilde{s}_μ represents certain smooth approximation to s_μ , was derived in [40, Corollary 2.2. and Theorem 3.3] in a neighbourhood of the diagonal relying on the theory of Fourier integral operators. Now, let G be a semisimple Lie group with finite center, Γ a discrete cocompact subgroup, and K a maximal compact subgroup of G . Let $\widehat{\Gamma}$ denote the set consisting of characters of Γ of finite order. For $\chi \in \widehat{\Gamma}$, introduce on $L^2(\Gamma_\chi \backslash G)$ the Hecke operators $\mathcal{T}_{\Gamma\beta\Gamma}^\chi$

$$(\mathcal{T}_{\Gamma\beta\Gamma}^\chi f)(x) := [\Gamma : \Gamma_\chi]^{-1} \sum_{\alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma} \overline{\chi(\alpha)} f(\alpha \cdot x),$$

where β belongs to a certain set containing the commensurator $C(\Gamma)$ of Γ . Based on the asymptotics for the kernel of $\tilde{s}_\mu \circ \Pi_\sigma$ mentioned above, we deduce in Proposition 4.1 for any small $\delta > 0$ and some constant $C > 0$ the equivariant bound

$$\begin{aligned} K_{\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, x) &\ll \frac{d_\sigma}{[\Gamma : \Gamma_\chi]} \mu^{\dim G/K-1} \sup_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_\infty M(x, \beta, \delta) \\ &+ \frac{d_\sigma}{[\Gamma : \Gamma_\chi]} \mu^{\frac{\dim G/K-1}{2}} \sup_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_\infty \int_\delta^C s^{-\frac{1}{2}} dM(s) \end{aligned}$$

uniformly in $x \in \Gamma_\chi \backslash G$ for the Schwartz kernel of $\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma$, where we introduced the lattice point counting function

$$M(\delta) := M(x, \beta, \delta) := \#\left\{ \alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma \mid \text{dist}(xK, \alpha \cdot xK)^{\dim G/K-1} < \delta \right\}$$

given in terms of the distance function on the Riemannian symmetric space G/K . In case that $K = T$ is a torus, a corresponding better estimate holds. From this, we obtain the subconvex bounds (1.7) and (1.8) by using known uniform upper bounds [27, 33] for $M(x, \beta, \delta)$ combined with arithmetic amplification. The bound (1.9) is inferred by analogous methods. In both cases, it is crucial to control the caustic behaviour of the kernels of $\tilde{s}_\mu \circ \Pi_\sigma$ and \tilde{s}_μ near the diagonal as $\mu \rightarrow +\infty$, respectively.

Let us close this introduction with some comments. There exist several variants of the bounds (1.5), beginning with [27, Appendix], where the non compact hyperbolic surface $\text{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$ is considered. On the other hand, bounds in the level aspect are shown in [47] for compact locally symmetric spaces of arithmetic type, while bounds in the eigenvalue and level aspect are derived for the modular surfaces $\Gamma_0(N) \backslash \mathbb{H}$ in [2, 48] and other papers. It is likely that our work can be extended to these settings, and we plan to deal with these questions in a future paper. Also, we intend to widen our results to Hecke-Maass forms of rank r , that is, simultaneous eigenfunctions of the Hecke operators and the full ring of invariant differential operators associated to the center of the universal enveloping algebra of

⁵Compare also [34, Section 4.4].

the complexification of the Lie algebra of G . For such forms, the exponent $-1/2m$ in (1.8) and (1.9) should be improvable by a factor r . Finally, we expect the factor $\sqrt{d_\sigma \sup_{u \leq \lfloor \frac{\dim \mathfrak{K}}{2} + 1 \rfloor} \|D^u \sigma\|_\infty}$ in (1.8) to be improvable to d_σ .

This paper is structured as follows. In Section 2, we introduce Hecke operators with character on semisimple Lie groups with finite center, in Section 3 we give a description of the asymptotic behaviour of spectral function of an elliptic operator by means of Fourier integral operators, and explain how convex bounds can be deduced from this in equivariant and non-equivariant situations. Based on these results, we derive spectral asymptotics for kernels of Hecke operators in Section 4. Relying on the latter, we finally prove subconvex bounds for arithmetic congruence lattices in $\mathrm{SL}(2, \mathbb{R})$, $\mathrm{SO}(3)$, and a large class of semisimple algebraic groups in Sections 5, 6, and 7, respectively. Throughout the paper, $\mathbb{N} := \{0, 1, 2, 3, \dots\}$ will denote the set of natural numbers, while $\mathbb{N}_* := \{1, 2, 3, \dots\}$.

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2. HECKE OPERATORS WITH CHARACTER ON SEMISIMPLE LIE GROUPS

To introduce our setting, let G be a real semisimple Lie group with finite center and Lie algebra \mathfrak{g} . Denote by $\langle X, Y \rangle := \mathrm{tr}(\mathrm{ad} X \circ \mathrm{ad} Y)$ the Cartan-Killing form on \mathfrak{g} and by θ a Cartan involution of \mathfrak{g} . Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the Cartan decomposition of \mathfrak{g} into the eigenspaces of θ , corresponding to the eigenvalues $+1$ and -1 , respectively, and denote the maximal compact subgroup of G with Lie algebra \mathfrak{k} by K . Put $\langle X, Y \rangle_\theta := -\langle X, \theta Y \rangle$. Then $\langle \cdot, \cdot \rangle_\theta$ defines a left-invariant Riemannian metric on G , which in general will possess some strictly positive sectional curvature, compare Milnor [35, p. 298 and p. 317]. Dividing by the K -action, the quotient G/K becomes a Riemannian symmetric space of non-positive sectional curvature. With respect to the left-invariant metric on G , a distance function $\mathrm{dist}(g, h)$ is defined on each connected component of G as the geodesic distance between two points g, h in that component. Note that $\mathrm{dist}(g_1 g, g_1 h) = \mathrm{dist}(g, h)$ for all $g_1 \in G$. In contrast to the Killing form, $\langle \cdot, \cdot \rangle_\theta$ is no longer $\mathrm{Ad}(G)$ -invariant, but still $\mathrm{Ad}(K)$ -invariant, so that $\mathrm{dist}(gk, hk) = \mathrm{dist}(g, h)$ for all $k \in K$.

Next, let $X_1, \dots, X_{\dim \mathfrak{p}}$ be an orthonormal basis of \mathfrak{p} and $Y_1, \dots, Y_{\dim \mathfrak{k}}$ an orthonormal basis for \mathfrak{k} with respect to $\langle \cdot, \cdot \rangle_\theta$. If Ω and Ω_K denote the Casimir elements of G and K , one has

$$(2.1) \quad \Omega = \sum_{i=1}^{\dim \mathfrak{p}} X_i^2 - \sum_{i=1}^{\dim \mathfrak{k}} Y_i^2, \quad \Omega_K = - \sum_{i=1}^{\dim \mathfrak{k}} Y_i^2,$$

and we put $\Theta := -\Omega + 2\Omega_K$. Then $dR(\Theta)$ is the Beltrami-Laplace operator Δ on G with respect to the left invariant metric defined by $\langle \cdot, \cdot \rangle_\theta$, while $\mathcal{C} := dR(\Omega)$ represents the Casimir operator, R being the right regular representation of G on $C^\infty(G)$, see [37, Section 3] and [8, Section 2.10]. Thus,

$$(2.2) \quad \Delta = -\mathcal{C} + 2dR(\Omega_K),$$

all three operators commuting with each other.

We consider now a discrete cocompact subgroup Γ of G , together with the set $\widehat{\Gamma}$ of its characters of finite order, and let $\chi \in \widehat{\Gamma}$. Then $\Gamma_\chi := \ker \chi$ is a subgroup of finite index in Γ and the quotient $\Gamma_\chi \backslash G$ a compact manifold without boundary. By requiring that the projection $G \rightarrow \Gamma_\chi \backslash G$ is a Riemannian submersion, we obtain a Riemannian structure on $\Gamma_\chi \backslash G$ which locally has the same curvature than G . Furthermore, the Riemannian structure on G/K induces a Riemannian metric on $\Gamma_\chi \backslash G/K$, becoming a locally symmetric space of negative curvature. In both cases, dist induces corresponding distances on $\Gamma_\chi \backslash G$ and $\Gamma_\chi \backslash G/K$.

Before we proceed, note that due to the compactness of $\Gamma_\chi \backslash G$, the right regular representation of G on $L^2(\Gamma_\chi \backslash G)$ decomposes into an orthogonal direct sum of countably many irreducible unitary representations with finite multiplicities, that is,

$$(2.3) \quad L^2(\Gamma_\chi \backslash G) \cong \bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma_\chi) \cdot \pi,$$

where \widehat{G} denotes the unitary dual of G and $m(\pi, \Gamma_\chi)$ is a non-negative integer, see [18]. Furthermore, both the spectra of Δ and \mathcal{C} in $L^2(\Gamma_\chi \backslash G)$ are discrete. Note that the eigenvalues of Δ are positive, while the ones of \mathcal{C} can be both negative and positive.

In what follows, we introduce Hecke operators on $\Gamma \backslash X$, where $X := G$ or G/K , following [21, Section 2], and consider the commensurator

$$C(\Gamma) := \{g \in G \mid \Gamma \text{ is commensurable with } g^{-1}\Gamma g\}$$

of Γ , where we say that two subgroups Γ_1 and Γ_2 are *commensurable* iff the indices $[\Gamma_1 : \Gamma_1 \cap \Gamma_2]$ and $[\Gamma_2 : \Gamma_1 \cap \Gamma_2]$ are finite. Let $\beta \in C(\Gamma)$. Since the mapping

$$(\Gamma \cap \beta^{-1}\Gamma\beta) \backslash \Gamma \ni (\Gamma \cap \beta^{-1}\Gamma\beta)\gamma \mapsto \Gamma\beta\gamma \in \Gamma \backslash \Gamma\beta\Gamma$$

is bijective, the double coset $\Gamma\beta\Gamma$ is a finite union of right cosets of Γ , that is, there exist representative elements $\beta_1, \beta_2, \dots, \beta_t$ in $\Gamma\beta\Gamma$ such that

$$\Gamma\beta\Gamma = \bigsqcup_{j=1}^t \Gamma\beta_j.$$

One can then associate to each double coset a linear operator $T_{\Gamma\beta\Gamma}$ on $L^2(\Gamma \backslash X)$ by setting

$$T_{\Gamma\beta\Gamma} : L^2(\Gamma \backslash X) \longrightarrow L^2(\Gamma \backslash X), \quad (T_{\Gamma\beta\Gamma}f)(x) := \sum_{j=1}^t f(\beta_j \cdot x),$$

where $\beta_j \cdot x \equiv \Gamma\beta_j \cdot \Gamma x := \Gamma\beta_j x$ depends on the choice of the representative x , but the sum does not depend on the choice of the representatives x and β_j . Summing up, one writes⁶

$$(T_{\Gamma\beta\Gamma}f)(x) = \sum_{\alpha \in \Gamma \backslash \Gamma\beta\Gamma} f(\alpha \cdot x),$$

and calls $T_{\Gamma\beta\Gamma}$ a *Hecke operator*.

If a subset U of $C(\Gamma)$ is decomposed into a finite disjoint union of double cosets of Γ , a linear operator T_U can be defined in the same manner according to

$$(2.4) \quad T_U := \sum_{k=1}^u T_{\Gamma\beta_k\Gamma} \quad U = \bigsqcup_{k=1}^u \Gamma\beta_k\Gamma, \quad \beta_k \in C(\Gamma).$$

More generally, one can introduce Hecke operators as follows. Write $H(\Gamma, C(\Gamma))$ for the space of left and right Γ -invariant \mathbb{C} -valued functions h on $C(\Gamma)$ such that the support of h is included in a finite union of double Γ -cosets. Endowed with the convolution product

$$h_1 * h_2(x) := \sum_{y \in \Gamma \backslash C(\Gamma)} h_1(y) h_2(xy^{-1}), \quad h_1, h_2 \in H(\Gamma, C(\Gamma)),$$

$H(\Gamma, C(\Gamma))$ becomes an associative algebra over \mathbb{C} with the characteristic function 1_Γ of Γ as unit element. For each $h \in H(\Gamma, C(\Gamma))$, a linear operator T_h on $L^2(\Gamma \backslash X)$ can then be defined by

$$(T_h f)(x) := \sum_{\alpha \in \Gamma \backslash C(\Gamma)} h(\alpha) f(\alpha \cdot x),$$

and one has $T_{h_1 * h_2} = T_{h_1} \circ T_{h_2}$. If U is as above and h is the characteristic function of U , then it is obvious that T_h equals T_U . We call $H(\Gamma, C(\Gamma))$ the *Hecke algebra* and refer the reader to [21,

⁶Here and above $\alpha \equiv \Gamma\alpha$ (resp. $\beta_j \equiv \Gamma\beta_j$) and $x \equiv \Gamma x$ are considered both as right cosets of Γ and representatives in G , and the products αx (resp. $\beta_j x$) are taken in G .

Section 2] for details. Next, let us introduce Hecke operators with character. Assume that G is a subgroup of another group G' . Let Ξ be a sub-semigroup of G' containing Γ . We suppose that χ can be extended to Ξ , and that there exists a homomorphism $\psi : \Xi \rightarrow C(\Gamma)$ such that $\psi|_{\Gamma}$ is the identity map, and $\alpha x \alpha^{-1} = \psi(\alpha) x \psi(\alpha)^{-1}$ holds for any $\alpha \in \Xi$, $x \in G$. In particular, for $\alpha, \beta \in \Xi$ we have $\chi(\alpha\beta) = \chi(\alpha)\chi(\beta)$, while the inverse element α^{-1} does not always belong to Ξ .

Example 2.1. One of the main examples we are having in mind is $G' := \mathrm{GL}(n, \mathbb{R})$ with

$$\Xi := \{\alpha = (\alpha_{ij}) \in M(n, \mathbb{Z}) \mid \det(\alpha) > 0, (\alpha_{11}, N) = 1, \alpha_{j1} \equiv 0 \pmod{N} \ (2 \leq j \leq n)\},$$

$$\chi((\alpha_{ij})) := \omega(\alpha_{11}), \quad G := \mathrm{SL}(n, \mathbb{R}), \quad \Gamma := G \cap \Xi, \quad \psi(\alpha) := \det(\alpha)^{-1/n} \alpha,$$

where ω is a Dirichlet character on $(\mathbb{Z}/N\mathbb{Z})^\times$.

Let us now define a left action of Ξ on G by setting $\alpha \cdot x := \psi(\alpha)x$. For a fixed $\beta \in \Xi$ we can then define the *Hecke operator with character*

$$(2.5) \quad \mathcal{T}_{\Gamma\beta\Gamma}^\chi : L^2(\Gamma_\chi \backslash X) \longrightarrow L^2(\Gamma_\chi \backslash X), \quad (\mathcal{T}_{\Gamma\beta\Gamma}^\chi f)(x) := [\Gamma : \Gamma_\chi]^{-1} \sum_{\alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma} \overline{\chi(\alpha)} f(\alpha \cdot x),$$

where we took into account that $\Gamma_\chi \backslash \Gamma\beta\Gamma \subset \Xi$. By definition we have $\mathcal{T}_{\Gamma\beta\Gamma}^\chi = T_h$ for some $h \in H(\Gamma_\chi, C(\Gamma))$ satisfying $h(\gamma_1 x \gamma_2) = \overline{\chi(\gamma_1 \gamma_2)} h(x)$ for any $\gamma_1, \gamma_2 \in \Gamma$ and $x \in C(\Gamma)$. Furthermore, for given $\beta_j \in \Xi$ and $h_j \in H(\Gamma_\chi, C(\Gamma))$ with $T_{h_j} = \mathcal{T}_{\Gamma\beta_j\Gamma}^\chi$, the convolution $h_1 * h_2$ also satisfies the latter condition. Thus, there exist elements $l \in \mathbb{N}$, $a_u \in \mathbb{C}$, and $\alpha_u \in \Xi$ such that

$$(2.6) \quad \mathcal{T}_{\Gamma\beta_1\Gamma}^\chi \circ \mathcal{T}_{\Gamma\beta_2\Gamma}^\chi = \sum_{u=1}^l a_u \mathcal{T}_{\Gamma\alpha_u\Gamma}^\chi.$$

Next, denote by $L_\chi^2(\Gamma \backslash X)$ the Hilbert space of measurable functions on X such that

$$(2.7) \quad f(\gamma x) = \chi(\gamma) f(x), \quad \gamma \in \Gamma, \quad x \in X,$$

and

$$(2.8) \quad \|f\| := \left(\int_{\Gamma \backslash X} |f(x)|^2 dx \right)^{1/2} < \infty,$$

which is well-defined since $|\chi(\gamma)| = 1$ for $\gamma \in \Gamma$, compare [36, p. 228].⁷ Notice that $f \in L_\chi^2(\Gamma \backslash X)$ implies $|f| \in L^2(\Gamma \backslash X)$. If χ is trivial, then $L_\chi^2(\Gamma \backslash X) = L^2(\Gamma \backslash X)$. Since Γ_χ is a normal subset of Γ we have

$$(2.9) \quad L^2(\Gamma_\chi \backslash X) \cong \bigoplus_{\chi' \in \widehat{\Gamma/\Gamma_\chi}} L_{\chi'}^2(\Gamma \backslash X),$$

where we regard $\widehat{\Gamma/\Gamma_\chi}$ as a subset of $\widehat{\Gamma}$, compare [36, Lemma 4.3.1]. In particular, because $\chi \in \widehat{\Gamma/\Gamma_\chi}$, $L_\chi^2(\Gamma \backslash X)$ is a closed subspace in $L^2(\Gamma_\chi \backslash X)$, and for a fixed $\beta \in \Xi$, the operator $\mathcal{T}_{\Gamma\beta\Gamma}^\chi$ restricts to the linear operator

$$(2.10) \quad T_{\Gamma\beta\Gamma}^\chi : L_\chi^2(\Gamma \backslash X) \longrightarrow L_\chi^2(\Gamma \backslash X), \quad (T_{\Gamma\beta\Gamma}^\chi f)(x) := (\mathcal{T}_{\Gamma\beta\Gamma}^\chi f)(x) = \sum_{\alpha \in \Gamma \backslash \Gamma\beta\Gamma} \overline{\chi(\alpha)} f(\alpha \cdot x).$$

Notice that for each χ' in $\widehat{\Gamma/\Gamma_\chi}$ with $\chi' \neq \chi$ and each function $f \in L_\chi^2(\Gamma \backslash X)$ we have

$$(2.11) \quad T_{\Gamma\beta\Gamma}^\chi f(x) = [\Gamma : \Gamma_\chi]^{-1} \sum_{\alpha_1 \in \Gamma_\chi \backslash \Gamma} \sum_{\alpha_2 \in \Gamma \backslash \Gamma\beta\Gamma} \overline{\chi(\alpha_1 \alpha_2)} \chi'(\alpha_1) f(\alpha_2 \cdot x) = 0$$

⁷Note that for $\gamma \in \Gamma_\chi$ condition (2.7) reads $f(\gamma x) = f(x)$. Therefore, instead of $L_\chi^2(\Gamma \backslash X)$ one could also consider the closed subspace of $L^2(\Gamma_\chi \backslash X)$ that consists of functions satisfying (2.7).

by the orthogonality relations for characters. Further, the projection of $L^2(\Gamma_\chi \backslash X)$ onto $L^2_\chi(\Gamma \backslash X)$ is given by the Hecke operator $T_{h(\chi)}$, where $h(\chi) \in H(\Gamma_\chi, \Xi)$ is the function

$$(2.12) \quad h(\chi) : x \mapsto [\Gamma : \Gamma_\chi]^{-1} \sum_{\alpha \in \Gamma_\chi \backslash \Gamma} \overline{\chi(\alpha)} \mathbf{1}_{\Gamma_\chi \alpha}(x),$$

$\mathbf{1}_{\Gamma_\chi \alpha}$ being the characteristic function of the coset $\Gamma_\chi \alpha$. Thus, one obtains the commutative diagram

$$(2.13) \quad \begin{array}{ccc} L^2(\Gamma_\chi \backslash X) & \xrightarrow{T_{\Gamma_\chi \beta \Gamma}^\chi} & L^2(\Gamma_\chi \backslash X) \\ \downarrow T_{h(\chi)} & & \downarrow T_{h(\chi)} \\ L^2_\chi(\Gamma \backslash X) & \xrightarrow{T_{\Gamma_\chi \beta \Gamma}^\chi} & L^2_\chi(\Gamma \backslash X) \end{array}$$

and in view of (2.11) we have $T_{h(\chi)} \circ T_{\Gamma_\chi \beta \Gamma}^\chi \circ T_{h(\chi)} = T_{\Gamma_\chi \beta \Gamma}^\chi$.

