ADDENDUM TO "THE EQUIVARIANT SPECTRAL FUNCTION OF AN INVARIANT ELLIPTIC OPERATOR"

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Abstract. Let M be a compact boundaryless Riemannian manifold, carrying an effective and isometric action of a torus T, and P_0 an invariant elliptic classical pseudodifferential operator on M. In this note, we strengthen the asymptotics for the equivariant (or reduced) spectral function of P_0 derived in [5], which are already sharp in the eigenvalue aspect, to become almost sharp in the isotypic aspect. In particular, this leads to hybrid equivariant L^p-bounds for eigenfunctions that are almost sharp in the eigenvalue and isotypic aspect.

Contents

1. Introduction

Let M be a closed n-dimensional Riemannian manifold with an effective and isometric action of a compact Lie group G. In this paper, we strenghten the asymptotics derived in [5] for the equivariant (or reduced) spectral function of an invariant elliptic operator on M, which are already sharp in the eigenvalue aspect, to become also almost sharp in the isotypic aspect in case that $G = T$ is a torus, that is, a compact connected Abelian Lie group. In particular, if T acts on M with orbits of the same dimension, we obtain hybrid equivariant L^p-bounds for eigenfunctions that are almost sharp up to a logarithmic factor.

To explain our results, consider an elliptic classical pseudodifferential operator

$$P_0 : \mathrm{C}^\infty(M) \longrightarrow \mathrm{L}^2(M)$$

of degree m on M acting on the Hilbert space of square integrable functions on M with the space of smooth functions on M as domain. We assume that P_0 is positive and symmetric, so that it has a unique self-adjoint extension P, which has discrete spectrum. Let $\{E_\lambda\}$ be a spectral resolution of P, and denote by $e(x, y, \lambda)$ the *spectral function* of P which is given by the Schwartz kernel of E_λ. Further, assume that M carries an effective and isometric action of a compact Lie group G with Lie algebra $\mathfrak{g}$ and orbits of dimension less or equal $n-1$. Suppose that P commutes with the left-regular representation $(\pi, \mathrm{L}^2(M))$ of G so that each eigenspace of P becomes a unitary G-module. If $\widehat{G}$ denotes the set of equivalence classes of irreducible unitary representations of G, the Peter-Weyl theorem asserts that

$$\mathrm{L}^2(M)=\bigoplus_{\gamma\in\widehat{G}}\mathrm{L}^2_\gamma(M), \tag{1.1}$$

a Hilbert sum decomposition, where $\mathrm{L}^2_\gamma(M):=\Pi_\gamma \mathrm{L}^2(M)$ denotes the γ-isotypic component, and Π_γ the corresponding projection. Let $e_\gamma(x,y,\lambda)$ be the spectral function of the operator $P_\gamma:=\Pi_\gamma\circ P\circ\Pi_\gamma$, which is also called the *reduced spectral function* of P. Further, let $\mathbb{J}:T^*M\to\mathfrak{g}^*$ denote the momentum map of the Hamiltonian G-action on T^*M, induced by the action of G on M, and write $\Omega:=\mathbb{J}^{-1}(\{0\})$. In [5, Theorem 4.3], the *equivariant local Wey law*

$$\left|e_\gamma(x,x,\lambda)-\lambda^{\frac{n-\kappa_x}{m}}\frac{d_\gamma[\pi_{\gamma|G_x}:\mathbf{1}]}{(2\pi)^{n-\kappa_x}}\int_{\{(x,\xi)\in\Omega,\,p(x,\xi)<1\}}\frac{d\xi}{\operatorname{vol}\mathcal{O}_{(x,\xi)}}\right|\leq C_{x,\gamma}\,\lambda^{\frac{n-\kappa_x-1}{m}},\quad x\in M,$$

was shown as $\lambda\to+\infty$, where $\kappa_x:=\dim\mathcal{O}_x$ is the dimension of the G-orbit through x, d_γ denotes the dimension of an irreducible G-representation π_γ belonging to γ and $[\pi_{\gamma|G_x}:\mathbf{1}]$ the multiplicity of the trivial representation in the restriction of π_γ to the isotropy group G_x of x, while $C_{x,\gamma}>0$ is a constant satisfying

$$C_{x,\gamma}=O_x\big(d_\gamma\sup_{l\leq\lfloor\kappa_x/2+3\rfloor}\left\|\mathcal{D}^l\gamma\right\|_\infty\big), \tag{1.2}$$

and D^l are differential operators on G of order l. Both the leading term and the constant $C_{x,\gamma}$ in general depend in a highly non-uniform way on $x\in M$, exhibiting a caustic behaviour in the neighborhood of singular orbits. A precise description of this caustic behaviour was achieved in [5] by relying on the results [4] on singular equivariant asymptotics obtained via resolution of singularities. More precisely, consider the stratification $M=M(H_1)\dot\cup\ldots\dot\cup M(H_L)$ of M into orbit types, arranged in such a way that $(H_i)\leq(H_j)$ implies $i\geq j$, and let Λ be the maximal length that a maximal totally ordered subset of isotropy types can have. Write $M_{\text{prin}}:=M(H_L)$, M_{except}, and M_{sing} for the union of all orbits of principal, exceptional, and singular type, respectively, so that

$$M=M_{\text{prin}}\,\dot\cup\,M_{\text{except}}\,\dot\cup\,M_{\text{sing}},$$

and denote by $\kappa:=\dim G/H_L$ the dimension of an orbit of principal type. Then, by [5, Theorem 7.7] one has for $x\in M_{\text{prin}}\cup M_{\text{except}}$ and $\lambda\to+\infty$ the *singular equivariant local Weyl law*

$$\begin{aligned}\Big|e_\gamma(x,x,\lambda)-\frac{d_\gamma\lambda^{\frac{n-\kappa}{m}}}{(2\pi)^{n-\kappa}}\sum_{N=1}^{\Lambda-1}\sum_{i_1<\cdots<i_N}\prod_{l=1}^{N}|\tau_{i_l}|^{\dim G-\dim H_{i_l}-\kappa}\mathcal{L}^{0,0}_{i_1\dots i_N}(x,\gamma)\Big|\\ \leq\widetilde{C}_\gamma\lambda^{\frac{n-\kappa-1}{m}}\sum_{N=1}^{\Lambda-1}\sum_{i_1<\cdots<i_N}\prod_{l=1}^{N}|\tau_{i_l}|^{\dim G-\dim H_{i_l}-\kappa-1},\end{aligned}$$

where the multiple sums run over all possible maximal totally ordered subsets $\{(H_{i_1}),\dots,(H_{i_N})\}$ of singular isotropy types, the coefficients $\mathcal{L}^{0,0}_{i_1\dots i_N}$ are explicitly given and bounded functions in x, and $\tau_{i_j}=\tau_{i_j}(x)\in(-1,1)$ are desingularization parameters that arise in the resolution process satisfying $|\tau_{i_j}|\approx\operatorname{dist}(x,M(H_{i_j}))$, while $\widetilde{C}_\gamma>0$ is a constant independent of x and λ that fulfills

$$\widetilde{C}_\gamma=O\big(d_\gamma\sup_{l\leq\lfloor\kappa/2+3\rfloor}\left\|\mathcal{D}^l\gamma\right\|_\infty\big). \tag{1.3}$$

As a major consequence, the above expansions lead to equivariant bounds for eigenfunctions. In the non-singular case, that is, when only principal and exceptional orbits are present, and consequently all G-orbits have the same dimension κ, the hybrid L^q-estimates

$$\|u\|_{\mathrm{L}^q(M)}\leq\begin{cases}C_\gamma\,\lambda^{\frac{\delta_{n-\kappa}(q)}{m}}\,\|u\|_{\mathrm{L}^2}, & \frac{2(n-\kappa+1)}{n-\kappa-1}\leq q\leq\infty,\\ C_\gamma\,\lambda^{\frac{(n-\kappa-1)(2-q')}{4mq'}}\,\|u\|_{\mathrm{L}^2}, & 2\leq q\leq\frac{2(n-\kappa+1)}{n-\kappa-1},\end{cases} \tag{1.4}$$

were shown in [5, Theorem 5.4] for any eigenfunction $u \in \mathrm{L}^2_\gamma(M)$ of P with eigenvalue λ, where $\frac{1}{q}+\frac{1}{q'}=1$, $\delta_n(p) := \max\left(n\left|1/2-1/p\right|-1/2, 0\right)$, and $C_\gamma > 0$ is a constant independent of λ satisfying the estimate

$$C_\gamma \ll \sqrt{d_\gamma \sup_{l \le \lfloor \kappa/2+1 \rfloor} \|D^l \gamma\|_\infty}, \tag{1.5}$$

provided that the co-spheres $S^*_x M$ are strictly convex. Note that for the proof of L^p-bounds it is necessary to describe the caustic behaviour of the relevant spectral kernels as $\mu \to +\infty$ in a neighborhood of the diagonal, which makes things considerably more envolved. In case that singular orbits are present, one has the pointwise bound

$$\sum_{\substack{\lambda_j \in (\lambda, \lambda+1], \\ e_j \in \mathrm{L}^2_\gamma(M)}} |e_j(x)|^2 \le \begin{cases} C\, \lambda^{\frac{n-1}{m}}, & x \in M_{\mathrm{sing}}, \\ \widetilde{C}_\gamma\, \lambda^{\frac{n-\kappa-1}{m}} \sum\limits_{N=1}^{\Lambda-1} \sum\limits_{i_1<\dots<i_N} \prod\limits_{l=1}^{N} |\tau_{i_l}|^{\dim G - \dim H_{i_l} - \kappa - 1}, & x \in M - M_{\mathrm{sing}}, \end{cases} \tag{1.6}$$

for a constant $C>0$ independent of γ, where $\{e_j\}_{j\ge 0}$ is an orthonormal basis of $\mathrm{L}^2(M)$ compatible with the decomposition (1.1), showing that eigenfunctions tend to concentrate along lower dimensional orbits.

