REDUCED WEYL ASYMPTOTICS FOR PSEUDODIFFERENTIAL OPERATORS ON BOUNDED DOMAINS I THE FINITE GROUP CASE

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Abstract. Let $G \subset \mathrm{O}(n)$ be a group of isometries acting on n-dimensional Euclidean space $\mathbb{R}^n$, and $\mathbf{X}$ a bounded domain in $\mathbb{R}^n$ which is transformed into itself under the action of G. Consider a symmetric, classical pseudodifferential operator A_0 in $\mathrm{L}^2(\mathbb{R}^n)$ with G-invariant Weyl symbol, and assume that it is semi-bounded from below. We show that the spectrum of the Friedrichs extension A of the operator $\mathrm{res} \circ A_0 \circ \mathrm{ext} : \mathrm{C}_c^\infty(\mathbf{X}) \to \mathrm{L}^2(\mathbf{X})$ is discrete, and derive asymptotics for the number $N_\chi(\lambda)$ of eigenvalues of A less or equal λ and with eigenfunctions in the χ-isotypic component of $\mathrm{L}^2(\mathbf{X})$, giving also an estimate for the remainder term in both cases where G is a finite, or, more generally, a compact group. In particular, we show that the multiplicity of each unitary irreducible representation in $\mathrm{L}^2(\mathbf{X})$ is asymptotically proportional to its dimension.

1. Statement of the problem

Let $G \subset \mathrm{O}(n)$ be a compact group of isometries acting on Euclidean space $\mathbb{R}^n$, and $\mathbf{X}$ a bounded domain in $\mathbb{R}^n$ which is transformed into itself under the action of G. Consider the regular representation of G

$$\mathrm{T}(g)\varphi(x) = \varphi(g^{-1}x)$$

in the Hilbert spaces $\mathrm{L}^2(\mathbb{R}^n)$, and $\mathrm{L}^2(\mathbf{X})$, respectively, and endow them with some G-invariant scalar product $(\cdot,\cdot)$, so that the representation T becomes unitary. As a consequence of the Peter-Weyl Theorem, the representation T decomposes into isotypic components according to

$$\mathrm{L}^2(\mathbb{R}^n) = \bigoplus_{\chi \in \hat{G}} \mathcal{H}_\chi, \qquad \mathrm{L}^2(\mathbf{X}) = \bigoplus_{\chi \in \hat{G}} \mathrm{res}\, \mathcal{H}_\chi,$$

where $\hat{G}$ denotes the set of irreducible characters of G, and $\mathrm{res} : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbf{X})$ is the natural restriction operator. Similarly, $\mathrm{ext} : \mathrm{C}_c^\infty(\mathbf{X}) \to \mathrm{L}^2(\mathbb{R}^n)$ will denote the natural extension operator. Let A_0 be a symmetric, classical pseudodifferential operator in $\mathrm{L}^2(\mathbb{R}^n)$ of order $2m$ with G-invariant Weyl symbol a and principal symbol a_{2m}, and assume that $(A_0\, u,\, u) \geq c \left\|u\right\|_m^2$ for some $c > 0$ and all $u \in \mathrm{C}_c^\infty(\mathbf{X})$, where $\left\|\cdot\right\|_s$ is a norm in the Sobolev space $\mathrm{H}^s(\mathbb{R}^n)$. Consider further the Friedrichs extension of the lower semi-bounded operator

$$\mathrm{res} \circ A_0 \circ \mathrm{ext} : \mathrm{C}_c^\infty(\mathbf{X}) \longrightarrow \mathrm{L}^2(\mathbf{X}),$$

which is a self-adjoint operator in $\mathrm{L}^2(\mathbf{X})$, and denote it by A. Finally, let $\partial\, \mathbf{X}$ be the boundary of $\mathbf{X}$, which is not assumed to be smooth, and assume that for some sufficiently small $\varrho > 0$, $\mathrm{vol}\,(\partial\, \mathbf{X})_\varrho \leq C\varrho$, where $(\partial\, \mathbf{X})_\varrho = \{x \in \mathbb{R}^n : \mathrm{dist}\,(x, \partial\, \mathbf{X}) < \varrho\}$.

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Since A commutes with the action of G due to the invariance of a, the eigenspaces of A are unitary G-modules that decompose into irreducible subspaces. In 1972, Arnol'd [1] conjectured that by studying the asymptotic behaviour of the spectral counting function

$$N_\chi(\lambda) = d_\chi \sum_{t \leq \lambda} \mu_\chi(t)$$

where $\mu_\chi(\lambda)$ is the multiplicity of the irreducible representation of dimension d_χ corresponding to the character χ in the eigenspace of A with eigenvalue λ, one should be able to show that the multiplicity of each unitary irreducible representation in the above decomposition of $\mathrm{L}^2(\mathbf{X})$ is asymptotically proportional to its dimension.

The asymptotic distribution of eigenvalues was first studied by Weyl [16] for certain second order differential operators in $\mathbb{R}^n$ using variational techniques. Another approach, which also gives an asymptotic description for the eigenfunctions, was introduced by Carleman [3]. His idea was to study the kernel of the resolvent, combined with a Tauberian argument. Minakshishundaram and Pleijel [13] showed that one can study the Laplace transform of the spectral function as well, and extended the results of Weyl to closed manifolds, and Gårding [6] generalized Carleman's approach to higher order elliptic operators on bounded sets in $\mathbb{R}^n$. Hörmander [10] then extended these results to elliptic differential operators on closed manifolds using the theory of Fourier integral operators. Further developments in this direction were given by Duistermaat and Guillemin, Helffer and Robert, and Ivrii. The first ones to study Weyl asymptotics for elliptic operators on closed Riemannian manifolds in the presence of a compact group of isometries in a systematic way were Donnelly [4] together with Brüning and Heintze [2], giving first order Weyl asymptotics for the spectral distribution function for each of the isotypic components, together with an estimate for the remainder in some special cases. Later, Guillemin and Uribe [7] described the relation between the spectrum of the considered operators, and the reduction of the corresponding bicharacteristic flow, and Helffer and Robert [8, 9] studied the situation in $\mathbb{R}^n$. Our approach is based on the Weyl calculus of pseudodifferential operators developed by Hörmander [11], and the method of approximate spectral projections, first introduced by Tulovskii and Shubin [15]. This method is somehow more closely related to the original work of Weyl, and starts from the observation that the asymptotic distribution function $N(\lambda)$ for the eigenvalues of an elliptic, self-adjoint operator is given by the trace of the orthogonal projection on the space spanned by the eigenvectors corresponding to eigenvalues $\leq \lambda$. By introducing suitable approximations to these spectral projections in terms of pseudodifferential operators, one can then derive asymptotics for $N(\lambda)$, and also obtain estimates for the remainder term. Nevertheless, due to the presence of the boundary, the original method of Shubin and Tulovskii cannot be applied to our situation, and one is forced to use more elaborate techniques, which were subsequently developed by Feigin [5] and Levendorskii [12]. Recently, Bronstein and Ivrii have obtained even sharp estimates for the remainder term in the case of differential operators on manifolds with boundaries satisfying the conditions specified above.

This paper is structured as follows. Part I provides the foundations of the calculus of approximate spectral projection operators, and addresses the case where G is a finite group of isometries. The case of a compact group of isometries will be the subject of Part II. The main result of Part I is the following

Theorem 1. *Let G be a finite group of isometries. Then the spectrum of A is discrete, and the number $N_\chi(\lambda)$ of eigenvalues of A, counting multiplicities, less or equal λ and with eigenfunctions in the χ-isotypic component* $\operatorname{res}\mathcal{H}_\chi$ *of* $\mathrm{L}^2(\mathbf{X})$*, is given by*

$$N_\chi(\lambda) = d_\chi \sum_{t \leq \lambda} \mu_\chi(t) = \frac{d_\chi^2}{|G|} \gamma \lambda^{n/2m} + O(\lambda^{(n-\varepsilon)/2m})$$

for arbitrary $\varepsilon \in (0, \frac{1}{2})$*, where* $|G|$ *denotes the cardinality of* G*,* d_χ *the dimension of the irreducible representation of* G *corresponding to the character* χ*, and*

$$\gamma = \frac{1}{n(2\pi)^n} \int_{\mathbf{X}} \int_{S^{n-1}} (a_{2m}(x,\xi))^{-n/2m} dx\, d\xi.$$

Consequently, the multiplicity in $\mathrm{L}^2(\mathbf{X})$ *of the irreducible representation corresponding to the character* χ *is given asymptotically by* $\frac{d_\chi}{|G|}\gamma\lambda^{n/2m}$ *as* $\lambda \to \infty$*.*

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2. The Weyl Calculus for Pseudodifferential Operators in $\mathbb{R}^n$

We first introduce the relevant symbol classes, as defined in [11], and recall some theorems of Weyl calculus that will be needed in the sequel. We then study the pullback of symbols, and the composition of pseudodifferential operators with linear transformations. Thus, let g be a slowly varying Riemannian metric in $\mathbb{R}^l$, regarded as a positive definite quadratic form, and assume that m is a positive, g-continuous function on $\mathbb{R}^l$ (see Definitions 2.1 and 2.2 in [11]).

Definition 1. *The class of symbols* $S(g,m)$ *is defined as the set of all functions* $u \in \mathrm{C}^\infty(\mathbb{R}^l)$ *such that, for every integer* $k \geq 0$*,*

$$\nu_k(g,m;u) = \sup_{x\in\mathbb{R}^l} \sup_{t_j\in\mathbb{R}^l} |u^{(k)}(x;t_1,\dots,t_k)| \Big/ \Big(\prod_{j=1}^k g_x(t_j)^{1/2} m(x) \Big) < \infty.$$

Here $u^{(k)}$ stands for the k-th differential of u. Note that with the topology defined by the above semi-norms, $S(g,m)$ becomes a Fréchet space. Consider now $\mathbb{R}^l = \mathbb{R}^n \oplus \mathbb{R}^n$, regarded as a symplectic space with the symplectic form

$$\sigma(x,\xi;y,\eta) = \langle \xi, y\rangle - \langle x, \eta\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual Euclidean product of two vectors. Thus, $\sigma = \sum d\xi_j \wedge dx_j$. Assume that g is σ-temperate, and that m is σ, g-temperate (see Definition 4.1 in [11]). If $a \in S(g,m)$ is interpreted as a Weyl symbol, the corresponding pseudodifferential operator is given by

$$\mathrm{Op}^w(a)u(x) = \int\int e^{i(x-y)\xi} a\Big(\frac{x+y}{2}, \xi\Big) u(y) dy\, đ\xi,$$

where $đ\xi = (2\pi)^{-n} d\xi$. Here and it what follows, it will be understood that each integral is to be performed over $\mathbb{R}^n$, unless otherwise specified. According to [11], Theorem 5.2, $\mathrm{Op}^w(a)$ defines a continuous linear map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, and from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, and the corresponding class of operators will be denoted by $\mathcal{L}(g,m)$. Moreover, one has the the following result concerning the L^2-continuity of pseudodifferential operators.

Theorem 2. *Let* g *be a* σ*-temperate metric in* $\mathbb{R}^n \oplus \mathbb{R}^n$*,* g^σ *the dual metric to* g *with respect to* σ*, and* $g \leq g^\sigma$*. Let* $a \in S(g,m)$*, and assume that* m *is* σ*,* g*-temperate. Then* $\mathrm{Op}^w(a) : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ *is a continuous operator if, and only if,* m *is bounded.*

Proof. See [11], Theorem 5.3. □

The composition of pseudodifferential operators is described by the Main Theorem of Weyl Calculus.

Theorem 3. *Let g be a σ-temperate metric in $\mathbb{R}^n \oplus \mathbb{R}^n$, and $g \leq g^\sigma$. Assume that $a_1 \in S(g, m_1)$, $a_2 \in S(g, m_2)$, where m_1, m_2 are σ, g-temperate functions. Then the composition of $\mathrm{Op}^w(a_1)$ with $\mathrm{Op}^w(a_2)$ in each of the spaces $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$ is a pseudodifferential operator with Weyl symbol $\sigma^w(\mathrm{Op}^w(a_1)\mathrm{Op}^w(a_2))$ in the class $S(g, m_1 m_2)$. Moreover,*

$$\begin{aligned}\sigma^w(\mathrm{Op}^w(a_1)\mathrm{Op}^w(a_2))(x,\xi) - \sum_{j<N} \big(\frac{1}{2} i\sigma(D_x, D_\xi; D_y, D_\eta)\big)^j a_1(x,\xi) a_2(y,\eta)_{|(y,\eta)=(x,\xi)}/j! \\ \in S(g, h_\sigma^N m_1 m_2)\end{aligned} \tag{1}$$

for every integer N, where $D_j = -i\,\partial_j$, $D = (D_1, \dots, D_n)$, and

$$h_\sigma^2(x,\xi) = \sup_{y,\eta} \frac{g_{x,\xi}(y,\eta)}{g^\sigma_{x,\xi}(y,\eta)}.$$

Proof. See [11], Theorems 4.2 and 5.2. ☐

Note that g can always be written in the form $g(y,\eta) = \sum(\lambda_j y_j^2 + \mu_j \eta_j^2)$. Then $g^\sigma(x,\xi) = \sum(y_j^2/\lambda_j + \eta_j^2/\mu_j)$, so that

$$h_\sigma(x,\xi) = \max(\lambda_j \mu_j)^{1/2}. \tag{2}$$

The following proposition describes the asymptotic expansion of symbols, see [12], Theorem 3.3.

Proposition 1. *Let $a_j \in S(g, h_\sigma^{N_j} m)$ be a sequence of symbols such that $0 = N_1 < N_2 < \dots \to \infty$. Then there exists a symbol $a \in S(g, m)$ such that*

a) $\operatorname{supp} a \subset \bigcup_j \operatorname{supp} a_j$;
b) $a - \sum_{j=1}^{l-1} a_j \in S(g, h_\sigma^{N_l} m), \quad l > 1.$

In this case, one writes $a \sim \sum a_j$.

We will further write

$$S^{-\infty}(g,m) = \bigcap_{N=1}^{\infty} S(g, h_\sigma^N m),$$

and denote the corresponding operator class by $\mathcal{L}^{-\infty}(g,m)$. We introduce now certain hypoelliptic symbols which will be needed in the sequel. They were introduced by Levendorskii in [12].

Definition 2. *The class of symbols $SI(g,m)$ consists of all $a \in S(g,m)$ that can be represented in the form $a = a_1 + a_2$, where $cm < |a_1|$ and $a_2 \in S(g, h_\sigma^\varepsilon m)$ for some constants $c, \varepsilon > 0$. The corresponding class of operators is denoted by $\mathcal{LI}(g,m)$. If instead of $cm < |a_1|$ one has $cm < a_1$, one writes $a \in SI^+(g,m)$ and $\mathcal{LI}^+(g,m)$, respectively.*

For a proof of the following lemmas, we refer the reader to [12], Lemma 5.5, Lemma 8.1, and Lemma 8.2.

Lemma 1. *Let $a \in SI(g,m)$. Then there exists a symbol $b \in SI(g, m^{-1})$ such that*

$$\mathrm{Op}^w(a)\mathrm{Op}^w(b) - \mathbf{1} \in \mathcal{L}^{-\infty}(g,1), \qquad Op^w(b)\mathrm{Op}^w(a) - \mathbf{1} \in \mathcal{L}^{-\infty}(g,1).$$

The operator $\mathrm{Op}^w(b)$ is called a parametrix for $\mathrm{Op}^w(a)$.

Lemma 2. *If $a \in SI^+(g,m)$, then there exists a symbol $b \in S(g, m^{1/2})$ such that*

$$\mathrm{Op}^w(a) - \mathrm{Op}^w(b)^* \mathrm{Op}^w(b) \in \mathcal{L}^{-\infty}(g,m),$$

where $\mathrm{Op}^w(b)^*$ is the adjoint of $\mathrm{Op}^w(b)$.

Lemma 3. *Let $\varepsilon > 0$, and $a_t \in S(g, h_\sigma^\varepsilon)$, $t \in \mathbb{R}$, be a family of symbols depending on a parameter. Furthermore, assume that the corresponding seminorms $\nu_k(g, h_\sigma^\varepsilon; a_t)$ are bounded by some constants independent of t, and let $c > 0$ be arbitrary. Then there exists a subspace $L \subset \mathrm{L}^2(\mathbb{R}^n)$ of finite codimension such that*

$$\|\mathrm{Op}^w(a_t)u\|_{\mathrm{L}^2} \leq c\,\|u\|_{\mathrm{L}^2} \qquad \text{for all } u \in L \text{ and all } t \in \mathbb{R}.$$

Remark 1. Lemma 3 is a consequence of the fact that, for $a \in S(g, 1)$, one has the uniform bound

$$\|\mathrm{Op}^w(a)\|_{\mathrm{L}^2} \leq C \max_{k \leq N} \nu_k(g, 1; a),$$

where $C > 0$ and $N \in \mathbb{N}$ depend only on the constants characterizing g, but not on a (see the proof of the sufficiency in Theorem 5.3 in [11], and Theorem 4.2 in [12]).

In general, the pullback of symbols under C^∞ mappings is described by the following

Lemma 4. *Let g_1, g_2 be slowly varying metrics on $\mathbb{R}^l$, respectively $\mathbb{R}^{l'}$, and $\chi \in \mathrm{C}^\infty(\mathbb{R}^l, \mathbb{R}^{l'})$. Then*

$$\chi^* S(g_2, 1) \subset S(g_1, 1)$$

if, and only if, for every $k > 0$,

$$g_{2\chi(x)}(\chi^{(k)}(x; t_1, \dots, t_k)) \leq C_k \prod_{j=1}^k g_{1x}(t_j), \qquad x, t_1, \dots, t_k \in \mathbb{R}^l.$$

In particular, if m is g_2-continuous, then $\chi^* m$ is g_1-continuous and $\chi^* S(g_2, m) \subset S(g_1, \chi^* m)$.

Proof. See [11], Lemma 8.1. □

In all our applications, we will be dealing mainly with metrics g on $\mathbb{R}^{2n}$ of the form

$$g_{x,\xi}(y, \eta) = (1 + |x|^2 + |\xi|^2)^\delta |y|^2 + (1 + |x|^2 + |\xi|^2)^{-\varrho} |\eta|^2, \tag{3}$$

where $1 \geq \varrho > \delta \geq 0$. The conditions of Theorem 3 are satisfied then, and $h_\sigma^2(x, \xi) = (1 + |x|^2 + |\xi|^2)^{\delta - \varrho}$ by (2). For the rest of this section, assume that g is of the form (3), and put $h(x, \xi) = (1 + |x|^2 + |\xi|^2)^{-1/2}$. In this case, the space of symbols $S(g, m)$ can also be characterized as follows.

Definition 3. *Let g be of the form (3), and m a g-continuous function. The class $\Gamma_{\varrho,\delta}(m, \mathbb{R}^{2n})$, $0 \leq \delta < \varrho \leq 1$, consists of all functions $u \in \mathrm{C}^\infty(\mathbb{R}^{2n})$ which for all multiindices α, β satisfy the estimates*

$$|\,\partial_\xi^\alpha\, \partial_x^\beta\, u(x, \xi)| \leq C_{\alpha\beta}\, m(x, \xi)\, (1 + |x|^2 + |\xi|^2)^{(-\varrho|\alpha| + \delta|\beta|)/2}.$$

In particular, we will write $\Gamma^l_{\varrho,\delta}(\mathbb{R}^{2n})$ for $\Gamma_{\varrho,\delta}(h^{-l}, \mathbb{R}^{2n})$, where $l \in \mathbb{R}$.