3. THE SPECTRAL FUNCTION OF AN ELLIPTIC OPERATOR AND CONVEX BOUNDS FOR EIGENFUNCTIONS

The main tool underlying our analysis is the spectral function of an elliptic operator on a smooth manifold, which contains essential information on the spectrum. For large spectral parameters, an asymptotic description of it can be derived within the theory of Fourier integral operators, yielding in particular convex bounds for eigenfunctions. In what follows, we shall briefly recall the main arguments in non-equivariant and equivariant situations, and provide the results that will be needed later.

3.1. The spectral function and convex bounds for eigenfunctions. Let M be a closed Riemannian manifold of dimension d and P_0 an elliptic classical pseudodifferential operator on M of degree m , which is assumed to be positive and symmetric. Denote its unique self-adjoint extension by P , and let $\{\phi_j\}_{j \geq 0}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of P with eigenvalues $\{\lambda_j\}_{j \geq 0}$ repeated according to their multiplicity. Let $p(x, \xi)$ be the principal symbol of P_0 , which is strictly positive and homogeneous in ξ of degree m as a function on $T^*M \setminus \{0\}$, that is, the cotangent bundle of M without the zero section. Here and in what follows (x, ξ) denotes an element in $T^*Y \simeq Y \times \mathbb{R}^d$ with respect to the canonical trivialization of the cotangent bundle over a chart domain $Y \subset M$. Consider further the m -th root $Q := \sqrt[m]{P}$ of P given by the spectral theorem. It is well known that Q is a classical pseudodifferential operator of order 1 with principal symbol $q(x, \xi) := \sqrt[m]{p(x, \xi)}$ and the first Sobolev space as domain. Again, Q has discrete spectrum, and its eigenvalues are given by $\mu_j := \sqrt[m]{\lambda_j}$. The spectral function $e(x, y, \lambda)$ of P can then be described by studying the spectral function of Q , which in terms of the basis $\{\phi_j\}$ is given by

$$(3.1) \quad e(x, y, \mu) := \sum_{\mu_j \leq \mu} \phi_j(x) \overline{\phi_j(y)},$$

and belongs to $C^\infty(M \times M)$ as a function of x and y for any $\mu \in \mathbb{R}$. Let s_μ be the spectral projection onto the sum of eigenspaces of Q with eigenvalues in the interval $(\mu, \mu + 1]$, and denote its Schwartz kernel by

$$s_\mu(x, y) := e(x, y, \mu + 1) - e(x, y, \mu).$$

To obtain an asymptotic description of the spectral function of Q , one first derives a description of $s_\mu(x, y)$ by approximating s_μ by Fourier integral operators. To do so, let $\varrho \in \mathcal{S}(\mathbb{R}, \mathbb{R}_+)$ be such that $\hat{\varrho}(0) = 1$ and $\text{supp } \hat{\varrho} \in (-\delta/2, \delta/2)$ for an arbitrarily small $\delta > 0$, and define the approximate spectral projection operator

$$\tilde{s}_\mu u := \sum_{j=0}^{\infty} \varrho(\mu - \mu_j) E_j u, \quad u \in L^2(M),$$

where E_j denotes the orthogonal projection onto the subspace spanned by ϕ_j . Clearly,

$$(3.2) \quad K_{\tilde{s}_\mu}(x, y) := \sum_{j=0}^{\infty} \varrho(\mu - \mu_j) \phi_j(x) \overline{\phi_j(y)} \in C^\infty(M \times M)$$

constitutes the kernel of \tilde{s}_μ . Now, notice that for $\mu, \tau \in \mathbb{R}$ one has

$$\varrho(\mu - \tau) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\varrho}(t) e^{-it\tau} e^{it\mu} dt,$$

where $\hat{\varrho}(t)$ denotes the Fourier transform of ϱ , so that for $u \in L^2(M)$ we obtain

$$(3.3) \quad \tilde{s}_\mu u = \frac{1}{2\pi} \sum_{j=0}^{\infty} \int_{\mathbb{R}} \hat{\varrho}(t) e^{it\mu} e^{-it\mu_j} dt E_j u = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\varrho}(t) e^{it\mu} U(t) u dt,$$

where $U(t)$ stands for the one-parameter group

$$U(t) := e^{-itQ} = \int e^{-it\mu} dE_\mu^Q, \quad t \in \mathbb{R},$$

of unitary operators in $L^2(M)$ given by the Fourier transform of the spectral measure, $\{E_\mu^Q\}$ being a spectral resolution of Q . The central result of Hörmander [22] then says that $U(t) : L^2(M) \rightarrow L^2(M)$ can be approximated by Fourier integral operators.

More precisely, let $\{(\kappa_\iota, Y_\iota)\}_{\iota \in I}$, $\kappa_\iota : Y_\iota \xrightarrow{\cong} \tilde{Y}_\iota \subset \mathbb{R}^d$, be an atlas for M , $\{f_\iota\}$ a corresponding partition of unity and $\hat{v}(\eta) := \mathcal{F}(v)(\eta) := \int_{\mathbb{R}^d} e^{-i\langle \tilde{y}, \eta \rangle} v(\tilde{y}) d\tilde{y}$ the Fourier transform of $v \in C_c^\infty(\tilde{Y}_\iota)$. Write $d\eta := d\eta/(2\pi)^d$, and introduce on \tilde{Y}_ι the operator

$$[\tilde{U}_\iota(t)v](\tilde{x}) := \int_{\mathbb{R}^d} e^{i\psi_\iota(t, \tilde{x}, \eta)} a_\iota(t, \tilde{x}, \eta) \hat{v}(\eta) d\eta,$$

where $a_\iota \in S_{\text{phg}}^0$ is a classical polyhomogeneous symbol satisfying $a_\iota(0, \tilde{x}, \eta) = 1$ and ψ_ι the defining phase function given as the solution of the Hamilton-Jacobi equation

$$\frac{\partial \psi_\iota}{\partial t} + q\left(x, \frac{\partial \psi_\iota}{\partial \tilde{x}}\right) = 0, \quad \psi_\iota(0, \tilde{x}, \eta) = \langle \tilde{x}, \eta \rangle,$$

see [24, Page 254]. Let us remark that ψ_ι is homogeneous in η of degree 1, so that Taylor expansion for small t gives

$$\psi_\iota(t, \tilde{x}, \eta) = \psi_\iota(0, \tilde{x}, \eta) + t \frac{\partial \psi_\iota}{\partial t}(0, \tilde{x}, \eta) + O(t^2|\eta|) = \langle \tilde{x}, \eta \rangle - tq_\iota(\tilde{x}, \eta) + O(t^2|\eta|),$$

where we wrote $q_\iota(\tilde{x}, \eta) := q(\kappa_\iota^{-1}(\tilde{x}), \eta)$. In other words, there exists a smooth function ζ_ι which is homogeneous in η of degree 1 and satisfies

$$\psi_\iota(t, \tilde{x}, \eta) = \langle \tilde{x}, \eta \rangle - t\zeta_\iota(t, \tilde{x}, \eta), \quad \zeta_\iota(0, \tilde{x}, \eta) = q_\iota(\tilde{x}, \eta).$$

Put now $\bar{U}_\iota(t)u := [\tilde{U}_\iota(t)(u \circ \kappa_\iota^{-1})] \circ \kappa_\iota$ for any $u \in C_c^\infty(Y_\iota)$. Consider further test functions $\bar{f}_\iota \in C_c^\infty(Y_\iota)$ satisfying $\bar{f}_\iota \equiv 1$ on $\text{supp } f_\iota$, and define

$$\bar{U}(t) := \sum_{\iota} F_\iota \bar{U}_\iota(t) \bar{F}_\iota,$$

where F_ι, \bar{F}_ι denote the multiplication operators with f_ι and \bar{f}_ι , respectively. Then Hörmander showed that for small $|t|$

$$(3.4) \quad \mathcal{R}(t) := U(t) - \bar{U}(t) \text{ is an operator with smooth kernel,}$$

compare [20, Page 134] and [45, Theorem 20.1]; in particular, the kernel $\mathcal{R}_t(x, y)$ of $\mathcal{R}(t)$ is smooth as a function of t .

Approximating in (3.3) the operator $U(t)$ by $\bar{U}(t)$, one obtains a description for the kernel of \tilde{s}_μ as the double oscillatory integral

$$(3.5) \quad K_{\tilde{s}_\mu}(x, y) = \frac{\mu^d}{(2\pi)^{d+1}} \sum_{\iota} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\mu[t-Rt]} I_\iota(\mu, R, t, x, y) dR dt$$

up to terms of order $O(\mu^{-\infty})$ which are uniform in x and y , where

$$I_\iota(\mu, R, t, x, y) := \int_{\Sigma_{\iota, x}^{R, t}} e^{i\mu\Phi_{\iota, x, y}(\omega)} \hat{\rho}(t) f_\iota(x) a_\iota(t, \kappa_\iota(x), \mu\omega) \bar{f}_\iota(y) b(q(x, \omega)) d\Sigma_{\iota, x}^{R, t}(\omega),$$

$$\Phi_{\iota, x, y}(\omega) := \langle \kappa_\iota(x) - \kappa_\iota(y), \omega \rangle,$$

and

$$(3.6) \quad \Sigma_{\iota, x}^{R, t} := \{ \omega \in \mathbb{R}^d \mid \zeta_\iota(t, \kappa_\iota(x), \omega) = R \},$$

while $0 \leq b \in C_c^\infty(1/2, 3/2)$ is a test function such that $b \equiv 1$ in a neighborhood of 1, compare [40, Eq. (2.8)]. Here $d\Sigma_{\iota, x}^{R, t}(\omega)$ denotes the quotient of Lebesgue measure in \mathbb{R}^d by Lebesgue measure in \mathbb{R} with respect to $\zeta_\iota(t, \tilde{x}, \omega)$. Furthermore, for sufficiently small $\delta > 0$ one can assume that the R -integration is over a compact set, and R and t are close to 1 and 0, respectively. From (3.5), an asymptotic description can be inferred as $\mu \rightarrow +\infty$ by means of the stationary phase principle. In fact, one has the following

Proposition 3.1. *Suppose that the cospheres $S_x^*M := \{(x, \xi) \in T^*M \mid p(x, \xi) = 1\}$ are strictly convex.⁸ Then, for any fixed $x, y \in M$, and $\tilde{N} = 0, 1, 2, 3, \dots$ one has the expansion*

$$K_{\tilde{s}_\mu}(x, y) = \mu^{d-1-\frac{\delta_{x, y}}{2}} \left[\sum_{r=0}^{\tilde{N}-1} \mathcal{L}_r(x, y, \mu) + \mathcal{R}_{\tilde{N}}(x, y, \mu) \right]$$

up to terms of order $O(\mu^{-\infty})$ as $\mu \rightarrow +\infty$, where

$$\delta_{x, y} := \begin{cases} 0, & y = x, \\ d-1, & y \neq x. \end{cases}$$

The coefficients in the expansion and the remainder $\mathcal{R}_{\tilde{N}}(x, y, \mu) = O_{x, y}(\mu^{-\tilde{N}})$ term can be computed explicitly; if $y = x$, they are uniformly bounded in x and y , while if $y \neq x$, they satisfy the bounds

$$(3.7) \quad \mathcal{L}_r(x, y, \mu) \ll \text{dist}(x, y)^{-(d-1)/2-r} \mu^{-r}, \quad \mathcal{R}_{\tilde{N}}(x, y, \mu) \ll \text{dist}(x, y)^{-(d-1)/2-\tilde{N}} \mu^{-\tilde{N}},$$

where $\text{dist}(x, y)$ denotes the geodesic distance between two points belonging to the same connected component. Otherwise, $\text{dist}(x, y) := \infty$.

Proof. If $x = y$, one has $I_\iota(\mu, R, t, x, y) = O(1)$ uniformly in all parameters since $a_\iota \in S_{phg}^0$ is a classical symbol of order 0, so that

$$|\partial_\omega^\alpha a_\iota(t, \kappa_\iota(x), \mu\omega)| = |\mu|^{|\alpha|} |(\partial_\omega^\alpha a_\iota)(t, \kappa_\iota(x), \mu\omega)| \leq C|\omega|^{-|\alpha|}.$$

Consequently, the dependence of the amplitude on μ does not interfere with the asymptotics, compare [14, Proposition 1.2.4]. Applying the stationary phase principle [23, Theorem 7.7.5] to the (R, t) -integral in (3.5) with $t(1-R)$ as phase function then yields the assertion for $x = y$, the unique critical point being $(R_0, t_0) = (1, 0)$ in this case. Let us now assume that $x \neq y$. By assumption, the cospheres S_x^*M are strictly convex, so that for small $|t|$, the hypersurfaces $\Sigma_{\iota, x}^{R, t}$ will be strictly convex, too.

⁸This condition is required only in the case $x \neq y$. For example, it holds if $P_0 = \Delta$ equals the Beltrami-Laplace operator, since then $p(x, \xi) = \|\xi\|_x^2$.

Applying [40, Lemma 3.5] to the integrals $I_\ell(\mu, R, t, x, y)$ with $\left\langle \frac{\kappa_\ell(x) - \kappa_\ell(y)}{\|\kappa_\ell(x) - \kappa_\ell(y)\|}, \omega \right\rangle$ as phase function and $\nu := \mu \|\kappa_\ell(x) - \kappa_\ell(y)\|$ as asymptotic parameter yields for any $\tilde{N} \in \mathbb{N}$ the expansion

$$I_\ell(\mu, R, t, x, y) = \sum_{\omega_0 \in \text{Crit} \Phi_{\ell, x, y}} e^{i\mu \Phi_{\ell, x, y}(\omega_0)} \left[\sum_{r=0}^{\tilde{N}-1} \mathcal{Q}_{\ell, r}(R, t, x, y, \omega_0) \mu^{-(d-1)/2-r} + \mathcal{R}_{\ell, \tilde{N}}(R, t, x, y, \omega_0, \mu) \right],$$

where the coefficients and the remainder are smooth in R and t , and satisfy the bounds

$$\begin{aligned} \partial_{R, t}^\alpha \mathcal{Q}_{\ell, r}(R, t, x, y, \omega_0) &\ll \|\kappa_\ell(x) - \kappa_\ell(y)\|^{-(d-1)/2-r}, \\ \partial_{R, t}^\alpha \mathcal{R}_{\ell, \tilde{N}}(R, t, x, y, \omega_0, \mu) &\ll (\|\kappa_\ell(x) - \kappa_\ell(y)\| \mu)^{-(d-1)/2-\tilde{N}}, \end{aligned}$$

uniformly in R and t . Regarding the value of $\Phi_{\ell, x, y}$ on its critical set, one computes for $\omega_0 \in \text{Crit} \Phi_{\ell, x, y}$

$$\Phi_{\ell, x, y}(\omega_0) = \pm \|\kappa_\ell(x) - \kappa_\ell(y)\| R / \|\text{grad}_\eta \zeta_\ell(t, \kappa_\ell(x), \omega_0)\|,$$

since $\kappa_\ell(x) - \kappa_\ell(y)$ must be colinear to $\text{grad}_\eta \zeta(t, \kappa_\ell(x), \omega_0)$. Notice that due to the fact that $\zeta(t, \kappa_\ell(x), \eta)$ is homogeneous of degree 1 in η , the gradient $\text{grad}_\eta \zeta(t, \kappa_\ell(x), \omega_0)$ only depends on the direction of ω_0 , and is therefore independent of R . From this and (3.5) we deduce for $K_{\tilde{s}_\mu}(x, y)$ as $\mu \rightarrow +\infty$ the expansion

$$\frac{\mu^d}{(2\pi)^{d+1}} \sum_{\ell, \omega_0} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\mu[t - Rt + \Phi_{\ell, x, y}(\omega_0)]} \left[\sum_{r=0}^{\tilde{N}} \mathcal{Q}_{\ell, r}(R, t, x, y, \omega_0) \mu^{-(d-1)/2-r} + \mathcal{R}_{\ell, \tilde{N}}(R, t, x, y, \omega_0, \mu) \right] dR dt$$

up to terms of order $O(\mu^{-\infty})$. Again, we apply the stationary phase principle to the (R, t) -integrals, where now the phase function reads $t(1 - R) + \Phi_{\ell, x, y}(\omega_0)$. The determinant of the matrix of its second derivatives is given by

$$-(1 \pm O(\|\kappa_\ell(x) - \kappa_\ell(y)\|))^2.$$

By choosing the charts Y_ℓ sufficiently small so that $\|\kappa_\ell(x) - \kappa_\ell(y)\| \ll 1$, we can therefore achieve that in a sufficiently small neighborhood of $(R, t) = (1, 0)$, which is where the amplitude of the (R, t) -integral is supported, the phase function $t(1 - R) + \Phi_{\ell, x, y}(\omega_0)$ has, if at all, only non-degenerate, hence isolated, critical points. If we now apply the stationary phase theorem, the assertion follows in the case $x \neq y$ as well. \square

Remark 3.2. By Cauchy-Schwarz and the positivity of the test function ϱ we infer from the previous proposition for $\tilde{N} = 0$ that

$$|K_{\tilde{s}_\mu}(x, y)| \leq \sqrt{\sum_{j \geq 0} \varrho(\mu - \mu_j) |\phi_j(x)|^2} \sqrt{\sum_{j \geq 0} \varrho(\mu - \mu_j) |\phi_j(y)|^2} = \sqrt{K_{\tilde{s}_\mu}(x, x)} \sqrt{K_{\tilde{s}_\mu}(y, y)} = O(\mu^{d-1})$$

uniformly in $x, y \in M$. Also, note that the asymptotics in the proposition off the diagonal are only meaningful if $\text{dist}(x, y)^{-1}$ is small with respect to μ .