The aim of this note is to sharpen the above results in the isotypic aspect in case that $G=T$ is a torus, and show that instead of the bounds (1.2) and (1.3) one has the better estimates

$$C_{x,\gamma} = O_x\Big(\sup_{l\le 1} \left\|D^l\gamma\right\|_\infty\Big), \qquad \widetilde{C}_\gamma = O\Big(\sup_{l\le 1} \left\|D^l\gamma\right\|_\infty\Big), \qquad \gamma \in \mathcal{W}_\lambda,$$

where $\mathcal{W}_\lambda$ denotes the set of representations

$$\mathcal{W}_\lambda := \left\{\gamma \in \widehat{T}' \mid |\gamma| \le \frac{\lambda^{1/m}}{\log \lambda}\right\}.$$

Here $\widehat{T}' \subset$ stands for the subset of representations occuring in the Peter-Weyl decomposition (1.1), and we denoted the differential of a character $\gamma \in \widehat{T}$, which corresponds to an integral linear form $\gamma : \mathfrak{t} \to i\mathbb{R}$, by the same letter. Similarly, it will be shown that the constant C_γ in (1.5) actually satisfies the bound

$$C_\gamma \ll 1, \qquad \gamma \in \mathcal{W}_\lambda.$$

By the equivariant Weyl law [4] and Gauss' law, $|\gamma|$ can grow at most of rate $\lambda^{1/m}$. Thus, the bounds (1.4) hold for *almost any* eigenfunction $u \in \mathrm{L}^2(M)$ with C_γ independent of γ, which is consistent with recent results of Tacy [7]. As will be discussed, the improved bounds are almost sharp in this sense, being already attained for SO(2)-actions on the 2-sphere and the 2-torus. For their proof, a careful examination of the remainder in the stationary phase expansion of the relevant spectral kernels is necessary. These bounds are crucial for deriving hybrid subconvex bounds for Hecke-Maass forms on compact arithimetic quotients of semisimple Lie groups in the eigenvalue and isotypic aspect [6].

Through the whole document, the notation $O(\mu^k), k \in \mathbb{R} \cup \{\pm\infty\}$, will mean an upper bound of the form $C\mu^k$ with a constant $C>0$ that is uniform in all relevant variables, while $O_\aleph(\mu^k)$ will denote an upper bound of the form $C_\aleph\, \mu^k$ with a constant $C_\aleph > 0$ that depends on the indicated variable $\aleph$. In the same way, we shall write $a \ll_\aleph b$ for two real numbers a and b, if there exists a constant $C_\aleph > 0$ depending only on $\aleph$ such that $|a| \le C_\aleph b$, and similarly $a \ll b$, if the bound is uniform in all relevant variables. Finally, $\mathbb{N}$ will denote the set of natural numbers $0, 1, 2, 3, \dots$.

2. The reduced spectral function of an invariant elliptic operator

Let M be a closed connected Riemannian manifold of dimension n with Riemannian volume density dM, and P_0 an elliptic classical pseudodifferential operator on M of degree m which is positive and symmetric. The principal symbol $p(x,\xi)$ of P_0 constitutes a strictly positive function on $T^*M \setminus \{0\}$, where T^*M denotes the cotangent bundle of M. The operator P_0 has a unique self-adjoint extension

P, its domain being the m-th Sobolev space $H^m(M)$. It is well known that there exists an orthonormal basis $\{e_j\}_{j\geq 0}$ of $\mathrm{L}^2(M)$ consisting of eigenfunctions of P with eigenvalues $\{\lambda_j\}_{j\geq 0}$ repeated according to their multiplicity, and that $Q := \sqrt[m]{P}$ constitutes a classical pseudodifferential operator of order 1 with principal symbol $q(x,\xi) := \sqrt[m]{p(x,\xi)}$ and domain $H^1(M)$. Again, Q has discrete spectrum, and its eigenvalues are given by $\mu_j := \sqrt[m]{\lambda_j}$. The spectral function $e(x,y,\lambda)$ of P can then be described by studying the spectral function of Q, which in terms of the basis $\{e_j\}$ is given by

$$e(x,y,\mu) := \sum_{\mu_j \leq \mu} e_j(x)\overline{e_j(y)}, \qquad \mu \in \mathbb{R},$$

and belongs to $\mathrm{C}^\infty(M \times M)$ as a function of x and y. Let χ_μ be the spectral projection onto the sum of eigenspaces of Q with eigenvalues in the interval $(\mu, \mu+1]$, and denote its Schwartz kernel by $\chi_\mu(x,y) := e(x,y,\mu+1) - e(x,y,\mu)$. To obtain an asymptotic description of the spectral function of Q let $\varrho \in \mathcal{S}(\mathbb{R}, \mathbb{R}_+)$ be such that $\varrho(0) = 1$ and supp $\hat{\varrho} \in (-\delta/2, \delta/2)$ for a given $\delta > 0$, and define the approximate spectral projection operator

$$\widetilde{\chi}_\mu u := \sum_{j=0}^{\infty} \varrho(\mu - \mu_j) E_j u, \qquad u \in \mathrm{L}^2(M), \tag{2.1}$$

where E_j denotes the orthogonal projection onto the subspace spanned by e_j. Clearly, $K_{\widetilde{\chi}_\mu}(x,y) := \sum_{j=0}^\infty \varrho(\mu-\mu_j) e_j(x)\overline{e_j(y)} \in \mathrm{C}^\infty(M \times M)$ constitutes the kernel of $\widetilde{\chi}_\mu$. As Hörmander [2] showed, $\widetilde{\chi}_\mu$ can be approximated by Fourier integral operators yielding an asymptotic formula for the kernels of $\widetilde{\chi}_\mu$ and χ_μ, and finally for the spectral function of Q and P.

Now, assume that M carries an effective and isometric action of a compact Lie group G. Let P commute with the left-regular representation $(\pi, \mathrm{L}^2(M))$ of G. Consider the Peter-Weyl decomposition of $\mathrm{L}^2(M)$, and let Π_γ be the projection onto the isotypic component belonging to $\gamma \in \widehat{G}$, which is given by the Bochner integral

$$\Pi_\gamma = d_\gamma \int_G \overline{\gamma(g)} \pi(g)\, d_G(g),$$

where d_γ is the dimension of an unitary irreducible representation of class γ, and $d_G(g) \equiv dg$ Haar measure on G, which we assume to be normalized such that $\operatorname{vol} G = 1$. If G is finite, d_G is simply the counting measure. In addition, let us suppose that the orthonormal basis $\{e_j\}_{j\geq 0}$ is compatible with the Peter-Weyl decomposition in the sense that each vector e_j is contained in some isotypic component $\mathrm{L}^2_\gamma(M)$. In order to describe the spectral function of the operator $Q_\gamma := \Pi_\gamma \circ Q \circ \Pi_\gamma = Q \circ \Pi_\gamma = \Pi_\gamma \circ Q$ given by

$$e_\gamma(x,y,\mu) := \sum_{\mu_j \leq \mu,\, e_j \in \mathrm{L}^2_\gamma(M)} e_j(x)\overline{e_j(y)}, \tag{2.2}$$

we consider the composition $\chi_\mu \circ \Pi_\gamma$ with kernel $K_{\chi_\mu \circ \Pi_\gamma}(x,y) = e_\gamma(x,y,\lambda+1) - e_\gamma(x,y,\lambda)$, together with the corresponding equivariant approximate spectral projection

$$(\widetilde{\chi}_\mu \circ \Pi_\gamma) u = \sum_{j\geq 0,\, e_j \in \mathrm{L}^2_\gamma(M)} \varrho(\mu - \mu_j) E_j u. \tag{2.3}$$

Its kernel can be written as

$$K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x,y) := \sum_{j\geq 0, e_j \in \mathrm{L}^2_\gamma(M)} \varrho(\mu-\mu_j) e_j(x)\overline{e_j(y)} \in \mathrm{C}^\infty(M \times M).$$