An easy computation then shows that $S(g, m) = \Gamma_{\varrho,\delta}(m, \mathbb{R}^{2n})$. For future reference, note that $u \in S(g, m)$ implies $\partial_\xi^\alpha\, \partial_x^\beta\, u \in S(g, m\, h^{\varrho|\alpha| - \delta|\beta|})$. The pullback of symbols for metrics of the form (3) can now be described as follows.

Lemma 5. *Let $\delta + \varrho \geq 1$, and g be a metric of the form (3). Assume that $\chi(x, \xi) = (y(x), \eta(x, \xi))$ is a diffeomorphism in $\mathbb{R}^{2n}$ such that η is linear in ξ, and the derivatives of y and η are bounded for $|\xi| < 1$. Furthermore, let*

$$\frac{1}{C} g_{x,\xi}(t) \leq g_{\chi(x,\xi)}(t) \leq C g_{x,\xi}(t), \qquad \frac{1}{C} m(x, \xi) \leq \chi^* m(x, \xi) \leq C m(x, \xi),$$

where m is a g-continuous function, and $C > 0$ is a suitable constant. Then $\chi^* S(g, m) \subset S(g, \chi^* m)$.

Proof. Instead of verifying the necessary and sufficient condition in Lemma 4, we will prove the statement directly. Let $b \in S(g,m)$, and let $s, t \dots$ be k vectors in $\mathbb{R}^{2n}$. The k-th differential

$$(b\circ\chi)^{(k)}(x,\xi;s,t\dots) = \langle t, D\rangle\langle s, D\rangle \dots (b\circ\chi)(x,\xi)$$

is given by a sum of terms of the form $s_i t_j \dots \partial^\alpha (b\circ\chi)(x,\xi)$, where we can assume that all the coefficients $s_i, t_j, \dots$ are different from zero; in particular, $(b\circ\chi)^{(1)}(x,\xi) = b^{(1)}(\chi(x,\xi))\chi^{(1)}(x,\xi)$, where

$$\chi^{(1)}(x,\xi) = \begin{pmatrix} y^{(1)}(x) & 0 \\ A(x,\xi) & B(x) \end{pmatrix},$$

A being linear in ξ. The derivatives $\partial^\alpha(b\circ\chi)(x,\xi)$ are sums of expressions of the form

$$(\partial^\beta b)(\chi(x,\xi))(\partial^{\gamma_1}\chi_{i_1})(x,\xi)\dots(\partial^{\gamma_l}\chi_{i_l})(x,\xi),$$

where $\gamma_1 + \dots + \gamma_l = \alpha$ and $l = |\beta|$. Since additional powers of ξ only appear in companion with additional derivatives of b with respect to η that originate from derivatives of $b\circ\chi$ with respect to x, each of the terms of $(b\circ\chi)^{(k)}(x,\xi;s,t,\dots)$ can be estimated from above by some constant times an expression of the form

$$|s_i t_j \dots (\partial_y^{\beta'}\partial_\eta^{\beta''} b)(\chi(x,\xi))P^d(x,\xi)|,$$

where $P^d(x,\xi)$ is a homogeneous polynomial in ξ of degree d which is bounded for $|\xi|<1$, and

$$d = |\beta''| - N'' = |\beta''| - k + N';$$

here $N' = |\alpha'|$ and $N'' = |\alpha''|$ denote the number of x- and ξ-components in the product $s_i t_j \dots$, respectively. Indeed, if we differentiate in $\partial^\alpha(b\circ\chi)(x,\xi)$ first with respect to ξ we get

$$\partial_\xi^{\alpha''}(b\circ\chi)(x,\xi) = \sum_{\eta_{j_1},\dots,\eta_{j_{|\alpha''|}}=1}^{n} (\partial_{\eta_{j_1}}\dots\partial_{\eta_{j_{|\alpha''|}}} b)(\chi(x,\xi))\frac{\partial\,\eta_{j_1}}{\partial\,\xi_{i_1}}(x)\dots\frac{\partial\,\eta_{j_{|\alpha''|}}}{\partial\,\xi_{i_{|\alpha''|}}}(x),$$

where $\partial_\xi^{\alpha''} = \partial_{\xi_{i_1}}\dots\partial_{\xi_{i_{|\alpha''|}}}$, and differentiating now with respect to x yields the assertion. Note that $N'' \leq |\beta''|$. In order to prove the assertion of the lemma, we have to show that

$$\sup_{x,\xi}\sup_{s,t,\dots}\frac{|s_i t_j \dots (\partial_y^{\beta'}\partial_\eta^{\beta''} b)(\chi(x,\xi))P^d(x,\xi)|}{g_{x,\xi}^{1/2}(s)g_{x,\xi}^{1/2}(t)\dots m(x,\xi)} < \infty, \tag{4}$$

where it suffices to consider only those $s,t,\dots$ whose only non-zero components are $s_i, t_j \dots$. Since $N' \geq d$, there are d vectors $p, q, \dots$ among the vectors $s, t, \dots$ contributing with x-components to the product $s_i t_j p_k q_l \dots$. Furthermore, let $w, z, \dots$ be d vectors such that $w_{n+k} = p_k$, $z_{n+l} = q_l, \dots$, their other components being zero. We then obtain the estimate

$$\frac{|s_i t_j p_k q_l \dots (\partial_x^{\beta'}\partial_\xi^{\beta''} b)(\chi(x,\xi))|}{m(\chi(x,\xi))g^{1/2}_{\chi(x,\xi)}(s)g^{1/2}_{\chi(x,\xi)}(t)\dots g^{1/2}_{\chi(x,\xi)}(w)g^{1/2}_{\chi(x,\xi)}(z)\dots}\cdot\frac{|P^d(\xi)g^{1/2}_{x,\xi}(w)g^{1/2}_{x,\xi}(z)\dots|}{g^{1/2}_{x,\xi}(p)g^{1/2}_{x,\xi}(q)\dots}$$
$$\leq C(1+|x|^2+|\xi|^2)^{d(1-\delta-\varrho)/2}$$

for all $x, \xi, s, t, \dots$. Indeed,

$$g^{1/2}_{x,\xi}(w) = |p_k|(1+|x|^2+|\xi|^2)^{-\varrho/2}, \qquad g^{1/2}_{x,\xi}(p) = |p_k|(1+|x|^2+|\xi|^2)^{\delta/2}, \dots.$$

On the other hand, besides the d vectors $p, q\dots$ there are still $N' - d = k - |\beta''| \geq |\beta'|$ vectors among the remaining vectors $s, t\dots$ contributing with x-components to the product $s_i t_j \dots$. Since the corresponding quotients $|r_l|/g^{1/2}_{\chi(x,\xi)}(r)$ can be estimated from above by some constant, we can assume that there are precisely $|\beta'|$ of them. Also note that there are exactly $d + N'' = |\beta''|$ vectors among the vectors $s, t\dots w, z\dots$ contributing with ξ-components to $s_i t_j \dots$. We can therefore assume that the components of $s, t \dots w, z, \dots$ are prescribed by the multiindex $\beta = (\beta', \beta'')$ in such a way that

$$s_i t_j p_k q_l \dots (\partial_x^{\beta'} \partial_\xi^{\beta''} b)(\chi(x,\xi)) = b^{(|\beta|)}(\chi(x,\xi); s, t \dots w, z \dots).$$

The desired estimate (4) now follows by using the assumptions that $b \in S(g,m)$ and $\delta + \varrho \geq 1$. □

If $a \in S(g,m)$ is regarded as a right, respectively left symbol, the corresponding pseudodifferential operators are given by

$$\mathrm{Op}^l(a)u(x) = \int\int e^{i(x-y)\xi} a(x,\xi)u(y)dy\, đ\xi, \quad \mathrm{Op}^r(a)u(x) = \int\int e^{i(x-y)\xi} a(y,\xi)u(y)dy\, đ\xi,$$

where g is assumed to be of the form (3). By [11], Theorem 4.5, the three sets of operators $\mathrm{Op}^w(a)$, $\mathrm{Op}^l(a)$, and $\mathrm{Op}^r(a)$ coincide. Theorem 3 can then also be formulated in terms of left and right symbols. In what follows, we would like to treat left, right, and Weyl symbols on the same grounding by introducing the notion of the τ-symbol. To do so, we introduce yet another class of amplitudes which is closely related to the space $\Gamma^l_{\varrho,\delta}(\mathbb{R}^{2n})$, compare [14], Chapter 4.

Definition 4. *The class $\Pi^l_{\varrho,\delta}(\mathbb{R}^{3n})$ consists of all functions $u \in \mathrm{C}^\infty(\mathbb{R}^{3n})$ which for a suitable $l' \in \mathbb{R}$ satisfy the estimates*

$$|\,\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma u(x,y,\xi)| \leq C_{\alpha\beta\gamma}(1+|x|^2+|y|^2+|\xi|^2)^{(l-\varrho|\alpha|+\delta|\beta+\gamma|)/2}(1+|x-y|^2)^{(l'+\varrho|\alpha|+\delta|\beta+\gamma|)/2}.$$

The relationship between the spaces $\Pi^l_{\varrho,\delta}(\mathbb{R}^{3n})$ and $\Gamma^l_{\varrho,\delta}(\mathbb{R}^{2n})$ is described by the following lemma.

Lemma 6. *Let $0 \leq \delta < \varrho \leq 1$, and $p : \mathbb{R}^{2n} \to \mathbb{R}^n$ be a linear map such that $(x,y) \mapsto (p(x,y), x-y)$ is an isomorphism. Let $a(w,\eta) \in \Gamma^l_{\varrho,\delta}(\mathbb{R}^{2n})$, and define*

$$b(x,y,\xi) = a(p(x,y), \psi(x,y)\xi),$$

where $\psi : \Xi \to \mathrm{GL}(n,\mathbb{R})$ is a C^∞ mapping on some open subset $\Xi \subset \mathbb{R}^{2n}$, having bounded derivatives. If $\delta + \varrho \geq 1$, then $b \in \Pi^l_{\varrho,\delta}(\Xi \times \mathbb{R}^n)$.

Proof. We will proof the assertion by induction on $|\alpha+\beta+\gamma|$. First note that $\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma b(x,y,\xi)$ is given by a sum of terms of the form

$$(\partial_\eta^{\alpha'} \partial_w^{\beta'} a)(p(x,y), \psi(x,y)\xi) P^d(x,y,\xi), \tag{5}$$

where $P^d(x,y,\xi)$ is a polynomial in ξ of degree d. Each of these summands can be estimated from above by

$$C(1+|p(x,y)|^2+|\psi(x,y)\xi|^2)^{(l-\varrho|\alpha'|+\delta|\beta'|)/2}|P^d(\xi)|,$$

where $P^d(\xi)$ is a polynomial in ξ of degree d with constant coefficients, and $C > 0$ is a constant. We assert that the inequality

$$-\varrho|\alpha'| + \delta|\beta'| + d \leq -\varrho|\alpha| + \delta|\beta+\gamma| \tag{6}$$

holds for all $|\alpha+\beta+\gamma| = N$, and all occurring combinations of α', β', and d. It is not difficult to verify the assertion for $N = 1$. Let us now assume that (6) holds for $|\alpha+\beta+\gamma| = N$. Differentiating (5) with respect to ξ_j yields

$$\sum_{i=1}^n (\partial_{\eta_i} \partial_\eta^{\alpha'} \partial_w^{\beta'} a)(p(x,y), \psi(x,y)\xi)\psi(x,y)_{ij} P^d(x,y,\xi)$$
$$+ (\partial_\eta^{\alpha'} \partial_w^{\beta'} a)(p(x,y), \psi(x,y)\xi)\, \partial_{\xi_j} P^d(x,y,\xi),$$

and we get the inequalities

$$
\begin{aligned}
&-\varrho(|\alpha'|+1)+\delta|\beta'|+d \le -\varrho(|\alpha|+1)+\delta|\beta+\gamma|,\\
&-\varrho|\alpha'|+\delta|\beta'|+d-1 \le -\varrho(|\alpha|+1)+\delta|\beta+\gamma|.
\end{aligned}
$$

Similarly, differentiation with respect to, say x_j, gives

$$
\begin{aligned}
&\sum_{i=1}^{n}(\partial_{w_i}\,\partial_\eta^{\alpha'}\,\partial_w^{\beta'}\,a)(p(x,y),\psi(x,y)\xi)(\partial_{x_j}\,p_i)(x,y)P^d(x,y,\xi)\\
&\quad+\sum_{i=1}^{n}(\partial_{\eta_i}\,\partial_\eta^{\alpha'}\,\partial_w^{\beta'}\,a)(p(x,y),\psi(x,y)\xi)\,\partial_{x_j}(\psi(x,y)\xi)_i P^d(x,y,\xi)\\
&\quad+(\partial_\eta^{\alpha'}\,\partial_w^{\beta'}\,a)(p(x,y),\psi(x,y)\xi)(\partial_{x_j}\,P^d)(x,y,\xi),
\end{aligned}
$$

and we arrive at the inequalities

$$
\begin{aligned}
&-\varrho|\alpha'|+\delta(|\beta'|+1)+d \le -\varrho|\alpha|+\delta(|\beta+\gamma|+1),\\
&-\varrho(|\alpha'|+1)+\delta|\beta'|+d+1 \le -\varrho|\alpha|+\delta|\beta+\gamma|-\varrho+1 \le -\varrho|\alpha|+\delta(|\beta+\gamma|+1),\\
&-\varrho|\alpha'|+\delta|\beta'|+d \le -\varrho|\alpha|+\delta(|\beta+\gamma|+1),
\end{aligned}
$$

where, in particular, we made use of the assumption $\delta+\varrho\ge 1$. This proves (6) for $|\alpha+\beta+\gamma|=N+1$. Summing up, we get the estimate

$$
\begin{aligned}
|\,\partial_\xi^\alpha\,\partial_x^\beta\,\partial_y^\gamma\,b(x,y,\xi)| &\le C_1(1+|p(x,y)|+|\xi|)^{l-\varrho|\alpha|+\delta|\beta+\gamma|}\\
&\le C_2(1+|(p(x,y)|+|x-y|+|\xi|)^{l-\varrho|\alpha|+\delta|\beta+\gamma|}(1+|x-y|)^{|l|+\varrho|\alpha|+\delta|\beta+\gamma|},
\end{aligned}
$$

where the latter inequality follows by using the easily verified inequality

$$
\frac{(1+|p(x,y)|+|\xi|)^s}{(1+|p(x,y)|+|x-y|+|\xi|)^s}\le C(1+|x-y|)^{|s|},\qquad s\in\mathbb{R},
$$

compare the proof of Proposition 23.3 in [14]. Since $|x|+|y|$ and $|p(x,y)|+|x-y|$ define equivalent metrics, the assertion of the lemma follows. □

Proposition 2. *Let $a(x,y,\xi)\in\Pi^l_{\varrho,\delta}(\mathbb{R}^{3n})$, where $1\ge\varrho>\delta\ge 0$. Then the oscillatory integral*

$$
Au(x)=\int\int e^{i(x-y)\xi}a(x,y,\xi)u(y)dy\,đ\xi \tag{7}
$$

defines a continuous linear operator from $S(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, and from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

Proof. Consider first the case $a\in C_c^\infty(\mathbb{R}^{3n})$, and assume that $u\in C^\infty(\mathbb{R}^n)$ has bounded derivatives. Then the integration in (7) is carried out over a compact set, and partial integration gives

$$
Au(x)=\int\int e^{i(x-y)\xi}\,\langle x-y\rangle^{-M}\,\langle D_\xi\rangle^M\,\langle D_y\rangle^N\,[\langle\xi\rangle^{-N}\,a(x,y,\xi)u(y)]dy\,đ\xi,
$$

where M,N are even non-negative integers, and $\langle x\rangle$ stands for $(1+x_1^2+\cdots+x_n^2)^{1/2}$. Let now $a\in\Pi^l_{\varrho,\delta}(\mathbb{R}^{3n})$, and assume that M,N are such that $l-N(1-\delta)<-n$, $l+l'+2\delta N-M<-n$. The latter integral then becomes absolutely convergent, defining a continuous function of x, and represents the regularization of the oscillatory integral (7). Increasing M and N we will obtain integrals which are convergent also after differentiation with respect to x. In view of the inequality $\langle x\rangle^k\le\langle y\rangle^k\,\langle x-y\rangle^k$, where $k>0$, one finally sees that A defines a continuous map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, which, by duality, can be extended to a continuous map from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. □

We can now introduce the notion of the τ-symbol. In what follows, m will be a g-continuous function.

Corollary 1. *Let $a \in S(g,m) = \Gamma_{\varrho,\delta}(m,\mathbb{R}^{2n})$, $0 \le 1-\varrho \le \delta < \varrho \le 1$, and $\tau \in \mathbb{R}$. Then*

$$Au(x) = \int\int e^{i(x-y)\xi} a((1-\tau)x + \tau y, \xi) u(y) dy\, đ\xi$$

defines a continuous operator in $\mathcal{S}(\mathbb{R}^n)$, respectively $\mathcal{S}'(\mathbb{R}^n)$. In this case, a is called the τ-symbol of A, and the operator A is denoted by $\mathrm{Op}^\tau(a)$.

Proof. For simplicity, we restrict ourselves to the case $m = h^{-l}$. By Lemma 6 we then have $b(x,y,\xi) = a((1-\tau)x+\tau y,\xi) \in \Pi^l_{\varrho,\delta}(\mathbb{R}^{3n})$, and the assertion follows with the previous proposition. The case of a general m is proved in a similar way. □

Our next aim is to prove the following

Theorem 4. *Let $0 \le 1-\varrho \le \delta < \varrho \le 1$, $\tau, \tau' \in \mathbb{R}$ be arbitrary, $a(x,\xi) \in S(g,m) = \Gamma_{\varrho,\delta}(m,\mathbb{R}^{2n})$, and assume that $\kappa : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map. Furthermore, assume that m is invariant under κ in the sense that $m(\kappa^{-1}(x), {}^t\kappa\,(\xi)) = m(x,\xi)$, and set $A = \mathrm{Op}^{\tau'}(a)$. Then*

$$A_1 u = [A(u\circ\kappa)]\circ\kappa^{-1}, \qquad u \in \mathcal{S}(\mathbb{R}^n),$$

defines a pseudodifferential operator with a uniquely defined τ-symbol $\sigma^\tau(A_1) \in S(g,m)$.