As the previous proposition shows, the kernel of \tilde{s}_μ exhibits a caustic behaviour⁹ in a neighbourhood of the diagonal since for $x = y$ the integrals $I_\ell(\mu, R, t, x, y)$ no longer oscillate and are of order $O(\mu^0)$. From Proposition 3.1, similar asymptotics for s_μ can be deduced. By looking at asymptotics on the diagonal, one obtains Weyl's law for the spectral function of Q and convex L^∞ -bounds for eigenfunctions, since $\|s_\mu\|_{L^2 \rightarrow L^\infty}^2 \equiv \sup_{x \in M} s_\mu(x, x)$, yielding for any eigenfunction the convex bound

$$\|\phi_j\|_\infty \ll \lambda_j^{\frac{d-1}{2m}},$$

compare [22, Theorem 5.1] and [46, Eq. (3.2.6)]. Nevertheless, in order to prove subconvex bounds, we shall also need asymptotics off the diagonal, so that the full caustic behaviour of $K_{\tilde{s}_\mu}(x, y)$ near the diagonal becomes relevant.

⁹For the terminology, see Appendix A in [40].

3.2. The reduced spectral function and equivariant convex bounds for eigenfunctions.

Keeping the notation of Section 3.1, assume now that M carries an effective and isometric action of a compact Lie group K , and consider the right regular representation π of K on $L^2(M)$ with corresponding Peter-Weyl decomposition

$$(3.8) \quad L^2(M) = \bigoplus_{\sigma \in \widehat{K}} L_\sigma^2(M), \quad L_\sigma^2(M) := \Pi_\sigma L^2(M),$$

where \widehat{K} denotes the unitary dual of K and

$$\Pi_\sigma := d_\sigma \int_K \overline{\sigma(k)} \pi(k) dk$$

the orthogonal projector onto the σ -isotypic component, dk being Haar measure and d_σ the dimension of an irreducible representation of K in the class $\sigma \in \widehat{K}$. Further, suppose that P commutes with π , and that the orthonormal basis $\{\phi_j\}_{j \geq 0}$ is compatible with the decomposition (3.8) in the sense that each ϕ_j lies in some $L_\sigma^2(M)$. Then every eigenspace of P is invariant under π , and decomposes into irreducible K -modules spanned by eigenfunctions. In order to study eigenfunctions of P of a certain K -type, one is interested in the spectral function of the operator $Q_\sigma := \Pi_\sigma \circ Q \circ \Pi_\sigma = \Pi_\sigma \circ Q = Q \circ \Pi_\sigma$, also called the *reduced spectral function*, given by

$$(3.9) \quad e_\sigma(x, y, \mu) = \sum_{\mu_j \leq \mu, \phi_j \in L_\sigma^2(M)} \phi_j(x) \overline{\phi_j(y)}.$$

For this, one considers the composition $s_\mu \circ \Pi_\sigma$, or rather $\tilde{s}_\mu \circ \Pi_\sigma$, whose kernel has the spectral expansion

$$(3.10) \quad K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y) = \sum_{j \geq 0, \phi_j \in L_\sigma^2(M)} \varrho(\mu - \mu_j) \phi_j(x) \overline{\phi_j(y)}.$$

Similarly to (3.5), it was shown in [40, Eq. (2.8)] that by approximating $U(t)$ in (3.3) by the Fourier integral operator $\bar{U}(t)$ one obtains a description for the kernel of $\tilde{s}_\mu \circ \Pi_\sigma$ as the double oscillatory integral

$$(3.11) \quad K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y) = \frac{\mu^d d_\sigma}{(2\pi)^{d+1}} \sum_t \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\mu[t-Rt]} I_\nu^\sigma(\mu, R, t, x, y) dR dt$$

up to terms of order $O(\mu^{-\infty})$ which are uniform in x and y , where

$$\begin{aligned} I_\nu^\sigma(\mu, R, t, x, y) &:= \int_K \int_{\Sigma_{\nu, x}^{R, t}} e^{i\mu \Phi_{\nu, x, y}(\omega, k)} \hat{g}(t) \overline{\sigma(k)} f_\nu(x) \\ &\quad \cdot a_\nu(t, \kappa_\nu(x), \mu\omega) \bar{f}_\nu(y \cdot k^{-1}) b(q(x, \omega)) J_\nu(k, y) d\Sigma_{\nu, x}^{R, t}(\omega) dk, \\ \Phi_{\nu, x, y}(\omega, k) &:= \langle \kappa_\nu(x) - \kappa_\nu(y \cdot k^{-1}), \omega \rangle, \end{aligned}$$

$J_\nu(k, y)$ being a Jacobian. Write $\mathcal{O}_x := x \cdot K$ for the K -orbit through $x \in M$. We then have the following

Proposition 3.3. *Suppose that K acts on M with orbits of the same dimension $\kappa \leq d-1$ and that the cospheres $S_x^*M := \{(x, \xi) \in T^*M \mid p(x, \xi) = 1\}$ are strictly convex. Then, for any fixed $x, y \in M$, $\sigma \in \widehat{K}$, and $\tilde{N} = 0, 1, 2, 3, \dots$ one has the expansion*

$$K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y) = \mu^{d - \frac{\varepsilon_{x, y}}{2} - 1} d_\sigma \left[\sum_{r=0}^{\tilde{N}-1} \mathcal{L}_r^\sigma(x, y, \mu) + \mathcal{R}_{\tilde{N}}^\sigma(x, y, \mu) \right]$$

up to terms of order $O(\mu^{-\infty})$ as $\mu \rightarrow +\infty$, where

$$\varepsilon_{x, y} := \begin{cases} 2\kappa, & y \in \mathcal{O}_x, \\ d-1 + \kappa, & y \notin \mathcal{O}_x. \end{cases}$$

The coefficients in the expansion and the remainder term can be computed explicitly; if $y \in \mathcal{O}_x$, they satisfy the bounds

$$\mathcal{L}_r^\sigma(x, y, \mu) \ll \sup_{u \leq 2r} \|D^u \sigma\|_\infty \mu^{-r}, \quad \mathcal{R}_{\tilde{N}}^\sigma(x, y, \mu) \ll \sup_{u \leq 2\tilde{N} + \lfloor \frac{\kappa}{2} + 1 \rfloor} \|D^u \sigma\|_\infty \mu^{-\tilde{N}},$$

uniformly in x and y , where D^u denote differential operators on K of order u , and if $y \notin \mathcal{O}_x$, the bounds

$$\begin{aligned} \mathcal{L}_r^\sigma(x, y, \mu) &\ll \sup_{u \leq 2r} \|D^u \sigma\|_\infty \cdot \text{dist}(x, \mathcal{O}_y)^{-\frac{d-\kappa-1}{2}-r} \mu^{-r}, \\ \mathcal{R}_{\tilde{N}}^\sigma(x, y, \mu) &\ll \sup_{u \leq 2\tilde{N} + \lfloor \frac{\kappa}{2} + 1 \rfloor} \|D^u \sigma\|_\infty \cdot \text{dist}(x, \mathcal{O}_y)^{-\frac{d-\kappa-1}{2}-\tilde{N}} \mu^{-\tilde{N}}, \end{aligned}$$

where $\text{dist}(x, \mathcal{O}_y) := \min \{ \text{dist}(x, z) \mid z \in \mathcal{O}_y \}$. If $K = T$ is a torus, let $\hat{T}' \subset \hat{T}$ be the subset of representations occurring in the decomposition (3.8), and identify \hat{T} with the set of integral linear forms on \mathfrak{t} . Then the remainder estimates can be improved to

$$\mathcal{R}_{\tilde{N}}^\sigma(x, y, \mu) \ll \sup_{u \leq 2\tilde{N}} \|D^u \sigma\|_\infty \mu^{-\tilde{N}}, \quad \mathcal{R}_{\tilde{N}}^\sigma(x, y, \mu) \ll \sup_{u \leq 2\tilde{N}} \|D^u \sigma\|_\infty \cdot \text{dist}(x, \mathcal{O}_y)^{-\frac{d-\kappa-1}{2}-\tilde{N}} \mu^{-\tilde{N}},$$

respectively, provided that $\sigma \in \mathcal{V}_\mu := \{ \sigma' \in \hat{T}' \mid |\sigma'| \leq C\mu / \log \mu \}$ for some constant $C > 0$.

Remark 3.4.

(1) Proposition 3.3 implies for $\tilde{N} = 0$ by Cauchy-Schwarz that

$$\begin{aligned} |K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y)| &\leq \sqrt{\sum_{j \geq 0, \phi_j \in \mathcal{L}_\sigma^2(M)} \varrho(\mu - \mu_j) |\phi_j(x)|^2} \sqrt{\sum_{j \geq 0, \phi_j \in \mathcal{L}_\sigma^2(M)} \varrho(\mu - \mu_j) |\phi_j(y)|^2} \\ &= \sqrt{K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, x)} \sqrt{K_{\tilde{s}_\mu \circ \Pi_\sigma}(y, y)} = O\left(d_\sigma \mu^{d-\kappa-1} \sup_{u \leq \lfloor \kappa/2 + 1 \rfloor} \|D^u \sigma\|_\infty\right) \end{aligned}$$

uniformly in $x, y \in M$ and $\sigma \in \hat{K}$, while taking $\tilde{N} = 1$ would yield an estimate of order

$$O\left(d_\sigma \mu^{d-\kappa-1} \left(\|\sigma\|_\infty + \sup_{u \leq \lfloor \kappa/2 + 3 \rfloor} \|D^u \sigma\|_\infty \mu^{-1} \right)\right).$$

If $K = T$ is a torus, better remainder estimates hold.

(2) Note that the asymptotics of Proposition 3.3 in the case $y \notin \mathcal{O}_x$ are only meaningful if $\text{dist}(x, \mathcal{O}_y)^{-1}$ is small with respect to μ .

Proof of Proposition 3.3. This proposition is essentially a consequence of [40, Theorem 3.3] and [41, Theorem 3.2]. In particular, an asymptotic expansion of $K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, x)$ was given in [40, Proposition 4.2] and, with an improved remainder estimate in the toric case, in [40, Proposition 4.1]. To obtain an asymptotic expansion off the diagonal from (3.11), we shall first apply the stationary phase theorem to the integrals $I_\ell^\sigma(\mu, R, t, x, y)$, and then to the (R, t) -integral. If $x \notin Y_\ell$ or $\mathcal{O}_y \cap Y_\ell = \emptyset$, $I_\ell^\sigma(\mu, R, t, x, y) = 0$. Otherwise, [40, Theorem 3.3] implies for sufficiently small Y_ℓ , fixed $R, t \in \mathbb{R}$, and any $\tilde{N} \in \mathbb{N}$ the asymptotic expansion

$$(3.12) \quad I_\ell^\sigma(\mu, R, t, x, y) = \mu^{-\frac{\text{codim Crit } \Phi_{\ell, x, y}}{2}} e^{i\mu \Phi_{\ell, x, y}^0(R, t)} \left[\sum_{r=0}^{\tilde{N}-1} \mathcal{L}_{\ell, r}^\sigma(R, t, x, y) \mu^{-r} + \tilde{\mathcal{R}}_{\ell, \tilde{N}}^\sigma(R, t, x, y, \mu) \right],$$

where $\text{Crit } \Phi_{\ell, x, y}$ denotes the critical set of $\Phi_{\ell, x, y}$, and

$$\text{codim Crit } \Phi_{\ell, x, y} = \begin{cases} 2\kappa, & y \in \mathcal{O}_x, \\ d-1+\kappa, & y \notin \mathcal{O}_x. \end{cases}$$

The coefficients and the remainder term are given by distributions depending smoothly on R, t with support in $\text{Crit } \Phi_{\iota, x, y}$ and $\Sigma_{\iota, x}^{R, t} \times K$, respectively. Furthermore, they and their derivatives with respect to R, t satisfy for $y \notin \mathcal{O}_x$ the bounds

$$\begin{aligned} \partial_{R, t}^\alpha \mathcal{L}_{\iota, r}^\sigma(R, t, x, y) &\ll \sup_{u \leq 2r} \|D^u \sigma\|_\infty \cdot \text{dist}(x, \mathcal{O}_y)^{-\frac{d-\kappa-1}{2}-r}, \\ \partial_{R, t}^\alpha \tilde{\mathcal{R}}_{\iota, \tilde{N}}^\sigma(R, t, x, y, \mu) &\ll \sup_{u \leq 2\tilde{N} + \lfloor \frac{\kappa}{2} + 1 \rfloor} \|D^u \sigma\|_\infty \cdot \text{dist}(x, \mathcal{O}_y)^{-\frac{d-\kappa-1}{2}-\tilde{N}} \mu^{-\tilde{N}} \end{aligned}$$

while for $y \in \mathcal{O}_x$ one has

$$\partial_{R, t}^\alpha \mathcal{L}_{\iota, r}^\sigma(R, t, x, y) \ll \sup_{u \leq 2r} \|D^u \sigma\|_\infty, \quad \partial_{R, t}^\alpha \tilde{\mathcal{R}}_{\iota, \tilde{N}}^\sigma(R, t, x, y, \mu) \ll \sup_{u \leq 2\tilde{N} + \lfloor \frac{\kappa}{2} + 1 \rfloor} \|D^u \sigma\|_\infty \mu^{-\tilde{N}}$$

uniformly in x and y . If $K = T$ is a torus and $\sigma \in \mathcal{V}_\mu$, the remainder estimates can be improved [41, Theorem 3.2] to contain only derivatives of σ up to order $2\tilde{N}$. Finally,

$$\Phi_{\iota, x, y}^0(R, t) = R c_{x, y}(t), \quad c_{x, y}(t) = \pm \frac{\|\kappa_\iota(x) - \kappa_\iota(y \cdot k_0^{-1})\|}{\text{grad}_\eta \zeta_\iota(t, \kappa_\iota(x), \omega_0)},$$

denotes the constant value(s) of $\Phi_{\iota, x, y}$ on (the components of) its critical set, where (ω_0, k_0) is some point in $\text{Crit } \Phi_{\iota, x, y}$. If $y \in \mathcal{O}_x$ one has $\Phi_{\iota, x, y}^0(R, t) = 0$. As already noted in the proof of Proposition 3.1, a_ι is a polyhomogeneous symbol of order 0, so that the dependence of the amplitude on μ does not interfere with the asymptotics. Putting (3.11) and (3.12) together we obtain

$$\begin{aligned} K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y) &= \mu^{d - \frac{\text{codim } \text{Crit } \Phi_{\iota, x, y}}{2}} d_\sigma \sum_\iota \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\mu[t-Rt]} e^{i\mu\Phi_{\iota, x, y}^0(R, t)} \left[\sum_{r=0}^{\tilde{N}-1} \mathcal{L}_{\iota, r}^\sigma(R, t, x, y) \mu^{-j} \right. \\ &\quad \left. + \tilde{\mathcal{R}}_{\iota, \tilde{N}}^\sigma(R, t, x, y, \mu) \right] dR dt \end{aligned}$$

up to terms of order $O(\mu^{-\infty})$ uniform in x and y . We now apply the stationary phase principle [23, Theorem 7.7.5] to the (R, t) -integral. If $y \in \mathcal{O}_x$, the phase function simply reads $t(1-R)$, and the only critical point is $(R_0, t_0) = (1, 0)$, which is non-degenerate, the determinant of the Hessian being -1 . Therefore, the necessary conditions for an application of the principle are fulfilled, yielding the assertion of the proposition in this case. In case that $y \notin \mathcal{O}_x$, the phase function is given by $t(1-R) + \Phi_{\iota, x, y}^0(R, t)$, and the determinant of the matrix of its second derivatives is given by

$$-(1 - c'_{x, y}(t))^2 \approx -(1 \pm O(\|\kappa_\iota(x) - \kappa_\iota(y \cdot k_0^{-1})\|))^2.$$

By choosing the charts Y_ι sufficiently small so that $\|\kappa_\iota(x) - \kappa_\iota(y \cdot k_0^{-1})\| \ll 1$, we can achieve that in a sufficiently small neighborhood of $(R, t) = (1, 0)$ the phase function $t(1-R) + \Phi_{\iota, x, y}^0(R, t)$ has, if at all, only non-degenerate, hence isolated, critical points. If we now apply the stationary phase theorem, the proposition follows. \square

From Proposition 3.3 equivariant convex bounds for eigenfunctions can be easily inferred. Indeed, recall that the test function $\varrho \in \mathcal{S}(\mathbb{R}, \mathbb{R}^+)$ was chosen such that $\hat{\varrho}(0) = 1$ and $\text{supp } \hat{\varrho} \subset (-\delta/2, \delta/2)$ for some arbitrary $\delta > 0$. By choosing δ sufficiently small, one can even achieve that $\varrho > 0$ on $[-1, 1]$, compare [15, Proof of Lemma 2.3]. But then

$$\begin{aligned} \min_{\nu \in [-1, 1]} \varrho(\nu) &\underbrace{\sum_{\mu_j \in (\mu, \mu+1], \phi_j \in L_\sigma^2(M)} |\phi_j(x)|^2}_{=K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, x)} \leq \underbrace{\sum_{j \geq 0, \phi_j \in L_\sigma^2(M)} \varrho(\mu - \mu_j) |\phi_j(x)|^2}_{=K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, x)} \\ &= O\left(d_\sigma \mu^{d-\kappa-1} \sup_{u \leq \lfloor \kappa/2 + 1 \rfloor} \|D^u \sigma\|_\infty\right), \end{aligned}$$

yielding a corresponding bound for $K_{s_\mu \circ \Pi_\sigma}(x, x)$, compare [40, Remark 4.4 (2)]. In view of the equality $\|s_\mu \circ \Pi_\sigma\|_{L^2 \rightarrow L^\infty}^2 \equiv \sup_{x \in M} K_{s_\mu \circ \Pi_\sigma}(x, x)$, one finally obtains the equivariant convex bound

$$\|\phi_j\|_\infty \ll \lambda_j^{\frac{d-\kappa-1}{2m}} \left(\sqrt{d_\sigma \sup_{u \leq \lfloor \kappa/2+1 \rfloor} \|D^u \sigma\|_\infty} \right)$$

for any $\phi_j \in L^2_\sigma(M)$ and $\sigma \in \widehat{K}$, see [40, Proposition 5.1 and Eq. (5.4)]. As in the non-equivariant case, the kernels $K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y)$ exhibit a caustic behaviour in their dependence on x, y , which will be crucial for the derivation of equivariant subconvex bounds. In case that $K = T$ is a torus and $\sigma \in \mathcal{V}_\mu$, the bounds above are independent of σ , see [41, Proposition 5.1].