By using Fourier integral operator methods, it was shown in [5] that the kernel of $\widetilde{\chi}_\mu \circ \Pi_\gamma$ can be expressed as follows. Let $\{(\kappa_\iota, Y_\iota)\}_{\iota \in I}$, $\kappa_\iota : Y_\iota \xrightarrow{\simeq} \widetilde{Y}_\iota \subset \mathbb{R}^n$, be an atlas for M, $\{f_\iota\}$ a corresponding partition of unity, and $\{\bar{f}_\iota\}$ a set of test functions with compact support in Y_ι satisfying $\bar{f}_\iota \equiv 1$ on supp f_ι. Consider further a test function $0 \leq \alpha \in \mathrm{C}^\infty_{\mathrm{c}}(1/2, 3/2)$ such that $\alpha \equiv 1$ in a neighborhood of 1, and set

$$
\begin{aligned}
I^\gamma_\iota(\mu, R, s, x, y) := \int_G \int_{\Sigma^{R,s}_{\iota,x}} & e^{i\mu\Phi_{\iota,x,y}(\omega,g)}\, \hat{\varrho}(s)\overline{\gamma(g)} f_\iota(x) \\
& \cdot a_\iota(s, \kappa_\iota(x), \mu\omega) \bar{f}_\iota(g \cdot y)\alpha(q(x,\omega)) J_\iota(g, y)\, d\Sigma^{R,s}_{\iota,x}(\omega)\, dg,
\end{aligned}
\tag{2.4}
$$

where $\Phi_{\iota,x,y}(\omega, g) := \langle \kappa_\iota(x) - \kappa_\iota(g \cdot y), \omega\rangle$, $a_\iota \in S^0_{\mathrm{phg}}$ is a suitable classical polyhomogeneous symbol satisfying $a_\iota(0, \tilde{x}, \eta) = 1$, $J_\iota(g, y)$ a Jacobian, and

$$
\Sigma^{R,s}_{\iota,x} := \{\omega \in \mathbb{R}^n \mid \zeta_\iota(s, \kappa_\iota(x), \omega) = R\}
\tag{2.5}
$$

is a smooth compact hypersurface given in terms of a smooth function ζ_ι which is homogeneous in η of degree 1 and satisfies $\zeta_\iota(0, \tilde{x}, \eta) = q(\kappa_\iota^{-1}(\tilde{x}), \eta)$. Then, by [5, Corollary 2.2] one has for $\mu \geq 1$, $x, y \in M$, and each $\tilde{N} \in \mathbb{N}$ the asymptotic expansion

$$
K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, y) = \Big(\frac{\mu}{2\pi}\Big)^{n-1} \frac{d_\gamma}{2\pi} \sum_\iota \Big[\sum_{j=0}^{\tilde{N}-1} D^{2j}_{R,s} I^\gamma_\iota(\mu, R, s, x, y)_{|(R,s)=(1,0)}\, \mu^{-j} + \mathcal{R}^\gamma_\iota(\mu, x, y)\Big]
\tag{2.6}
$$

up to terms of order $O(|\mu|^{-\infty} \|\gamma\|_\infty)$ which are uniform in x, y, where $D^{2j}_{R,s}$ are known differential operators of order $2j$ in R, s, and

$$
|\mathcal{R}^\gamma_\iota(\mu, x, y)| \leq C\mu^{-\tilde{N}} \sum_{|\beta| \leq 2\tilde{N}+3} \sup_{R,s} \big| \partial^\beta_{R,s} I^\gamma_\iota(\mu, R, s, x, y)\big|
$$

for some constant $C > 0$. On the other hand, $K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, y)$ is rapidly decaying as $\mu \to -\infty$ and uniformly bounded in x, y by $\|\gamma\|_\infty$.

3. Equivariant asymptotics of oscillatory integrals

Let the notation be as in the previous section. As we have seen there, the question of describing the spectral function in the equivariant setting reduces to the study of oscillatory integrals of the form

$$
I^\gamma_{x,y}(\mu) := \int_G \int_{\Sigma^{R,s}_x} e^{i\mu\Phi_{x,y}(\omega,g)} \overline{\gamma(g)} a(x, y, \omega, g)\, d\Sigma^{R,s}_x(\omega)\, dg, \qquad \mu \to +\infty,
\tag{3.1}
$$

with $\Sigma^{R,s}_x$ as in (2.5) and phase function

$$
\Phi_{x,y}(\omega, g) := \langle \kappa(x) - \kappa(g \cdot y), \omega\rangle\,,
$$

where we have skipped the index ι for simplicity of notation, and $a \in \mathrm{C}^\infty_{\mathrm{c}}$ is an amplitude that might depend on μ and other parameters such that $(x, y, \omega, g) \in \operatorname{supp} a$ implies $x, g \cdot y \in Y$. In what follows, we shall write ${}^yG := \{g \in G \mid g \cdot y \in Y\}$, as well as

$$
I^\gamma_x(\mu) := I^\gamma_{x,x}(\mu), \qquad \Phi_x := \Phi_{x,x}.
\tag{3.2}
$$

Let us assume in the following that G is a continuous group, and write $\kappa(x) = (\tilde{x}_1, \dots, \tilde{x}_n)$ so that the canonical local trivialization of T^*Y reads

$$
Y \times \mathbb{R}^n \ni (x, \eta) \quad \equiv \quad \sum_{k=1}^n \eta_k (d\tilde{x}_k)_x \in T^*_x Y.
$$

With respect to this trivialization, we shall identify $\Sigma^{R,s}_{x'}$ with a subset in $T^*_x Y$ for eventually different x and x', if convenient. Let $\Omega := \mathbb{J}^{-1}(\{0\})$ be the zero level set of the momentum map $\mathbb{J} : T^*M \to \mathfrak{g}^*$ of the underlying Hamiltonian G-action on T^*M. Let $\mathcal{O}_x := G \cdot x$ denote the G-orbit and $G_x := \{g \in G \mid g \cdot x = x\}$ the stabilizer or isotropy group of a point $x \in M$. Throughout the paper, it is assumed that

$$
\dim \mathcal{O}_x \leq n - 1 \qquad \text{for all } x \in M.
$$

Let further $N_y\mathcal{O}_x$ be the normal space to the orbit $\mathcal{O}_x$ at a point $y \in \mathcal{O}_x$, which can be identified with $\mathrm{Ann}(T_y\mathcal{O}_x)$ via the underlying Riemannian metric. For $x \in Y$ and $\mathcal{O}_y \cap Y \neq \emptyset$ let

$$\mathrm{Crit}\,\Phi_{x,y} := \Big\{(\omega, g) \in \Sigma_x^{R,s} \times {}^yG \mid d(\Phi_{x,y})_{(\omega,g)} = 0\Big\}$$

be the critical set of $\Phi_{x,y}$. With M_{prin}, M_{except}, and M_{sing} denoting the principal, exceptional, and singular stratum, respectively, it was shown in [5, Lemma 3.1] that

- if $y \in \mathcal{O}_x$, the set $\mathrm{Crit}\,\Phi_{x,y}$ is clean and given by the smooth submanifold
$$\mathcal{J} = \big\{(\omega, g) \mid (g \cdot y, \omega) \in \Omega,\, x = g \cdot y\big\} = V_{\mathcal{J}} \times G_{\mathcal{J}}$$
of codimension $2\dim \mathcal{O}_x$, with $V_{\mathcal{J}} = \Sigma_x^{R,s} \cap N_x\mathcal{O}_x$ and $G_{\mathcal{J}} = \{g \in G \mid x = g \cdot y\} \subset {}^yG$.
- if $y \notin \mathcal{O}_x$,
$$\mathrm{Crit}\,\Phi_{x,y} = \Big\{(\omega, g) \mid (g \cdot y, \omega) \in \Omega,\, \kappa(x) - \kappa(g \cdot y) \in N_\omega \Sigma_x^{R,s}\Big\};$$
furthermore, assume that G acts on M with orbits of the same dimension κ, that is, $M = M_{\mathrm{prin}} \cup M_{\mathrm{except}}$, and that the co-spheres S_x^*M are strictly convex. Then, either $\mathrm{Crit}\,\Phi_{x,y}$ is empty, or, choosing Y sufficiently small, $\mathrm{Crit}\,\Phi_{x,y}$ is clean and of codimension $n-1+\kappa$, its finitely many connected components being of the form
$$\mathcal{J} = V_{\mathcal{J}} \times G_{\mathcal{J}}$$
with $V_{\mathcal{J}} = \{\omega_{\mathcal{J}}\}$ and $G_{\mathcal{J}} = g_{\mathcal{J}} \cdot G_y \subset {}^yG$ for some $\omega_{\mathcal{J}} \in \Sigma_x^{R,s}$ and $g_{\mathcal{J}} \in G$.

From this an asymptotic expansion for the integrals $I^\gamma_{x,y}(\mu)$ was deduced in [5, Theorem 3.3], yielding a corresponding asymptotic formula for $K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x, y)$. In this paper, we improve the estimate for the remainder in the isotypic aspect in case that $G = T$ is a torus, which we assume from now on.