Proof. Let us consider first the case $m = h^{-l}$. Putting $\kappa_1 = \kappa^{-1}$, one sees that A_1 is a Fourier integral operator given by

$$\begin{aligned} A_1u(x) &= \int\int e^{i(\kappa_1(x)-y)\cdot\xi} a((1-\tau')\kappa_1(x) + \tau' y, \xi) u(\kappa(y)) dy\, đ\xi \\ &= \int\int e^{i(\kappa_1(x)-\kappa_1(y))\cdot\xi} a((1-\tau')\kappa_1(x) + \tau'\kappa_1(y),\xi) |\det \kappa_1'(y)| u(y) dy\, đ\xi, \end{aligned}$$

and performing the change of variables $\xi \mapsto {}^t\kappa\,(\xi)$, we get

$$A_1u(x) = \int\int e^{i(x-y)\cdot\xi} a_1(x,y,\xi) u(y) dy\, đ\xi,$$

where we put $a_1(x,y,\xi) = a((1-\tau')\kappa_1(x) + \tau'\kappa_1(y), {}^t\kappa\,(\xi))|\det\kappa_1||\det {}^t\kappa|$. Applying Lemma 6 with $p(x,y) = (1-\tau')\kappa_1(x) + \tau'\kappa_1(y)$, one obtains $a_1(x,y,\xi) \in \Pi^l_{\varrho,\delta}(\mathbb{R}^{3n})$ for arbitrary $a \in \Gamma^l_{\varrho,\delta}(\mathbb{R}^{2n}) = S(g,h^{-l})$. Next, let us introduce the coordinates $v = (1-\tau)x + \tau y$, $w = x - y$, and expand $a_1(x,y,\xi) = a_1(v+\tau w, v-(1-\tau)w,\xi)$ into a Taylor series at $w = 0$, compare [14], pages 180-182. This yields

$$a_1(x,y,\xi) = \sum_{|\beta+\gamma|\le N-1} \frac{(-1)^{|\gamma|}}{\beta!\gamma!} \tau^{|\beta|}(1-\tau)^{|\gamma|}(x-y)^{\beta+\gamma}(\partial_x^\beta\,\partial_y^\gamma\, a_1)(v,v,\xi) + r_N(x,y,\xi),$$

where

$$r_N(x,y,\xi) = \sum_{|\beta+\gamma|=N} c_{\beta\gamma}(x-y)^{\beta+\gamma}\int_0^1 (1-t)^{N-1}(\partial_x^\beta\,\partial_y^\gamma\, a_1)(v+t\tau w, v-t(1-\tau)w,\xi)dt,$$

$c_{\beta\gamma}$ being constants. Since the operator with amplitude $(x-y)^{\beta+\gamma}(\partial_x^\beta\,\partial_y^\gamma\, a_1)(v,v,\xi)$ coincides with the one with amplitude $(-1)^{|\beta+\gamma|}(\partial_\xi^{\beta+\gamma}\, D_x^\beta D_y^\gamma a_1)(v,v,\xi)$, we can write A_1 also as $A_1 = B_N + R_N$, where B_N is the operator with τ-symbol

$$b_N(x,\xi) = \sum_{|\beta+\gamma|\le N-1} \frac{1}{\beta!\gamma!}\tau^{|\beta|}(1-\tau)^{|\gamma|}\,\partial_\xi^{\beta+\gamma}(-D_x)^\beta D_y^\gamma a_1(x,y,\xi)_{|y=x},$$

and R_N has amplitude $r_N(x,y,\xi)$. Similarly, we can assume that R_N is given by a sum of terms having amplitudes of the form

$$\int_0^1 (\partial_\xi^{\beta+\gamma}\,\partial_x^\beta\,\partial_y^\gamma\,a_1)(v+t\tau w, v-t(1-\tau)w,\xi)(1-t)^{N-1}dt,$$

where $|\beta+\gamma|=N$. In view of the estimate

$$|(\partial_\xi^{\beta+\gamma}\,\partial_x^\beta\,\partial_y^\gamma\,a_1)(v+t\tau w, v-t(1-\tau)w,\xi)| \le C(1+|v|+|wt|+|\xi|)^{l-N(\varrho-\delta)}(1+|tw|)^{l'+N(\varrho+\delta)},$$

for some l' and $|\beta+\gamma|=N$, one can then show that $r_N(x,y,\xi)\in\Pi_{\varrho,\delta}^{l-N(\varrho-\delta)}(\mathbb{R}^{3n})$, where, by assumption, $\varrho-\delta>0$. Define now A_1' as the pseudodifferential operator with τ-symbol

$$a_1'(x,\xi)\sim\sum_{N=0}^\infty (b_N(x,\xi)-b_{N-1}(x,\xi)). \tag{8}$$

Then A_1-A_1' has kernel and τ-symbol belonging to $\mathcal{S}(\mathbb{R}^{2n})$. Since $b_N(x,\xi)\in S(g,h^{-l})$ for all N, the assertion of the theorem follows in view of the uniqueness of the τ-symbol, and $\sigma^\tau(A_1)\in\Gamma^l_{\varrho,\delta}(\mathbb{R}^{2n})$. Let us consider now the case of a general m. By examining the proof of Lemma 6, we see that $a_1(x,y,\xi)$ must satisfy an estimate of the form

$$\begin{aligned}|\,\partial_\xi^\alpha\,\partial_x^\beta\,\partial_y^\gamma\,a_1(x,y,\xi)| &\le C_1\,m(p(x,y),\,{}^t\kappa\,(\xi))\,(1+|p(x,y)|+|\xi|)^{-\varrho|\alpha|+\delta|\beta+\gamma|}\\ &\le C_2\,m(p(x,y),\,{}^t\kappa\,(\xi))\,(1+|x|+|y|+|\xi|)^{-\varrho|\alpha|+\delta|\beta+\gamma|}(1+|x-y|)^{\varrho|\alpha|+\delta|\beta+\gamma|}.\end{aligned}$$

Consequently,

$$\begin{gathered}|(\partial_\xi^{\beta+\gamma}\,\partial_x^\beta\,\partial_y^\gamma\,a_1)(v+t\tau w, v-t(1-\tau)w,\xi)|\le\\ C\,m(p(v+t\tau w, v-t(1-\tau)w),\,{}^t\kappa\,(\xi))\,(1+|v|+|wt|+|\xi|)^{-N(\varrho-\delta)}(1+|tw|)^{l'+N(\varrho+\delta)},\end{gathered}$$

where $|\beta+\gamma|=N$, and we can again define $A_1'=\mathrm{Op}^\tau(a_1')$ by the asymptotic expansion (8), such that A_1-A_1' has kernel and τ-symbol belonging to $\mathcal{S}(\mathbb{R}^{2n})$. The assertion of the theorem now follows by noting that $b_N(x,\xi)\in S(g,m)=\Gamma_{\varrho,\delta}(m,\mathbb{R}^{2n})$ for all N, due to the invariance of m. In particular, one has the asymptotic expansion

$$\sigma^\tau(A_1)(x,\xi)-\sum_{|\beta+\gamma|\le N-1}\frac{1}{\beta!\gamma!}\tau^{|\beta|}(1-\tau)^{|\gamma|}\,\partial_\xi^{\beta+\gamma}(-D_x)^\beta D_y^\gamma a_1(x,y,\xi)_{|y=x}\in S(g,h_\sigma^N m) \tag{9}$$

for arbitrary integers N, where the first summand is given by $a_1(x,x,\xi)=a(\kappa^{-1}(x),\,{}^t\kappa\,(\xi))$. □

Theorem 4 allows us, in particular, to express the τ-symbol of an operator in terms of its τ'-symbol. More generally, one has the following

Corollary 2. *In the setting of Theorem 4 assume that, in addition,* $a(\kappa^{-1}(x),\,{}^t\kappa\,(\xi))=a(x,\xi)$, *and* $\det\kappa=\pm1$. *Then* $A_1=A$, *and the* τ*-symbol of* $A=\mathrm{Op}^{\tau'}(a)$ *is given by*

$$\sigma^\tau(A)(x,\xi)\sim\sum_{\beta,\gamma}\frac{1}{\beta!\gamma!}\tau^{|\beta|}(1-\tau)^{|\gamma|}(\tau'-1)^{|\beta|}\tau'^{|\gamma|}\,\partial_\xi^{\beta+\gamma}\,D_x^{\beta+\gamma}a(x,\xi). \tag{10}$$

Proof. With $a_1(x,y,\xi)$ defined as in the proof of Theorem 4, we have $a_1(x,y,\xi)=a((1-\tau')x+\tau'y,\xi)$, so that $A_1=\mathrm{Op}^{\tau'}(a)=A$. The corollary then follows with the asymptotic expansion (9). □

3. The approximate spectral projection operators

Let $G \subset \mathrm{O}(n)$ be a compact group of isometries acting on Euclidean space $\mathbb{R}^n$, and $\mathbf{X}$ a bounded domain in $\mathbb{R}^n$ which is invariant under G. Consider the regular representation T in the Hilbert spaces $\mathrm{L}^2(\mathbb{R}^n)$ and $\mathrm{L}^2(\mathbf{X})$, respectively, and endow them with a G-invariant scalar product, so that T becomes unitary. Let A_0 be a symmetric, classical pseudodifferential operator of order $2m$ with principal symbol a_{2m} as defined in [14], and regard it as an operator in $\mathrm{L}^2(\mathbb{R}^n)$ with domain $\mathrm{C}_{\mathrm{c}}^\infty(\mathbb{R}^n)$. Furthermore, assume that A_0 is G-invariant, i.e. that it commutes with the operators $T(g)$ for all $g \in G$, and that

$$(A_0 u, u) \geq c \|u\|_m^2, \qquad u \in \mathrm{C}_{\mathrm{c}}^\infty(\mathbf{X}), \tag{11}$$

for some $c > 0$, where $(\cdot, \cdot)$ denotes the scalar product in $\mathrm{L}^2(\mathbb{R}^n)$, and $\|\cdot\|_s$ is a norm in the Sobolev space $H^s(\mathbb{R}^n)$. Consider next the decomposition of $\mathrm{L}^2(\mathbb{R}^n)$ and $\mathrm{L}^2(\mathbf{X})$ into isotypic components,

$$\mathrm{L}^2(\mathbb{R}^n) = \bigoplus_{\chi \in \hat{G}} \mathcal{H}_\chi, \qquad \mathrm{L}^2(\mathbf{X}) = \bigoplus_{\chi \in \hat{G}} \operatorname{res} \mathcal{H}_\chi,$$

where $\hat{G}$ is the set of all irreducible characters of G, and res denotes the restriction of functions defined on $\mathbb{R}^n$ to $\mathbf{X}$. Similary, $\operatorname{ext} : \mathrm{C}_{\mathrm{c}}^\infty(\mathbf{X}) \to \mathrm{L}^2(\mathbf{X})$ will denote the natural extension operator. The $\mathcal{H}_\chi$ are closed subspaces, and the corresponding projection operators are given by

$$P_\chi = d_\chi \int_G \overline{\chi(k)} T(k) dk,$$

where d_χ is the dimension of the irreducible representation corresponding to the character χ, and dk denotes Haar measure on G. If G is just finite, dk is the counting measure, and one simply has

$$P_\chi = \frac{d_\chi}{|G|} \sum_{k \in G} \overline{\chi(k)} T(k).$$

Since $T(k)$ is unitary, one computes for $u, v \in \mathrm{L}^2(\mathbb{R}^n)$

$$(u, P_\chi v) = d_\chi \int_G \chi(k)(u, T(k)v) dk = d_\chi \int_G \overline{\chi(k^{-1})}(T(k^{-1})u, v) dk = (P_\chi u, v),$$

where we made use of $\overline{\chi(g)} = \chi(g^{-1})$. Hence P_χ is self-adjoint. Let now A be the Friedrichs extension of the lower semi-bounded operator

$$\operatorname{res} \circ A_0 \circ \operatorname{ext} : \mathrm{C}_{\mathrm{c}}^\infty(\mathbf{X}) \longrightarrow \mathrm{L}^2(\mathbf{X}).$$

A is a self-adjoint operator in $\mathrm{L}^2(\mathbf{X})$, and is itself lower semi-bounded. Its spectrum is real, and consists of the point spectrum and the continuous spectrum. Recall that, in general, a symmetric operator S in a separable Hilbert space is called lower semi-bounded, if there exists a real number c such that

$$(Su, u) \geq c \|u\|^2 \qquad \text{for all } u \in \mathcal{D}(S),$$

where $\mathcal{D}(S)$ denotes the domain of S. Now, if V is a subspace contained in $\mathcal{D}(S)$, the quantity

$$\mathcal{N}(S, V) = \sup_{L \subset V} \{\dim L : (S\,u,\, u) < 0 \quad \forall\, 0 \neq u \in \mathrm{L}\},$$

can be used to give a qualitative description of the spectrum of S. More precisely, one has the following classical variational result of Glazman.

Lemma 7. *Let S be a self-adjoint, lower semi-bounded operator in a separable Hilbert space, and define $N(\lambda, S)$ to be equal to the number of eigenvalues of S, counting multiplicities, less or equal λ, if $(-\infty, \lambda)$ contains no points of the essential spectrum, and equal to ∞, otherwise. Then*

$$N(\lambda, S) = \mathcal{N}(S - \lambda \mathbf{1}, \mathcal{D}(S)).$$

Proof. See [12], Lemma A.1. □

In particular, the lemma above allows one to determine whether S has essential spectrum or not, where the latter is given by the continuous spectrum and the eigenvalues of infinite multiplicity. Let us now return to the situation above. Since A commutes with the action of G on $\mathrm{L}^2(\mathbf{X})$, the eigenspaces of A are unitary G-modules that decompose into irreducible subspaces. Let therefore $N_\chi(\lambda)$ be equal to the number of eigenvalues of A, counting multiplicities, less or equal λ and with eigenfunctions in $\operatorname{res}\mathcal{H}_\chi$, if $(-\infty,\lambda)$ contains no points of the essential spectrum, and equal to ∞, otherwise. One has then the following

Lemma 8. $N_\chi(\lambda)=\mathcal{N}(A_0-\lambda\mathbf{1},\mathcal{H}_\chi\cap \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X}))$.

Proof. Let A_χ be the Friedrichs extension of $\operatorname{res}\circ A_0\circ \operatorname{ext}: \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X})\cap\mathcal{H}_\chi \longrightarrow \operatorname{res}\mathcal{H}_\chi$. Then $N_\chi(\lambda)=N(\lambda,A_\chi)$, and the assertion follows with [12], Lemma A.2. □

In order to estimate $\mathcal{N}(A_0-\lambda\mathbf{1},\mathcal{H}_\chi\cap \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X}))$, we will apply the method of approximate spectral projection operators. It was first introduced by Tulovskii and Shubin, and later developed and generalized by Feigin and Levendorskii, and we will mainly follow [12] in our construction. Thus, let us consider on $\mathbb{R}^{2n}$ the metric

$$g_{x,\xi}(y,\eta)=|y|^2+h(x,\xi)^2|\eta|^2, \qquad h(x,\xi)=(1+|x|^2+|\xi|^2)^{-1/2}. \tag{12}$$

which is clearly of the form (3). Our symbol classes will be mainly of the form $S(h^{-2\delta}g,p)=\Gamma_{1-\delta,\delta}(p,\mathbb{R}^{2n})$ where p is a $\sigma,h^{-2\delta}g$-temperate function, and $0\leq\delta<1/2$. In this case,

$$h^2_\sigma(x,\xi)=(1+|x|^2+|\xi|^2)^{2\delta-1},$$

by equation (2), which amounts to $h_\sigma=h^{1-2\delta}$. Also note that $u\in S(h^{-2\delta}g,p)$ implies $\partial^\alpha_\xi\,\partial^\beta_x\,u\in S(h^{-2\delta}g,h^{(1-\delta)|\alpha|-\delta|\beta|}p)$. In particular, $S(h^{-2\delta}g,h^{-l})=\Gamma^l_{1-\delta,\delta}(\mathbb{R}^{2n})$, where $l\in\mathbb{R}$. The symbols and functions used will also depend on the spectral parameter λ. Nevertheless, their membership to specific symbol classes will be uniform in λ, which means that the values of their seminorms in the corresponding symbol classes will be bounded by some constant independent of λ. Now, if a denotes the left symbol of the classical pseudodifferential operator A_0, clearly $a\in S(g,h^{-2m},K\times\mathbb{R}^n)$ for any compact set $K\subset\mathbb{R}^n$, so that $\sigma^l(A_0-\lambda\mathbf{1})\in S(g,\tilde q^2_\lambda,K\times\mathbb{R}^n)$ uniformly in $\lambda\geq 1$, where

$$\tilde q^2_\lambda(x,\xi)=h^{-2m}(x,\xi)+\lambda \tag{13}$$

is a σ,g-temperate function. But for $u\in \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X})$, the quadratic form $((A_0-\lambda\mathbf{1})u,u)$ entering in the definition of $\mathcal{N}(A_0-\lambda\mathbf{1},\mathcal{H}_\chi\cap \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X}))$ depends only on values of $\sigma^l(A_0-\lambda\mathbf{1})$ on $\mathbf{X}\times\mathbb{R}^n$. By changing the latter symbol outside $\mathbf{X}\times\mathbb{R}^n$ we can achieve that $\sigma^l(A_0-\lambda\mathbf{1})\in S(g,\tilde q^2_\lambda)$ uniformly in $\lambda\geq 1$. In view of Corollary 2 we can therefore assume that $A_0-\lambda\mathbf{1}$ can be represented as a pseudodifferential operator with Weyl symbol $\tilde a_\lambda=\sigma^w(A_0-\lambda\mathbf{1})\in S(g,\tilde q^2_\lambda)$. In particular, we may take $\sigma^w(A_0)\in S(g,h^{-2m})$. But by equation (11) and Lemma 13.1 in [12] we even have

$$a_{2m}(x,\xi)\geq c \qquad \text{for all } (x,\xi)\in\mathbf{X}\times S^{n-1} \text{ and some constant } c>0.$$

Since $a-a_{2m}\in S(g,h^{-2m+1},K\times\mathbb{R}^n)$, we can therefore assume that $A_0\in\mathcal{LI}^+(g,h^{-2m})$, obtaining

Lemma 9. *Let A_0 be a classical pseudodifferential operator satisfying* (11). *Then A_0 and $A_0-\lambda\mathbf{1}$ can be represented as pseudodifferential operators with Weyl symbols $\sigma^w(A_0)\in SI^+(g,h^{-2m})$ and $\tilde a_\lambda\in SI^+(g,\tilde q^2_\lambda)$, respectively.*

Note that if $\sigma^w(A_0)$, and consequently also $\tilde a_\lambda$, are G-invariant in the sense that

$$\sigma^w(A_0)(\sigma_g(x,\xi))=\sigma^w(A_0)(x,\xi), \qquad \tilde a_\lambda(\sigma_g(x,\xi))=\tilde a_\lambda(x,\xi),$$

where σ_g is the symplectic transformation given by $\sigma_g(x,\xi)=(\kappa_g(x),{}^t\kappa'_g(x)^{-1}(\xi))=(\kappa_g(x),\kappa_g(\xi))$, and $\kappa_g(x)=gx$ denotes the action of g, the operators A_0 and $A_0-\lambda\mathbf{1}$ will commute with the action of G by Corollary 2. We can therefore formulate the assumption about the G-invariance of A_0 also in terms of its Weyl symbol, and shall henceforth assume that the Weyl symbol and the principal symbol a_{2m} of A_0 are invariant under σ_g for all $g \in G$. In order to define the approximate spectral projection operators, we will introduce now the relevant symbols. Having in mind Lemma 5, let $a_\lambda \in S(g,1)$, and $d \in S(g,d)$ be G-invariant symbols which, on $\mathbf{X}_\varrho \times \{\xi : |\xi| > 1\}$, $\mathbf{X}_\varrho = \{x : \mathrm{dist}\,(x, \mathbf{X}) < \varrho\}$, are given by

$$\begin{aligned} a_\lambda(x,\xi) &= \frac{1}{1+\lambda|\xi|^{-2m}}\Big(1 - \frac{\lambda}{a_{2m}(x,\xi)}\Big), \\ d(x,\xi) &= |\xi|^{-1}, \end{aligned}$$

where $\varrho > 0$ is some fixed constant, and in addition assume that d is positive and that $d(x,\xi) \to 0$ as $|x| \to \infty$. We need to define smooth approximations to the Heaviside function, and to certain characteristic functions on $\mathbf{X}$. Thus, let $\tilde\chi$ be a smooth function on the real line satisfying $0 \leq \tilde\chi \leq 1$, and

$$\tilde\chi(s) = \begin{cases} 1 & \text{for } s < 0, \\ 0 & \text{for } s > 1. \end{cases}$$

Let $C_0 > 0$ and $\delta \in (1/4, 1/2)$ be constants, and put $\omega = 1/2 - \delta$. We then define the G-invariant function

$$\chi_\lambda = \tilde\chi \circ ((a_\lambda + 4h^{\delta-\omega} + 8C_0 d)\, h^{-\delta}), \tag{14}$$

where $0 < \delta - \omega < 1/2$.