4. SPECTRAL ASYMPTOTICS FOR KERNELS OF HECKE OPERATORS

Keep the notation of Sections 2 and 3. The main goal of this paper consists in proving subconvex bounds for Hecke–Maass forms of rank 1 on the compact d -dimensional Riemannian manifold $M = \Gamma_\chi \backslash G$. To this purpose, we shall first derive asymptotics for kernels of Hecke operators in the eigenvalue and isotypic aspect. Recall that K acts on G and M from the right in an isometric and effective way, the isotropy group of a point $\Gamma_\chi g \in \Gamma_\chi \backslash G$ being conjugate to the finite group $gKg^{-1} \cap \Gamma_\chi$. Hence, all K -orbits in $\Gamma_\chi \backslash G$ are either principal or exceptional, and of dimension $\dim K$. Since the maximal compact subgroups of G are precisely the conjugates of K , exceptional K -orbits arise from elements in Γ_χ of finite order. Consider now the right regular representation π of K on $L^2(\Gamma_\chi \backslash G)$ together with the corresponding Peter-Weyl decomposition (3.8), and suppose that P commutes with π and the Hecke operators $\mathcal{T}_{\Gamma\beta\Gamma}^\chi$, which commute with the right regular K -representation as well. To describe the growth of simultaneous eigenfunctions of P and $\mathcal{T}_{\Gamma\beta\Gamma}^\chi$ in the σ -isotypic component

$$(4.1) \quad L^2_{\sigma, \chi}(\Gamma \backslash G) := L^2_\sigma(\Gamma_\chi \backslash G) \cap L^2_\chi(\Gamma \backslash G), \quad \chi \in \widehat{\Gamma}, \sigma \in \widehat{K},$$

of $L^2_\chi(\Gamma \backslash G)$, we are interested in spectral asymptotics for the Schwartz kernel of the operator

$$\Pi_\sigma \circ T_{h(\chi)} \circ \mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ Q \circ T_{h(\chi)} \circ \Pi_\sigma = \mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ Q \circ \Pi_\sigma : L^2(\Gamma_\chi \backslash G) \longrightarrow L^2(\Gamma_\chi \backslash G).$$

Let $\{\phi_j\}_{j \geq 0}$ be an orthonormal basis of $L^2(\Gamma_\chi \backslash G)$ consisting of simultaneous eigenfunctions of P and $\mathcal{T}_{\Gamma\beta\Gamma}^\chi$ compatible with the decompositions (2.9) and (3.8). Applying the Hecke operators $\mathcal{T}_{\Gamma\beta\Gamma}^\chi$ to the spectral expansion (3.9) of the spectral function of $Q \circ \Pi_\sigma$ yields

$$(4.2) \quad \sum_{\mu_j \leq \mu, \phi_j \in L^2_\sigma(M)} \lambda_j(\chi, \beta) \phi_j(x) \overline{\phi_j(y)} = [\Gamma : \Gamma_\chi]^{-1} \sum_{\substack{\mu_j \leq \mu, \phi_j \in L^2_\sigma(\Gamma_\chi \backslash G) \\ \alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma}} \overline{\chi(\alpha)} \phi_j(\alpha \cdot x) \overline{\phi_j(y)}.$$

In order to get an asymptotic description of the right-hand side of (4.2), we consider the composition $\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma$ with the approximate spectral projection \tilde{s}_μ . Clearly, its Schwartz kernel can be written as

$$(4.3) \quad K_{\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, y) := \frac{1}{[\Gamma : \Gamma_\chi]} \sum_{\alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma} \overline{\chi(\alpha)} K_{\tilde{s}_\mu \circ \Pi_\sigma}(\alpha \cdot x, y),$$

$K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y)$ being as in (3.10), and by Remark 3.4 (1) one immediately deduces

$$K_{\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, x) \ll \frac{|\Gamma_\chi \backslash \Gamma\beta\Gamma|}{[\Gamma : \Gamma_\chi]} d_\sigma \mu^{d - \dim K - 1} \sup_{u \leq \lfloor \dim K/2+1 \rfloor} \|D^u \sigma\|_\infty$$

uniformly in x , or if $K = T$ is a torus,

$$K_{\mathcal{T}_{\Gamma\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, x) \ll \frac{|\Gamma_\chi \backslash \Gamma\beta\Gamma|}{[\Gamma : \Gamma_\chi]} \mu^{d - \dim K - 1}, \quad \sigma \in \mathcal{V}_\mu.$$

Nevertheless, to obtain subconvex bounds, more subtle estimates are necessary that take into account the caustic behaviour of the kernels $K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y)$ near the diagonal.

Proposition 4.1. *Let $\chi \in \widehat{\Gamma}$, $\sigma \in \widehat{K}$, $x \in \Gamma_\chi \backslash G$, and $\tilde{N} \in \mathbb{N}$ be arbitrary. Assume that the cospheres $S_x^*(\Gamma_\chi \backslash G)$ are strictly convex. Then, for any $0 < \delta \ll 1$,*

$$K_{\mathcal{T}_{\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, x) \ll \frac{d_\sigma}{[\Gamma : \Gamma_\chi]} \sup_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_\infty \left\{ \mu^{d - \dim K - 1} M(x, \beta, \delta) + \mu^{\frac{d - \dim K - 1}{2}} \int_\delta^C s^{-\frac{1}{2}} dM(s) \right\}$$

uniformly in $x \in \Gamma_\chi \backslash G$ for some sufficiently large¹⁰ constant $C > 0$ up to terms of order $O(|\Gamma_\chi \backslash \Gamma \beta \Gamma| \mu^{-\infty})$, where we set

$$(4.4) \quad M(\delta) := M(x, \beta, \delta) := \# \{ \alpha \in \Gamma_\chi \backslash \Gamma \beta \Gamma : \text{dist}(xK, \alpha \cdot xK)^{d - \dim K - 1} < \delta \}$$

and $\text{dist}(xK, \alpha \cdot xK) \equiv \text{dist}(\Gamma xK, \Gamma \alpha xK)$. If $K = T$ is a torus and $\sigma \in \mathcal{V}_\mu$, the better estimate

$$K_{\mathcal{T}_{\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, x) \ll \frac{1}{[\Gamma : \Gamma_\chi]} \left\{ \mu^{d - \dim K - 1} M(x, \beta, \delta) + \mu^{\frac{d - \dim K - 1}{2}} \int_\delta^C s^{-\frac{1}{2}} dM(s) \right\}$$

holds.

Proof. By Proposition 3.3 one deduces as $\mu \rightarrow +\infty$

$$\begin{aligned} & \sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma \beta \Gamma, \\ \text{dist}(xK, \alpha \cdot xK)^{d - \dim K - 1} \geq \delta}} \overline{\chi(\alpha)} K_{\tilde{s}_\mu \circ \Pi_\sigma}(\alpha \cdot x, x) \\ & \ll d_\sigma \mu^{(d - \dim K - 1)/2} \sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma \beta \Gamma, \\ \text{dist}(xK, \alpha \cdot xK)^{d - \dim K - 1} \geq \delta}} \text{dist}(xK, \alpha \cdot xK)^{-(d - \dim K - 1)/2} \sup_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_\infty. \end{aligned}$$

Furthermore, by Remark 3.4 (1) one has the uniform bound

$$K_{\tilde{s}_\mu \circ \Pi_\sigma}(x, y) = O\left(d_\sigma \mu^{d - \dim K - 1} \sup_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_\infty\right).$$

In view of (4.3) we therefore obtain

$$\begin{aligned} [\Gamma : \Gamma_\chi] K_{\mathcal{T}_{\beta\Gamma}^\chi \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, x) &= \sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma \beta \Gamma, \\ \text{dist}(xK, \alpha \cdot xK)^{d - \dim K - 1} < \delta}} \overline{\chi(\alpha)} K_{\tilde{s}_\mu \circ \Pi_\sigma}(\alpha \cdot x, x) \\ &+ \sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma \beta \Gamma, \\ \text{dist}(xK, \alpha \cdot xK)^{d - \dim K - 1} \geq \delta}} \overline{\chi(\alpha)} K_{\tilde{s}_\mu \circ \Pi_\sigma}(\alpha \cdot x, x) \\ &\ll d_\sigma \mu^{d - \dim K - 1} \sup_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_\infty M(x, \beta, \delta) \\ &+ d_\sigma \mu^{(d - \dim K - 1)/2} \sup_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_\infty \int_\delta^C s^{-1/2} dM(s) \end{aligned}$$

by definition of the Stieltjes integral. In case that $K = T$ is a torus, corresponding better estimates hold, and the assertion follows. \square

5. SUBCONVEX BOUNDS ON $\Gamma_\chi \backslash \text{SL}(2, \mathbb{R})$ FOR ARITHMETIC CONGRUENCE LATTICES

In this section, we shall use the kernel asymptotics derived in the previous section to prove subconvex bounds on the quotient $\Gamma \backslash \text{SL}(2, \mathbb{R})$, where Γ is an arithmetic congruence lattice as considered by Iwaniec and Sarnak [27].

¹⁰It suffices to take C larger than the diameter of any component of $\Gamma_\chi \backslash G/K$.

5.1. Arithmetic congruence lattices. To introduce the setting, let A be an indefinite quaternion division algebra over \mathbb{Q} . Hence, there exist two square-free integers a and b such that $a > 0$ and

$$A = \mathbb{Q} + \mathbb{Q}\omega + \mathbb{Q}\Omega + \mathbb{Q}\omega\Omega$$

where $\omega^2 = a$, $\Omega^2 = b$, and $\omega\Omega = -\Omega\omega$. For each element $x = x_0 + x_1\omega + x_2\Omega + x_3\omega\Omega$, its conjugate is defined as $\bar{x} := x_0 - x_1\omega - x_2\Omega - x_3\omega\Omega$, and its trace and norm as $\text{tr}(x) := x + \bar{x}$ and $N(x) := x\bar{x}$, respectively. Let \mathcal{R} be an *order* of A , that is, \mathcal{R} is a finitely generated free \mathbb{Z} -module, \mathcal{R} is a subring of A containing 1, and $\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Q} = A$. For each prime number p , set $A_p := A \otimes \mathbb{Q}_p$ and $\mathcal{R}_p := \mathcal{R} \otimes \mathbb{Z}_p$. Let d_A be the product of all primes p such that A_p is a division algebra. Then d_A is called the *discriminant* of A . d_A is greater than 1 and square free, and A_p is isomorphic to $M(2, \mathbb{Q}_p)$ if p does not divide q . Throughout this section, we assume that \mathcal{R} is an *Eichler order of level L* , where L is a natural number such that $(d_A, L) = 1$. Hence, \mathcal{R} satisfies

- (1) \mathcal{R}_p is the maximal order of A_p if p divides d_A , or
- (2) \mathcal{R}_p is conjugate to $\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ L\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$.

Note that any Eichler order is included in a maximal order. Particularly, \mathcal{R} is maximal when $L = 1$. Now, choose an embedding $\theta : A \rightarrow M(2, \mathbb{Q}(\sqrt{a})) \subset M(2, \mathbb{R})$ by setting

$$\theta(x_0 + x_1\omega + x_2\Omega + x_3\omega\Omega) = \begin{pmatrix} x_0 - x_1\omega & x_2 + x_3\omega \\ b(x_2 - x_3\omega) & x_0 + x_1\omega \end{pmatrix}.$$

For each natural number $n \in \mathbb{N}_*$, we set

$$\mathcal{R}(n) := \{\alpha \in \mathcal{R} \mid N(\alpha) = n\}.$$

Then $\Gamma := \theta(\mathcal{R}(1))$ becomes a cocompact lattice of $G := \text{SL}(2, \mathbb{R})$. Note that $\text{tr}(x) = \text{tr}(\theta(x))$ and $N(x) = \det(\theta(x))$ hold for any x in A . In what follows, we identify A with $\theta(A)$. Especially, we will often use Γ instead of $\mathcal{R}(1)$.

Next, let χ be a Dirichlet character on $(\mathbb{Z}/L\mathbb{Z})^\times$. In view of the product isomorphism $(\mathbb{Z}/L\mathbb{Z})^\times \cong \prod_{p|L} (\mathbb{Z}_p/L\mathbb{Z}_p)^\times$ given by the diagonal embedding $a \mapsto (a)_p$, a character χ_p can be defined on $(\mathbb{Z}_p/L\mathbb{Z}_p)^\times$ by restriction of χ to each factor. Set

$$\Xi_{\mathcal{R}} := \{\alpha \in \mathcal{R} \mid N(\alpha) > 0, (N(\alpha), L) = 1\} \quad \text{and} \quad \mathcal{R}_L := \{(x_p)_{p|L} \mid x_p \in \mathcal{R}_p, N(x_p) \notin p\mathbb{Z}_p\},$$

and define a character χ_L on the semigroup \mathcal{R}_L by

$$\mathcal{R}_L \ni \left(\begin{pmatrix} a_p & b_p \\ Lc_p & d_p \end{pmatrix} \right)_{p|L} \mapsto \prod_{p|L} \overline{\chi_p(a_p)} \in \mathbb{C}.$$

Composing χ_L and the diagonal embedding $\Xi_{\mathcal{R}} \subset \mathcal{R}_L$, we obtain a character χ on the sub-semigroup $\Xi_{\mathcal{R}}$ of A^\times . By the inclusion $\Gamma \subset \Xi_{\mathcal{R}}$, χ becomes a character on Γ which is called a *Nebentypus character*. Notice that there are only finitely many Nebentypus characters for each fixed Eichler order \mathcal{R} . Now, because Γ and $\alpha^{-1}\Gamma\alpha$ are commensurable [38, Proposition 4.1], an inclusion map $\psi : \mathcal{R}(n) \rightarrow C(\Gamma)$ is given by

$$\psi(\alpha) = n^{1/2}\alpha, \quad \alpha \in \mathcal{R}(n).$$

Since the subset $\mathcal{R}(n)$ is left and right Γ -invariant, and it is known that $\Gamma \backslash \mathcal{R}(n)$ is finite [36, Section 5.3], we can introduce the *Hecke operators*

$$(T_n^\chi f)(x) := \sum_{\alpha \in \Gamma \backslash \mathcal{R}(n)} \overline{\chi(\alpha)} f(\alpha \cdot x), \quad f \in L_\chi^2(\Gamma \backslash G).$$

Indeed, since $\psi(\mathcal{R}(n))$ is given as a disjoint union of double cosets $\sqcup_j \Gamma \alpha_j \Gamma$, the operator T_n^χ coincides with the sum $\sum_j T_{\Gamma \alpha_j \Gamma}^\chi$ of Hecke operators defined in (2.10). In particular, we have also the Hecke operators

$$\mathcal{T}_n^\chi : L^2(\Gamma_\chi \backslash G) \longrightarrow L^2(\Gamma_\chi \backslash G), \quad (\mathcal{T}_n^\chi f)(x) := \frac{1}{[\Gamma : \Gamma_\chi]} \sum_{\alpha \in \Gamma_\chi \backslash \mathcal{R}(n)} \overline{\chi(\alpha)} f(\alpha \cdot x),$$

compare (2.5) and Diagram 2.13. Furthermore, set

$$q := d_A L.$$

For natural numbers n such that $(n, q) = 1$, the T_n^χ are self-dual, commute with the Beltrami–Laplace operator Δ on G , and satisfy the composition rule [36, Section 5.3]

$$(5.1) \quad T_n^\chi T_m^\chi = \sum_{d|(m,n)} d \cdot \chi(d) T_{nm/d^2}^\chi.$$

Next, recall that the group $\mathrm{GL}(2, \mathbb{R})^+ := \{x \in M(2, \mathbb{R}) \mid \det(x) > 0\}$ acts transitively on the upper half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ by fractional transformations¹¹

$$g \cdot z = \frac{az + b}{cz + d}, \quad z \in \mathbb{H}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

by which \mathbb{H} becomes isomorphic to the homogeneous space G/K , where $K := \mathrm{SO}(2)$, and we define

$$(5.2) \quad j(g, z) := (cz + d)(\det g)^{1/2}.$$

In what follows, we shall identify $\Gamma_\chi \backslash G/K \simeq \Gamma_\chi \backslash \mathbb{H}$ with a subset in \mathbb{H} , and endow it with the standard hyperbolic distance on \mathbb{H} given by [25, Section 1.1]

$$\mathrm{dist}_{\mathbb{H}}(z, w) := \mathrm{arcosh}(1 + u(z, w)/2) = \ln \left(\frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|} \right) \approx |z - w|,$$

where

$$(5.3) \quad u(z, w) := \frac{|z - w|^2}{\mathrm{Im} z \mathrm{Im} w}, \quad z, w \in \mathbb{H}.$$

By this, $\Gamma_\chi \backslash \mathbb{H}$ becomes a compact hyperbolic surface. Note that $\mathrm{dist}_{\mathbb{H}}$ agrees with the distance function dist introduced at the beginning of Section 2. Furthermore, one has the following important result of Iwaniec and Sarnak.

Lemma 5.1. *For arbitrary $\varepsilon > 0$ one has*

$$\#\{\alpha \in \mathcal{R}(n) : u(z, \alpha \cdot z) < \delta\} \ll_\varepsilon (\delta + \delta^{1/4}) n^{1+\varepsilon} + n^\varepsilon$$

uniformly in z .