For this, recall that the exponential function exp is a covering homomorphism of $\mathfrak{t}$ onto T, and its kernel L a lattice in $\mathfrak{t}$. Let $\widehat{T}$ denote the *set of characters of* T, that is, of all continuous homomorphisms of T into the circle, which we identify with the unitary dual of T. The differential of a character $\gamma : T \to S^1$, denoted by the same letter, is a linear form $\gamma : \mathfrak{t} \to i\mathbb{R}$ which is *integral* in the sense that $\gamma(L) \subset 2\pi i\,\mathbb{Z}$. On the other hand, if γ is an integral linear form, one defines

$$t^\gamma = e^{\gamma(X)}, \qquad t = \exp X \in T,$$

setting up an identification of $\widehat{T}$ with the integral linear forms on $\mathfrak{t}$ via $\gamma(t) \equiv t^\gamma$. Further, all irreducible representations of T are 1-dimensional. We now make the following

Definition 3.1. *Denote by $\widehat{T}' \subset \widehat{T}$ the subset of representations occuring in the decomposition* (1.1) *of* $\mathrm{L}^2(M)$*, and let $\{\mathcal{V}_\mu\}_{\mu\in(0,\infty)}$ be a family of finite subsets $\mathcal{V}_\mu \subset \widehat{T}'$ such that*

$$\max_{\gamma\in\mathcal{V}_\mu} |\gamma| \leq C\frac{\mu}{\log \mu}$$

for a constant $C > 0$ independent of μ.

Our main result is the following improvement of the remainder and coefficient estimates in [5, Theorem 3.3].

Theorem 3.2. *Assume that T is a torus acting on M with orbits of dimension less or equal $n-1$, and let $\mathcal{V}_\mu$ be as in the previous definition.*

(a) *Let $y \in \mathcal{O}_x$. Then, for every $\gamma \in \widehat{T}$ and $\tilde{N} = 0, 1, 2, \dots$ one has the asymptotic formula*
$$I^\gamma_{x,y}(\mu) = (2\pi/\mu)^{\dim \mathcal{O}_x}\left[\sum_{k=0}^{\tilde{N}-1} \mathcal{Q}_k(x, y)\mu^{-k} + \mathcal{R}_{\tilde{N}}(x, y, \mu)\right], \qquad \mu \to +\infty,$$
where the coefficients and the remainder depend smoothly on R and s. The coefficients satisfy the bounds
$$|\mathcal{Q}_k(x, y)| \leq C_{k,\Phi_{x,y}}\, vol(\mathrm{supp}\ a(x, y, \cdot, \cdot) \cap \mathcal{C}_{x,y}) \sup_{l\leq k} \big\|(D^{2l}_\omega D^l_t \gamma a)(x, y, \cdot, \cdot)\big\|_\infty$$

while the remainder satisfies

$$|\mathcal{R}_{\tilde N}(x,y,\mu)| \leq \widetilde{C}_{\tilde N,\Phi_{x,y}} vol(\text{supp } a(x,y,\cdot,\cdot)) \cdot \sup_{l\leq 2\tilde N+\dim \mathcal{O}_x+1} \left\|(D^l_\omega D^l_t a)(x,y,\cdot,\cdot)\right\|_\infty \sup_{l\leq \tilde N} \left\|D^l_t \gamma\right\|_\infty \mu^{-\tilde N}, \qquad \gamma \in \mathcal{V}_\mu.$$

The bounds are uniform in R,s for suitable constants $C_{k,\Phi_{x,y}}>0$ and $\widetilde{C}_{\tilde N,\Phi_{x,y}}>0$, where D^l_ω and D^l_t denote differential operators of order l on $\Sigma^{R,s}_x$ and T, respectively. As functions in x and y, $\mathcal{Q}_k(x,y)$ and $\mathcal{R}_{\tilde N}(x,y,\mu)$ are smooth on $Y\cap M_{\text{prin}}$, and the constants $C_{k,\Phi_{x,y}}$ and $\widetilde{C}_{\tilde N,\Phi_{x,y}}$ are uniformly bounded in x and y if $M=M_{\text{prin}}\cup M_{\text{except}}$.

(b) *Let $y\notin \mathcal{O}_x$. Assume that $M=M_{\text{prin}}\cup M_{\text{except}}$ and that the co-spheres S^*_xM are strictly convex. Then, for sufficiently small Y and every $\tilde N=0,1,2,\dots$ one has the asymptotic formula*

$$I^\gamma_{x,y}(\mu) = \sum_{\mathcal{J}\in\pi_0(\text{Crit }\Phi_{x,y})} (2\pi/\mu)^{\frac{n-1+\kappa}{2}} e^{i\mu\, {}^0\Phi^{\mathcal{J}}_{x,y}} \left[\sum_{k=0}^{\tilde N-1} \mathcal{Q}_{\mathcal{J},k}(x,y)\mu^{-k} + \mathcal{R}_{\mathcal{J},\tilde N}(x,y,\mu)\right]$$

as $\mu\to+\infty$, where $\kappa:=\dim M/T$ and ${}^0\Phi^{\mathcal{J}}_{x,y}$ stands for the constant values of $\Phi_{x,y}$ on the connected components $\mathcal{J}$ of its critical set. The coefficients $\mathcal{Q}_{\mathcal{J},k}(x,y)$ and the remainder term $\mathcal{R}_{\mathcal{J},\tilde N}(x,y,\mu)$ depend smoothly on R,s, and $x,y\in Y\cap M_{\text{prin}}$. Furthermore, they satisfy bounds analogous to the ones in (a), where now derivatives in t up to order $2k$ and $2\tilde N$ can occur, and the constants $C_{k,\Phi_{x,y}}$ and $\widetilde{C}_{\tilde N,\Phi_{x,y}}$ are no longer uniformly bounded, but satisfy

$$C_{k,\Phi_{x,y}} \ll dist(y,\mathcal{O}_x)^{-(n-1-\kappa)/2-k}, \qquad \widetilde{C}_{\tilde N,\Phi_{x,y}} \ll dist(y,\mathcal{O}_x)^{-(n-1-\kappa)/2-\tilde N}.$$

Proof. The asymptotic expansion for the integral $I^\gamma_{x,y}(\mu)$, the smoothness of the coefficients $\mathcal{Q}_k(x,y)$, $\mathcal{Q}_{\mathcal{J},k}(x,y)$, and the remainder terms in the parameters R,s, and $x,y\in Y\cap M_{\text{prin}}$, as well as corresponding bounds for the coefficients and the remainder term were shown in [5, Theorem 3.3]. To improve on the remainder estimate concerning its dependence on γ as $\mu\to+\infty$, we rewrite $I^\gamma_{x,y}(\mu)$ up to a volume factor as

$$I^\gamma_{x,y}(\mu) \equiv \int_{\mathfrak{t}}\int_{\Sigma^{R,s}_x} e^{i\mu\Phi_{x,y}(\omega,\exp(-X))} e^{-\gamma(X)} a(x,y,\omega,X)\, d\Sigma^{R,s}_x(\omega)\, dX, \qquad \gamma\in\widehat{T},$$

where we can assume that a is compactly supported with respect to $X\in\mathfrak{t}$ in a small open connected subset ${}^y\mathfrak{t}\subset\mathfrak{t}$ by choosing Y small. If we were to apply the stationary and non-stationary phase principles to $I^\gamma_{x,y}(\mu)$ with $\Phi_{x,y}$ as phase function, which was the way we followed in [5], this would involve derivatives of the amplitude $\overline{\gamma}a$ and generate non-optimal powers in γ in the remainder estimate. Instead, note that the character $\gamma(t)=e^{\gamma(X)}\in S^1$ constitutes itself a phase, which can oscillate rather quickly as γ increases. To deal with these oscillations, we shall absorb them into the phase function, and define for arbitrary $\xi\in\mathfrak{t}^*$

$$\Phi^\xi_{x,y}(\omega,X) := \Phi_{x,y}(\omega,\mathrm{e}^{-X}) - \xi(X), \qquad t=\exp X\in T.$$

The idea is then to apply the stationary and non-stationary phase principles to the integrals $I^\gamma_{x,y}(\mu)$ with phase function $\Phi^\xi_{x,y}(\omega,X)$ and $\xi=\gamma/i\mu$ as parameter, compare [3, Theorem 7.7.6], to obtain remainder estimates that are optimal in $\gamma\in\mathcal{V}_\mu$. If $\{X_1,\dots,X_d\}$ denotes a basis of $\mathfrak{t}$, the X-derivatives of $\Phi^\xi_{x,y}(\omega,X)$ read

$$\sum_{k=1}^n \omega_k(d\tilde x_k)_{\mathrm{e}^{-X}\cdot y}(\widetilde{X}_j) - \xi(X_j) = [\mathbb{J}(\mathrm{e}^{-X}\cdot y,\omega)-\xi](X_j),$$

so that

$$\text{Crit }\Phi^\xi_{x,y} = \left\{(\omega,X) \mid \kappa(x)-\kappa(\mathrm{e}^{-X}\cdot y)\in N_\omega(\Sigma^{R,s}_x), \quad (\mathrm{e}^{-X}\cdot y,\omega)\in\mathbb{J}^{-1}(\{\xi\})\right\}.$$