Lemma 10. *$\chi_\lambda \in S(h^{-2\delta}g, 1) = \Gamma^0_{1-\delta,\delta}(\mathbb{R}^{2n})$ uniformly in λ.*

Proof. We first note that

$$h^{-\delta} \in S(g, h^{-\delta}), \qquad (a_\lambda + 4h^{\delta-\omega} + 8C_0 d) \in S(g,1),$$

since $d \in S(g,d) \subset S(g,1)$, and $h^{\delta-\omega} \in S(g, h^{\delta-\omega}) \subset S(g,1)$. Now, each of the derivatives of χ_λ with respect to x and ξ can be estimated by a sum of derivatives of $a_\lambda + 4h^{\delta-\omega} + 8C_0 d)\, h^{-\delta}$. But because of $\partial_\xi^\alpha\, \partial_x^\beta (a_\lambda + 4h^{\delta-\omega} + 8C_0 d) \in S(g, h^{|\alpha|})$, $\partial_\xi^\alpha\, \partial_x^\beta\, h^{-\delta} \in S(g, h^{-\delta+|\alpha|})$, we obtain

$$|\,\partial_\xi^\alpha\, \partial_x^\beta\, \chi_\lambda(x,\xi)| \leq C_{\alpha,\beta} h^{(1-\delta)|\alpha|} = C_{\alpha,\beta}(1 + |x|^2 + |\xi|^2)^{-(1-\delta)|\alpha|/2},$$

where $C_{\alpha,\beta}$ is independent of λ. We therefore obtain $\chi_\lambda \in \Gamma^0_{1-\delta,0}(\mathbb{R}^{2n}) \subset \Gamma^0_{1-\delta,\delta}(\mathbb{R}^{2n})$ uniformly in λ, and the assertion follows. $\square$

Next, let U be a subset in $\mathbb{R}^{2n}$, $c > 0$, and put

$$U(c,g) = \big\{(x,\xi) \in \mathbb{R}^{2n} : \exists (y,\eta) \in U : g_{(x,\xi)}(x-y, \xi-\eta) < c\big\}\,;$$

according to Levendorskii [12], Corollary 1.2, there exists a smoothened characteristic function $\psi_c \in S(g,1)$ belonging to the set U and the parameter c, such that $\mathrm{supp}\,\psi_c \subset U(2c,g)$, and $\psi_{c|U(c,g)} = 1$. Let now

$$\mathcal{M}_\lambda = \big\{(x,\xi) \in \mathbb{R}^{2n} : a_\lambda < 4h^{\delta-\omega} + 8C_0 d\big\}\,. \tag{15}$$

Both $\mathcal{M}_\lambda$ and $\partial\,\mathbf{X} \times \mathbb{R}^n$ are invariant under σ_k for all $k \in G$, as well as $(\partial\,\mathbf{X} \times \mathbb{R}^n)(c, h^{-2\delta}g)$, and $\mathcal{M}_\lambda(c, h^{-2\delta}g)$, due to the invariance of $a_{2m}(x,\xi)$, and the considered metrics and symbols. Now, let $\tilde\eta_c, \psi_{\lambda,c} \in S(h^{-2\delta}g, 1)$ be smoothened characteristic functions corresponding to the parameter c, and the sets $\partial\,\mathbf{X} \times \mathbb{R}^n$ and $\mathcal{M}_\lambda$, respectively. According to Lemma 5, we can assume that they are invariant under σ_k for all $k \in G$; otherwise consider $\int_G \tilde{\eta}_c \circ \sigma_k \, dk$, $\int_G \psi_{\lambda,c} \circ \sigma_k \, dk$, respectively. We then define the functions

$$\eta_{\lambda,-c}(x,\xi) = \begin{cases} 0, & x \notin \mathbf{X}, \\ (1-\tilde{\eta}_c(x,\xi))\psi_{\lambda,1/c}(x,\xi), & x \in \mathbf{X}, \end{cases} \tag{16}$$

$$\eta_c(x,\xi) = \begin{cases} \tilde{\eta}_c(x,\xi), & x \notin \mathbf{X}, \\ 1, & x \in \mathbf{X}. \end{cases} \tag{17}$$

Only the support of $\psi_{\lambda,c}$ depends on λ, but not its growth properties, so that $\eta_c, \eta_{\lambda,-c} \in S(h^{-2\delta}g, 1)$ uniformly in λ. Furthermore, since $\tilde{\eta}_{2c} = 1$ on $\operatorname{supp} \tilde{\eta}_c$, and $\psi_{\lambda,1/c} = 1$ on $\operatorname{supp} \psi_{\lambda,1/2c}$, on has $\eta_{\lambda,-c} = 1$ on $\operatorname{supp} \eta_{\lambda,-2c}$, which implies $\eta_{\lambda,-2c}\eta_{\lambda,-c} = \eta_{\lambda,-2c}$. Similarly, one verifies $\eta_c\eta_{2c} = \eta_c$. We are now ready to define the approximate spectral projection operators.

Definition 5. *The approximate spectral projection operators of the first kind are defined by*

$$\tilde{\mathcal{E}}_\lambda = \operatorname{Op}^w(\eta_{\lambda,-2}) \operatorname{Op}^w(\chi_\lambda) \operatorname{Op}^w(\eta_{\lambda,-2}),$$

while the approximate spectral projection operators of the second kind are

$$\mathcal{E}_\lambda = \tilde{\mathcal{E}}_\lambda^2(3 - 2\tilde{\mathcal{E}}_\lambda).$$

Remark 2. $\tilde{\mathcal{E}}_\lambda$ is a smooth approximation to the spectral projection operator E_λ of A using Weyl calculus, while $\mathcal{E}_\lambda$ is an approximation to $E_\lambda^2(3 - 2E_\lambda) = E_\lambda$. Note that, since $\eta_{\lambda,-2}$ and χ_λ are G-invariant, Corollary 2 implies that the operators $\operatorname{Op}^w(\eta_{\lambda,-2})$, $\operatorname{Op}^w(\chi_\lambda)$, and consequently also $\tilde{\mathcal{E}}_\lambda$ and $\mathcal{E}_\lambda$, commute with the action $T(g)$ of G. The choice of $\mathcal{E}_\lambda$ was originally due to the fact that its trace class norm can be estimated from above by the operator norm of $3 - 2\tilde{\mathcal{E}}_\lambda$, and the Hilbert-Schmidt-norm of $\tilde{\mathcal{E}}_\lambda$, which are easier to handle. This construction was first used by Feigin [5].

Both $\tilde{\mathcal{E}}_\lambda$ and $\mathcal{E}_\lambda$ are integral operators with kernels in $\mathcal{S}(\mathbb{R}^{2n})$. Indeed, the asymptotic expansion (1), together with Proposition 1, imply that the Weyl symbols of $\tilde{\mathcal{E}}_\lambda$ and $\mathcal{E}_\lambda$ can be written in the form $a+r$, where a has compact support, and $r \in S^{-\infty}(h^{-2\delta}g, 1)$, because χ_λ has compact support in ξ, and $\eta_{\lambda,-2}$ has x-support in $\mathbf{X}$. Thus, $\sigma^w(\tilde{\mathcal{E}}_\lambda)$ and $\sigma^w(\mathcal{E}_\lambda)$ are rapidly decreasing Schwartz functions, and the same holds for the corresponding τ-symbols. By Lemma 7.2 in [11], this also implies that $\tilde{\mathcal{E}}_\lambda$ and $\mathcal{E}_\lambda$ are of trace class and, in particular, compact operators in $\mathrm{L}^2(\mathbb{R}^n)$. In addition, by Theorem 3, and the asymptotic expansion (10), one has $\sigma^\tau(\tilde{\mathcal{E}}_\lambda), \sigma^\tau(\mathcal{E}_\lambda) \in S(h^{-2\delta}g, 1)$ uniformly in λ. On the other hand, the functions $\eta_{\lambda,-2}$ and χ_λ are real valued, which by general Weyl calculus implies that $\operatorname{Op}^w(\eta_{\lambda,-2})$, $\operatorname{Op}^w(\chi_\lambda)$, and consequently also $\tilde{\mathcal{E}}_\lambda$, and $\mathcal{E}_\lambda$, are self-adjoint operators in $\mathrm{L}^2(\mathbb{R}^n)$. By construction, $\mathcal{E}_\lambda$ commutes with the projection P_χ, so that $P_\chi\mathcal{E}_\lambda = \mathcal{E}_\lambda P_\chi$ is a self-adjoint operator of trace class as well. Although the decay properties of $\sigma^\tau(\mathcal{E}_\lambda)$ are independent of λ, its support does depend on λ, which will result in estimates for the trace of $P_\chi\mathcal{E}_\lambda$ in terms of λ that will be used in order to prove Theorem 1. In particular, the estimate for the remainder term in Theorem 1 is determined by the particular choice of the range $(1/4, /1/2)$ for the parameter δ, which guarantees that $1 - \delta > \delta$. By the general theory of compact, self-adoint operators, zero is the only accumulation point of the point spectra of $\tilde{\mathcal{E}}_\lambda$ and $\mathcal{E}_\lambda$, as well as the only point that could possibly belong to the continuous spectrum. The following proposition and its corollary give uniform bounds for the number of eigenvalues away from zero. They are based on certain L^2-estimates for pseudodifferential operators.

Proposition 3. *The number of eigenvalues of $\tilde{\mathcal{E}}_\lambda$ lying outside the interval $[-\frac{1}{4}, \frac{5}{4}]$ is bounded by some constant independent of λ.*

Proof. Since $\chi_\lambda, \eta_{\lambda,-c} \in S(h^{-2\delta}g, 1)$, Theorem 3 yields $\sigma^w(\tilde{\mathcal{E}}_\lambda) \in S(h^{-2\delta}g, 1)$ uniformly in λ. Furthermore, taking into account the asymptotic expansion (1), we have

$$\sigma^w(\tilde{\mathcal{E}}_\lambda) = \eta^2_{\lambda,-2}\chi_\lambda + r_\lambda,$$

where $r_\lambda \in S(h^{-2\delta}g, h^{1-2\delta})$. Now, since $0 \le \chi_\lambda, \eta^2_{\lambda,-2} \le 1$, for each $\varepsilon > 0$ there exists a constant $c > 0$ such that $\varepsilon + \eta^2_{\lambda,-2}\chi_\lambda \ge c$ and $(1+\varepsilon) - \eta^2_{\lambda,-2}\chi_\lambda \ge c$. Consequently, the symbols of $\varepsilon\mathbf{1} + \tilde{\mathcal{E}}_\lambda$ and $(1+\varepsilon)\mathbf{1} - \tilde{\mathcal{E}}_\lambda$ admit a representation of the form $a_1 + a_2$, where $a_1 \ge c$, $a_2 \in S(h^{-2\delta}g, h^{1-2\delta})$; thus

$$\varepsilon\mathbf{1} + \tilde{\mathcal{E}}_\lambda \in \mathcal{LI}^+(h^{-2\delta}g, 1), \qquad (1+\varepsilon)\mathbf{1} - \tilde{\mathcal{E}}_\lambda \in \mathcal{LI}^+(h^{-2\delta}g, 1)$$

uniformly in λ. According to Lemma 2, this implies that for each λ there exist two operators T_1, T_2 such that $\varepsilon\mathbf{1} + \tilde{\mathcal{E}}_\lambda \ge T_1$ and $(1+\varepsilon)\mathbf{1} - \tilde{\mathcal{E}}_\lambda \ge T_2$, and $T_i \in \mathcal{L}^{-\infty}(g, 1)$ uniformly in λ. Therefore, by Lemma 3, there exist two subspaces $L_i \subset \mathrm{L}^2(\mathbb{R}^n)$ of finite codimension such that $\|T_i u\|_{\mathrm{L}^2} \le \varepsilon \|u\|_{\mathrm{L}^2}$ for $u \in L_i$ and all λ, which implies, via Cauchy-Schwartz, that $-\varepsilon \|u\|^2_{\mathrm{L}^2} \le (T_i u, u) \le \varepsilon \|u\|^2_{\mathrm{L}^2}$ on L_i. Putting everything together we arrive at the L^2-estimates

$$(\tilde{\mathcal{E}}_\lambda u, u) \ge ((T_1 - \varepsilon\mathbf{1})u, u) \ge -2\varepsilon \|u\|^2_{\mathrm{L}^2},$$
$$(\tilde{\mathcal{E}}_\lambda u, u) \le (((1+\varepsilon)\mathbf{1} - T_2)u, u) \le (1+2\varepsilon) \|u\|^2_{\mathrm{L}^2},$$

where $u \in L_1 \cap L_2$, and taking $\varepsilon = \frac{1}{8}$ yields the desired result, since $\operatorname{codim} L_1 \cap L_2 < \infty$. □

Corollary 3. *The number of eigenvalues of $\mathcal{E}_\lambda$ lying outside the interval $[0,1]$ is bounded by some constant independent of λ.*

Proof. If $\tilde{\nu}_i$ denote the eigenvalues of $\tilde{\mathcal{E}}_\lambda$, then the eigenvalues of $\mathcal{E}_\lambda$ are given by $\nu_i = \tilde{\nu}_i^2(3-2\tilde{\nu}_i)$. □

Let now $N_\chi^{\mathcal{E}_\lambda}$ denote the number of eigenvalues of $\mathcal{E}_\lambda$ which are $\ge 1/2$, and whose eigenfunctions are contained in the χ-isotypic component $\mathcal{H}_\chi$ of $\mathrm{L}^2(\mathbb{R}^n)$. Since zero is the only accumulation point of the point spectrum of $\mathcal{E}_\lambda$, $N_\chi^{\mathcal{E}_\lambda}$ is clearly finite. The next lemma will show that it can be estimated by the trace of the operator $P_\chi\mathcal{E}_\lambda$, and its square, so that it is natural to expect that it will provide a good approximation for $N_\chi(\lambda) = \operatorname{tr} P_\chi E_\lambda = \mathcal{N}(A_0 - \lambda\mathbf{1}, \mathcal{H}_\chi \cap \mathrm{C}^\infty_c(\mathbf{X}))$.

Lemma 11. *There exist constants $c_1, c_2 > 0$ independent of λ such that*

$$2\operatorname{tr}(P_\chi\mathcal{E}_\lambda)^2 - \operatorname{tr} P_\chi\mathcal{E}_\lambda - c_1 \le N_\chi^{\mathcal{E}_\lambda} \le 3\operatorname{tr} P_\chi\mathcal{E}_\lambda - 2\operatorname{tr}(P_\chi\mathcal{E}_\lambda)^2 + c_2. \tag{18}$$

Proof. Since $\mathcal{E}_\lambda \in \mathcal{L}(h^{-2\delta}g, 1)$, Theorem 2 implies that $\mathcal{E}_\lambda$ is L^2-continuous. Moreover, by Remark 1, there is a constant C independent of λ such that $\|\mathcal{E}_\lambda\|_{\mathrm{L}^2} \le C$; hence all eigenvalues of the operators $\mathcal{E}_\lambda$ are bounded by C. Let now $\nu_{i,\chi}$ denote the eigenvalues of $\mathcal{E}_\lambda$ with eigenfunctions in $\mathcal{H}_\chi$. Taking into account Corollary 3 and the previous remark, we obtain the estimate

$$N_\chi^{\mathcal{E}_\lambda} \le \sum_{\nu_{i,\chi}\ge 1/2} \nu_{i,\chi} + \sum_{1/2\le\nu_{i,\chi}\le 1} (1-\nu_{i,\chi}) + c_1 \le \sum_{\nu_{i,\chi}\ge 1/2} \nu_{i,\chi} + 2\sum_{1/2\le\nu_{i,\chi}\le 1} \nu_{i,\chi}(1-\nu_{i,\chi}) + c_1,$$

where $c_1 > 0$, like all other constants $c_i > 0$ occurring in this proof, can be chosen independent of λ. Consequently, the right hand side can be estimated from above by $3\operatorname{tr} P_\chi\mathcal{E}_\lambda - 2\operatorname{tr} P_\chi\mathcal{E}_\lambda \cdot P_\chi\mathcal{E}_\lambda + c_2$. In the same way one computes

$$\begin{aligned} N_\chi^{\mathcal{E}_\lambda} &= \sum_{\nu_{i,\chi}\ge 1/2} \nu_{i,\chi} + \sum_{\nu_{i,\chi}\ge 1/2} (1-\nu_{i,\chi}) \ge \sum_i \nu_{i,\chi} - \sum_{0\le\nu_{i,\chi}\le 1/2} \nu_{i,\chi} - c_3 \\ &\ge \sum_i \nu_{i,\chi} - 2\sum_{0\le\nu_{i,\chi}\le 1/2} \nu_{i,\chi}(1-\nu_{i,\chi}) - c_3 \ge \sum_i \nu_{i,\chi} - 2\sum_i \nu_{i,\chi}(1-\nu_{i,\chi}) - c_4, \end{aligned}$$

where the right hand side can be estimated from below by $2\operatorname{tr} P_\chi\mathcal{E}_\lambda \cdot P_\chi\mathcal{E}_\lambda - \operatorname{tr} P_\chi\mathcal{E}_\lambda - c_4$. This completes the proof of (58). □

As the next section will show, $N_\chi^{\mathcal{E}_\lambda}$ will provide us with a lower bound for the spectral counting function $N_\chi(\lambda)$. Nevertheless, in order to obtain an upper bound as well, it will be necessary to introduce new approximations to the spectral projection operators. Namely, let

$$\chi_\lambda^+ = \tilde{\chi}(a_\lambda^+ h^{-\delta}), \qquad a_\lambda^+ = a_\lambda - 4h^{\delta-\omega} - 8C_0 d,$$

where $\tilde{\chi}$ is defined as in (14). As in Lemma 10, one verifies that $\chi_\lambda^+ \in S(h^{-2\delta}g, 1)$ uniformly in λ.