Proof. See [27, Lemma 1.3]. □

5.2. Equivariant subconvex bounds. With the notation of the previous section, we shall first derive subconvex bounds for Hecke–Maass forms in $L^2_\chi(\Gamma \backslash G)$ in the eigenvalue and isotypic aspect for the Beltrami–Laplace operator Δ . For this, let $\{\phi_j\}_{j \geq 0}$ be an orthonormal basis of $L^2(\Gamma_\chi \backslash G)$ consisting of simultaneous eigenfunctions of $P_0 = \Delta$ and \mathcal{T}_n^χ compatible with the decompositions (2.9) and (3.8), where $X = G$ and $M = \Gamma_\chi \backslash G$, respectively, so that with $(n, q) = 1$

$$(5.4) \quad \Delta \phi_j = \lambda_j \phi_j, \quad \mathcal{T}_n^\chi \phi_j = \lambda_j(n) \phi_j.$$

Note that $\lambda_j(n) = 0$ if $\phi_j \notin L^2_\chi(\Gamma \backslash G)$. Further, let $\sigma \in \widehat{K}$ be a fixed K -type, and $L^2_{\sigma, \chi}(\Gamma \backslash G)$ be defined as in (4.1). When σ is trivial, the space $L^2_{\sigma, \chi}(\Gamma \backslash G)$ can be identified with $L^2_\chi(\Gamma \backslash G/K)$. In what follows, we shall make the identification $\mathrm{SO}(2) \simeq S^1 \subset \mathbb{C}$, so that the characters of K are given by the exponentials

$$\sigma_l(e^{i\theta}) := e^{il\theta}, \quad \theta \in [0, 2\pi), \quad l \in \mathbb{Z}.$$

Since all irreducible representations of K are one-dimensional, Proposition 4.1 yields for any $x \in \Gamma_\chi \backslash G$ and σ_l with $|l| \ll \mu / \log \mu$ the estimate

$$K_{\mathcal{T}_n^\chi \circ \bar{s}_\mu \circ \Pi_{\sigma_l}}(x, x) \ll \frac{1}{[\Gamma : \Gamma_\chi]} \left[\mu M(x, n, \delta) + \mu^{\frac{1}{2}} \int_\delta^C s^{-\frac{1}{2}} dM(s) \right]$$

¹¹ Note that the center $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $a \in \mathbb{R}^*$, acts trivially on \mathbb{H} .

up to terms of order $O(|\Gamma_\chi \backslash \mathcal{R}(n)| \mu^{-\infty})$, where we set

$$M(\delta) := M(x, n, \delta) := \#\{\alpha \in \Gamma_\chi \backslash \mathcal{R}(n) : u(xK, \alpha \cdot xK) < \delta^2\},$$

regarding $xK \in \Gamma_\chi \backslash G/K \simeq \Gamma_\chi \backslash \mathbb{H}$ as an element in \mathbb{C} , and took into account that for suitable constants $c_1, c_2 > 0$

$$(5.5) \quad c_1 u(z, w) \leq \text{dist}(z, w)^2 \equiv \text{dist}_{\mathbb{H}}(z, w)^2 \leq c_2 u(z, w), \quad z, w \in \Gamma_\chi \backslash \mathbb{H}.$$

In order to derive a uniform bound for $K_{\mathcal{T}_n^\chi \circ \tilde{s}_\mu \circ \Pi_{\sigma_l}}(x, x)$, note that by Lemma 5.1 one has with $N(s) := s^{-1/2}$ and $\delta = \mu^{-1}$

$$\begin{aligned} \int_\delta^C s^{-1/2} dM(s) &= \underbrace{N(C)M(C)}_{\ll_\varepsilon n^{1+\varepsilon}} - \underbrace{N(\delta)M(\delta)}_{\ll_\varepsilon \delta^{-1/2}[(\delta^2 + \delta^{1/2})n^{1+\varepsilon} + n^\varepsilon]} - \underbrace{\int_\delta^C M(s)N'(s) ds}_{\ll_\varepsilon (s^{3/2} + \log s)n^{1+\varepsilon} + s^{-1/2}n^\varepsilon|_\delta^C} \\ &\ll_\varepsilon n^{1+\varepsilon} + \mu^{1/2}n^\varepsilon + n^{1+\varepsilon} \log \mu \ll_\varepsilon (\mu^{1/2} + n \log \mu)n^\varepsilon, \end{aligned}$$

Taking everything together we have shown

Theorem 5.2. *For any $n \in \mathbb{N}_*$, $\mu > 0$, and σ_l with $|l| \ll \mu / \log \mu$ the uniform bound*

$$K_{\mathcal{T}_n^\chi \circ \tilde{s}_\mu \circ \Pi_{\sigma_l}}(x, x) \ll_\varepsilon (\mu + n\mu^{1/2} \log \mu) n^\varepsilon$$

holds up to terms of order $O(n^{1+\varepsilon} \mu^{-\infty})$, where $x \in \Gamma_\chi \backslash G$, and $\chi \in \widehat{\Gamma}$ is a Nebentypus character. \square

Remark 5.3. The previous theorem is the non-spherical analogon of [27, Lemma 1.2]. Note that the bounds for the point pair invariants on \mathbb{H} used by Iwaniec and Sarnak in order to show [27, Lemma 1.2] are better than ours by a factor $(1 + u(\alpha \cdot z, z))^{-5/4}$ in the Stieltjes integral, but the lattice point counting function considered by them is unbounded, while ours is a priori bounded.

Following the original approach of Iwaniec and Sarnak, we shall now make use of arithmetic amplification to deduce from Theorem 5.2 equivariant subconvex bounds. Since we will later choose $n \ll \mu^A$ for some $A \in \mathbb{N}$, we can neglect the contributions of order $O(n^{1+\varepsilon} \mu^{-\infty})$ in the following. Thus, let $\chi \in \widehat{\Gamma}$, $\sigma_l \in \widehat{K}$ be arbitrary, and $\{\phi_j\}_{j \in \mathbb{N}}$ as in (5.4). Writing $\eta_j(n) := \lambda_j(n) / \sqrt{n}$ we deduce with (3.10), (4.3), (5.1), and (5.4) that¹²

$$\begin{aligned} &\sum_{j \geq 0, \phi_j \in L_{\sigma_l}^2(\Gamma_\chi \backslash G)} \varrho(\mu - \mu_j) \phi_j(x) \overline{\phi_j(y)} \eta_j(m) \eta_j(n) \\ &= \sum_{d|(n,m)} \sum_{j \geq 0, \phi_j \in L_{\sigma_l}^2(\Gamma_\chi \backslash G)} \varrho(\mu - \mu_j) \phi_j(x) \overline{\phi_j(y)} \chi(d) \eta_j\left(\frac{nm}{d^2}\right) = \sum_{d|(n,m)} \frac{d\chi(d)}{\sqrt{nm}} K_{\mathcal{T}_{nm/d^2}^\chi \circ \tilde{s}_\mu \circ \Pi_{\sigma_l}}(x, y). \end{aligned}$$

If one replaces μ by $\mu \log \mu$ in Theorem 5.2 one obtains for any $\sigma_l \in \widehat{K}$

$$K_{\mathcal{T}_n^\chi \circ \tilde{s}_\mu \circ \Pi_{\sigma_l}}(x, x) \ll_\varepsilon \log \mu (\mu + n\mu^{1/2} \log \mu) n^\varepsilon,$$

yielding for arbitrary $N \in \mathbb{N}_*$ and σ_l

$$\begin{aligned} &\sum_{j \geq 0, \phi_j \in L_{\sigma_l}^2(\Gamma_\chi \backslash G)} \varrho(\mu - \mu_j) |\phi_j(x)|^2 \left| \sum_{n \leq N} z_n \eta_j(n) \right|^2 = \sum_{n, m \leq N} \sum_{d|(n,m)} \frac{d\chi(d)}{\sqrt{nm}} z_n \overline{z_m} K_{\mathcal{T}_{nm/d^2}^\chi \circ \tilde{s}_\mu \circ \Pi_{\sigma_l}}(x, x) \\ &\ll_\varepsilon N^\varepsilon \mu^\varepsilon \sum_{n, m \leq N} \sum_{d|(n,m)} \frac{d}{\sqrt{nm}} |z_n z_m| \left(\mu + \frac{nm}{d^2} \mu^{1/2} \right), \end{aligned}$$

¹²Since the T_n^χ are adjoint operators for all n with $(n, q) = 1$, one has $\eta_j(n) = \overline{\eta_j(n)}$, compare [36, Theorem 5.3.8]. More generally, $(T_{\Gamma_\alpha \Gamma}^\chi)^* = T_{\Gamma_\alpha' \Gamma}^\chi$, where $\alpha' = \det(\alpha) \alpha^{-1}$.

since $|\chi(d)| = 1$, where $z_n \in \mathbb{C}$ are arbitrary complex numbers. A simple computation then gives

$$(5.6) \quad \sum_{j \geq 0, \phi_j \in L_{\sigma_l}^2(\Gamma_\chi \backslash G)} \varrho(\mu - \mu_j) |\phi_j(x)|^2 \left| \sum_{n \leq N} z_n \eta_j(n) \right|^2 \ll_\varepsilon N^\varepsilon \mu^\varepsilon \left[\mu \sum_{n \leq N} |z_n|^2 + N \mu^{1/2} \left(\sum_{n \leq N} |z_n| \right)^2 \right].$$

We thus arrive at

Proposition 5.4. *For any $\mu > 0$, $\sigma_l \in \widehat{K}$, $\chi \in \widehat{\Gamma}$, and $N \in \mathbb{N}_*$ one has the estimate*

$$\sum_{\substack{\mu \leq \sqrt{\lambda_j} \leq \mu+1, \\ \phi_j \in L_{\sigma_l}^2(\Gamma_\chi \backslash G)}} |\phi_j(x)|^2 \left| \sum_{n \leq N} z_n \eta_j(n) \right|^2 \ll_\varepsilon N^\varepsilon \mu^\varepsilon \left[\mu \sum_{n \leq N} |z_n|^2 + N \mu^{1/2} \left(\sum_{n \leq N} |z_n| \right)^2 \right]$$

uniformly in $x \in \Gamma_\chi \backslash G$.

Proof. As explained at the end of Section 3, the test function $\varrho \in \mathcal{S}(\mathbb{R}, \mathbb{R}^+)$ can be chosen such that $\varrho > 0$ on $[-1, 1]$. The proposition now follows from (5.6) and the estimate

$$\begin{aligned} \sum_{\substack{\mu \leq \sqrt{\lambda_j} \leq \mu+1, \\ \phi_j \in L_{\sigma_l}^2(\Gamma_\chi \backslash G)}} |\phi_j(x)|^2 \left| \sum_{n \leq N} z_n \eta_j(n) \right|^2 \cdot \min \{ \varrho(\mu) \mid \mu \in [-1, 1] \} \\ \leq \sum_{j \geq 0, \phi_j \in L_{\sigma_l}^2(\Gamma_\chi \backslash G)} \varrho(\mu - \mu_j) |\phi_j(x)|^2 \left| \sum_{n \leq N} z_n \eta_j(n) \right|^2. \end{aligned}$$

□

Next, one proceeds as follows. Let $j_0 \geq 0$ be fixed such that $\phi_{j_0} \in L_{\sigma_l, \chi}^2(\Gamma \backslash G)$, and consider the *amplifier*

$$(5.7) \quad z_n := \begin{cases} \eta_{j_0}(p), & n = p \leq \sqrt{N}, \\ -1 & n = p^2 \leq N, \\ 0, & \text{otherwise,} \end{cases}$$

where p is a prime not dividing q . Note that (2.13) and (5.1) imply

$$\eta_j(p)^2 - \eta_j(p^2) = \begin{cases} 1 & \text{if } \phi_j \in L_\chi^2(\Gamma \backslash G), \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\left| \sum_{n \leq N} z_n \eta_{j_0}(n) \right| = \sum_{p \leq \sqrt{N}, p \nmid q} 1 = O(\sqrt{N} / \log N^{1/2}),$$

by the Prime Number Theorem. Writing $\lambda_j = 1/4 + r_j^2$ and taking $\mu = r_{j_0}$ Proposition 5.4 then gives

$$(5.8) \quad |\phi_{j_0}(x)|^2 \ll_\varepsilon N^{\varepsilon-1} r_{j_0}^\varepsilon \left[r_{j_0} \left(\sum_{p \leq \sqrt{N}} |\eta_{j_0}(p)|^2 + \sqrt{N} \right) + N r_{j_0}^{1/2} \left(\sum_{p \leq \sqrt{N}} |\eta_{j_0}(p)| + N^{1/2} \right)^2 \right].$$

As a next step, note that Jacquet-Langlands correspondence [29] and the study of Rankin-Selberg convolutions ([26, Theorem 8.3] and [16, Proposition 19.6]) imply for any $j \in \mathbb{N}$ with $\phi_j \in L_\chi^2(\Gamma \backslash G)$ the bound

$$(5.9) \quad \sum_{n \leq N} |\eta_j(n)|^2 \ll_\varepsilon N^{1+\varepsilon} r_j^\varepsilon,$$

where n moves over natural numbers prime to q . Here we used the facts that the Strong Multiplicity One Theorem holds for $\mathrm{GL}(2)$ and each automorphic representation factors as a tensor product of local representations. Consequently, with (5.9) and Cauchy's inequality one deduces

$$|\phi_{j_0}(x)|^2 \ll_\varepsilon N^\varepsilon r_{j_0}^\varepsilon (r_{j_0} N^{-1/2} + r_{j_0}^{1/2} N).$$

Choosing $N = r_{j_0}^{1/3}$ finally gives

$$|\phi_{j_0}(x)|^2 \ll_\varepsilon r_{j_0}^{\frac{5}{6} + \varepsilon}$$

uniformly in $x \in \Gamma_\chi \backslash G$. Thus, we have shown our first main result.

Theorem 5.5. *Let $G = \mathrm{SL}(2, \mathbb{R})$ and Γ be a congruence arithmetic lattice in G . Let further $\chi \in \widehat{\Gamma}$ be a Nebentypus character. Then, for any Hecke–Maass form $\phi_j \in L_\chi^2(\Gamma \backslash G)$ with $\Delta \phi_j = \lambda_j \phi_j$ and $\|\phi_j\|_{L^2} = 1$ one has*

$$\|\phi_j\|_\infty \ll_\varepsilon \lambda_j^{\frac{5}{24} + \varepsilon}$$

for any $\varepsilon > 0$. □

For trivial σ_l and χ , Theorem 5.5 is due to Iwaniec–Sarnak [27, Theorem 0.1]. In fact, their method can also be used for non-trivial χ , as was done in [2, Section 10] for non-compact arithmetic surfaces. Thus, we recover

Corollary 5.6 (Iwaniec–Sarnak). *For any Hecke–Maass form $\phi_j \in L_\chi^2(\Gamma \backslash \mathbb{H})$ with $\|\phi_j\|_{L^2} = 1$ and Beltrami–Laplace eigenvalue λ_j one has*

$$\|\phi_j\|_\infty \ll_\varepsilon \lambda_j^{\frac{5}{24} + \varepsilon}$$

for any $\varepsilon > 0$.

Proof. If $\sigma_l = \mathrm{id}$ is trivial, $L_{\sigma_l, \chi}^2(\Gamma \backslash G) \simeq L_\chi^2(\Gamma \backslash \mathbb{H})$, and the assertion follows from the previous theorem. Note that since all K -orbits in G have the same volume, each eigenfunction of the Beltrami–Laplace operator on $\mathbb{H} \simeq G/K$ lifts to a unique K -invariant eigenfunction of the Beltrami–Laplace operator on G . □

5.3. Automorphic forms on $\mathrm{SL}(2, \mathbb{R})$ and representation-theoretic interpretation. In what follows, we would like to discuss our results within the theory of automorphic forms and their representation-theoretic meaning. For this, let us first recall the concept of an automorphic form on $G = \mathrm{SL}(2, \mathbb{R})$ for a discrete co-compact subgroup Γ , compare [8, Section 5].

Definition 5.7. *A smooth function $f : G \rightarrow \mathbb{C}$ is called an automorphic form on G for Γ iff:*

- (A1) $f(\gamma g) = f(g)$ for all $\gamma \in \Gamma$ and $g \in G$,
- (A2) f is K -finite on the right, where $K = \mathrm{SO}(2)$,
- (A3) f is \mathcal{Z} -finite, where \mathcal{Z} denotes the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ of the complexification of \mathfrak{g} .

Note that (A2) means that f is a finite sum of functions f_l belonging to a specific K -type σ_l , while (A3) is equivalent to the existence of a polynomial $p(\mathcal{C})$ in the Casimir operator $\mathcal{C} = dR(\Omega)$ that annihilates f , the notation being as in Section 2. Now, $\mathfrak{g} = \{X \in M(2, \mathbb{R}) \mid \mathrm{tr}(X) = 0\}$, while a Cartan involution is given by $-{}^tX$. With respect to the basis

$$X'_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X'_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y'_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

of $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, the modified Killing form $\langle \cdot, \cdot \rangle_\theta$ is represented by the matrix

$$\frac{1}{8} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consequently, a corresponding orthonormal basis of \mathfrak{g} is given by $X_1 := X'_1/2\sqrt{2}$, $X_2 := X'_2/2\sqrt{2}$, $Y_1 := Y'_1/2\sqrt{2}$, so that the Casimir element reads

$$\Omega = X_1^2 + X_2^2 - Y_1^2 \equiv \frac{1}{8}\text{id},$$

compare (2.1). Note that our normalization of \mathcal{C} differs from the one in [8, p. 20], where $\Omega \equiv \frac{1}{2}\text{id}$. Writing $p(\mathcal{C}) = \prod_i (\mathcal{C} - \mu_i)$ and $\mu_{\sigma_l} = l^2/8$ for the eigenvalue of $dR(\Omega_K)$ on the σ_l -isotypic component one sees that

$$p(\mathcal{C})f_l = \prod_i (2dR(\Omega_K) - \Delta - \mu_i)f_l = \prod_i (2\mu_{\sigma_l} - \Delta - \mu_i)f_l =: q_l(\Delta)f_l,$$

where we took into account (2.2). Thus, $p(\mathcal{C})f = 0$ iff $q_l(\Delta)f_l = 0$ for all l , by orthogonality. Since $q_l(\Delta)$ is an elliptic differential operator of the same order than $p(\mathcal{C})$, and any subspace defined by a K -type and a Casimir eigenvalue is finite dimensional by Harish-Chandra's theorem [9, Theorem 1.7], we see that f is essentially given by a finite sum of Hecke–Maass forms in the sense of this paper.