A repetition of the arguments given in [5, Proof of Lemma 3.1] then shows that for sufficiently small $|\xi|$

- if $y \in \mathcal{O}_x$, the set $\mathrm{Crit}\, \Phi^\xi_{x,y}$ is clean and given by the smooth submanifold
$$\mathcal{J} = \big\{(\omega, X) \mid (\,\mathrm{e}^{-X} \cdot y, \omega) \in \mathbb{J}^{-1}(\{\xi\}),\, x = \,\mathrm{e}^{-X} \cdot y\big\}$$
of codimension $2 \dim \mathcal{O}_x$;
- if $y \notin \mathcal{O}_x$ and T acts on M with orbits of the same dimension κ and the co-spheres $S^*_x M$ are strictly convex, then either $\mathrm{Crit}\, \Phi^\xi_{x,y}$ is empty, or, choosing Y sufficiently small, $\mathrm{Crit}\, \Phi^\xi_{x,y}$ is clean and of codimension $n - 1 + \kappa$,

which would also just follow from [5, Proof of Lemma 3.1] and the implicit function theorem. In addition, note that for $(\omega, X) \in \mathrm{Crit}\, \Phi^\xi_{x,y}$

$$\mathcal{M}_{x,y}(\omega, X) := \text{Trans Hess}\, \Phi^\xi_{x,y}(\omega, X) \,\text{is independent of}\, \xi.$$

Next, notice that under the assumptions in (a) and (b), respectively, there is an open tubular neighborhood U_0 of $\mathrm{Crit}\, \Phi_{x,y}$ and a constant $\mu_0 > 0$ such that for all $\mu \geq \mu_0$ and $\gamma \in \mathcal{V}_\mu$

- $\mathrm{Crit}\, \Phi^{\gamma/i\mu}_{x,y} \subset U_0$,
- $\mathrm{Crit}\, \Phi^{\gamma/i\mu}_{x,y}$ is clean, that is, $\Phi^{\gamma/i\mu}_{x,y}$ is a Morse-Bott function.

Let U_1 and U_2 be two further open tubular neighborhoods of $\mathrm{Crit}\, \Phi_{x,y}$ and $\mu_0 > \mu_1 > \mu_2 > 0$ be such that $U \subset U_1 \subset U_2$ are proper inclusions and the pairs (U_1, μ_1), (U_2, μ_2) have the same properties than (U_0, μ_0). Let $u \in \mathrm{C}^\infty(U_2, \mathbb{R}^+)$ be a test function with $u_{|U_1} \equiv 1$ and define

$$\begin{aligned} {}^1 I^\gamma_{x,y}(\mu) &:= \int_{\mathfrak{t}} \int_{\Sigma^{R,s}_x} e^{i\mu \Phi^{\gamma/i\mu}_{x,y}(\omega, X)} u(\omega, X) a(x, y, \omega, X)\, d\Sigma^{R,s}_x(\omega)\, dX, \\ {}^2 I^\gamma_{x,y}(\mu) &:= I^\gamma_{x,y}(\mu) -{}^1 I^\gamma_{x,y}(\mu). \end{aligned}$$

By construction, for $\gamma \in \mathcal{V}_\mu$ and $\mu \geq \mu_0$ all critical sets $\mathrm{Crit}\, \Phi^{\gamma/i\mu}_{x,y}$ have a minimal, non-vanishing[1] distance to ∂U_1, so that

$$|\,\mathrm{grad}\, \Phi^{\gamma/i\mu}_{x,y}| \geq C > 0 \quad \text{on supp}\,(1-u) a(x, y, \cdot, \cdot) \text{ for all } \gamma \in \mathcal{V}_\mu \text{ with } \mu \geq \mu_0.$$

An application of the non-stationary phase principle [3, Theorem 7.7.1] with respect to the phase function $\Phi^{\gamma/i\mu}_{x,y}$ then yields for every $k \in \mathbb{N}$ the uniform bound

$${}^2 I^\gamma_{x,y}(\mu) = O_{k,a}(\mu^{-k}) \qquad \text{for all } \gamma \in \mathcal{V}_\mu \text{ with } \mu \geq \mu_0.$$

It remains to estimate the integral ${}^1 I^\gamma_{x,y}(\mu)$ by means of the stationary phase principle with $\xi = \gamma/i\mu$ as parameter, for which we shall follow [3, Theorem 7.7.5] and its proof. Assume as we may that U_2 is sufficiently small, and introduce normal tubular coordinates on U_2 in form of an atlas $\{(\zeta_\iota, \mathcal{Y}_\iota)\}_{\iota \in I}$ such that

(1) $\text{supp}\, u a(x, y, \cdot, \cdot) \subset \bigcup_\iota \mathcal{Y}_\iota$,
(2) $\zeta^{-1}_\iota(m', m'') \in \mathrm{Crit}\, \Phi^\xi_{x,y}$ iff $\mathbb{R}^{d''} \ni m'' = m''_\xi$, where
$$d'' = \begin{cases} 2 \dim \mathcal{O}_x & \text{in case (a)} \\ n - 1 + \kappa & \text{in case (b).} \end{cases}$$
(3) the $\mathfrak{t}$-coordinates are given by standard Euclidean coordinates, so that in each chart
$$X = \sum_\alpha m'_{\mathfrak{t},\alpha} X'_\alpha + \sum_\beta m''_{\mathfrak{t},\beta} X''_\beta$$
for a suitable basis $\{X'_\alpha, X''_\beta\}$ of $\mathfrak{t}$.

[1] At least on the intersection of the support of $a(x, y, \cdot, \cdot)$ and U_1.

Let $\{p_\iota\}$ be a partition of unity subordinated to the covering $\{\mathcal{Y}_\iota\}$, and write $a_\iota(x,y,\omega,X) := p_\iota(\omega,X)a(x,y,\omega,X)$ as well as $a_\iota(x,y,m) := a_\iota(x,y,\zeta_\iota^{-1}(m))\beta_\iota(m)$, β_ι being a Jacobian. Denote the product of $u\circ\zeta_\iota^{-1}$ with the Taylor expansion of $a_\iota(x,y,\cdot)$ in the variable m'' at the point m''_ξ of order $2k$ by $T_\iota^\xi(x,y,m)$, which is smooth and bounded in ξ. Let $\mathcal{M}_{x,y}(\omega,X)$ be as above and set $\mathcal{M}^\iota_{x,y}(m',m''_\xi) := (\mathcal{M}_{x,y}\circ\zeta_\iota^{-1})(m',m''_\xi)$. Since for sufficiently small $|m''-m''_\xi|$

$$\frac{|m''-m''_\xi|}{|\operatorname{grad}_{m''}\Phi^\xi_{x,y}(m',m''_\xi)|} \ll \left\|\mathcal{M}^\iota_{x,y}(m',m''_\xi)^{-1}\right\| \ll 1$$

for all ξ, [3, Theorem 7.7.1] yields with respect to $\Phi^\xi_{x,y}(m) := (\Phi^\xi_{x,y}\circ\zeta_\iota^{-1})(m)$ for any $k\in\mathbb{N}$

$${}^1I^\gamma_{x,y}(\mu) = \sum_\iota \int_{\mathbb{R}^{d'}}\int_{\mathbb{R}^{d''}} e^{i\mu\Phi^{i\gamma/\mu}_{x,y}(m)} T_\iota^{i\gamma/\mu}(x,y,m)\,dm''\,dm' + O_{k,a}(\mu^{-k})$$

uniformly in γ. Next, note that for fixed m'

$$m'' \longmapsto \left\langle \mathcal{M}^\iota_{x,y}(m',m''_\xi)(m''-m''_\xi),(m''-m''_\xi)\right\rangle \tag{3.3}$$

defines a non-degenerate quadratic form, and introduce the auxiliary function

$$H^\xi(m) := \Phi^\xi_{x,y}(m) - \Phi^\xi_{x,y}(m',m''_\xi) - \left\langle \mathcal{M}^\iota_{x,y}(m',m''_\xi)(m''-m''_\xi),(m''-m''_\xi)\right\rangle/2,$$

which vanishes of third order at $m''=m''_\xi$. The function

$${}^s\Phi^\xi_{x,y}(m) := \left\langle \mathcal{M}^\iota_{x,y}(m',m''_\xi)(m''-m''_\xi),(m''-m''_\xi)\right\rangle/2 + sH^\xi(m)$$

interpolates between $\Phi^\xi_{x,y}(m)-\Phi^\xi_{x,y}(m',m''_\xi) = {}^1\Phi^\xi_{x,y}(m)$ and the quadratic form (3.3), and we define

$$\mathcal{I}(s) := \int_{\mathbb{R}^{d''}} e^{i\mu\,{}^s\Phi^\xi_{x,y}(m)} T_\iota^\xi(x,y,m)\,dm''.$$