Definition 6. *The approximate spectral projection operators of the third kind are*

$$\tilde{\mathcal{F}}_\lambda = \mathrm{Op}^w(\eta_2^2 \chi_\lambda^+),$$

while the approximate spectral projection operators of the fourth kind are

$$\mathcal{F}_\lambda = \tilde{\mathcal{F}}_\lambda^2(3 - 2\tilde{\mathcal{F}}_\lambda).$$

Like the projection operators of the first and second kind, $\tilde{\mathcal{F}}_\lambda$ and $\mathcal{F}_\lambda$ are self-adjoint operators in $\mathrm{L}^2(\mathbb{R}^n)$ with kernels in $\mathcal{S}(\mathbb{R}^{2n})$, and therefore of trace class. Since $\mathcal{F}_\lambda$ commutes with $T(k)$, $P_\chi \mathcal{F}_\lambda$ is a self-adjoint operator of trace class, too. Let $M_\chi^{\mathcal{F}_\lambda}$ denote the number of eigenvalues of $\mathcal{F}_\lambda$ which are $\geq 1/2$, and whose eigenfunctions are contained in the χ-isotypic compoment $\mathcal{H}_\chi$. Since Proposition 3 and Corollary 3 hold for $\tilde{\mathcal{F}}_\lambda$ and $\mathcal{F}_\lambda$ as well, we obtain

Lemma 12. *There exist constants $c_1, c_2 > 0$ independent of λ such that*

$$2\,\mathrm{tr}(P_\chi \mathcal{F}_\lambda)^2 - \mathrm{tr}\, P_\chi \mathcal{F}_\lambda - c_1 \leq M_\chi^{\mathcal{F}_\lambda} \leq 3\,\mathrm{tr}\, P_\chi \mathcal{F}_\lambda - 2\,\mathrm{tr}(P_\chi \mathcal{F}_\lambda)^2 + c_2. \tag{19}$$

Proof. The proof is a verbatim repetition of the proof of Lemma 11 with $\mathcal{E}_\lambda$ replaced by $\mathcal{F}_\lambda$. □

4. Estimates from below for $N_\chi(\lambda)$

In this section, we shall estimate the spectral counting function $N_\chi(\lambda) = \mathcal{N}(A_0 - \lambda \mathbf{1}, \mathcal{H}_\chi \cap \mathrm{C}_c^\infty(\mathbf{X}))$ from below. More precisely, by adapting techniques developed in [12] to our situation, we will show the following

Theorem 5. *Let $N_\chi^{\mathcal{E}_\lambda}$ be the number of eigenvalues of $\mathcal{E}_\lambda$ which are $\geq 1/2$, and whose eigenfunctions are contained in the χ-isotypic component $\mathcal{H}_\chi$. Then there exists a constant $C > 0$ independent of λ such that*

$$\mathcal{N}(A_0 - \lambda \mathbf{1}, \mathcal{H}_\chi \cap \mathrm{C}_c^\infty(\mathbf{X})) \geq N_\chi^{\mathcal{E}_\lambda} - C. \tag{20}$$

As a first step towards the proof, let $\tilde{q}_\lambda$ be defined as in (13), and $q_\lambda \in SI(g, \tilde{q}_\lambda^{-1})$ be a G-invariant symbol which, on $\mathbf{X}_\varepsilon \times \{\xi : |\xi| > 1\}$ is given by

$$q_\lambda(x,\xi) = \big(a_{2m}(x,\xi)(1 + |\xi|^{-2m}\lambda)\big)^{-1/2},$$

and consider the G-invariant function $\pi = (h^{\delta-\omega} + C_0 d)^{-1/2} \in SI(g, \pi)$, together with the operators

$$\Pi = \mathrm{Op}^w(\pi), \qquad Q_\lambda = \mathrm{Op}^w(q_\lambda).$$

Since $\pi \tilde{q}_\lambda^{-1}$ is bounded, ΠQ_λ is a continuous operator in $\mathrm{L}^2(\mathbb{R}^n)$. The parametrices of Π and Q_λ, which exist according to Lemma 1, will be denoted by R_Π and R_{Q_λ}. Furthermore, an examination of the proof of Lemma 1 shows that if $a \in SI(g, m)$ is G-invariant, then the Weyl symbol b of the parametrix of $\mathrm{Op}^w(a)$ can be assumed to be G-invariant. Consequently, the parametrices R_Π and R_{Q_λ} commute with the operators $T(k)$.

Lemma 13. *Let $L_\chi^{\mathcal{E}_\lambda} = \mathrm{Span}\{u \in \mathcal{S}(\mathbb{R}^n) \cap \mathcal{H}_\chi : \mathcal{E}_\lambda u = \nu u, \nu \geq \frac{1}{2}\}$ and $\tilde{L}_\chi^{\mathcal{E}_\lambda} = \mathrm{Op}^l(\eta_{\lambda,-1})\, Q_\lambda\, \Pi L_\chi^{\mathcal{E}_\lambda}$. Then*

$$\dim \tilde{L}_\chi^{\mathcal{E}_\lambda} \geq \dim L_\chi^{\mathcal{E}_\lambda} - C = N_\chi^{\mathcal{E}_\lambda} - C \tag{21}$$

for some constant $C > 0$ independent of λ.

Proof. Let us first note that since $\eta_{\lambda,-1}$ has support in $\mathbf{X}\times\mathbb{R}^n$, and $\mathrm{Op}^l(\eta_{\lambda,-1})\,Q_\lambda\,\Pi$ commutes with P_χ, we have $\tilde L_\chi^{\mathcal{E}_\lambda}\subset \mathrm{C}_{\mathrm{c}}^\infty(\mathbf{X})\cap\mathcal{H}_\chi$. Next, we will prove that

$$R_\Pi R_{Q_\lambda}\,\mathrm{Op}^l(\eta_{\lambda,-1})\,Q_\lambda\,\Pi\mathcal{E}_\lambda=\mathcal{E}_\lambda+T, \tag{22}$$

where $T\in\mathcal{L}^{-\infty}(g,1)$. Indeed, the Weyl symbol of $\mathrm{Op}^l(\eta_{\lambda,-1})\,Q_\lambda\,\Pi\mathcal{E}_\lambda$ is given by a linear combination of products of derivatives of the Weyl symbols of Q_λ, Π, $\mathcal{E}_\lambda$, and $\mathrm{Op}^l(\eta_{\lambda,-1})$. By the asymptotic expansion (10),

$$\sigma^w(\mathrm{Op}^l(\eta_{\lambda,-1}))(x,\xi)\sim\sum_\beta\frac{1}{\beta!}\Big(\frac{-1}{2}\Big)^{|\beta|}\,\partial_\xi^\beta\,D_x^\beta\eta_{\lambda,-1}(x,\xi).$$

Now, equation (51) implies that, up to terms of order $-\infty$, the support of $\sigma^w(\mathcal{E}_\lambda)$ is contained in $\operatorname{supp}\eta_{\lambda,-2}$, and we shall express this by writing $\operatorname{supp}_\infty\sigma^w(\mathcal{E}_\lambda)\subset\operatorname{supp}\eta_{\lambda,-2}$. For the same reason, we must have $\operatorname{supp}_\infty\sigma^w(\mathrm{Op}^l(\eta_{\lambda,-1})\,Q_\lambda\,\Pi\mathcal{E}_\lambda)\subset\operatorname{supp}\eta_{\lambda,-2}$. But $\eta_{\lambda,-1}=1$ on $\operatorname{supp}\eta_{\lambda,-2}$ implies that all terms in the expansion of $\sigma^w(\mathrm{Op}^l(\eta_{\lambda,-1}))$ vanish on $\operatorname{supp}\eta_{\lambda,-2}$, except for the zero order terms. Proposition 1 then yields

$$\sigma^w(\mathrm{Op}^l(\eta_{\lambda,-1}))(x,\xi)=\eta_{\lambda,-1}(x,\xi)$$

on $\operatorname{supp}\eta_{\lambda,-2}$, up to a term of order $-\infty$. On this set, the Weyl symbol of $\mathrm{Op}^l(\eta_{\lambda,-1})\,Q_\lambda\,\Pi\mathcal{E}_\lambda$ therefore reduces to $\eta_{\lambda,-1}=1$ times a linear combination of products of derivatives of the Weyl symbols of Q_λ, Π and $\mathcal{E}_\lambda$ supported in $\operatorname{supp}\eta_{\lambda,-2}$, which corresponds to the Weyl symbol of $Q_\lambda\,\Pi\mathcal{E}_\lambda$, plus an additional term of order $-\infty$. Thus,

$$\mathrm{Op}^l(\eta_{\lambda,-1})\,Q_\lambda\,\Pi\mathcal{E}_\lambda=Q_\lambda\,\Pi\mathcal{E}_\lambda+\tilde T,\qquad \tilde T\in\mathcal{L}^{-\infty}(g,\pi\tilde q_\lambda^{-1}), \tag{23}$$

and (22) follows by taking into account the definition of the parametrix. Now, $\mathcal{E}_\lambda: L_\chi^{\mathcal{E}_\lambda}\to L_\chi^{\mathcal{E}_\lambda}$ is clearly surjective, and

$$\|\mathcal{E}_\lambda u\|\geq\frac{1}{2}\,\|u\|\,,\qquad u\in L_\chi^{\mathcal{E}_\lambda},$$

implies that $\mathcal{E}_\lambda$ is injective on $L_\chi^{\mathcal{E}_\lambda}$ as well. Equation (22) therefore means that on $L_\chi^{\mathcal{E}_\lambda}$

$$R_\Pi R_{Q_\lambda}\,\mathrm{Op}^l(\eta_{\lambda,-1})\,Q_\lambda\,\Pi=\mathbf{1}_{L_\chi^{\mathcal{E}_\lambda}}+T\mathcal{E}_\lambda^{-1}. \tag{24}$$

According to Lemma 3, there exists a subspace of finite codimension M such that $\|Tu\|\leq\|u\|/8$ for all $u\in M$ and all λ. This gives

$$\left\|T\mathcal{E}_\lambda^{-1}u\right\|\leq 2\,\|Tu\|\leq\frac{1}{4}\,\|u\|\qquad\text{for all } u\in L_\chi^{\mathcal{E}_\lambda}\cap M.$$

Let now $v,w\in L_\chi^{\mathcal{E}_\lambda}\cap M$, and assume that $\mathrm{Op}^l(\eta_{\lambda,-1})\,Q_\lambda\,\Pi v=\mathrm{Op}^l(\eta_{\lambda,-1})\,Q_\lambda\,\Pi w$. By (24) we deduce $w+T\mathcal{E}_\lambda^{-1}w=v+T\mathcal{E}_\lambda^{-1}v$ and consequently $\left\|(\mathbf{1}+T\mathcal{E}_\lambda^{-1})(v-w)\right\|=0$. But for $u\in M\cap L_\chi^{\mathcal{E}_\lambda}$ one computes

$$\left\|(\mathbf{1}+T\mathcal{E}_\lambda^{-1})u\right\|\geq\|u\|-\left\|T\mathcal{E}_\lambda^{-1}u\right\|\geq\Big(1-\frac{1}{4}\Big)\,\|u\|\,;$$

hence $\mathbf{1}+T\mathcal{E}_\lambda^{-1}$ is injective, and $v=w$. Thus we have shown that

$$\mathrm{Op}^l(\eta_{\lambda,-1})\,Q_\lambda\,\Pi: L_\chi^{\mathcal{E}_\lambda}\cap M\longrightarrow\tilde L_\chi^{\mathcal{E}_\lambda} \tag{25}$$

is injective, and the assertion of the lemma follows with $C=\operatorname{codim}M<\infty$. □

Since $\tilde L_\chi^{\mathcal{E}_\lambda}\subset\mathrm{C}_{\mathrm{c}}^\infty(\mathbf{X})\cap\mathcal{H}_\chi$, the next proposition will provide us with a suitable reference subspace in order to prove Theorem 5. Its dimension will be estimated from below with the help of the preceding lemma. Note that the parametrices of Π and Q_λ were needed to show the injectivity of (25).

Proposition 4. *There exists a subspace* $L \subset \tilde{L}_\chi^{\mathcal{E}_\lambda}$ *such that* $\dim L \geq \dim L_\chi^{\mathcal{E}_\lambda} - C$ *for some constant* $C > 0$ *independent of* λ*, and*

$$((A_0 - \lambda \mathbf{1})u, u)_{\mathrm{L}^2} < 0 \quad \textit{for all } 0 \neq u \in L.$$

Note that, by construction, $\tilde{L}_\chi^{\mathcal{E}_\lambda} \subset \mathrm{C}_\mathrm{c}^\infty(\mathbf{X}) \cap \mathcal{H}_\chi$, while $L_\chi^{\mathcal{E}_\lambda} \not\subset \mathrm{C}_\mathrm{c}^\infty(\mathbf{X})$. It is this proposition that accomplishes the transition from $\mathbb{R}^n$ to $\mathbf{X}$, which, according to (23), is achieved by a perturbation of order $-\infty$.

Proof. Let $v \in L_\chi^{\mathcal{E}_\lambda}$ and $w = \mathrm{Op}^l(\eta_{\lambda,-1})\, Q_\lambda\, \Pi \mathcal{E}_\lambda v \in \tilde{L}_\chi^{\mathcal{E}_\lambda}$. Equation (23) implies that

$$w = Q_\lambda\, \Pi \mathcal{E}_\lambda v + \tilde{T} v, \qquad \tilde{T} \in \mathcal{L}^{-\infty}(g, \pi \tilde{q}_\lambda^{-1}).$$

Consequently, one computes

$$\begin{aligned} ((A_0 - \lambda\mathbf{1})w, w) &= \big(\Pi^*\, Q_\lambda^* [A_0 - \lambda \mathbf{1} + 4 R_{Q_\lambda}^* \mathrm{Op}^w(h^{\delta-\omega} + C_0 d) R_{Q_\lambda}]\, Q_\lambda\, \Pi \mathcal{E}_\lambda v, \mathcal{E}_\lambda v\big) \\ &\quad - 4(\Pi^*\, Q_\lambda^* R_{Q_\lambda}^* \mathrm{Op}^w(h^{\delta-\omega} + C_0 d) R_{Q_\lambda}\, Q_\lambda\, \Pi \mathcal{E}_\lambda v, \mathcal{E}_\lambda v) + (Tv, v) \\ &=: (D_1 \mathcal{E}_\lambda v, \mathcal{E}_\lambda v) - 4(D_2 \mathcal{E}_\lambda v, \mathcal{E}_\lambda v) + (Tv, v), \end{aligned} \tag{26}$$

where T is of order $-\infty$. Now, since $Q_\lambda R_{Q_\lambda} - \mathbf{1} \in \mathcal{L}^{-\infty}(g, 1)$, we have

$$D_2 = \mathbf{1} + K_2, \qquad K_2 \in \mathcal{L}(g, h);$$

indeed, by definition, the Weyl symbol of Π is equal to $\pi = (h^{\delta-\omega} + C_0 d)^{-1/2} \in SI(g, \pi)$. Now, according to Lemma 9, $A_0 - \lambda \mathbf{1} = \mathrm{Op}^w(\tilde{a}_\lambda)$, where $\tilde{a}_\lambda \in SI^+(g, \tilde{q}_\lambda^2)$. Thus,

$$D_1 = \Pi^* [Q_\lambda^*\, \mathrm{Op}^w(\tilde{a}_\lambda)\, Q_\lambda + 4\, \mathrm{Op}^w(h^{\delta-\omega} + C_0 d)]\, \Pi + K_1, \tag{27}$$

where $K_1 \in \mathcal{L}^{-\infty}(g, 1)$. Furthermore, we can assume that $q_\lambda \in S(g, \tilde{q}_\lambda^{-1})$ is such that $a_\lambda = (a_{2m} - \lambda) q_\lambda^2 \in S(g, 1)$, and using Theorem 3 one computes

$$a_\lambda - \sigma^w(Q_\lambda^*\, \mathrm{Op}^w(\tilde{a}_\lambda)\, Q_\lambda) = a_\lambda - q_\lambda^2 \tilde{a}_\lambda + r = q_\lambda^2 (a_{2m} - \lambda - \tilde{a}_\lambda) + r \in S(g, d), \tag{28}$$

where $r \in S(g, h)$. But this implies $a_\lambda - \sigma^w(Q_\lambda^* \mathrm{Op}^w(\tilde{a}_\lambda) Q_\lambda) + 4 C_0 d \geq cd$ for some sufficiently large C_0 and some $c > 0$; hence

$$a_\lambda - \sigma^w(Q_\lambda^*\, \mathrm{Op}^w(\tilde{a}_\lambda)\, Q_\lambda) + 4 C_0 d \in SI^+(g, d). \tag{29}$$

Using Lemma 2, we conclude from (29) that there exists a $T_4 \in \mathcal{L}^{-\infty}(g, d)$ such that

$$Q_\lambda^*\, \mathrm{Op}^w(\tilde{a}_\lambda)\, Q_\lambda \leq \mathrm{Op}^w(a_\lambda) + 4 C_0\, \mathrm{Op}^w(d) + T_4. \tag{30}$$

Together with $\|\mathcal{E}_\lambda v\|^2 \geq \frac{1}{4} \|v\|^2$, equations (26) - (30) therefore yield the estimate

$$\begin{aligned} ((A_0 - \lambda \mathbf{1})w, w) &= (Tv, v) - 4(K_2 \mathcal{E}_\lambda v, \mathcal{E}_\lambda v) - 4(\mathcal{E}_\lambda v, \mathcal{E}_\lambda v) + (K_1 \mathcal{E}_\lambda v, \mathcal{E}_\lambda v) \\ &\quad + (\Pi^* [Q_\lambda^*\, \mathrm{Op}^w(\tilde{a}_\lambda)\, Q_\lambda + 4 C_0\, \mathrm{Op}^w(d) + 4\, \mathrm{Op}^w(h^{\delta-\omega})]\, \Pi \mathcal{E}_\lambda v, \mathcal{E}_\lambda v) \\ &\leq (\Pi^* [\mathrm{Op}^w(a_\lambda) + 8 C_0\, \mathrm{Op}^w(d) + 4\, \mathrm{Op}^w(h^{\delta-\omega})]\, \Pi \mathcal{E}_\lambda v, \mathcal{E}_\lambda v) - \|v\|^2 + (K_3 v, v), \end{aligned}$$

where $K_3 \in S(h^{-2\delta} g, h)$. We therefore set

$$a_\lambda^- := a_\lambda + 8 C_0 d + 4 h^{\delta-\omega} \in S(g, 1),$$

and obtain the estimate

$$((A_0 - \lambda \mathbf{1})w, w) \leq (\Pi^*\, \mathrm{Op}^w(a_\lambda^-)\, \Pi \mathcal{E}_\lambda v, \mathcal{E}_\lambda v) - \|v\|^2 + (K_3 v, v). \tag{31}$$

Thus, it remains to show that $\mathcal{E}_\lambda^*\, \Pi^*\, \mathrm{Op}^w(a_\lambda^-)\, \Pi \mathcal{E}_\lambda - \mathbf{1} + K_3$ is negative definite on some subspace of finite codimension. In order to do so, we will show that $\mathcal{E}_\lambda^*\, \Pi^*\, \mathrm{Op}^w(a_\lambda^-)\, \Pi \mathcal{E}_\lambda - \mathbf{1} + K_3 \leq -\mathbf{1} + K_4$, where $K_4 \in \mathcal{L}(h^{-2\delta} g, h^\omega)$ and $\omega > 0$. As it shall become apparent in the following discussion, the key to this is contained in the fact that, although $a_\lambda^- \in S(g,1)$, there exists a $K_5 \in \mathcal{L}(h^{-2\delta}g, h^\delta)$ such that $\mathrm{Op}^w(\chi_\lambda a_\lambda^- \chi_\lambda) \leq K_5$! Now,

$$\begin{aligned}\Pi\mathcal{E}_\lambda &= \Pi\tilde{\mathcal{E}}_\lambda\mathcal{D}_\lambda = \Pi\,\mathrm{Op}^w(\eta_{\lambda,-2})\,\mathrm{Op}^w(\chi_\lambda)\,\mathrm{Op}^w(\eta_{\lambda,-2})\mathcal{D}_\lambda \\ &= [\Pi\,\mathrm{Op}^w(\eta_{\lambda,-2}),\, \mathrm{Op}^w(\chi_\lambda)]\,\mathrm{Op}^w(\eta_{\lambda,-2})\mathcal{D}_\lambda \\ &+ \mathrm{Op}^w(\chi_\lambda)\,\Pi\,\mathrm{Op}^w(\eta_{\lambda,-2})\,\mathrm{Op}^w(\eta_{\lambda,-2}))\mathcal{D}_\lambda =: W_1 + W_2,\end{aligned}$$

where we put $\mathcal{D}_\lambda = \tilde{\mathcal{E}}_\lambda(3 - 2\tilde{\mathcal{E}}_\lambda)$. Since Π and $\tilde{\mathcal{E}}_\lambda$ are self-adjoint, we obtain

$$\mathcal{E}_\lambda\,\Pi\,\mathrm{Op}^w(a_\lambda^-)\,\Pi\mathcal{E}_\lambda = W_2^*\,\mathrm{Op}^w(a_\lambda^-)W_2 + R \tag{32}$$

where $R = W_1^*\,\mathrm{Op}^w(a_\lambda^-)W_2 + W_2^*\,\mathrm{Op}^w(a_\lambda^-)W_1 + W_1^*\,\mathrm{Op}^w(a_\lambda^-)W_1$ is given by a sum of terms which contain either $[\Pi\,\mathrm{Op}^w(\eta_{\lambda,-2}),\, \mathrm{Op}^w(\chi_\lambda)]$, or its adjoint $[\mathrm{Op}^w(\chi_\lambda),\, \mathrm{Op}^w(\eta_{\lambda,-2})\,\Pi]$, as factors. Now, the crucial remark is that

$$\mathrm{supp}_\infty\,\sigma^w([\Pi\,\mathrm{Op}^w(\eta_{\lambda,-2}),\, \mathrm{Op}^w(\chi_\lambda)]) \subset \mathrm{supp}_{\mathrm{diff}}\,\chi_\lambda \subset \left\{(x,\xi) : |a_\lambda^-(x,\xi)| \leq h^\delta(x,\xi)\right\}, \tag{33}$$

where $\mathrm{supp}_{\mathrm{diff}}\,\chi_\lambda = \left\{(x,\xi) : \exists k > 0 : \chi_\lambda^{(k)}(x,\xi) \neq 0\right\}$. To see this, first note that by Theorem 3 and Proposition 1, we have the trivial inclusion $\mathrm{supp}_\infty\,\sigma^w([\Pi\,\mathrm{Op}^w(\eta_{\lambda,-2}),\, \mathrm{Op}^w(\chi_\lambda)]) \subset \mathrm{supp}\,\chi_\lambda$. But since the terms in the asymptotic expansion of the Weyl symbol of $[\Pi\,\mathrm{Op}^w(\eta_{\lambda,-2}),\, \mathrm{Op}^w(\chi_\lambda)]$ are of order ≥ 1, they vanish unless $(x,\xi) \in \mathrm{supp}_{\mathrm{diff}}\,\chi_\lambda$, and one obtains the first inclusion. The second inclusion follows by noting the implications

$$\chi_\lambda^{(k)} = 0\,\forall\, k > 0 \quad \Leftarrow \quad \chi_\lambda = 0 \text{ or } \chi_\lambda = 1 \quad \Leftarrow \quad a_\lambda^- h^{-\delta} \geq 1 \text{ or } a_\lambda^- h^{-\delta} \leq 0.$$