To interpret our results in terms of the representation theory of G , let us first notice that, since $-I_2$ belongs to Γ , one has

$$\mathbb{L}_{\sigma_l, \chi}^2(\Gamma \backslash G) = \{0\} \quad \text{if} \quad \sigma_l(-I_2) \neq \chi(-I_2).$$

Hence, in case that $\mathbb{L}_{\sigma_l, \chi}^2(\Gamma \backslash G) \neq \{0\}$, l must be even if $\chi(-I_2) = 1$, and odd otherwise. Now, according to the decomposition (2.3) the following irreducible unitary representations of G can appear in $\mathbb{L}_{\sigma_l, \chi}^2(\Gamma \backslash G)$, see [8, Section 15]:

- (1) If $l = 0$,
 - (a) the trivial representation,
 - (b) the unitary principal series $H(0, s)$ with $s \in i\mathbb{R}_{\geq 0}$,
 - (c) the complementary series $I(0, s)$ with $s \in (0, 1)$
 can appear.
- (2) If l is even and $l \neq 0$,
 - (a) the discrete series D_s with $s \in \mathbb{Z} - \{0\}$, $|s| < |l|$, $\text{sgn}(s) = \text{sgn}(l)$, s odd,
 - (b) the unitary principal series $H(0, s)$ with $s \in i\mathbb{R}_{\geq 0}$,
 - (c) the complementary series $I(0, s)$ with $s \in (0, 1)$
 can appear.
- (3) If l is odd,
 - (a) the discrete series D_s with $s \in \mathbb{Z} - \{0\}$, $|s| < |l|$, $\text{sgn}(s) = \text{sgn}(l)$, s even,
 - (b) the unitary principal series $H(1, s)$ with $s \in i\mathbb{R}_{> 0}$,
 - (c) the limits of discrete series $D_{+,0}$ (resp. $D_{-,0}$) with $l > 0$ (resp. $l < 0$)
 can appear.

Note that in each of the above unitary representations the σ_l -isotypic component is 1-dimensional. By the above list, one sees that representations occurring in $\mathbb{L}_{\sigma_l, \chi}^2(\Gamma \backslash G)$ are in general different from the spherical case $L^2(\Gamma \backslash \mathbb{H})$. Hence, Theorem 5.5 implies subconvex bounds for new classes of automorphic representations, in particular for the discrete series D_s and their limits $D_{\pm,0}$, as well as the principal series $H(1, s)$.

To rephrase our results in terms of the spectrum of the Casimir operator \mathcal{C} , note that by (2.2) we have with $\mu_{\sigma_l} = l^2/8$

$$(5.10) \quad \Delta\phi_j = \lambda_j\phi_j = (-\mu_j + l^2/4)\phi_j, \quad \phi_j \in \mathbb{L}_{\sigma_l, \chi}^2(\Gamma \backslash G),$$

μ_j being the Casimir eigenvalue of ϕ_j . Now, in the cases relevant to us the Casimir eigenvalues can take the following values [8, p. 158 and p. 163]:¹³

¹³Note that our normalization of the Casimir operator differs from the one in [8] by a factor 1/4.

- (1) On $(D_s)^\infty$, $\mathcal{C} \equiv (s^2 - 1) \text{id} / 8$, where $s \in \mathbb{Z} - \{0\}$. For $|s| < |l|$ and $\text{sgn}(s) = \text{sgn}(l)$ one has $0 < s^2 - 1 < l^2 - 1$.
- (2) On $H(0, s)^\infty$ or $H(1, s)^\infty$, $\mathcal{C} \equiv (s^2 - 1) \text{id} / 8$. For $s \in i\mathbb{R}_{\geq 0}$ one has $s^2 - 1 \leq -1$.
- (3) On $I(0, s)^\infty$, $\mathcal{C} \equiv (s^2 - 1) \text{id} / 8$. For $s \in (0, 1)$ one has $-1 < s^2 - 1 < 0$.
- (4) On $(D_{+,0})^\infty$ or $(D_{-,0})^\infty$, $\mathcal{C} \equiv -\text{id} / 8$.

Here H^∞ denotes the subspace of differentiable vectors in a Hilbert representation H . Consequently, the subconvex bound in Theorem 5.5 can be restated as follows.

Theorem 5.8. *Let $G = \text{SL}(2, \mathbb{R})$, $K = \text{SO}(2)$, and Γ be a congruence arithmetic lattice in G . Let further $\sigma_l \in \widehat{K}$ and $\chi \in \widehat{\Gamma}$ be a Nebentypus character. Then, for any Hecke eigenform $\phi \in L^2_{\sigma_l, \chi}(\Gamma \backslash G)$ satisfying $\|\phi\|_{L^2} = 1$ and $\mathcal{C}\phi = \frac{s^2-1}{8}\phi$ one has*

$$\|\phi\|_\infty \ll_\varepsilon (1 - s^2 + 2l^2)^{\frac{5}{24} + \varepsilon}$$

for any $\varepsilon > 0$. □

Classically, an *automorphic form of weight $l \in \mathbb{N}$ and Nebentypus character χ* was first introduced as a holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ satisfying

$$f(\gamma \cdot z) = \chi(\gamma) j(\gamma, z)^l f(z), \quad \gamma \in \Gamma, z \in \mathbb{H},$$

where $j(g, z)$ is as in (5.2). Its lift $\tilde{f}(g) := f(g \cdot i) j(g, i)^{-l}$ constitutes an automorphic form on G in the sense of Definition 5.7; it is of K -type σ_l and satisfies $\mathcal{C}\tilde{f} = \frac{1}{4}(l^2/2 - l)\tilde{f}$, see [8, Sections 5.14 and 5.15]. In particular, if $l > 1$, \tilde{f} belongs to the discrete series representation D_{l-1} in $L^2_{\sigma_l, \chi}(\Gamma \backslash G)$. If, in addition, \tilde{f} is a Hecke eigenform with $\|\tilde{f}\|_2 = 1$, one deduces from Theorem 5.8

$$(5.11) \quad \|f\|_\infty \ll_\varepsilon l^{\frac{5}{12} + \varepsilon},$$

since $\|f\|_p \equiv \|\tilde{f}\|_p$ for all p , compare [36, p. 219], yielding subconvex bounds for classical automorphic forms on \mathbb{H} in the weight aspect. This is consistent with Godement's formula [19], by which one has the convex bound $\|f\|_\infty \ll l^{\frac{1}{2}}$, see [12, 10]. Furthermore, a corresponding subconvex bound was proven in [12], the exponent there being $\frac{1}{2} - \frac{1}{33} = \frac{31}{66}$. Thus, our results do imply new results about holomorphic modular forms on \mathbb{H} . Note that in the case $\Gamma = \text{SL}(2, \mathbb{Z})$ one can even show [52] that $l^{\frac{1}{4} - \varepsilon} \ll_\varepsilon \|f\|_\infty \ll_\varepsilon l^{\frac{1}{4} + \varepsilon}$ by using the Fourier expansion of f and Deligne's bound [13], though this method is not available for cocompact arithmetic subgroups. In the non-cocompact case, hybrid bounds in the eigenvalue and the level aspect were considered in [2].

6. SUBCONVEX BOUNDS ON $\text{SO}(3)$

In this section, we shall derive equivariant and non-equivariant subconvex bounds on $\text{SO}(3)$ in the setting of [31, 32]. They are proven in an analogous way than the ones proven in Section 5 using results of [49]. To begin, consider the quaternion algebra

$$H(R) := \{a_0 + a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \mid a_0, a_1, a_2, a_3 \in R\}$$

over a given commutative ring R , where $\mathbf{i}^2 = \mathbf{j}^2 = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, and recall that for an element $a := a_0 + a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \in H(R)$ its conjugate is given by $\bar{a} := a_0 - a_1 \mathbf{i} - a_2 \mathbf{j} - a_3 \mathbf{k} \in H(R)$ while its norm reads $N(a) := a\bar{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2$. Note that $H(\mathbb{R})$ corresponds to the field of Hamilton's quaternions and $H(\mathbb{Z})$ to the ring of Lipschitz integers [11]. Write $H(R)^1 := \{a \in H(R) \mid N(a) = 1\}$ and put $G := H(\mathbb{R})^1$. As a group G , can be identified with $\text{SU}(2)$ via the mapping

$$G \ni a_0 + a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \longmapsto \begin{pmatrix} a_0 + a_1 i & a_2 + a_3 i \\ -a_2 + a_3 i & a_0 - a_1 i \end{pmatrix} \in \text{SU}(2).$$

G is compact, while $H(\mathbb{Z})^1 = \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$ is finite, so that by choosing the lattice $\Gamma := \{\pm 1\}$ in G we have

$$\Gamma \backslash G \cong \text{SO}(3)$$

via the adjoint action of G on its Lie algebra. Next, we introduce Hecke operators on $\mathrm{SO}(3)$ following [31, 32]. Thus, for each $\alpha \in H(\mathbb{R}) \setminus \{0\}$ and $x \in G$ set

$$\alpha \cdot x := N(\alpha)^{-1/2} \alpha x \in G.$$

As in Section 2, one can associate to each double coset $\Gamma\alpha\Gamma$, $\alpha \in H(\mathbb{Q})$, a Hecke operator $T_{\Gamma\alpha\Gamma}$. We then have $T_{\Gamma\alpha_1\Gamma} \circ T_{\Gamma\alpha_2\Gamma} = T_{\Gamma\alpha_2\alpha_1\Gamma}$. Further, setting ¹⁴

$$\mathcal{R}(n) := \{a \in H(\mathbb{Z}) \mid N(a) = n, \quad a \equiv 1 \pmod{2}\},$$

one can define the Hecke operator

$$(\mathcal{T}_n f)(x) := \frac{1}{2} \sum_{\alpha \in \mathcal{R}(n)} f(\alpha \cdot x), \quad f \in L^2(\Gamma \backslash G).$$

For natural numbers $r, s \equiv 1 \pmod{4}$, one has $\mathcal{T}_r \mathcal{T}_s = \sum_{d \mid (r,s)} d \mathcal{T}_{rs/d^2}$, see [32, Remark 1] and [49, p. 331]. Since Hecke operators commute with the right regular representation of G on $L^2(\Gamma \backslash G)$, we may replace $L^2(\Gamma \backslash G)$ by $L^2(\Gamma \backslash G/K)$ for any subgroup K of G in the above argument on Hecke operators. Choose $K = \mathrm{SO}(2) \simeq \mathbb{C}^1$, and denote the corresponding characters by $\sigma_l : e^{i\theta} \mapsto e^{il\theta}$, $l \in \mathbb{Z}$. Let Δ denote the Beltrami–Laplace operator on G . Since Δ and \mathcal{T}_n commute, there exists an orthonormal basis $\{\phi_j\}_{j \geq 0}$ of $L^2(\Gamma \backslash G)$ consisting of simultaneous eigenfunctions compatible with the decompositions (2.9) and (3.8), where $X = G$ and $M = \Gamma \backslash G$, respectively. Further, note that the action of K on $\Gamma \backslash G$ is isometric and non-singular. We then can prove the following equivariant subconvex bounds.

Theorem 6.1. *Let $G = \mathrm{SU}(2)$, $K = \mathrm{SO}(2)$, $\Gamma = \{\pm 1\}$, and $\sigma_l \in \widehat{K} \simeq \mathbb{Z}$. Then, for any Hecke–Maass form $\phi_j \in L^2_{\sigma_l}(\Gamma \backslash G)$ with Beltrami–Laplace eigenvalue λ_j and $\|\phi_j\|_{L^2} = 1$ one has*

$$\|\phi_j\|_{\infty} \ll_{\varepsilon} \lambda_j^{\frac{5}{24} + \varepsilon},$$

$\varepsilon > 0$ being arbitrary.

Remark 6.2. This theorem is a generalization of [49, Theorem 1.1], where the case $L^2_{\sigma_0}(\Gamma \backslash G) \cong L^2(\Gamma \backslash G/K) \cong L^2(S^2)$ is treated, S^2 being the 2-sphere. In the papers [5, 6], hybrid L^{∞} -norms for general arithmetic quotients of 2-spheres in the eigenvalue and level aspect are studied. However, the exponents for the spectral parameter are a little greater than $5/24$ there, namely $\frac{1}{24} - \frac{1}{27} = \frac{1}{216}$.

Proof. By [49, Lemma 2.1] one has

$$(6.1) \quad \#\{\alpha \in \mathcal{R}(n) \mid \mathrm{dist}(x, \alpha \cdot x) < \delta\} \ll_{\varepsilon} \begin{cases} \delta^{\frac{1}{2}} n^{1+\varepsilon} + n^{\varepsilon} & \text{if } \delta < 1/n, \\ n^{\frac{1}{2}+\varepsilon} + \delta^{\frac{2}{3}} n^{1+\varepsilon} & \text{otherwise.} \end{cases}$$

Further, Proposition 4.1 also holds in the present case, since Proposition 3.3 is true for arbitrary compact manifolds and symmetry groups. By repeating the arguments given in Section 5.2 we therefore get for any $|l| \ll \mu / \log \mu$, $\mu > n$, and $n \equiv 1 \pmod{4}$ the uniform bound

$$K_{\mathcal{T}_n \circ \tilde{s}_{\mu} \circ \Pi_{\sigma_l}}(x, x) \ll_{\varepsilon} (\mu + n\mu^{1/2} \log \mu) n^{\varepsilon}$$

up to negligible terms. Now, by the Dirichlet prime number theorem on arithmetic progressions it is well-known that

$$\#\{p < x \mid p \text{ is a prime, } p \equiv 1 \pmod{4}\} \sim \frac{1}{2} \frac{x}{\log x}.$$

Furthermore, the Ramanujan conjecture proved by Deligne [13] together with the Jacquet–Langlands correspondence for $\mathrm{GL}(2)$ [29] implies that the Hecke eigenvalues $\lambda_j(p)$ of \mathcal{T}_p are bounded from above by $2p^{\frac{1}{2}+\varepsilon}$ for prime levels. Hence, the argument of Iwaniec–Sarnak already used in Section 5.2, but now applied to $L^2(\mathrm{SO}(3))$, yields

$$\|\phi_j\|_{\infty}^2 \ll_{\varepsilon} \mu^{\varepsilon} N^{\varepsilon} (\mu N^{-1/2} + \mu^{1/2} N).$$

¹⁴ $a_0 + a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \equiv 1 \pmod{2}$ means that a_0 is odd and a_1, a_2, a_3 are even. Note that $\mathcal{R}(n)$ is empty unless $n \equiv 1 \pmod{4}$.

The theorem now follows by taking $N = \mu^{1/3}$. \square

Note that the right regular representation of $\Gamma \backslash G \cong \mathrm{SO}(3)$ on $L^2(\Gamma \backslash G)$ decomposes according to

$$L^2(\Gamma \backslash G) \cong \bigoplus_{k \in \mathbb{N}} \mathcal{M}_k, \quad \mathcal{M}_k \cong \pi_k^{\oplus 2k+1},$$

where π_k denotes the irreducible representation of $\mathrm{SO}(3)$ of dimension $2k + 1$. In particular, the Beltrami–Laplace eigenvalue corresponding to π_k is $k(k+1)$, and the restriction of π_k to K is isomorphic to $\widehat{\bigoplus_{l=-k}^k \sigma_l}$. Hence, if we choose an orthonormal sequence $\{\psi_j\}_{j \geq 0}$ in $L^2(\Gamma \backslash G)$ consisting of Hecke–Maass forms with $\psi_j \in (\sigma_{l_j})^{\oplus 2k_j+1} \subset \mathcal{M}_{k_j}$, where $|l_j| \leq k_j$, Theorem 6.1 yields

$$\|\psi_j\|_\infty \ll_\varepsilon k_j^{\frac{5}{12} + \varepsilon}.$$

7. SUBCONVEX BOUNDS ON $\Gamma \backslash G$ FOR SEMISIMPLE GROUPS AND ARITHMETIC CONGRUENCE SUBGROUPS

7.1. General framework. In what follows, we shall develop a general framework to prove subconvex bounds of Hecke–Maass forms on semisimple groups. For this, let us return to the general setting of Sections 2, 3 and 4. Write $\mathfrak{L}^2 := L^2_\chi(\Gamma \backslash G)$ or $L^2_{\sigma, \chi}(\Gamma \backslash G)$, and consider on this space the family of Hecke operators $T_{\Gamma\beta\Gamma}^\chi$ introduced in (2.10), together with the corresponding \mathbb{C} -module

$$H_\Xi^\chi := \langle \mathcal{T}_{\Gamma\beta\Gamma}^\chi \mid \beta \in \Xi \rangle$$

generated by them. In what follows, we assume that there exists a submodule \mathcal{H} of H_Ξ^χ such that there is an orthonormal basis $\{\phi_j\}_{j \in \mathbb{N}}$ of $L^2(\Gamma_\chi \backslash G)$ compatible with the decomposition (2.9), and in case that P commutes with the right regular K -representation, also with the decomposition (3.8), consisting of simultaneous eigenfunctions of P and all $\mathcal{T} \in \mathcal{H}$ with $P\phi_j = \lambda_j\phi_j$. As before, such simultaneous eigenfunctions will be called *Hecke–Maass forms of rank 1*. We also suppose that \mathcal{T}^* belongs to \mathcal{H} for each $\mathcal{T} \in \mathcal{H}$ and that the cospheres $S_x^*(\Gamma_\chi \backslash G) := \{(x, \xi) \in T^*(\Gamma_\chi \backslash G) \mid p(x, \xi) = 1\}$ are strictly convex for all $x \in \Gamma_\chi \backslash G$. Further, consider the lattice point counting functions $\mathfrak{M}(x, \beta, \delta) := \mathcal{M}(x, \beta, \delta)$ or $M(x, \beta, \delta)$ corresponding to \mathfrak{L}^2 , respectively; that is,

$$\mathcal{M}(\delta) := \mathcal{M}(x, \beta, \delta) := \#\{\alpha \in \Gamma_\chi \backslash \Gamma\beta\Gamma : \mathrm{dist}(x, \alpha \cdot x) < \delta\},$$

where $\mathrm{dist}(\alpha \cdot x, x) \equiv \mathrm{dist}(\Gamma\alpha x, \Gamma x)$, and $M(x, \beta, \delta)$ is as in (4.4). We then have the following

Lemma 7.1. *Fix a character χ in $\widehat{\Gamma}$ such that $[\Gamma : \Gamma_\chi] < \infty$. Let ϕ_{j_0} be a Hecke–Maass form in \mathfrak{L}^2 with corresponding spectral eigenvalue λ_{j_0} . Let \mathcal{P}' be an infinite set and $\mathfrak{N}' : \mathcal{P}' \rightarrow \mathbb{N}$ a mapping such that*

$$(7.1) \quad \#\{v \in \mathcal{P}' \mid N/2 < \mathfrak{N}'(v) < N\} \gg N/\log N.$$

Assume that for each element $v \in \mathcal{P}'$ there exists a Hecke operator $\mathcal{T}'_v \in \mathcal{H}$ satisfying $\mathcal{T}'_v \phi_{j_0} = \phi_{j_0}$, and that for any $N \in \mathbb{N}$ and any $x \in \Gamma_\chi \backslash G$ we have a suitable finite subset $Q'_{N,x} \subset \mathcal{P}'$ such that $\#Q'_{N,x} \ll \log N$. Write

$$\mathcal{T}'_{N,x} := \sum_{v \in Q'_{N,x}} \mathcal{T}'_v \quad \text{where} \quad Q'_{N,x} := \{v \in \mathcal{P}' \mid N/2 < \mathfrak{N}'(v) < N, v \notin Q'_{N,x}\}.$$