Taylor expansion then yields

$$\Big|\mathcal{I}(1) - \sum_{l=0}^{2k-1}\mathcal{I}^{(l)}(0)/l!\Big| \ll \sup_{0\le s\le 1}|\mathcal{I}^{(2k)}(s)|/L!.$$

Now, differentiation with respect to s gives

$$\mathcal{I}^{(l)}(s) = \int_{\mathbb{R}^{d''}} e^{i\mu\,{}^s\Phi^\xi_{x,y}(m)}(i\mu H^\xi(m))^l\,T_\iota^\xi(x,y,m)\,dm''.$$

In view of the uniform bounds

$$\frac{|m''-m''_\xi|}{|\operatorname{grad}_{m''}\,{}^s\Phi^\xi_{x,y}(m',m''_\xi)|} \ll \left\|\mathcal{M}^\iota_{x,y}(m',m''_\xi)^{-1}\right\| \ll 1 \qquad \text{for all } \xi \text{ and } s$$

and[2]

$$\left|D^\alpha_{m''}[H^\xi(m)^{2k}\,T_\iota^\xi(x,y,m)]\right| \ll |m''-m''_\xi|^{6k-|\alpha|} \quad \text{for all } \xi$$

we obtain from [3, Theorem 7.7.1] with k replaced by $3k$ there the important uniform bound

$$\mathcal{I}^{(2k)}(s) = O(\mu^{-k}) \qquad \text{for all } \gamma\in\mathcal{V}_\mu \text{ with } \mu\ge\mu_0 \text{ and all } s.$$

Next, denote by $\mathcal{H}^\xi(m)$ the Taylor expansion of $H^\xi(m)$ of order $3k$, and notice that one has

$$(H^\xi)^l - (\mathcal{H}^\xi)^l = O(|m''-m''_\xi|^{2k+2l})$$

[2]Note that $D^\alpha_{m''}H^\xi(m) = D^\alpha_{m''}\Phi_{x,y}(m)$ for $|\alpha|\ge 3$, while for $|\alpha|\le 2$ Taylor expansion at m''_ξ implies

$$|D^\alpha_{m''}H^\xi(m)| \ll |m''-m''_\xi|^{3-|\alpha|}\sum_{|\beta|=3}\sup|D^\beta_{m''}\Phi_{x,y}(m)| \ll |m''-m''_\xi|^{3-|\alpha|}$$

uniformly in ξ since $H^\xi(m)$ depends on ξ only via the term $\xi\big(\sum_\alpha m'_{\mathfrak{t},\alpha}X'_\alpha + \sum_\beta m''_{\mathfrak{t},\beta}X''_\beta\big)$, which vanishes when differentiated more than one time.

uniformly in ξ. Applying again [3, Theorem 7.7.1] gives

$$\mathcal{I}^{(l)}(0) = \int_{\mathbb{R}^{d''}} e^{i\mu\, {}^0\Phi^{\xi}_{x,y}(m)} (i\mu \mathcal{H}^{\xi}(m))^l \, T^{\xi}_{\iota}(x,y,m) \, dm'' + O_{k,a}(\mu^{-k})$$

uniformly in ξ. The assertion now follows by taking into account [3, Lemma 7.7.3] and the final arguments in the proof of [3, Theorem 7.7.5]. Note that the Taylor expansion $\mathcal{H}^{\xi}$ starts with terms of degree 3 and depends on ξ in that the coefficients are evaluated at $m'' = m''_{\xi}$. Consequently, when applied to $\mathcal{I}^{(l)}(0)$ the remainder estimate in [3, Lemma 7.7.3] can be uniformly estimated in ξ. The final remainder estimate results from the above uniform estimates, and local contributions of higher order where additional derivatives of γ arise. The local terms are unique, and coincide with the ones with phase function $\Phi_{x,y}$ and amplitude $\overline{\gamma} a$ considered in [5, Theorem 3.3], from which the corresponding bounds are deduced. The fact that in case (a) only t-derivatives of order k and $\tilde{N}$ occur, follows from the particular form of the transversal Hessian, [5, Proof of Theorem 3.3]. □

Similarly, one derives

Theorem 3.3. *Consider the integrals $I^{\gamma}_{x,y}(\mu)$ defined in (3.1). Assume that the torus T acts on M with orbits of the same dimension $\kappa \leq n-1$, and that the co-spheres $S^*_x M$ are strictly convex. Then, for sufficiently small Y and arbitrary $\tilde{N}_1, \tilde{N}_2 \in \mathbb{N}$ one has the asymptotic formula*

$$\begin{aligned} &I^{\gamma}_{x,y}(\mu) \\ &= \sum_{\mathcal{J} \in \pi_0(\mathrm{Crit}\,\Phi_{x,y})} \frac{e^{i\mu\, {}^0\Phi^{\mathcal{J}}_{x,y}}}{\mu^{\kappa} (\mu \left\|\kappa(x) - \kappa(g_{\mathcal{J}} \cdot y)\right\| + 1)^{\frac{n-1-\kappa}{2}}} \Bigg[\sum_{k_1,k_2=0}^{\tilde{N}_1-1, \tilde{N}_2-1} \frac{\mathcal{Q}_{\mathcal{J},k_1,k_2}(x,y)}{\mu^{k_1} (\mu \left\|\kappa(x) - \kappa(g_{\mathcal{J}} \cdot y)\right\| + 1)^{k_2}} \\ &\qquad + \mathcal{R}_{\mathcal{J},\tilde{N}_1,\tilde{N}_2}(x,y,\mu) \Bigg] \end{aligned}$$

as $\mu \to +\infty$. The coefficients and the remainder term depend smoothly on R, t, while ${}^0\Phi^{\mathcal{J}}_{x,y} := R\, c_{x, g_{\mathcal{J}} \cdot y}(t)$ denotes the constant value of $\Phi_{x,y}$ on $\mathcal{J}$. Furthermore, the coefficients are uniformly bounded in R, s, x, and y by derivatives of γ up to order $2k_1$, and the remainder term

$$\mathcal{R}_{\mathcal{J},\tilde{N}_1,\tilde{N}_2}(x,y,\mu) = O_{\mathcal{J},\tilde{N}_1,\tilde{N}_2}\left(\mu^{-\tilde{N}_1} (\mu \left\|\kappa(x) - \kappa(g_{\mathcal{J}} \cdot y)\right\| + 1)^{-\tilde{N}_2} \right)$$

by derivatives of γ up to order $2\tilde{N}_1$, provided that $\gamma \in \mathcal{V}_{\mu}$.

Proof. The proof is essentially the same than the one of [5, Theorem 3.4], using the arguments given in the proof of the previous theorem. □

4. The equivariant local Weyl law

We shall now prove an improved version of the equivariant local Weyl derived in [5]. For this, we first prove the following refinement of [5, Proposition 4.1].

Proposition 4.1 (Point-wise asymptotics for the kernel of the equivariant approximate projection). *For any fixed $x \in M$, $\gamma \in \widehat{T}$, and $\tilde{N} \in \mathbb{N}$ one has as $\mu \to +\infty$*

$$\begin{aligned} K_{\widetilde{\chi}_{\mu} \circ \Pi_{\gamma}}(x,x) &= \sum_{j \geq 0,\, e_j \in \mathrm{L}^2_{\gamma}(M)} \varrho(\mu - \mu_j) |e_j(x)|^2 \\ &= \left(\frac{\mu}{2\pi}\right)^{n - \dim \mathcal{O}_x - 1} \frac{d_{\gamma}}{2\pi} \left[\sum_{k=0}^{\tilde{N}-1} \mathcal{L}_k(x,\gamma) \mu^{-k} + \mathcal{R}_{\tilde{N}}(x,\gamma) \right] \end{aligned} \tag{4.1}$$

with coefficients and remainder depending smoothly on $x \in M_{\mathrm{prin}}$. They satisfy the bounds

$$|\mathcal{L}_k(x,\gamma)| \leq C_{k,x} \sup_{l \leq k} \left\| D^l \gamma \right\|_{\infty},$$

as well as

$$|\mathcal{R}_{\tilde{N}}(x,\gamma)| \leq \tilde{C}_{\tilde{N},x} \sup_{l\leq \tilde{N}} \left\|D^l\gamma\right\|_\infty \mu^{-\tilde{N}}, \qquad \gamma \in \mathcal{V}_\mu,$$

where D^l denotes a differential operator on T of order l, and the constants $C_{k,x}$, $\tilde{C}_{\tilde{N},x}$ are uniformly bounded in x if $M = M_{\mathrm{prin}} \cup M_{\mathrm{except}}$. In particular, the leading coefficient is given by

$$\mathcal{L}_0(x,\gamma) = \hat{\varrho}(0)[\pi_{\gamma|T_x} : \mathbf{1}]\, vol\,[(\Omega \cap S^*_x M)/T],$$

where $S^*M := \{(x,\xi) \in T^*M \mid p(x,\xi) = 1\}$. If $\mu \to -\infty$, the function $K_{\widetilde{\chi}_\mu \circ \Pi_\gamma}(x,x)$ is rapidly decreasing in μ.