While computing the Weyl symbol of R, we can therefore replace a_λ^- with

$$b_\lambda^- = a_\lambda^-\theta_\lambda, \qquad \theta_\lambda = \theta\Big(\frac{1}{2}a_\lambda^- h^{-\delta}\Big), \tag{34}$$

where $\theta \in \mathrm{C_c^\infty}(\mathbb{R})$ is a real valued function taking values between 0 and 1, which is equal 1 on $[-1,1]$, and which vanishes outside $[-2,2]$, so that $\theta_\lambda = 1$ on $\left\{(x,\xi) : |a_\lambda^-(x,\xi)| \leq h^\delta(x,\xi)\right\}$. Indeed, this replacement adds at most a term of order $-\infty$ to the Weyl symbol of R. Now, the adventage of performing this replacement resides in the fact that, on $\mathrm{supp}\,\theta_\lambda$, one has $|a_\lambda^-| \leq 4h^\delta$, which together with

$$|\,\partial_\xi^\alpha\,\partial_x^\beta\,a_\lambda^-(x,\xi)| \leq C(1+|x|^2+|\xi|^2)^{\frac{-|\alpha|}{2}} \leq C(1+|x|^2+|\xi|^2)^{\frac{-\delta-(1-\delta)|\alpha|+\delta|\beta|}{2}}, \quad |\alpha| \geq 1,$$

i.e. $\nu_k(h^{-2\delta}g, h^\delta; a_\lambda^-) < \infty$, $k \geq 1$, yields $a_\lambda^- \in S(h^{-2\delta}g, h^\delta, \mathrm{supp}\,\theta_\lambda)$, in contraposition to $a_\lambda^- \in S(g,1)$. Consequently, $b_\lambda^- \in S(h^{-2\delta}g, h^\delta)$, and we obtain

$$R \in \mathcal{L}(h^{-2\delta}g, h^\delta\pi^2) \subset \mathcal{L}(h^{-2\delta}g, h^\omega), \tag{35}$$

since $W_1, W_2 \in \mathcal{L}(h^{-2\delta}g, \pi)$, $\mathcal{D}_\lambda \in \mathcal{L}(h^{-2\delta}g, 1)$, and $h^\delta\pi^2 = h^\delta(h^{\delta-\omega}+C_0 d)^{-1} = (h^{-\omega}+C_0h^{-\delta}d)^{-1} \sim h^\omega$. Equations (31), (32), and (35) therefore yield the estimate

$$((A_0 - \lambda\mathbf{1})w, w) \leq (W_2^*\,\mathrm{Op}^w(a_\lambda^-)W_2 v, v) - \|v\|^2 + (K_4 v, v), \tag{36}$$

where $K_4 = K_3 + R \in \mathcal{L}(h^{-2\delta}g, h^\omega)$. To examine $W_2^*\mathrm{Op}^w(a_\lambda^-)W_2$ more closely, let us consider the operator

$$S = \mathrm{Op}^w(\chi_\lambda)\mathrm{Op}^w(a_\lambda^-)\mathrm{Op}^w(\chi_\lambda) - \mathrm{Op}^w(\chi_\lambda a_\lambda^- \chi_\lambda).$$

By the usual argument, the asymptotic expansion (1) and Proposition 1 yield $\mathrm{supp}_\infty\,\sigma^w(S) \subset \mathrm{supp}_{\mathrm{diff}}\,\chi_\lambda$. In the computation of the Weyl symbol of S we can therefore again replace a_λ^- with b_λ^-, getting at most an additional term of order $-\infty$. Since $\mathrm{Op}^w(\chi_\lambda) \in \mathcal{L}(h^{-2\delta}g, 1)$ by Lemma 10, we obtain

$$S \in \mathcal{L}(h^{-2\delta}g, h^\delta). \tag{37}$$

Now, by construction, $a_\lambda^- \chi_\lambda \leq h^\delta$, since $0 \leq \chi_\lambda \leq 1$ and $\chi_\lambda = 0$ for $a_\lambda^- h^{-\delta} > 1$, so that one infers

$$|\,\partial_\xi^\alpha \,\partial_x^\beta (\chi_\lambda a_\lambda^- \chi_\lambda)(x,\xi)| \leq C(1+|x|^2+|\xi|^2)^{(-\delta-(1-\delta)|\alpha|+\delta|\beta|)/2}$$

for some constant $C > 0$. But this implies $\mathrm{Op}^w(\chi_\lambda a_\lambda^- \chi_\lambda) \in \mathcal{L}(h^{-2\delta} g, h^\delta)$. Using (36) and (37), we therefore get

$$((A_0 - \lambda \mathbf{1})w, w) \leq (W_3^* \,\mathrm{Op}^w(\chi_\lambda a_\lambda^- \chi_\lambda) W_3 v, v) - \|v\|^2 + (K_5 v, v) = -\|v\|^2 + (K_6 v, v),$$

with

$$\begin{aligned}
W_3 &= \Pi \,\mathrm{Op}^w(\eta_{\lambda,-2})\,\mathrm{Op}^w(\eta_{\lambda,-2}) \mathcal{D}_\lambda \in \mathcal{L}(h^{-2\delta} g, \pi), \\
K_5 &= K_4 + W_3^* [\mathrm{Op}^w(\chi_\lambda) \mathrm{Op}^w(a_\lambda^-) \mathrm{Op}^w(\chi_\lambda) - \mathrm{Op}^w(\chi_\lambda a_\lambda^- \chi_\lambda)] W_3 \in \mathcal{L}(h^{-2\delta} g, h^\omega), \\
K_6 &= K_5 + W_3^* \mathrm{Op}^w(\chi_\lambda a_\lambda^- \chi_\lambda) W_3 \in \mathcal{L}(h^{-2\delta} g, h^\omega).
\end{aligned}$$

Since $h_\sigma = h^{1-2\delta}$, Lemma 3 implies that the operator $-\mathbf{1} + K_6$ is negative definite on a subspace $U \subset \mathrm{L}^2(\mathbb{R}^n)$ of finite codimension which does not depend on λ. Putting $L := \mathrm{Op}^l(\eta_{\lambda,-1})\, Q_\lambda \,\Pi \mathcal{E}_\lambda (U \cap L_\chi^{\mathcal{E}_\lambda} \cap M) \subset \tilde{L}_\chi^{\mathcal{E}_\lambda}$ with M as in (25), we finally get

$$((A_0 - \lambda \mathbf{1})w, w) < 0 \quad \forall\, 0 \neq w \,\in L, \tag{38}$$

where $\dim U \cap L_\chi^{\mathcal{E}_\lambda} \cap M - \operatorname{codim} M \,\leq\, \dim M \cap \mathcal{E}_\lambda(U \cap L_\chi^{\mathcal{E}_\lambda} \cap M) \leq \dim L$, since $\mathcal{E}_\lambda$ is bijective on $L_\chi^{\mathcal{E}_\lambda}$, and $\dim L_\chi^{\mathcal{E}_\lambda} \,\leq \dim U \cap L_\chi^{\mathcal{E}_\lambda} \cap M + \operatorname{codim} U \cap M$. The assertion of the proposition now follows. ☐

We can now prove Theorem 5.

Proof of Theorem 5. Let $L \subset \tilde{L}_\chi^{\mathcal{E}_\lambda} \subset \mathrm{C_c^\infty}(\mathbf{X}) \cap \mathcal{H}_\chi$ be as in the previous proposition. Then (38) holds, and $\mathcal{N}(A_0 - \lambda \mathbf{1}, \mathcal{H}_\chi \cap \mathrm{C_c^\infty}(\mathbf{X})) \geq \dim L$. Furthermore, $\dim L \geq \dim L_\chi^{\mathcal{E}_\lambda} - C = N_\chi^{\mathcal{E}_\lambda} - C$, and the assertion of the theorem follows. ☐

5. Estimates from above for $N_\chi(\lambda)$

In this section, we will prove an estimate from above for $N_\chi(\lambda) = \mathcal{N}(A_0 - \lambda \mathbf{1}, \mathcal{H}_\chi \cap \mathrm{C_c^\infty}(\mathbf{X}))$ in terms of the number $M_\chi^{\mathcal{F}_\lambda}$ of eigenvalues of $\mathcal{F}_\lambda$ which are $\geq 1/2$, and whose eigenfunctions are contained in the χ-isotypic component $\mathcal{H}_\chi$. In order to do so, we first prove the following

Proposition 5. *There exists a constant $C > 0$ independent of λ such that*

$$\mathcal{N}(A_0 - \lambda \mathbf{1}, \mathcal{H}_\chi \cap \mathrm{C_c^\infty}(\mathbf{X})) \leq \mathcal{N}(\mathrm{Op}^l(\eta_1)\,\Pi\,\mathrm{Op}^w(a_\lambda^+)\,\Pi\,\mathrm{Op}^r(\eta_1) + \mathbf{1}, \mathcal{H}_\chi \cap \mathrm{C_c^\infty}(\mathbb{R}^n)) + C.$$

Note that this proposition accomplishes the transition from variational quantities related to $\mathbb{R}^n$ to quantities related to the bounded subdomain $\mathbf{X}$. Now, the proof of Proposition 5 relies on the following

Lemma 14. *There exists a subspace $L \subset \mathrm{C_c^\infty}(\mathbf{X})$ of finite codimension in $\mathrm{C_c^\infty}(\mathbf{X})$ such that*

$$((A_0 - \lambda \mathbf{1})\, u,\, u) \geq ([\mathrm{Op}^l(\eta_1)\,\Pi\,\mathrm{Op}^w(a_\lambda^+)\,\Pi\,\mathrm{Op}^r(\eta_1)\, + \mathbf{1}]\, R_\Pi\, R_{Q_\lambda}\, u,\, R_\Pi\, R_{Q_\lambda}\, u)$$

for all $0 \neq u \in L$, and all λ.

Proof of Proposition 5. Let us assume Lemma 14 for a moment, and introduce the notation

$$A_\lambda[u] = (A_0 - \lambda \mathbf{1})u, u), \quad B_\lambda[u] = ([\mathrm{Op}^l(\eta_1)\,\Pi\,\mathrm{Op}^w(a_\lambda^+)\,\Pi\,\mathrm{Op}^r(\eta_1)\, + \mathbf{1}]u, u).$$

According to that lemma, there exists a subspace L in $\mathrm{C_c^\infty}(\mathbf{X})$ of finite codimension such that

$$A_\lambda[u] \geq B_\lambda[R_\Pi R_{Q_\lambda} u], \quad 0 \neq u \in L,$$

for all λ. Let now m be a positive σ, g -temperate function such that $1/m$ is bounded. Following [12], we introduce the weight spaces of Sobolev type

$$\mathcal{H}(g,m) = \operatorname{span}\left\{Tw : w \in \mathrm{L}^2(\mathbb{R}^n),\, T \in \mathcal{L}(g,1/m)\right\} \subset \mathrm{L}^2(\mathbb{R}^n),$$

and endow them with the strongest topology in which each of the operators $T : \mathrm{L}^2(\mathbb{R}^n) \to \mathcal{H}(g,m)$, $T \in \mathcal{L}(g,1/m)$, is continuous. It can then be shown that there exists an operator $\Lambda_m \in \mathcal{L}(g,m)$ such that $\Lambda_m : \mathcal{H}(g,m) \to \mathrm{L}^2(\mathbb{R}^n)$ is a topological isomorphism. In particular, $\mathcal{H}(g,m)$ becomes a Hilbert space with the norm $\|u\|_m = \|\Lambda_m u\|_{\mathrm{L}^2}$. Furthermore, we have the continuous embedding $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{H}(g,m)$, and if m_1 is a bounded, σ, g-tempered function, and $A \in \mathrm{L}(g, mm_1)$, then $A : \mathcal{H}(g,m) \to \mathcal{H}(g, m_1^{-1})$ defines a continuous map. Now, by Theorem 4 and the asymptotic expansion (9), $R_\Pi R_{Q_\lambda} \in \mathcal{LI}(g, \pi^{-1}\tilde{q}_\lambda)$, so that by Lemma 1 the operator $\Lambda_\pi R_\Pi R_{Q_\lambda} \Lambda^{-1}_{\tilde{q}_\lambda} \in \mathcal{LI}(g,1)$ has a parametrix $Z \in \mathcal{LI}(g,1)$ satisfying $Z\,\Lambda_\pi R_\Pi R_{Q_\lambda} \Lambda^{-1}_{\tilde{q}_\lambda} = \mathbf{1} + K$, where $K \in \mathcal{L}^{-\infty}(g,1)$. Since by Lemma 3 the kernel of $\mathbf{1}+K$ must be finite dimensional, $\operatorname{Ker} \Lambda_\pi R_\Pi R_{Q_\lambda} \Lambda^{-1}_{\tilde{q}_\lambda} < \infty$; consequently

$$r = \dim \operatorname{Ker}(R_\Pi R_{Q_\lambda} : \mathcal{H}(g,\tilde{q}_\lambda) \to H(g,\pi)) < \infty. \tag{39}$$

Next, let $U \subset \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X}) \cap \mathcal{H}_\chi$ be a subspace such that

$$A_\lambda[u] < 0, \qquad \forall\; 0 \neq u \in U.$$

Then, for all $0 \neq u \in V := U \cap L \cap \complement_{\mathcal{H}(g,\tilde{q}_\lambda)} (\operatorname{Ker} R_\Pi\, R_{Q_\lambda} : \mathcal{H}(g,\tilde{q}_\lambda) \to \mathcal{H}(g,\pi))$,

$$0 > B_\lambda[R_\Pi\, R_{Q_\lambda}\, u]. \tag{40}$$

Because $R_\Pi\, R_{Q_\lambda}$ is injective on V, (39) yields the inequality $\dim U \leq \dim V + C \leq \dim R_\Pi\, R_{Q_\lambda} V + C$ for some constant $C > 0$ independent of λ. Since $R_\Pi R_{Q_\lambda}$ commutes with the operators $T(k)$ of the representation of G, $R_\Pi R_{Q_\lambda} V \subset \mathcal{H}_\chi \cap \mathcal{H}(g,\pi)$, and we obtain the estimate

$$\dim U \leq \sup_{W \in \mathcal{H}(g,\pi)\cap\mathcal{H}_\chi} \{\dim W : B_\lambda[w] < 0 \quad \forall\, 0 \neq w \in W\} + C.$$

But $\mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \cap \mathcal{H}_\chi$ is dense in $\mathcal{H}(g,\pi) \cap \mathcal{H}_\chi$, and the assertion of the proposition follows. □

Let us now prove Lemma 14.