As a linear operator on \mathfrak{L}^2 , $\mathcal{T}'_{N,x} \circ (\mathcal{T}'_{N,x})^$ can be represented as*

$$\mathcal{T}'_{N,x} \circ (\mathcal{T}'_{N,x})^* = \sum_{u=1}^l a_u \mathcal{T}'_{\Gamma\alpha_u\Gamma}^\chi$$

for certain $l \in \mathbb{N}$, $a_u \in \mathbb{C}$, and $\alpha_u \in \Xi$ depending on x . Further, suppose that there exist numbers $0 < \kappa \ll 1$ and $A_1, A_2 > 2$ such that for each $N \gg 1$ and each $x \in \Gamma_\chi \backslash G$ one has

$$(7.2) \quad \sum_{u=1}^l |a_u| |\mathfrak{M}(x, \alpha_u, N^{-A_2})| \ll N^{2-2\kappa}, \quad \sum_{u=1}^l |a_u| |\Gamma_\chi \backslash \Gamma \alpha_u \Gamma| \ll N^{A_1}.$$

Then, if $\mathfrak{L}^2 = L_\chi^2(\Gamma \backslash G)$, there exists a constant $\delta > 0$ such that

$$\|\phi_{j_0}\|_\infty \ll \lambda_{j_0}^{\frac{\dim G - 1}{2m} - \delta},$$

while if $\mathfrak{L}^2 = L_{\sigma, \chi}^2(\Gamma \backslash G)$, there exists a constant $\delta > 0$, which does not depend on σ , such that

$$\|\phi_{j_0}\|_\infty \ll \sqrt{d_\sigma \max_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_\infty} \lambda_{j_0}^{\frac{\dim G/K - 1}{2m} - \delta}.$$

Finally, if $K = T$ is a torus and $\mathfrak{L}^2 = L_\chi^2(\Gamma \backslash G)$, there exists a constant $\delta > 0$ such that

$$\|\phi_{j_0}\|_\infty \ll \lambda_{j_0}^{\frac{\dim G/K - 1}{2m} - \delta}.$$

Proof. Let us consider first the case $\mathfrak{L}^2 = L_{\sigma, \chi}^2(\Gamma \backslash G)$. Set $\mu := \sqrt[m]{\lambda_{j_0}}$ and denote by $\lambda'_{j, N}$ the eigenvalue of $\mathcal{T}'_{N, x}$ for ϕ_j , so that

$$|\mathcal{T}'_{N, x} \phi_j(x)|^2 = \lambda'_{j, N} \overline{\lambda'_{j, N}} \phi_j(x) \overline{\phi_j(x)} = \mathcal{T}'_{N, x} \circ (\mathcal{T}'_{N, x})^* \phi_j(x) \overline{\phi_j(x)}.$$

Taking $\tilde{N} = 0$ in Proposition 3.3 we deduce that

$$\sum_{u=1}^l a_u \sum_{\substack{\alpha \in \Gamma_\chi \backslash \Gamma \alpha_u \Gamma, \\ \text{dist}(xK, \alpha \cdot xK)^{\dim G/K - 1} \geq N^{-A_2}}} \overline{\chi(\alpha)} K_{\tilde{s}_\mu \circ \Pi_\sigma}(\alpha \cdot x, x) \ll d_\sigma \max_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_\infty \mu^{\frac{\dim G/K - 1}{2}} N^A$$

up to terms of order $O(N^A \mu^{-\infty})$, where we set $A := \frac{1}{2}A_2 + A_1$. With the same arguments than at the end of Section 3 and in the proof of Proposition 4.1 one now deduces with $\mu_j := \sqrt[m]{\lambda_j}$ for any $x \in \Gamma_\chi \backslash G$

$$\begin{aligned} (\#\mathcal{Q}'_{N, x})^2 |\phi_{j_0}(x)|^2 &= |\mathcal{T}'_{N, x} \phi_{j_0}(x)|^2 \leq \sum_{\substack{\mu \leq \mu_j \leq \mu+1, \\ \phi_j \in L_{\sigma, \chi}^2(\Gamma_\chi \backslash G)}} |\mathcal{T}'_{N, x} \phi_j(x)|^2 \\ &\ll \sum_{\substack{j \geq 0, \\ \phi_j \in L_{\sigma, \chi}^2(\Gamma_\chi \backslash G)}} \varrho(\mu - \mu_j) |\mathcal{T}'_{N, x} \phi_j(x)|^2 = K_{\mathcal{T}'_{N, x} \circ (\mathcal{T}'_{N, x})^* \circ \tilde{s}_\mu \circ \Pi_\sigma}(x, x) \\ &\ll d_\sigma \max_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_\infty \mu^{\dim G/K - 1} N^{2-2\kappa} + d_\sigma \max_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_\infty \mu^{\frac{\dim G/K - 1}{2}} N^A \\ &= d_\sigma \max_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_\infty \left(\mu^{\dim G/K - 1} N^{2-2\kappa} + \mu^{\frac{\dim G/K - 1}{2}} N^A \right) \end{aligned}$$

up to terms of order $O(N^A \mu^{-\infty})$. Hence, the assertion follows from (7.1) by taking $N \sim \mu^B$, $B := \frac{\dim G/K - 1}{2(A - 2 + 2\kappa)}$. The case $\mathfrak{L}^2 = L_\chi^2(\Gamma \backslash G)$ is seen in a similar way taking into account Proposition 3.1 and the toric case in Proposition 3.3. \square

Remark 7.2. The assumptions of the previous lemma are primarily motivated by the work of Marshall [33] in the case that χ is trivial. One can easily verify that they are fulfilled in the setup of Section 5 when \mathcal{P}' is the totality of primes and \mathfrak{N}' is the inclusion mapping $\mathcal{P}' \subset \mathbb{N}$. Indeed, for each prime p with $(p, q) = 1$, there exists an element β_p in $\mathcal{R}(p^2)$ such that $\tilde{\mathcal{T}}_{p^2}^\chi := \mathcal{T}_{\Gamma \beta_p \Gamma}^\chi = \mathcal{T}_{p^2}^\chi - \mathcal{T}_{\Gamma(pI_2)\Gamma}^\chi$, see [36, p. 217]. Now, let $\tilde{\lambda}_j(p^2)$ be the eigenvalue of $\tilde{\mathcal{T}}_{p^2}^\chi$ belonging to the eigenfunction $\phi_j \in L_{\sigma, \chi}^2(\Gamma \backslash G)$.

For each ϕ_j in $L^2_{\sigma, \chi}(\Gamma \backslash G)$ and p as above, $|\lambda_j(p)| \leq p^{1/2}/2$ implies that $|\tilde{\lambda}_j(p^2)| \geq p/2$ by $|\tilde{\lambda}_j(p^2)| = |\lambda_j(p)^2 - (p+1)\chi(p)|$. Therefore, if we choose

$$\mathcal{T}'_p := \begin{cases} \frac{1}{\lambda_{j_0}(p)} \mathcal{T}_p^\chi & \text{if } |p^{-1/2} \lambda_{j_0}(p)| > 1/2, \\ \frac{1}{\lambda_{j_0}(p^2)} \tilde{\mathcal{T}}_{p^2}^\chi & \text{otherwise,} \end{cases}$$

the mentioned assumptions must hold by Lemma 5.1. This choice is essentially the same as in the work of Blomer-Maga [5, 6] on $\mathrm{SL}(n, \mathbb{Z}) \subset \mathrm{PGL}(n, \mathbb{R})$ in the case $n = 2$.

Remark 7.3. When using Lemma 7.1, we shall take for \mathcal{P}' a subset of the totality of primes. Note that it is unnecessary to suppose that ϕ_j is an eigenfunction of the operators \mathcal{T}_p for all primes p to prove the subconvex bound in Section 5, as we did already see in Section 6. In fact, one can make the conditions on \mathcal{P}' and \mathcal{H} weaker. Namely, we can replace \mathcal{P}' by a smaller subset satisfying (7.1) and replace \mathcal{H} by the submodule $\langle \mathcal{T}'_v \mid v \in \mathcal{P}' \rangle$. This means that our concept of a Hecke-Maass form is much weaker than the usual one in Section 5. Such forms are not eigenfunctions of the center of the universal enveloping algebra in general, and can be obtained in abundance by functorial lifts of Hecke characters.

7.2. Equivariant subconvex bounds. In what follows, we shall derive equivariant subconvex bounds on arithmetic quotients for a large class of semisimple algebraic groups, extending the work of Marshall [33] to non-spherical situations. Thus, let \underline{G} be a connected semisimple algebraic group over a number field F . We write $\underline{G}(k)$ for the set of k -rational points in \underline{G} for a field $k \supset F$ and F_v for the completion of F by a place v of F . Following [33], we assume that there exists a real place v_0 of F such that

(WS) The group $\underline{G}(F_{v_0})$ is quasi-split, and not isogeneous to a product of odd special unitary groups.

We set $H := \mathrm{Res}_{F/\mathbb{Q}} \underline{G}$, where $\mathrm{Res}_{F/\mathbb{Q}}$ means the restriction of scalars from F to \mathbb{Q} . Then H is a connected semisimple algebraic group over \mathbb{Q} and $G := H(\mathbb{R})$ a real semisimple Lie group with finite center [38, Chapter 3]. Let K be a maximal compact subgroup in G , K_0 an open compact subgroup of $H(\mathbb{A}_{\mathrm{fin}})$, and put $\Gamma := H(\mathbb{Q}) \cap (H(\mathbb{R})K_0)$. In the following theorem, we will also impose the condition $H(\mathbb{A}) = H(\mathbb{Q})(H(\mathbb{R})K_0)$. This condition holds for any K_0 if H has the strong approximation property with respect to ∞ , that is, if H is simply connected as an algebraic group, and does not have any \mathbb{R} -simple component H' such that $H'(\mathbb{R})$ is compact, compare [38, Theorem 7.12]. As the second main result of this paper we obtain

Theorem 7.4. *Suppose that $H(\mathbb{A}) = H(\mathbb{Q})(H(\mathbb{R})K_0)$ and $H(\mathbb{Q}) \backslash H(\mathbb{A})$ is compact, so that $\Gamma \backslash G$ is compact as well. Further, assume that H satisfies the condition (WS). Now, let P_0 be an elliptic left-invariant differential operator on G of degree m which gives rise to a positive and symmetric operator on $\Gamma \backslash G$ that commutes with the right regular K -representation and has strictly convex cospheres $S_x^*(\Gamma \backslash G)$. Then, there exist a submodule \mathcal{H} of $H_{\Xi=H(\mathbb{Q})}^{\chi=1}$ and a constant $\delta > 0$ such that*

- (1) *there is an orthonormal basis $\{\phi_j\}_{j \in \mathbb{N}}$ of $L^2(\Gamma \backslash G)$ which consists of simultaneous eigenfunctions for the unique self-adjoint extension P of P_0 and all $\mathcal{T} \in \mathcal{H}$;*
- (2) *for each $\phi_j \in L^2_\sigma(\Gamma \backslash G)$ with spectral eigenvalue λ_j one has*

$$\|\phi_j\|_\infty \ll \sqrt{d_\sigma \sup_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_\infty} \lambda_j^{\frac{\dim G/K - 1}{2m} - \delta}.$$

If $K = T$ is a torus, one has the stronger estimate

$$\|\phi_j\|_\infty \ll \lambda_j^{\frac{\dim G/K - 1}{2m} - \delta}.$$

Remark 7.5. The equivariant subconvex bound of the previous theorem can be rephrased using the Cartan-Weyl classification of unitary irreducible representations of compact groups. In fact, assume that K is a compact connected semisimple Lie group, \mathfrak{k} its Lie algebra, and $T \subset K$ a maximal torus with Lie algebra \mathfrak{t} . Denote by $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}$ the complexifications of \mathfrak{k} and \mathfrak{t} , respectively. Then $\mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$, and we write $\Sigma(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ for the corresponding system of roots and Σ^+

for a set of positive roots. Now, as a consequence of the Cartan-Weyl classification of irreducible finite-dimensional representations of reductive Lie algebras over \mathbb{C} one has the identification

$$\widehat{K} \simeq \{\Lambda \in \mathfrak{t}_{\mathbb{C}}^* : \Lambda \text{ is dominant integral and } T\text{-integral}\},$$

compare [51], and we write $\Lambda_{\sigma} \in \mathfrak{t}_{\mathbb{C}}^*$ for the *highest weight* corresponding to $\sigma \in \widehat{K}$ under this isomorphism. Weyl's dimension formula then implies that $d_{\sigma} = O(|\Lambda_{\sigma}|^{|\Sigma^+|})$, while from Weyl's character formula one infers that if D^u is a differential operator on K of order u ,

$$(7.3) \quad \|D^u \sigma\|_{\infty} = O(|\Lambda_{\sigma}|^{u+|\Sigma^+|}), \quad |\Lambda_{\sigma}| \rightarrow \infty,$$

compare [39, Eq. (3.5)]. Consequently, the bound in Theorem 7.4 can be rewritten as

$$\|\phi_j\|_{\infty} \ll \sqrt{|\Lambda_{\sigma}|^{2|\Sigma^+| + \lfloor \frac{\dim K}{2} \rfloor + 1}} \lambda_j^{\frac{\dim G/K - 1}{2m} - \delta}.$$

Proof of Theorem 7.4. By translating the results in [33, Section 3] to our non-adelic setting, one verifies that the assumptions of Lemma 7.1 are fulfilled under the hypothesis of the theorem. Note that it is unnecessary to relate the subgroup K to the specific maximal connected compact subgroup considered in [33], because the assumptions in question are concerned only with the structure of the Hecke algebra and the lattice point counting function $M(x, \alpha, \delta)$.

Let us explain this in a more detailed way. Since $H(\mathbb{A}) = H(\mathbb{Q})(H(\mathbb{R})K_0)$ one has

$$H(\mathbb{Q}) \backslash H(\mathbb{A}) / K_0 \cong \Gamma \backslash G.$$

Now, any function φ in $L^2_{\sigma}(\Gamma \backslash G)$ can be identified with a function $\varphi_{\mathbb{A}}$ in $L^2_{\sigma}(H(\mathbb{Q}) \backslash H(\mathbb{A}) / K_0)$ by setting

$$(7.4) \quad \varphi_{\mathbb{A}}(\gamma g k) := \varphi(g), \quad \gamma \in H(\mathbb{Q}), \quad g \in G, \quad k \in K_0.$$

For each double coset $K_0 \alpha K_0$ with $\alpha \in H(\mathbb{A}_{\text{fin}})$, a linear operator $T_{K_0 \alpha K_0}$ on $L^2_{\sigma}(H(\mathbb{Q}) \backslash H(\mathbb{A}) / K_0)$ can then be defined by setting

$$(T_{K_0 \alpha K_0} \phi_{\mathbb{A}})(x) := \sum_{h \in K_0 \alpha K_0 / K_0} \phi_{\mathbb{A}}(xh), \quad \phi_{\mathbb{A}} \in L^2(H(\mathbb{Q}) \backslash H(\mathbb{A}) / K_0).$$

Moreover, there exist finitely many elements β_1, \dots, β_m in $H(\mathbb{Q})$ such that

$$(7.5) \quad H(\mathbb{Q}) \cap (H(\mathbb{R})K_0 \alpha^{-1} K_0) = \bigsqcup_{i=1}^m \Gamma \beta_i \Gamma,$$

the intersection being non-empty due to the assumption $H(\mathbb{A}) = H(\mathbb{Q})(H(\mathbb{R})K_0)$.¹⁵ This implies that for all $\varphi \in L^2_{\sigma}(\Gamma \backslash G)$

$$(7.6) \quad T_{K_0 \alpha K_0} \varphi_{\mathbb{A}} = \sum_{i=1}^m T_{\Gamma \beta_i \Gamma} \varphi.$$

Hence, any adelic Hecke operator $T_{K_0 \alpha K_0}$ can be regarded as a sum of non-adelic Hecke operators via the identification $\varphi \equiv \varphi_{\mathbb{A}}$.