Proof. We only have to prove the bounds for the coefficients and the remainder, since all other asssertions have been shown in [5]. Let the notation be as in Section 2, and $R, s \in \mathbb{R}$, $x \in Y_\iota$ be fixed. As a direct consequence of Theorem 3.2 (a) we have for any $\tilde{N} \in \mathbb{N}$

$$\partial^\beta_{R,s} I^\gamma_\iota(\mu,R,s,x,x) = (2\pi/\mu)^{\dim \mathcal{O}_x} \left[\sum_{k=0}^{\tilde{N}-1} \mathcal{L}^k_{\iota,\beta}(R,s,x,\gamma)\mu^{-k} + \mathcal{R}^{\tilde{N}}_{\iota,\beta}(R,s,x,\gamma,\mu)\right],$$

where the coefficients and the remainder term are explicitly given and depend smoothly on R, s, and $x \in Y \cap M_{\mathrm{prin}}$. Furthermore, both the coefficients $\mathcal{L}^k_{\iota,\beta}(R,s,x,\gamma)$ and the remainder are bounded by expressions involving derivatives of γ up to order k and $\tilde{N}$, respectively, which are uniformly bounded in x if $M = M_{\mathrm{prin}} \cup M_{\mathrm{except}}$. Equation (2.6) then implies the asymptotic expansion (4.1) with the specified estimate for the remainder. □

We can now sharpen [5, Theorem 4.3] in the isotypic aspect as follows.

Theorem 4.2 (**Equivariant local Weyl law**). *Let M be a closed connected Riemannian manifold M of dimension n carrying an isometric and effective action of a torus T, and P_0 a T-invariant elliptic classical pseudodifferential operator on M of degree m. Let $p(x,\xi)$ be its principal symbol, and assume that P_0 is positive and symmetric. Denote its unique self-adjoint extension by P, and for a given $\gamma \in \widehat{T}$ let $e_\gamma(x,y,\lambda)$ be its reduced spectral function. Further, let $\mathbb{J} : T^*M \to \mathfrak{t}^*$ be the momentum map of the T-action on M, and put $\Omega := \mathbb{J}^{-1}(\{0\})$. Then, for fixed $x \in M$ one has*

$$\left| e_\gamma(x,x,\lambda) - \frac{[\pi_{\gamma|T_x} : \mathbf{1}]}{(2\pi)^{n-\kappa_x}} \lambda^{\frac{n-\kappa_x}{m}} \int_{\{\xi |\, (x,\xi)\in\Omega,\, p(x,\xi)<1\}} \frac{d\xi}{vol\,\mathcal{O}_{(x,\xi)}} \right| \leq C_{x,\gamma}\, \lambda^{\frac{n-\kappa_x-1}{m}} \tag{4.2}$$

as $\lambda \to +\infty$, where $\kappa_x := \dim \mathcal{O}_x$ and $[\pi_{\gamma|T_x} : \mathbf{1}] \in \{0,1\}$ denotes the multiplicity of the trivial representation in the restriction of π_γ to the isotropy group T_x of x. Furthermore, for arbitrary $\gamma \in \mathcal{W}_\lambda := \left\{\gamma \in \widehat{T}' \mid |\gamma| \leq \frac{\lambda^{1/m}}{\log \lambda}\right\}$

$$C_{x,\gamma} = O_x\Big(\sup_{l\leq 1} \left\|D^l\gamma\right\|_\infty\Big) = O_x(|\gamma|) \tag{4.3}$$

is a constant that depends smoothly on $x \in M_{\mathrm{prin}}$ and is uniformly bounded in x if $M = M_{\mathrm{prin}} \cup M_{\mathrm{except}}$.

Proof. This follows directly by taking $\tilde{N} = 1$ in (4.1) and integrating with respect to μ from $-\infty$ to $\sqrt[m]{\lambda}$ with the arguments given in [1, Proof of Eq. (2.25)]. □

Remark 4.3.

(1) With the same constant $C_{x,\gamma}$ as in (4.2) one also has the bound

$$|e_\gamma(x,y,\lambda+1) - e_\gamma(x,y,\lambda)| \leq \sqrt{C_{x,\gamma}\lambda^{\frac{n-\kappa_x-1}{m}}}\sqrt{C_{y,\gamma}\lambda^{\frac{n-\kappa_y-1}{m}}}, \qquad x,y \in M,\ \gamma \in \mathcal{W}_\lambda,$$

compare [5, Remark 4.4].

(2) As a consequence of Theorem 4.2, the constant $C_{x,\gamma}$ in [5, Corollary 4.6] can be improved accordingly, as well as all examples given in [5, Section 4].

5. Equivariant L^p-bounds of eigenfunctions for non-singular group actions

Let the notation be as in the previous sections. As a consequence of the improved point-wise asymptotics for the kernel of the equivariant approximate projection, one obtains in the non-singular case the following sharpened equivariant L^∞-bounds for eigenfunctions.

Proposition 5.1 (L^∞**-bounds for isotypic spectral clusters).** *Assume that T acts on M with orbits of the same dimension κ, and denote by χ_λ the spectral projection onto the sum of eigenspaces of P with eigenvalues in the interval $(\lambda, \lambda+1]$. Then, for any $\gamma \in \mathcal{W}_\lambda$,*

$$\|(\chi_\lambda \circ \Pi_\gamma)u\|_{\mathrm{L}^\infty(M)} \leq C(1+\lambda)^{\frac{n-\kappa-1}{2m}} \|u\|_{\mathrm{L}^2(M)}, \qquad u \in \mathrm{L}^2(M), \tag{5.1}$$

for a positive constant C independent of γ. In particular, we obtain

$$\|u\|_{\mathrm{L}^\infty(M)} \ll \lambda^{\frac{n-\kappa-1}{2m}}$$

for any eigenfunction $u \in \mathrm{L}^2_\gamma(M)$ of P with eigenvalue λ satisfying $\|u\|_{\mathrm{L}^2} = 1$ and $\gamma \in \mathcal{W}_\lambda$.

Proof. By Proposition 4.1 we have for $\gamma \in \mathcal{W}_\lambda$ the uniform bound

$$|K_{\widetilde{\chi}_\lambda \circ \Pi_\gamma}(y,y)| \ll (1+\lambda)^{\frac{n-\kappa-1}{m}}, \qquad y \in M = M_{\mathrm{prin}} \cup M_{\mathrm{except}}.$$

The assertion now follows by a repetition of the arguments in the proof of [5, Proposition 5.1 and Equation (5.4)]. □

Similarly, we are able to sharpen the L^p-bounds for isotypic spectral clusters derived in [5, Theorem 5.4] in the isotypic aspect.

Theorem 5.2 (L^p**-bounds for isotypic spectral clusters).** *Let M be a closed connected Riemannian manifold M of dimension n on which a torus T acts effectively and isometrically with orbits of the same dimension κ. Further, let P be the unique self-adjoint extension of a T-invariant elliptic positive symmetric classical pseudodifferential operator on M of degree m, and assume that its principal symbol $p(x,\xi)$ is such that the co-spheres $S^*_xM := \{(x,\xi) \in T^*M \mid p(x,\xi) = 1\}$ are strictly convex. Denote by χ_λ the spectral projection onto the sum of eigenspaces of P with eigenvalues in the interval $(\lambda, \lambda+1]$, and by Π_γ the projection onto the isotypic component $\mathrm{L}^2_\gamma(M)$, where $\gamma \in \widehat{T}$. Then, for $u \in \mathrm{L}^2(M)$ and arbitrary $\gamma \in \mathcal{W}_\lambda$*

$$\|(\chi_\lambda \circ \Pi_\gamma)u\|_{\mathrm{L}^q(M)} \leq \begin{cases} C\,\lambda^{\frac{\delta_{n-\kappa}(q)}{m}} \|u\|_{\mathrm{L}^2(M)}, & \frac{2(n-\kappa+1)}{n-\kappa-1} \leq q \leq \infty, \\ C\,\lambda^{\frac{(n-\kappa-1)(2-q')}{4mq'}} \|u\|_{\mathrm{L}^2(M)}, & 2 \leq q \leq \frac{2(n-\kappa+1)}{n-\kappa-1}, \end{cases} \tag{5.2}$$

for a positive constant C independent of γ, where $\frac{1}{q} + \frac{1}{q'} = 1$ and

$$\delta_{n-\kappa}(q) := \max\left((n-\kappa)\left|\frac{1}{2} - \frac{1}{q}\right| - \frac{1}{2}, 0\right).$$

In particular,

$$\|u\|_{\mathrm{L}^q(M)} \ll \begin{cases} \lambda^{\frac{\delta_{n-\kappa}(q)}{m}}, & \frac{2(n-\kappa+1)}{n-\kappa-1} \leq q \leq \infty, \\ \lambda^{\frac{(n-\kappa-1)(2-q')}{4mq'}}, & 2 \leq q \leq \frac{2(n-\kappa+1)}{n-\kappa-1}, \end{cases}$$

for any eigenfunction $u \in \mathrm{L}^2_\gamma(M)$ of P with eigenvalue λ satisfying $\|u\|_{\mathrm{L}^2} = 1$ and $\gamma \in \mathcal{W}_\lambda$.