Proof of Lemma 14. Let $u \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X})$. Then

$$\operatorname{Op}^r(\eta_c)\, u(x) = \int\int e^{i(x-y)\xi} \eta_c(y,\xi) u(y) dy\, đ\xi = u(x),$$

since η_c is equal one on $\mathbf{X} \times \mathbb{R}^n$. Now, for general $B \in \mathcal{L}(g,p)$, $\sigma^r(\operatorname{Op}^r(\eta_c)\, B)$ is given by an asymptotic expansion $\sum_j a_j$, where the first term is equal to $\eta_c \sigma^r(B)$. Consequently, $\sigma^r(\operatorname{Op}^r(\eta_c) B) = \eta_c \sigma^r(B) + (a - \eta_c \sigma^r(B)) + r$, with a as in Propostion 1, and $r \in S^{-\infty}(h^{-2\delta} g, p)$. But $a - \eta_c \sigma^r(B) = 0$ on $\mathbf{X} \times \mathbb{R}^n$, and we obtain

$$\operatorname{Op}^r(\eta_c)\, B\, u = B\, u + T\, u, \qquad T \in \mathcal{L}^{-\infty}(h^{-2\delta} g, p). \tag{41}$$

Using Lemma 9, and setting $\tilde{u} = R_\Pi\, R_{Q_\lambda} u$, one computes

$$\begin{aligned}
((A_0 - \lambda \mathbf{1})\, u,\, u) &= (\operatorname{Op}^w(\tilde{a}_\lambda)\, Q_\lambda\, \Pi\, \tilde{u},\, Q_\lambda\, \Pi\, \tilde{u}) + (T_1\, u,\, u) \\
&= (\Pi^*[Q^*_\lambda \operatorname{Op}^w(\tilde{a}_\lambda)\, Q_\lambda - 4\operatorname{Op}^w(h^{\delta-\omega} + C_0 d)]\, \Pi\, \tilde{u},\, \tilde{u}) \\
&\quad + 4(\Pi^* \operatorname{Op}^w(h^{\delta-\omega} + C_0 d) \Pi\, \tilde{u},\, \tilde{u}) + (T_1\, u,\, u) \\
&=: (\Pi^*\, D_1\, \Pi\, \operatorname{Op}^r(\eta_1)\, \tilde{u},\, \operatorname{Op}^r(\eta_1) \tilde{u}) + 4(D_2 \tilde{u},\, \tilde{u}) + (T_2\, u,\, u),
\end{aligned}$$

where we took (41) into account together with $R_\Pi R_{Q_\lambda} - R_\Pi R_{Q_\lambda} \in \mathcal{L}^{-\infty}(g, \tilde q_\lambda \pi^{-1})$, and $T_i \in \mathcal{L}^{-\infty}$. The reason for including $\mathrm{Op}^r(\eta_1)$ will become apparent in the proof of the next theorem. Now, by (28), $a_\lambda - \sigma^w(Q_\lambda^* \mathrm{Op}^w(\tilde a_\lambda) Q_\lambda) \in S(g,d)$, which implies that for sufficiently large C_0

$$D_1 - \mathrm{Op}^w(a_\lambda^+) = Q_\lambda^* \mathrm{Op}^w(\tilde a_\lambda) Q_\lambda + 4C_0 \mathrm{Op}^w(d) - \mathrm{Op}^w(a_\lambda) \in \mathcal{LI}^+(g,d),$$

so that according to Lemma 2, there exists a $T_3 \in \mathcal{L}^{-\infty}(g,d)$ such that $D_1 - \mathrm{Op}^w(a_\lambda^+) \geq T_3$. On the other hand, since $\pi^2 = (h^{\delta-\omega} + C_0 d)^{-1}$, $D_2 - \mathbf{1} \in \mathcal{L}(g,h)$, and we obtain

$$((A_0 - \lambda \mathbf{1})\, u,\, u) \geq (\mathrm{Op}^l(\eta_1)\, \Pi^*\, \mathrm{Op}^w(a_\lambda^+)\, \Pi\, \mathrm{Op}^r(\eta_1)\, \tilde u,\, \tilde u) + 2\, \|\tilde u\|^2 + (T_4\, u,\, u), \tag{42}$$

where $T_4 \in \mathcal{L}(g, \pi^{-2} \tilde q_\lambda^2 h)$; hereby we used the fact that $\mathrm{Op}^l(\eta_1)$ is the adjoint of $\mathrm{Op}^r(\eta_1)$, compare [14], page 26. Furthermore, since by (9) the Weyl symbol of $R_\Pi R_{Q_\lambda}$ is equal to $\pi^{-1} \tilde q_\lambda$ modulo terms of lower order,

$$(R_\Pi\, R_{Q_\lambda})^* R_\Pi\, R_{Q_\lambda} + T_4 \in \mathcal{LI}^+(g, \pi^{-2} \tilde q_\lambda^2).$$

Lemmas 1 - 3 now allow us to deduce the existence of a subspace $L \subset \mathrm{C_c^\infty}(\mathbf{X})$ of finite codimension in $\mathrm{L}^2(\mathbf{X})$ such that

$$\|\tilde u\|^2 + (T_4\, u,\, u) = ([(R_\Pi\, R_{Q_\lambda})^* R_\Pi\, R_{Q_\lambda} + T_4] u,\, u) > 0 \tag{43}$$

for all $0 \neq u \in L$, and all λ. Indeed, according to Lemma 2, $\Lambda_{\pi^2} [(R_\Pi\, R_{Q_\lambda})^* R_\Pi\, R_{Q_\lambda} + T_4] \Lambda^{-1}_{\tilde q_\lambda^2} \in \mathcal{LI}^+(g,1)$ can be written in the form $B^*B + T_5$, where $B \in \mathcal{LI}(g,1)$ and $T \in \mathcal{L}^{-\infty}(g,1)$. By a reasoning similar to the one that led to (39), one can infer from Lemma 1 that the kernel of B must be finite dimensional, and together with Lemma 3 conclude that there exists a subspace $\tilde L \subset \mathrm{L}^2(\mathbb{R}^n)$ of finite codimension such that

$$\|Bu\|_{\mathrm{L}^2} \geq c\, \|u\|_{\mathrm{L}^2}\,, \qquad \|T_5 u\|_{L^2} < \frac{c^2}{2}\, \|u\|_{\mathrm{L}^2}\,,$$

for all $u \in \tilde L$ and some constant $c > 0$. Thus, we obtain (43), and together with (42) we get

$$((A_0 - \lambda \mathbf{1})\, u,\, u) \geq ([\mathrm{Op}^l(\eta_1)\, \Pi^* \mathrm{Op}^w(a_\lambda^+)\, \Pi\, \mathrm{Op}^r(\eta_1) + \mathbf{1}]\, \tilde u,\, \tilde u)$$

for all $0 \neq u \in L$. This concludes the proof of the lemma. □

We are now in position to prove an estimate from above for $\mathcal{N}(A_0 - \lambda \mathbf{1}, \mathcal{H}_\chi \cap \mathrm{C_c^\infty}(\mathbf{X}))$.

Theorem 6. *Let $M_\chi^{\mathcal{F}_\lambda}$ be the number of eigenvalues of $\mathcal{F}_\lambda$ which are $\geq 1/2$, and whose eigenfunctions are contained in the χ-isotypic component $\mathcal{H}_\chi$. Then there exists a constant $C > 0$ independent of λ such that*

$$\mathcal{N}(A_0 - \lambda \mathbf{1}, \mathcal{H}_\chi \cap \mathrm{C_c^\infty}(\mathbf{X})) \leq M_\chi^{\mathcal{F}_\lambda} + C.$$

Proof. We shall continue with the notation introduced in the proof of Proposition 5. According to that proposition, it suffices to prove a similar estimate for $\mathcal{N}(\mathrm{Op}^l(\eta_1) \Pi \mathrm{Op}^w(a_\lambda^+) \Pi \mathrm{Op}^r(\eta_1) + \mathbf{1}, \mathcal{H}_\chi \cap \mathrm{C_c^\infty}(\mathbb{R}^n))$ from above. For this sake, we will show that there exists a subspace $L \subset U_\chi^{\mathcal{F}_\lambda} = \mathrm{Span}\, \{u \in \mathcal{S}(\mathbb{R}^n) \cap \mathcal{H}_\chi : \mathcal{F}_\lambda\, u = \nu\, u,\, \nu < 1/2\}$, whose codimension in $U_\chi^{\mathcal{F}_\lambda}$ is finite and uniformly bounded in λ, such that

$$B_\lambda[u] \geq 0 \quad \text{for all } u \in L.$$

Indeed, let us assume this statement for a moment. Since $\mathcal{F}_\lambda$ is a compact self-adjoint operator in $\mathrm{L}^2(\mathbb{R}^n)$, there exists an orthonormal basis of eigenfunctions $\{u_j\}_{j=1}^\infty$ in $\mathcal{S}(\mathbb{R}^n)$. But $\mathcal{F}_\lambda$ commutes with the action $T(g)$ of G, so that each of the eigenspaces of $\mathcal{F}_\lambda$ is an invariant subspace, and must therefore decompose into a sum of irreducible G-modules. Consequently, $\mathcal{H}_\chi$ has an orthonormal basis of eigenfunctions lying in $\mathcal{S}(\mathbb{R}^n) \cap \mathcal{H}_\chi$. Hence,

$$\mathcal{H}_\chi = U_\chi^{\mathcal{F}_\lambda} \oplus V_\chi^{\mathcal{F}_\lambda},$$

where $V_\chi^{\mathcal{F}_\lambda} = \mathrm{Span}\{u \in \mathcal{S}(\mathbb{R}^n) \cap \mathcal{H}_\chi : \mathcal{F}_\lambda u = \nu u, \, \nu \geq 1/2\}$. Now, if $W \subset \mathcal{S}(\mathbb{R}^n) \cap \mathcal{H}_\chi$ is a subspace with

$$B_\lambda[u] < 0 \quad \text{for all } 0 \neq u \in W,$$

then $L \cap W = \{0\}$, and therefore $W \subset V_\chi^{\mathcal{F}_\lambda} \oplus U$, where U is a finite dimensional subspace of $U_\chi^{\mathcal{F}_\lambda}$ whose dimension is bounded by some constant $C > 0$ independent of λ. Consequently, $\dim W \leq \dim V_\chi^{\mathcal{F}_\lambda} + C$. But this implies

$$\begin{aligned}
&\mathcal{N}((\mathrm{Op}^l(\eta_1)\Pi\mathrm{Op}^w(a_\lambda^+)\Pi\mathrm{Op}^r(\eta_1) + \mathbf{1}), \mathcal{H}_\chi \cap \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n)) \\
&\quad \leq \sup_{W \subset \mathcal{S}(R^n) \cap \mathcal{H}_\chi} \Big\{ \dim W : ((\mathrm{Op}^l(\eta_1)\Pi\mathrm{Op}^w(a_\lambda^+)\Pi\mathrm{Op}^r(\eta_1) + \mathbf{1})\, u,\, u) < 0 \quad \forall\, 0 \neq u \in W \Big\} \\
&\quad \leq \dim V_\chi^{\mathcal{F}_\lambda} + C = M_\chi^{\mathcal{F}_\lambda} + C,
\end{aligned}$$

and the assertion of the theorem follows with the previous proposition. Let us now show the existence of the subspace L. Take $v \in U_\chi^{\mathcal{F}_\lambda} \subset \mathrm{L}^2(\mathbb{R}^n)$, and put $\tilde{v} = (\mathbf{1} - \mathcal{F}_\lambda)v$. We then expect that $B_\lambda[\tilde{v}] \geq 0$. Now, one computes

$$\begin{aligned}
B_\lambda[\tilde{v}] &= ((\mathbf{1} - \mathcal{F}'_\lambda)\mathrm{Op}^l(\eta_1)\Pi\mathrm{Op}^w(a_\lambda^+)\Pi\mathrm{Op}^r(\eta_1)(\mathbf{1} - \mathcal{F}'_\lambda)\, v,\, v) + \|(\mathbf{1} - \mathcal{F}_\lambda)v\|^2 + (K_1\, v,\, v) \\
&\geq (Dv, v) + (\|v\| - \|\mathcal{F}_\lambda v\|)^2 + (K_1 v, v) \\
&\geq (Dv, v) + \frac{1}{4}\, \|v\|^2 + (K_1 v, v),
\end{aligned} \tag{44}$$

where we put $\mathcal{F}'_\lambda = \mathrm{Op}^w(\chi_\lambda^+)^2(3 - 2\mathrm{Op}^w(\chi_\lambda^+))$,

$$D = (\mathbf{1} - \mathcal{F}'_\lambda)\mathrm{Op}^l(\eta_1)\Pi\mathrm{Op}^w(a_\lambda^+)\Pi\mathrm{Op}^r(\eta_1)(\mathbf{1} - \mathcal{F}'_\lambda),$$

and $K_1 \in \mathcal{L}^{-\infty}$. Indeed, one has $\|\mathcal{F}_\lambda v\| \leq \frac{1}{2}\, \|v\|$, and $\mathrm{Op}^r(\eta_1)\mathcal{F}_\lambda - \mathrm{Op}^r(\eta_1)\mathcal{F}'_\lambda \in \mathcal{L}^{-\infty}(h^{-2\delta}g, 1)$, since the terms in the asymptotic expansions of the Weyl symbols of $\mathrm{Op}^r(\eta_1)\mathcal{F}_\lambda$ and $\mathrm{Op}^r(\eta_1)\mathcal{F}'_\lambda$ coincide because of $\eta_2 = 1$ on $\mathrm{supp}\, \eta_1$. Next we note that, similarly to (33),

$$\mathrm{supp}_\infty\, \sigma^w([\mathcal{F}'_\lambda, \mathrm{Op}^l(\eta_1)\Pi]) \subset \mathrm{supp}_{\mathrm{diff}}\, \chi_\lambda^+ \subset \big\{(x, \xi) : |a_\lambda^+(x, \xi)| \leq h^\delta(x, \xi)\big\}\,, \tag{45}$$

and we set

$$b_\lambda^+ = a_\lambda^+ \theta_\lambda, \qquad \theta_\lambda = \theta\Big(\frac{1}{2} a_\lambda^+ h^{-\delta}\Big),$$

with θ as in (34). An argument similar to that concerning b_λ^- shows that $b_\lambda^+ \in S(h^{-2\delta}g, h^\delta)$. Now, because of $b_\lambda^+ = a_\lambda^+$ on $\mathrm{supp}_\infty\, \sigma^w([\mathcal{F}'_\lambda, \mathrm{Op}^l(\eta_1)\Pi])$, we have

$$\begin{aligned}
(D\, v,\, v) &= ([(\mathbf{1} - \mathcal{F}'_\lambda), \mathrm{Op}^l(\eta_1)\Pi]\mathrm{Op}^w(b_\lambda^+)\Pi\mathrm{Op}^r(\eta_1)(\mathbf{1} - \mathcal{F}'_\lambda)\, v,\, v) \\
&\quad + (\mathrm{Op}^l(\eta_1)\Pi(\mathbf{1} - \mathcal{F}'_\lambda)\mathrm{Op}^w(a_\lambda^+)\Pi\mathrm{Op}^r(\eta_1)(\mathbf{1} - \mathcal{F}'_\lambda)\, v,\, v) + (K_2\, v,\, v),
\end{aligned}$$

where K_2 is of order $-\infty$. Since $[(\mathbf{1}-\mathcal{F}'_\lambda), \mathrm{Op}^l(\eta_1)\Pi]\mathrm{Op}^w(b_\lambda^+)\Pi\mathrm{Op}^r(\eta_1)(\mathbf{1}-\mathcal{F}'_\lambda) \in \mathcal{L}(h^{-2\delta}g, h^\delta\pi^2) \subset \mathcal{L}(h^{-2\delta}g, h^\omega)$, we therefore obtain

$$(D\, v,\, v) = (\mathrm{Op}^l(\eta_1)\Pi(\mathbf{1} - \mathcal{F}'_\lambda)\mathrm{Op}^w(a_\lambda^+)\Pi\mathrm{Op}^r(\eta_1)(\mathbf{1} - \mathcal{F}'_\lambda)\, v,\, v) + (K_3\, v,\, v),$$

where $K_3 \in \mathcal{L}(h^{-2\delta}g, h^\omega)$. Using a similar argument to commute $\Pi\mathrm{Op}^r(\eta_1)$ with $\mathbf{1} - \mathcal{F}'_\lambda$, we finally get

$$(D\, v,\, v) = (\mathrm{Op}^l(\eta_1)\Pi(\mathbf{1} - \mathcal{F}'_\lambda)\mathrm{Op}^w(a_\lambda^+)(\mathbf{1} - \mathcal{F}'_\lambda)\Pi\mathrm{Op}^r(\eta_1)\, v,\, v) + (K_3\, v,\, v), \tag{46}$$

where $K_3 \in \mathcal{L}(h^{-2\delta}g, h^\omega)$. Now, the asymptotic expansion of the Weyl symbol of the operator $(\mathbf{1} - \mathcal{F}'_\lambda)\mathrm{Op}^w(a_\lambda^+)(\mathbf{1} - \mathcal{F}'_\lambda)$ gives

$$\sigma^w((\mathbf{1} - \mathcal{F}'_\lambda)\mathrm{Op}^w(a_\lambda^+)(\mathbf{1} - \mathcal{F}'_\lambda)) = [1 - (\chi_\lambda^+)^2(3 - 2\chi_\lambda^+)]^2 a_\lambda^+ + r \tag{47}$$

with $\operatorname{supp}_\infty r \subset \operatorname{supp}_{\text{diff}} \chi_\lambda^+$. While computing r, we can therefore replace a_λ^+ by b_λ^+, so that $r \in S(h^{-2\delta} g, h^\delta)$. As a consequence, (46) and (47) yield

$$(D\, v,\, v) = (\operatorname{Op}^l(\eta_1)\Pi\operatorname{Op}^w\big([1-(\chi_\lambda^+)^2(3-2\chi_\lambda^+)]^2 a_\lambda^+\big)\Pi\operatorname{Op}^r(\eta_1)\, v,\, v) + (K_4\, v,\, v),$$

where $K_4 \in \mathcal{L}(h^{-2\delta} g, h^\omega)$. Hereby we used again the fact that $\pi^2 h^\delta \sim h^\omega$. Next, one verifies that $[1-(\chi_\lambda^+)^2(3-2\chi_\lambda^+)]^2 a_\lambda^+ + C_1 h^\delta \in SI^+(h^{-2\delta} g, [1-(\chi_\lambda^+)^2(3-2\chi_\lambda^+)]^2 a_\lambda^+ + C_1 h^\delta)$ for some $C_1 > 0$, since $\chi_\lambda^+ = 1$ for $a_\lambda^+ < 0$, so that $[1-(\chi_\lambda^+)^2(3-2\chi_\lambda^+)]^2 a_\lambda^+ \geq 0$. According to Lemma 2, we therefore have

$$\operatorname{Op}^w\big([1-(\chi_\lambda^+)^2(3-2\chi_\lambda^+)]^2 a_\lambda^+\big) \geq K_5 \in \mathcal{L}(h^{-2\delta} g, h^\delta),$$

and we arrive at the estimate

$$(D\, v,\, v) \geq (K_6\, v,\, v), \qquad K_6 \in \mathcal{L}(h^{-2\delta} g, h^\omega).$$

Together with (44) we finally obtain the estimate

$$B_\lambda[\tilde{v}] \geq \frac{1}{4}(v, v) + (K_7\, v,\, v), \qquad K_7 \in \mathcal{L}(h^{-2\delta} g, h^\omega).$$

Using the already familiar argument of Lemma 3, one infers the existence of a subspace $M \subset \mathrm{L}^2(\mathbb{R}^n)$ of finite codimension on which $\mathbf{1}/4 + K_7$ is positive definite. Putting $L := (1-\mathcal{F}_\lambda)(U_\chi^{\mathcal{F}_\lambda} \cap M) \subset U_\chi^{\mathcal{F}_\lambda}$ we therefore get

$$B_\lambda[w] \geq 0, \qquad \text{for all } w \in L.$$

Furthermore, since $\mathbf{1} - \mathcal{F}_\lambda$ is injective on $U_\chi^{\mathcal{F}_\lambda}$, $\operatorname{codim}_{U_\chi^{\mathcal{F}_\lambda}} L = \operatorname{codim}_{U_\chi^{\mathcal{F}_\lambda}} (M \cap U_\chi^{\mathcal{F}_\lambda}) \leq \operatorname{codim} M$, as desired. This completes the proof of the theorem. $\square$

Remark 3. The leading idea in the proof of the last theorem was that each $v \in U_\chi^{\mathcal{F}_\lambda}$ has to be, approximately, an eigenvector of the corresponding spectral projection operator of A with eigenvalue zero. For this reason, such a v cannot satisfy $(Av, v) < \lambda \|v\|^2$, nor be an element of W.