In order to apply Lemma 7.1 in the present context, we choose $\chi = 1$ and $\Xi = H(\mathbb{Q})$. Let \mathcal{P}' be the set denoted by \mathcal{P} in [33, Section 2.5], that is, an infinite subset of the totality of finite places of F . A map $\mathfrak{N}' : \mathcal{P}' \rightarrow \mathbb{N}$ is defined by the order of the residue field of F_v . Then, $\underline{G}(F_v)$ is split for each $v \in \mathcal{P}'$ and (7.1) holds by the prime ideal theorem and the Chebotarev density theorem. Now, put

$$(7.7) \quad \mathcal{H} := \langle T_{K_0 \alpha K_0} \mid \alpha \in \underline{G}(F_v), \quad v \in \mathcal{P}' \rangle.$$

Note that $\underline{G}(F_v) \subset H(\mathbb{Q}_p)$ if $v|p$, and by [36, Proof of Theorem 2.8.2 (2)] we have $(T_{K_0 \alpha K_0})^* = T_{K_0 \alpha^{-1} K_0} \in \mathcal{H}$. Since P_0 commutes with all Hecke operators and each automorphic representation of $H(\mathbb{A})$ factors as a tensor product of irreducible unitary representations of $\underline{G}(F_v)$ for all places v [17], there exists an orthonormal basis $\{\phi_j\}_{j \geq 0}$ of $L^2(\Gamma \backslash G)$ consisting of simultaneous eigenfunctions

¹⁵Without this assumption, the intersection in (7.5) might be empty, and the following arguments make no sense. Along the same lines, recall that Hecke operators on $\text{SO}(3)$ are only defined in the case $p \equiv 1 \pmod{4}$.

for P and all $\mathcal{T} \in \mathcal{H}$. Now, let $\phi_{j_0} \in L_\sigma^2(\Gamma \backslash G)$ be fixed. Applying the results in [33] to the function $\psi := \phi_{j_0, \mathbb{A}}$, that is also denoted by ψ there, one can verify the assumptions of Lemma 7.1 for ϕ_{j_0} . Indeed, by [33, Propositions 6.1], for each place $v \in \mathcal{P}'$ there exists a Hecke operator $\mathcal{T}_v \in \mathcal{H}$ such that $\mathcal{T}_v \psi = \psi$ holds and \mathcal{T}_v is a linear combination of operators $T_{K_0 \alpha K_0}$ with $\alpha \in \underline{G}(F_v)$. In view of (7.6) we can identify \mathcal{T}_v with a non-adelic Hecke operator \mathcal{T}'_v on $L_\sigma^2(\Gamma \backslash G)$ such that

$$\mathcal{T}'_v \phi_{j_0} = \mathcal{T}_v \phi_{j_0, \mathbb{A}} = \phi_{j_0, \mathbb{A}} = \phi_{j_0}.$$

Similarly, set $\mathcal{T}_{N,x} := \sum_{v \in \mathcal{Q}'_{N,x}} \mathcal{T}_v$, and denote the corresponding non-adelic Hecke operators by $\mathcal{T}'_{N,x}$, where $\mathcal{Q}'_{N,x}$ is chosen as the set denoted by \mathcal{Q}_N in [33, Section 3.4]. By the convolution on $H(\mathbb{A}_{\text{fin}})$, there exist $n \in \mathbb{N}$, $b_k \in \mathbb{C}$, and $\omega_k \in H(\mathbb{A}_{\text{fin}})$ such that

$$\mathcal{T}_{N,x} \circ (\mathcal{T}_{N,x})^* = \sum_{k=1}^n b_k T_{K_0 \omega_k K_0}.$$

The corresponding $l \in \mathbb{N}$, $a_u \in \mathbb{C}$, and $\alpha_u \in C(\Gamma)$ in the decomposition of $\mathcal{T}'_{N,x} \circ (\mathcal{T}'_{N,x})^*$ in Lemma 7.1 are then obtained from this equality via the identification (7.6). Finally, the upper bounds (7.2) in Lemma 7.1 can be verified using (7.5), (7.6) and the arguments in [33, Section 3], completing the proof of the theorem. \square

Example 7.6 (Equivariant subconvex bounds for $\text{SL}(n, \mathbb{R})$). Choose a central division algebra D of index n over \mathbb{Q} such that $D \otimes \mathbb{R} \cong M(n, \mathbb{R})$. It is well known that the equivalence classes of central division algebras over \mathbb{Q} are parameterized by the Brauer group $\text{Br}(\mathbb{Q})$, which can be realized as the set

$$\left\{ (a, x) \mid a \in \{0, 1/2\}, \quad x = (x_p) \in \bigoplus_p \mathbb{Q}/\mathbb{Z}, \quad a + \sum_p x_p = 0 \pmod{\mathbb{Z}} \right\},$$

where p runs over all primes, via the Brauer-Hasse-Noether theorem, compare [38, Theorem 1.12]. If we choose a prime p_1 and a parameter $(0, x)$ in $\text{Br}(\mathbb{Q})$ such that $x_{p_1} = a/n$ and a is prime to n , then there is a central division algebra D corresponding to $(0, x)$ and satisfying $D \otimes \mathbb{R} \cong M(n, \mathbb{R})$. A semisimple algebraic group H over \mathbb{Q} is then defined by $H := \text{SL}(1, D)$. Clearly, $G := H(\mathbb{R}) \cong \text{SL}(n, \mathbb{R})$, $K := \text{SO}(n)$, and $\Gamma := H(\mathbb{Q}) \cap (H(\mathbb{R})K_0)$ is cocompact for any open compact subgroup K_0 of $G(\mathbb{A}_{\text{fin}})$, while H satisfies the condition (WS) and is simply connected. Then, the assumptions of Theorem 7.4 hold, and for any Hecke–Maass form ϕ_j in $L_\sigma^2(\Gamma \backslash G)$ with eigenvalue λ_j one obtains the subconvex bound

$$\|\phi_j\|_\infty \ll \sqrt{d_\sigma \sup_{u \leq \lfloor \frac{n^2-n+4}{4} \rfloor} \|D^u \sigma\|_\infty} \lambda_j^{\frac{n^2+n-4}{4m} - \delta}$$

for some $\delta > 0$.

Example 7.7 (Equivariant subconvex bounds for $\text{SL}(n, \mathbb{C})$). Let D denote a central division algebra over a number field F . Then $H := \text{Res}_{F/\mathbb{Q}} \text{SL}(1, D)$ is a simply connected semisimple algebraic group over \mathbb{Q} . The Brauer-Hasse-Noether theorem ensures that there exist various division algebras D such that H satisfies the assumptions of Theorem 7.4. For example, by choosing suitable parameters in the Brauer group, one gets a central division algebra D of index n over an imaginary quadratic extension F of \mathbb{Q} such that $H(\mathbb{R})$ is isomorphic to $\text{SL}(n, \mathbb{C})$. Since $\text{SL}(n, \mathbb{C})$ satisfies the condition (WS), Theorem 7.4 yields subconvex bounds for $\Gamma \backslash \text{SL}(n, \mathbb{C})$, where Γ is defined by an open compact subgroup K_0 in $H(\mathbb{A}_{\text{fin}})$.

Example 7.8 (Equivariant subconvex bounds for $\text{SU}(n, n, \mathbb{R})$). There exists a central division algebra D over an imaginary quadratic extension F of \mathbb{Q} with a F/\mathbb{Q} -involution such that $H := \text{SU}(1, D)$ satisfies $G := H(\mathbb{R}) \cong \text{SU}(n, n, \mathbb{R})$, see [44, Theorem 8.1]. Since H obviously satisfies the assumptions of Theorem 7.4, one obtains equivariant subconvex bounds for $\Gamma \backslash G$ in this case.

7.3. Non-equivariant subconvex bounds. We shall now prove non-equivariant subconvex bounds on arithmetic quotients of semisimple algebraic groups without the condition (WS). Let H be a connected semisimple algebraic group over \mathbb{Q} and choose any open compact subgroup K_0 of $H(\mathbb{A}_{\text{fin}})$. We then have the following

Theorem 7.9. *Put $G := H(\mathbb{R})$ and $\Gamma := H(\mathbb{Q}) \cap (H(\mathbb{R})K_0)$. Assume that $H(\mathbb{A}) = H(\mathbb{Q})(H(\mathbb{R})K_0)$ and that $\Gamma \backslash G$ is compact. Let P_0 be an elliptic left-invariant differential operator on G of degree m that gives rise to a positive and symmetric operator on $\Gamma \backslash G$ with strictly convex cospheres $S_x^*(\Gamma \backslash G)$. Then, there exist a submodule \mathcal{H} of $H_{\Xi=H(\mathbb{Q})}^{\chi=1}$ and a constant $\delta > 0$ such that there is an orthonormal basis $\{\phi_j\}_{j \in \mathbb{N}}$ of $L^2(\Gamma \backslash G)$ consisting of simultaneous eigenfunctions for P and all $\mathcal{T} \in \mathcal{H}$, so that for each ϕ_j with spectral eigenvalue λ_j one has*

$$\|\phi_j\|_{\infty} \ll \lambda_j^{\frac{\dim G - 1}{2m} - \delta}.$$

Proof. To prove this theorem, we need an explicit distance on G . We may assume that H is a closed subgroup of $\text{SL}(m)$ over \mathbb{Q} for some sufficiently large $m \in \mathbb{N}$, so that $G = H(\mathbb{R})$ becomes a closed subgroup of $\text{SL}(m, \mathbb{R})$ with respect to the topology induced from the Euclidean topology on \mathbb{R}^{m^2} , compare [38, Chapter 3]. Note that $H(\mathbb{R})$ might consist of finitely many connected components with respect to the usual topology, even if H is connected in the sense of Zariski [38, Corollary 1]. One then defines on $M(m, \mathbb{R})$ the Euclidean distance

$$\text{dist}_1(x, y) := \|x - y\|, \quad \|x\| := \text{Tr}({}^t x x), \quad x, y \in M(m, \mathbb{R}),$$

obtaining a distance on G by the inclusions $G \subset \text{SL}(m, \mathbb{R}) \subset M(m, \mathbb{R})$. In fact, the distance dist is locally equivalent to the distance dist_1 . Indeed, dist_1 is equivalent to dist in a small neighborhood \mathcal{U} of the identity. Furthermore, for fixed $g \in G$ one computes

$$\text{dist}_1(gx, gy) \leq \|g\| \text{dist}_1(x, y), \quad \text{dist}_1(gx, gy) \geq \|g^{-1}\|^{-1} \text{dist}_1(x, y),$$

so that $\text{dist}_1(x, y)$ is equivalent to the distance $(x, y) \mapsto \text{dist}_1(gx, gy)$ on G . The assertion now follows by covering G by translates of \mathcal{U} .

The first assertion follows from the corresponding argument in Theorem 7.4. It remains to show that the assumptions in Lemma 7.1 are satisfied for the module \mathcal{H} given in (7.7), for which we shall follow the considerations in [33]. Let us choose the same norm $\|\cdot\|$ as in [33, Section 2.2] on the group of cocharacters of a maximal torus over $\overline{\mathbb{Q}}$, and regard $\|\cdot\|$ as a norm on the cocharacters of each \mathbb{Q}_p -torus by conjugation. Let \mathcal{P}' be the set denoted by \mathcal{P} in [33, Section 2.5] for $F = \mathbb{Q}$ and $\underline{G} = H$. Then \mathcal{P}' is an infinite set of prime numbers, (7.1) holds, and for each prime $p \in \mathcal{P}'$ the group $H(\mathbb{Q}_p)$ is split. Furthermore, a Hecke operator $\tau(p, \mu)$ is defined by the product of $p^{-\|\mu\|}$ with the characteristic function of $H(\mathbb{Z}_p)\mu(p)H(\mathbb{Z}_p)$, where μ is a cocharacter on a suitable maximal split torus \underline{T}_p in $H(\mathbb{Q}_p)$. In addition, several conditions are imposed on H , \mathcal{P}' , and \underline{T}_p , and we refer the reader to [33, Section 2] for details. By [33, Proposition 6.1], there exists for each $p \in \mathcal{P}'$ a Hecke operator \mathcal{T}_p such that

$$\begin{aligned} \mathcal{T}_p \phi_{j, \mathbb{A}} &= \phi_{j, \mathbb{A}}, \quad \mathcal{T}_p = \sum_{\|\mu\| \leq R} a(p, \mu) \tau(p, \mu), \quad \mathcal{T}_p \mathcal{T}_p^* = \sum_{\|\mu\| \leq R} b(p, \mu) \tau(p, \mu), \\ a(p, \mu) &\ll 1, \quad a(p, 0) = 0, \quad b(p, \mu) \ll 1 \end{aligned}$$

for some constant $R \in \mathbb{N}$. Now, choose a compact subset Ω of G such that $G = \Gamma\Omega$, let x be an element in Ω , and set $\mathcal{T}_{N, x} := \sum_{p \in \mathcal{P}', p \leq N} \mathcal{T}_p$.¹⁶ In order to verify the necessary conditions in Lemma 7.1 we proceed as in the proof of Theorem 7.4, and let \mathcal{T}'_p and $\mathcal{T}'_{N, x}$ be non-adelic Hecke operators corresponding to \mathcal{T}_p and $\mathcal{T}_{N, x}$, respectively. For $\gamma \in H(\mathbb{Q}) \subset \text{SL}(m, \mathbb{Q})$, let $\|\gamma\|_f$ denote the least common multiple of denominators of components of γ . By [33, Corollary 3.6], one has $\|\gamma\|_f \ll N^{A'_2}$ for some $A'_2 > 0$ if $\gamma \in H(\mathbb{Q}) \cap \text{supp}(\mathcal{T}_{N, x}(\mathcal{T}_{N, x})^*)$, where $\text{supp}(\mathcal{T}_{N, x}(\mathcal{T}_{N, x})^*)$ means the support of $\mathcal{T}_{N, x}(\mathcal{T}_{N, x})^*$ in $H(\mathbb{A}_{\text{fin}})$, and the second bound in (7.2) follows. Furthermore, for the distance dist_1 , one can show that for some $A_2 > A'_2$ the inequality $\text{dist}_1(\gamma x, x) < c_1 N^{-A_2}$ does not hold for any non-trivial

¹⁶Note that $\mathcal{T}_{N, x}$ does actually not depend on x , but we preferred to keep the notation of Lemma 7.1.

element $\gamma \in H(\mathbb{Q}) \cap \text{supp}(\mathcal{T}_{N,x}(\mathcal{T}_{N,x})^*)$, where we choose $c_1 > 0$ such that $c_1 \text{dist}(y_1, y_2) < \text{dist}_1(y_1, y_2)$ for all $y_1, y_2 \in \Gamma \backslash G$. Indeed, if $\gamma \in H(\mathbb{Q}) \cap \text{supp}(\mathcal{T}_{N,x}(\mathcal{T}_{N,x})^*)$ is such that $\text{dist}(\gamma x, x) < N^{-A_2}$, one necessarily must have $\gamma = 1$ together with $\mathcal{T}_p(x^{-1}\gamma x) = \mathcal{T}_p(1) = 0$. It follows that

$$\sum_{u=1}^l |a_u| |\mathcal{M}(x, \alpha_u, N^{-A_2})| \leq \sum_{p \leq N} \sum_{\|\mu\|^* \leq R} |b(p, \mu)| \tau(p, \mu)(1) \ll N$$

since

$$\mathcal{T}_{N,x}(\mathcal{T}_{N,x})^* = \sum_{p \leq N} \sum_{\|\mu\|^* \leq R} b(p, \mu) \tau(p, \mu) + \sum_{p \neq q \leq N} \sum_{\|\mu\|^*, \|\nu\|^* \leq R} a(p, \mu) a(q, \nu) \tau(p, \mu) \tau(q, -\nu),$$

yielding the first bound in (7.2). Thus, the proof is completed. \square

Example 7.10 (Non-equivariant subconvex bounds for division algebras). Let D be a central division algebra over a number field F and $H := \text{Res}_{F/\mathbb{Q}} \text{SL}(1, D)$. If H has no \mathbb{R} -simple components H' for which $H'(\mathbb{R})$ is compact, then Theorem 7.9 yields a subconvex bound for $\Gamma \backslash G$ with G and Γ as in the previous theorem. In particular $G = \text{SL}(n, \mathbb{R})$, there exists a constant $\delta > 0$ such that one has the subconvex bound

$$\|\phi_j\|_\infty \ll \lambda_j^{\frac{n^2-2}{2m}-\delta}$$

for any Hecke–Maass form ϕ_j in $L^2(\Gamma \backslash G)$ with eigenvalue λ_j . It is also possible to consider $H := \text{Res}_{F/\mathbb{Q}} \text{SU}(1, D)$, where D is a central division algebra over a quadratic extension E of F equipped with a E/F -involution. Under the assumption that $G := H(\mathbb{R})$ has no \mathbb{R} -simple components H' for which $H'(\mathbb{R})$ is compact, Theorem 7.9 then implies subconvex bounds for $\Gamma \backslash G$. Regarding the existence of division algebras with E/F -involutions satisfying our assumptions we refer the reader to [44, Theorem 8.1].

7.4. Automorphic forms and representation theoretic interpretation of the results. To close, let us indicate how our results fit into the general theory of automorphic forms [9]. With G , K , and Γ as above let us recall the following

Definition 7.11. *A smooth function $f : G \rightarrow \mathbb{C}$ satisfying*

- (A1) $f(\gamma g) = f(g)$ for all $g \in G$ and $\gamma \in \Gamma$,
- (A2) f is K -finite on the right,
- (A3) f is \mathcal{Z} -finite, where \mathcal{Z} denotes the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ of the complexification of the Lie algebra \mathfrak{g} of G ,

is called an automorphic form on G for Γ .

This definition implies that with respect to the decomposition (2.3) each automorphic form f is contained already in finitely many constituents, so that $f \in \bigoplus_{k=1}^l m(\pi_k, \Gamma) \cdot \pi_k$ for suitable π_k , compare [28, Corollaries 8.14 and 10.37]. Hence, $p(\mathcal{C}) := \prod_{k=1}^l (\mathcal{C} - \mu_k)$, where μ_k is the Casimir eigenvalue of π_k , represents a polynomial that annihilates f . On the other hand, (A2) implies that f is a finite sum of functions f_σ belonging to a specific K -type σ , and by (2.2) one deduces

$$p(\mathcal{C})f_\sigma = \prod_{k=1}^l (2dR(\Omega_K) - \Delta - \mu_k)f_\sigma = \prod_{k=1}^l (2\mu_\sigma - \Delta - \mu_k)f_\sigma =: q(\Delta)f_\sigma$$

where μ_σ denotes the eigenvalue of $dR(\Omega_K)$ on σ , and q is a polynomial. Thus, f is essentially given by a sum of Hecke–Maass forms in the sense of this paper, because $q(\Delta)$ is an elliptic differential operator and any subspace defined by a K -type and a Casimir eigenvalue is finite dimensional by Harish-Chandra’s theorem [9, Theorem 1.7]. Consequently, Theorems 7.4 and 7.9 can be rephrased as follows.

Theorem 7.12. *Let H be a connected semisimple algebraic group over \mathbb{Q} and K_0 an open compact subgroup of $H(\mathbb{A}_{\text{fin}})$. Set $G := H(\mathbb{R})$ and $\Gamma := H(\mathbb{Q}) \cap (H(\mathbb{R})K_0)$, and suppose that $H(\mathbb{A}) =$*

$H(\mathbb{Q})(H(\mathbb{R})K_0)$ and $H(\mathbb{Q})\backslash H(\mathbb{A})$ is compact. Then, there exist a submodule \mathcal{H} of $H_{\mathbb{R}}^{\chi=1}(\mathbb{Q})$ and a constant $\delta > 0$, which are independent of $\sigma \in \widehat{K}$, such that

- (1) there exists an orthonormal basis $\{\phi_j\}_{j \in \mathbb{N}}$ of $L_{\sigma}^2(\Gamma \backslash G)$ which consists of simultaneous eigenfunctions for the Casimir operator \mathcal{C} and all Hecke operators $\mathcal{T} \in \mathcal{H}$;
- (2) for each ϕ_j with Casimir eigenvalue μ_j one has

$$\|\phi_j\|_{\infty} \ll \sqrt{d_{\sigma} \sup_{u \leq \lfloor \frac{\dim K}{2} + 1 \rfloor} \|D^u \sigma\|_{\infty}} (-\mu_j + 2\mu_{\sigma})^{\frac{\dim G/K-1}{4}-\delta},$$

provided that $H = \text{Res}_{F/\mathbb{Q}} \underline{G}$ and (WS) is fulfilled, while in general

$$\|\phi_j\|_{\infty} \ll (-\mu_j + 2\mu_{\sigma})^{\frac{\dim G-1}{4}-\delta}$$

μ_{σ} being the eigenvalue of $dR(\Omega_K)$ on σ . If $K = T$ is a torus,

$$\|\phi_j\|_{\infty} \ll (-\mu_j + 2\mu_{\sigma})^{\frac{\dim G/K-1}{4}-\delta}.$$

□

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