Proof. The proof is a verbatim repetition of the proof of [5, Theorem 5.4] where instead of [5, Theorem 3.4] the improved estimates from Theorem 3.3 are used. □

As a consequence of the previous theorem, all examples given in [5, Section 5] can be sharpened in the isotypic aspect.

6. The singular equivariant local Weyl law. Caustics and concentration of eigenfunctions

Using the improved remainder estimates from Theorem 3.2 all results in [5, Section 7] can be sharpened. In particular, the singular equivariant local Weyl law proved in [5, Theorem 7.7] can be improved in the isotypic aspect. As before, let M be a closed connected Riemannian manifold and T a torus acting on M by isometries, and consider the decomposition of M into orbit types

$$M = M(H_1) \dot\cup \cdots \dot\cup M(H_L), \tag{6.1}$$

where we suppose that the isotropy types are numbered in such a way that $(H_i) \geq (H_j)$ implies $i \leq j$, (H_L) being the principal isotropy type. We then have the following

Theorem 6.1 (Singular equivariant local Weyl law). *Let M be a closed connected Riemannian manifold M of dimension n with an isometric and effective action of a torus T and P_0 a T-invariant elliptic classical pseudodifferential operator on M of degree m. Let $p(x,\xi)$ be its principal symbol, and assume that P_0 is positive and symmetric. Denote its unique self-adjoint extension by P, and for a given $\gamma \in \widehat{T}$ let $e_\gamma(x,y,\lambda)$ be its reduced spectral counting function. Write κ for the dimension of an T-orbit in M of principal type. Then, for $x \in M_{\mathrm{prin}} \cup M_{\mathrm{except}}$ one has the asymptotic formula*

$$\left| e_\gamma(x,x,\lambda) - \frac{\lambda^{\frac{n-\kappa}{m}}}{(2\pi)^{n-\kappa}} \sum_{N=1}^{\Lambda-1} \sum_{i_1<\cdots<i_N} \prod_{l=1}^{N} |\tau_{i_l}|^{\dim G - \dim H_{i_l} - \kappa} \mathcal{L}^{0,0}_{i_1\ldots i_N}(x,\gamma) \right|$$
$$\leq \widetilde{C}_\gamma\, \lambda^{\frac{n-\kappa-1}{m}} \sum_{N=1}^{\Lambda-1} \sum_{i_1<\cdots<i_N} \prod_{l=1}^{N} |\tau_{i_l}|^{\dim G - \dim H_{i_l} - \kappa - 1}$$

as $\lambda \to +\infty$, where the multiple sum runs over all possible totally ordered subsets $\{(H_{i_1}),\ldots,(H_{i_N})\}$ of singular isotropy types, and the coefficients satisfy the bounds $\mathcal{L}^{0,0}_{i_1\ldots i_N}(x,\gamma) \ll \|\gamma\|_\infty$ uniformly in x, while

$$\widetilde{C}_\gamma \ll \sup_{l \leq 1} \left\| D^l \gamma \right\|_\infty$$

is a constant independent of x and λ, the D^l are differential operators on T of order l, and the $\tau_{i_j} = \tau_{i_j}(x)$ parameters satisfying $|\tau_{i_j}| \approx dist\,(x, M(H_{i_j}))$.

Proof. The proof consists in a verbatim repetition of the proof of [5, Theorem 7.7] using the improved remainder estimate in Theorem 3.2 (a). □

As an immediate consequence this yields

Corollary 6.2 (Singular point-wise bounds for isotypic spectral clusters). *In the setting of Theorem 6.1 we have*

$$\sum_{\substack{\lambda_j \in (\lambda,\lambda+1],\\ e_j \in \mathrm{L}^2_\gamma(M)}} |e_j(x)|^2 \leq \begin{cases} C\,\lambda^{\frac{n-1}{m}}, & x \in M_{\mathrm{sing}}, \\ C_\gamma\, \lambda^{\frac{n-\kappa-1}{m}} \sum_{N=1}^{\Lambda-1} \sum_{i_1<\cdots<i_N} \prod_{l=1}^{N} |\tau_{i_l}|^{\dim G - \dim H_{i_l} - \kappa - 1}, & x \in M - M_{\mathrm{sing}}, \end{cases}$$

with $C > 0$ independent of γ. In particular, the bound holds for each individual $e_j \in \mathrm{L}^2_\gamma(M)$ with $\lambda_j \in (\lambda, \lambda+1]$.

□

Integrating the asymptotic formulae in Theorems 4.2 and 6.1 over $x \in M$ yields a sharpened remainder estimate for the equivariant Weyl law derived in [4]. In addition, as a consequence of the previous theorem, the example given in [5, Section 7] can be sharpened in the isotypic aspect.

7. Sharpness

By the arguments given in [5, Section 8] the remainder estimates in Theorems 4.2 and 6.1 are sharp in the spectral parameter λ, and already attained on the 2-dimensional sphere S^2. To see that they are almost sharp in the isotypic aspect, endow $M = S^2$ with the induced metric, and let Δ be the corresponding Laplace-Beltrami operator. The eigenvalues of $-\Delta$ are given by the numbers $\lambda_k = k(k+1)$ with $k = 0, 1, 2, 3, \dots$, and the corresponding $k(k+1)$-dimensional eigenspaces $\mathcal{H}_k$ are spanned by the classical spherical functions Y_{km}, $m \in \mathbb{Z}$, $|m| \leq k$. The Y_{kl} are orthonormal to each other, and by the spectral theorem we have the decomposition $\mathrm{L}^2(M) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$. Furthermore, by restricting the left regular representation of $\mathrm{SO}(3)$ in $\mathrm{L}^2(S^2)$ to the eigenspaces $\mathcal{H}_k$ one obtains realizations for all elements in the unitary dual $\widehat{\mathrm{SO}(3)} \simeq \{k = 0, 1, 2, 3, \dots\}$. Now, let $T = \mathrm{SO}(2)$ be isomorphic to the isotropy group of a point in $S^2 \simeq \mathrm{SO}(3)/\mathrm{SO}(2)$. The irreducible representations of $\mathrm{SO}(2)$ are 1-dimensional, and the corresponding characters are given by the exponentials $\theta \mapsto e^{im\theta}$, where $\theta \in [0, 2\pi) \simeq \mathrm{SO}(2)$, $m \in \mathbb{Z} \simeq \widehat{\mathrm{SO}(2)}$. Each $\mathcal{H}_k$ decomposes into $\mathrm{SO}(2)$ representations with multiplicity 1 according to $\mathcal{H}_k = \bigoplus_{|m| \leq k} \mathcal{H}_k^m$, where $\mathcal{H}_k^m$ is spanned by Y_{km}. Consequently, if $N_m(\lambda) := \int_{S^2} e_m(x, x, \lambda) dS^2(x)$ denotes the equivariant counting function of Δ we obtain the estimate

$$N_m(\lambda) = \sum_{k(k+1) \leq \lambda,\, |m| \leq k} 1 \approx \sum_{|m| \leq k \leq \sqrt{\lambda}} 1 \approx \sqrt{\lambda} - |m|, \tag{7.1}$$

as $\lambda \to +\infty$, showing that the remainder estimates in Theorems 4.2 and 6.1 are almost sharp both in the eigenvalue and in the isotypic aspect.

To see that the equivariant L^p-bounds in Section 5 are almost sharp in the eigenvalue and isotypic aspect, let us consider the standard 2-torus $M = T^2 \subset \mathbb{R}^3$ on which $G = \mathrm{SO}(2)$ acts by rotations around the symmetry axis. Then all orbits are 1-dimensional and of principal type. Proposition 5.1 then implies the bound

$$\|u\|_{\mathrm{L}^\infty(T^2)} = O(1), \qquad u \in \mathrm{L}^2(T^2),\, \|u\|_{\mathrm{L}^2} = 1,$$

for any eigenfunction of the Laplace-Beltrami operator Δ on T^2. Now, via the identification

$$\mathbb{R}^2/\mathbb{Z}^2 \xrightarrow{\simeq} T^2 \simeq S^1 \times S^1, (x_1, x_2) \longmapsto (e^{2\pi i x_1}, e^{2\pi i x_2}),$$

the standard orthonormal basis of eigenfunctions of Δ is given by $\left\{ e^{2\pi i k_1 x_1} e^{2\pi i k_2 x_2} \mid (k_1, k_2) \in \mathbb{Z}^2 \right\}$, showing that the bounds in Proposition 5.1 and Theorem 5.2 are almost sharp both in the eigenvalue and isotypic aspect.

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