6. Asymptotics for $\operatorname{tr} P_\chi \mathcal{E}_\lambda$ and $\operatorname{tr} P_\chi \mathcal{F}_\lambda$. The finite group case

For the rest of Part I, we shall concentrate on the case where G is a finite group. The compact group case will be treated in Part II. The two preceding sections showed that, in view of Lemmata 11 and 12, the spectral counting function $N_\chi(\lambda) = \mathcal{N}(A_0 - \lambda\mathbf{1}, \mathcal{H}_\chi \cap \mathrm{C}_c^\infty(\mathbf{X}))$ can be estimated from below and from above in terms of the traces of $P_\chi \mathcal{E}_\lambda$ and $P_\chi \mathcal{F}_\lambda$, and their squares. We will therefore now proceed to estimate these traces in terms of the reduced Weyl volume. For this sake, we introduce first certain sets associated to the support of the symbols of the approximate spectral projection operators; their significance will become apparent later. Thus, let

$$\begin{aligned}
W_\lambda &= \{(x,\xi) \in \mathbf{X} \times \mathbb{R}^n : a_\lambda < 0\},\\
A_{c,\lambda} &= \big\{(x,\xi) \in \mathbf{X} \times \mathbb{R}^n : a_\lambda < c(h^{\delta-\omega} + d)\big\}, \qquad B_{c,\lambda} = \mathbf{X} \times \mathbb{R}^n - A_{c,\lambda},\\
D_c &= (\partial\,\mathbf{X} \times \mathbb{R}^n)(c, h^{-2\delta} g),\\
F_\lambda &= \{(x,\xi) \in \mathbf{X} \times \mathbb{R}^n : \chi_\lambda = 0 \quad \text{or} \quad \eta_{\lambda,-2} = 0 \quad \text{or} \quad \chi_\lambda = \eta_{\lambda,-2} = 1\},\\
\mathcal{RV}_{c,\lambda} &= \big\{(x,\xi) \in \mathbf{X} \times \mathbb{R}^n : |a_\lambda| < c(h^{\delta-\omega} + d)\big\} \cup \big\{(x,\xi) \in D_c : x \in \mathbf{X},\, a_\lambda < c(h^{\delta-\omega} + d)\big\}.
\end{aligned}$$

Note that $D_c = \Big\{(x,\xi) \in \mathbb{R}^{2n} : \operatorname{dist}(x, \partial\,\mathbf{X}) < \sqrt{c}\big(1 + |x|^2 + |\xi|^2\big)^{-\delta/2}\Big\}$, since for

$$h^{-2\delta}(x,\xi) g_{(x,\xi)}(x-y, \xi-\eta) = (1 + |x|^2 + |\xi|^2)^\delta \Big[\frac{|\xi-\eta|^2}{1 + |x|^2 + |\xi|^2} + |x-y|^2\Big] < c$$

to hold for some $(y,\eta) \in \partial\,\mathbf{X} \times \mathbb{R}^n$, it is necessary and sufficient that $|x-y|^2(1 + |x|^2 + |\xi|^2)^\delta < c$ is satisfied for some $y \in \partial\,\mathbf{X}$. Now, recall that $|G| = \sum_{\chi \in \hat{G}} d_\chi^2$. We then have the following

Proposition 6. *For sufficiently large $c > 0$ we have*

$$| \operatorname{tr} P_\chi \mathcal{E}_\lambda - V_\chi(\mathbf{X} \times \mathbb{R}^n, a_\lambda)| \leq c \, vol \mathcal{RV}_{c,\lambda}, \tag{48}$$

where

$$V_\chi(\mathbf{X} \times \mathbb{R}^n, a_\lambda) = \frac{d_\chi^2}{|G|} \int \int_{\mathbf{X} \times \mathbb{R}^n} \iota_{(-\infty,0]}(a_\lambda(x,\xi)) dx \, d\xi = \frac{d_\chi^2}{(2\pi)^n |G|} \, vol W_\lambda \tag{49}$$

is the expected approximation given in terms of the reduced Weyl volume, and $\iota_{(-\infty,0]}$ denotes the characteristic function of the interval $(-\infty, 0]$. Furthermore, a similar estimate holds for tr $P_\chi \mathcal{E}_\lambda \cdot P_\chi \mathcal{E}_\lambda$*, too.*

Proof. The proof will require several steps. Let $\sigma^r(\mathcal{E}_\lambda)(x,\xi)$ denote the right symbol of $\mathcal{E}_\lambda$. Then, for $u \in \mathrm{C_c^\infty}(\mathbb{R}^n)$,

$$P_\chi \mathcal{E}_\lambda u(x) = \frac{d_\chi}{|G|} \sum_{h \in G} \overline{\chi(h)} \int \int e^{i(h^{-1}x - y)\xi} \sigma^r(\mathcal{E}_\lambda)(y,\xi) u(y) dy \, đ\xi.$$

The kernel of $P_\chi \mathcal{E}_\lambda$, which is a rapidly decreasing function, is given by

$$K_{P_\chi \mathcal{E}_\lambda}(x,y) = \frac{d_\chi}{|G|} \sum_{h \in G} \overline{\chi(h)} \int e^{i(h^{-1}x - y)\xi} \sigma^r(\mathcal{E}_\lambda)(y,\xi) \, đ\xi.$$

The trace of $P_\chi \mathcal{E}_\lambda$ can therefore be computed by

$$\begin{aligned} \operatorname{tr} P_\chi \mathcal{E}_\lambda &= \int K_{P_\chi \mathcal{E}_\lambda}(x,x) dx \\ &= \frac{d_\chi^2}{|G|} \operatorname{tr} \mathcal{E}_\lambda + \frac{d_\chi}{|G|} \sum_{h \neq e} \overline{\chi(h)} \int \int e^{i(h^{-1}x - x)\xi} \sigma^r(\mathcal{E}_\lambda)(x,\xi) dx \, đ\xi, \end{aligned}$$

where we made use of the relation $\chi(e) = d_\chi$, and the fact that $\operatorname{tr} \mathcal{E}_\lambda = \int \int \sigma^r(\mathcal{E}_\lambda)(x,\xi) dx \, đ\xi$. As a next step, we will prove that, for all $e \neq h \in G$, there exists a sufficiently large constant $c > 0$ such that

$$\left| \int \int e^{i(h^{-1}x - x)\xi} \sigma^r(\mathcal{E}_\lambda)(x,\xi) dx \, đ\xi \right| \leq c \operatorname{vol}(\mathcal{RV}_{c,\lambda}). \tag{50}$$

As already noticed, the decay properties of $\sigma^\tau(\mathcal{E}_\lambda)(x,\xi) \in S(h^{-2\delta}g, 1)$ are independent of λ for arbitrary $\tau \in \mathbb{R}$, while its support does depend on λ. Indeed, by Theorem 3 and Corollary 2, together with the asymptotic expansions (1) and (10) and Proposition 1,

$$\sigma^\tau(\mathcal{E}_\lambda) = (\eta_{\lambda,-2}^2 \chi_\lambda)^2 (3 - 2\eta_{\lambda,-2}^2 \chi_\lambda) + f_\lambda + r_\lambda, \tag{51}$$

where $r_\lambda \in S^{-\infty}(h^{-2\delta}g, 1)$, and $f_\lambda \in S(h^{-2\delta}g, h^{1-2\delta})$, everything uniformly in λ; in addition, $f_\lambda(x,\xi) = 0$ if $(x,\xi) \in F_\lambda$. To see this, note that $\sigma^\tau(\mathcal{E}_\lambda)(x,\xi)$ is given asymptotically as a linear combination of products of derivatives of $\sigma^w(\mathcal{E}_\lambda)$ at (x,ξ), which in turn is given asymptotically by a linear combination of terms involving derivatives of $\eta_{\lambda,-2}$, χ_λ. The τ-symbol of $\mathcal{E}_\lambda$ is therefore asymptotically given by

$$\sigma^\tau(\mathcal{E}_\lambda) - \sum_{0 \leq j < N} a_j \in S(h^{-2\delta}g, h^{(1-2\delta)N}), \qquad a_j \in S(h^{-2\delta}g, h^{(1-2\delta)j}),$$

where the first summand a_0 is equal to $(\eta_{\lambda,-2}^2 \chi_\lambda)^2 (3 - 2\eta_{\lambda,-2}^2 \chi_\lambda)$. Let now a be as in Proposition 1 such that $a \sim \sum_{j \geq 0} a_j$, and put $r_\lambda = \sigma^\tau(\mathcal{E}_\lambda) - a \in S^{-\infty}(h^{-2\delta}g, 1)$. Since $\operatorname{supp}(a - a_0) \subset \bigcup_{j \geq 1} \operatorname{supp} a_j$, $f_\lambda = a - (\eta_{\lambda,-2}^2 \chi_\lambda)^2 (3 - 2\eta_{\lambda,-2}^2 \chi_\lambda) \in S(h^{-2\delta}g, h^{1-2\delta})$ must vanish on F_λ, and we obtain (51). Now, since $|r_\lambda(x,\xi)| \leq C'(1+|x|^2+|\xi|^2)^{-N/2}$ for some constant C' independent of λ and N arbitrarily large, we get the uniform bound

$$\int\int |r_\lambda(x,\xi)| dx\, đ\xi \leq C;$$

note that the x-dependence of $h(x,\xi)$ is crucial at this point. For this reason, and in order to show (50), where now $\tau=1$, we can restrict ourselves to the study of

$$\int\int_{\mathbf{X}\times\mathbb{R}^n} e^{i(h^{-1}x-x)\xi}((\eta^2_{\lambda,-2}\chi_\lambda)^2(3-2\eta^2_{\lambda,-2}\chi_\lambda)+f_\lambda)(x,\xi) dx\, đ\xi, \tag{52}$$

where we took into account that $\eta_{\lambda,-2}$ has compact x-support in $\mathbf{X}$. Next, we examine the geometry of the action of G in more detail. Thus, let

$$\Sigma = \{x \in \mathbb{R}^n : gx = x \text{ for some } e \neq g \in G\}$$

denote the set of not necessarily simultaneous fixed points of G. In other words,

$$\Sigma = \bigcup_{e\neq g\in G} \Sigma_g, \qquad \Sigma_g = \{x\in\mathbb{R}^n : gx = x\}.$$

Note that every connected component of Σ_g is a closed, totally geodesic submanifold. We then have the following

Lemma 15. *There exists a constant $\kappa > 0$ such that $d(gx,x) \geq \kappa\, d(x,\Sigma_g)$ for all $x\in\mathbb{R}^n$, and arbitrary $e\neq g\in G$.*

Proof of Lemma 15. Let $x \in \mathbb{R}^n - \Sigma_g$ be an arbitrary point, and p the closest point to x belonging to Σ_g. Write $x = \exp_p t_0 X$, where $\exp_p$ denotes the exponential mapping of $\mathbb{R}^n$, and $(p,X)\in T_p(\mathbb{R}^n)$, $|X|=1$. Then $t_0 = d(x,\Sigma_g)$. Consider next the direct sum decomposition $T_p(\mathbb{R}^n) = U\oplus V$, where

$$U = \{(p,Y)\in T_p(\mathbb{R}^n) : dg_p(Y) = Y\},$$

and $V = U^\perp$. Since p is a fixed point of g, we also have the identity

$$g \exp_p Y = \exp_p dg_p(Y),$$

which implies $\exp_p tY \in \Sigma_g$ if, and only if, $(p,Y)\in U$, where $t\in\mathbb{R}$. Consequently, $U = T_p(\Sigma_g)$. Now, with $\exp_p tY = p+tY$, and x, p as above, one computes

$$|gx-x|^2 = |\exp_p t_0 dg_p(X) - \exp_p t_0 X|^2 = |p + t_0 dg_p(X) - p - t_0X|^2 = d^2(x,\Sigma_g)|dg_p(X)-X|^2.$$

Because of $(x-p)\perp\Sigma_g$, we must have $(p,X)\in T_p(\Sigma_g)^\perp = V$, and therefore $|dg_p(X)-X|^2\neq 0$. The latter expression depends continuously on $(p,X)\in\left\{(p,Y)\in T_p(\Sigma_g)^\perp : |Y|=1\right\}$, and is actually independent of p, so that it can be estimated from below by some positive constant uniformly for all x. The assertion of the proposition now follows. □

Returning now to our previous computations, we split the integral in (52) in an integral over

$$\mathcal{D} = \left\{(x,\xi)\in\mathbf{X}\times\mathbb{R}^n : \operatorname{dist}(x,\Sigma) \geq (1+|\xi|^2)^{-\delta/2}\right\},$$

and a second integral over the complement of $\mathcal{D}$ in $\mathbf{X}\times\mathbb{R}^n$. Since $\operatorname{supp}\chi_\lambda \subset \{(x,\xi) : a_\lambda + 4h^{\delta-\omega} + 8C_0 \leq h^\delta\}$, the integral over $\complement_{\Omega\times\mathbf{X}}\mathcal{D}$ can be estimated by a constant times the volume of the set $\{(x,\xi)\in\complement_{\Omega\times\mathbf{X}}\mathcal{D} : a_\lambda + 4h^{\delta-\omega}+8C_0 d \leq h^\delta\}$, which is contained in the set $\{(x,\xi)\in\mathbf{X}\times\mathbb{R}^n : \operatorname{dist}(x,\Sigma) < (1+|\xi|^2)^{-\delta/2}, a_\lambda \leq c(h^{\delta-\omega}+d)\}$ for some sufficiently large $c>0$. By examining the proof of Lemma 18, one sees that the volume of the latter can be estimated from above by

$$\int_{K\leq|\xi|\leq c_1\lambda^{1/2m}} \operatorname{vol}(\Sigma_{c_2|\xi|^{-\delta}}\cap\mathbf{X})\, d\xi + c_3$$

for some suitable constants $K, c_i > 0$, and consequently has the same asymptotic behaviour in λ as the volume of $\mathcal{RV}_{c,\lambda}$. In studying the asymptotic behaviour of the integral (52), we can therefore restrict the domain of integration to $\mathcal{D}$. By the previous lemma, there exists a constant $\kappa > 0$ such that

$$|h^{-1}x - x| \geq \kappa(1 + |\xi|^2)^{-\delta/2} \quad \text{for all } (x, \xi) \in \mathcal{D} \text{ and } e \neq h \in G.$$

Since $(\eta^2_{\lambda,-2}\chi_\lambda)^2(3 - 2\eta^2_{\lambda,-2}\chi_\lambda) + f_\lambda$ has compact support in ξ, this implies that

$$\frac{e^{i(h^{-1}x-x)\xi}}{|h^{-1}x - x|^2}\, \partial^\alpha_\xi ((\eta^2_{\lambda,-2}\chi_\lambda)^2(3 - 2\eta^2_{\lambda,-2}\chi_\lambda) + f_\lambda)(x, \xi)$$

is integrable on $\mathcal{D}$, as well as rapidly decreasing in ξ. Integrating by parts with respect to ξ we therefore get for (52) the expression

$$\int\int_{\mathcal{D}} \frac{e^{i(h^{-1}x-x)\xi}}{|h^{-1}x - x|^2} (-\partial^2_{\xi_1} - \cdots - \partial^2_{\xi_n})((\eta^2_{\lambda,-2}\chi_\lambda)^2(3 - 2\eta^2_{\lambda,-2}\chi_\lambda) + f_\lambda)(x, \xi) dx\, đ\xi; \tag{53}$$

in particular notice that, by Fubini's Theorem, the boundary contributions vanish. Now, if $(x, \xi) \in F_\lambda$, the function $(\eta^2_{\lambda,-2}\chi_\lambda)^2(3 - 2\eta^2_{\lambda,-2}\chi_\lambda) + f_\lambda$ is constant, so its derivatives with respect to ξ are zero, and we can restrict the integration in (53) to the set $\complement_{\mathbf{X}\times\mathbb{R}^n} F_\lambda \cap \mathcal{D}$, where $\complement_{\mathbf{X}\times\mathbb{R}^n} F_\lambda$ denotes the complement of $\mathcal{F}_\lambda$ in $\mathbf{X} \times \mathbb{R}^n$.

Lemma 16. *For sufficiently large $c > 0$, the set $\complement_{\mathbf{X}\times\mathbb{R}^n} F_\lambda$ is contained in $\mathcal{RV}_{c,\lambda}$.*

Proof. This assertion is already stated in [12], page 55. For the sake of completeness, we give a proof here. Thus, consider

$$E_\lambda = \left\{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : (x, \xi) \notin D_4, a_\lambda < -4h^{\delta-\omega} - 8C_0 d\right\},$$

and let $\mathcal{M}_\lambda$ be defined as in (15). Since $\operatorname{supp} \tilde{\eta}_2 \subset D_4$, and $\psi_{\lambda,1/2} = 1$ on $\mathcal{M}_\lambda(1/2, h^{-2\delta}g)$, it is clear that

$$E_\lambda \subset \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : \chi_\lambda = \eta_{\lambda,-2} = 1\} \subset F_\lambda, \tag{54}$$

and consequently $\complement_{\mathbf{X}\times\mathbb{R}^n} F_\lambda \subset \complement_{\mathbf{X}\times\mathbb{R}^n} E_\lambda$. Next, we are going to prove that, for sufficiently large c, $(x, \xi) \in B_{c,\lambda}$ implies $(x, \xi) \notin \mathcal{M}_\lambda(1, h^{-2\delta}g)$. Thus, assume $(x, \xi) \in B_{c,\lambda}$; on $\mathbf{X}_\varepsilon \times \{\xi : |\xi| > 1\}$ we have

$$c\Big(\frac{1}{|\xi|} + \frac{1}{(1 + |x|^2 + |\xi|^2)^{(\delta-\omega)/2)}}\Big) \leq \frac{|\xi|^{2m}}{|\xi|^{2m} + \lambda}\Big(1 - \frac{\lambda}{a_{2m}(x, \xi)}\Big).$$

Therefore, as c becomes large, $|\xi|$ must become large, too. On the other hand, if $(y, \eta) \in \mathcal{M}_\lambda$, $|\eta|$ must be bounded. For large c we therefore have $|\xi-\eta|^2 \sim |\xi|^2$, which means that $h^{-2\delta}(x, \xi)g_{(x,\xi)}(x-y, \xi - \eta) \sim (1 + |x|^2 + |\xi|^2)^\delta \to \infty$ as $c \to \infty$. Hence, for sufficiently large c, $(x, \xi) \notin \mathcal{M}_\lambda(1, h^{-2\delta}g)$. Since $\operatorname{supp} \psi_{\lambda,1/2} \subset \mathcal{M}_\lambda(1, h^{-2\delta}g)$, we arrive in this case at the inclusions

$$B_{c,\lambda} \subset \{(x, \xi) \in \mathbf{X} \times \mathbb{R}^n : \eta_{\lambda,-2}(x, \xi) = 0\} \subset F_\lambda, \tag{55}$$

and combining (54) and (55) we get

$$\complement_{\mathbf{X}\times\mathbb{R}^n} F_\lambda \subset A_{c,\lambda} \cap \complement_{\mathbf{X}\times\mathbb{R}^n} E_\lambda \subset \mathcal{RV}_{c,\lambda}, \tag{56}$$

as desired. □

As a consequence of the foregoing lemma, the integral in (53) is bounded from above by the volume of $\mathcal{RV}_{c,\lambda}$, times a constant independent of λ, since the integrand is uniformly bounded with respect to λ. Thus, we have shown (50). The assertion of the Proposition now follows by observing that

$$\Big|\operatorname{tr} \mathcal{E}_\lambda - \frac{\operatorname{vol} W_\lambda}{(2\pi)^n}\Big| \leq c \operatorname{vol} \mathcal{RV}_{c,\lambda}. \tag{57}$$

Indeed, similarly to our previous discussion of the integral $\int\int e^{i(h^{-1}x-x)\xi}\sigma^r(\mathcal{E}_\lambda)(x,\xi)dx\,d\!\!{}^-\xi$, the integral

$$\operatorname{tr}\mathcal{E}_\lambda = \int\int \sigma^r(\mathcal{E}_\lambda)(x,\xi)dx\,d\!\!{}^-\xi = \int\int((\eta^2_{\lambda,-2}\chi_\lambda)^2(3-2\eta^2_{\lambda,-2}\chi_\lambda)+f_\lambda+r_\lambda)(x,\xi)dx\,d\!\!{}^-\xi$$

can be split into three parts; the contribution coming from $r_\lambda(x,\xi)$ is bounded by some constant independent of λ, while the contribution coming from f_λ can be estimated in terms of the volume of $\mathcal{RV}_{c,\lambda}$, since $\operatorname{supp} f_\lambda \subset \complement_{\mathbf{X}\times\mathbb{R}^n}F_\lambda \subset \mathcal{RV}_{c,\lambda}$, by the previous lemma. Now, $(\eta^2_{\lambda,-2}\chi_\lambda)^2(3-2\eta^2_{\lambda,-2}\chi_\lambda)$ must be equal 1 on $W_\lambda\cap\complement_{\mathbf{X}\times\mathbb{R}^n}\mathcal{RV}_{c,\lambda}$, since according to (56) we have $\complement_{\mathbf{X}\times\mathbb{R}^n}\mathcal{RV}_{c,\lambda}\subset B_{c,\lambda}\cup E_\lambda$, and hence $W_\lambda\cap\complement_{\mathbf{X}\times\mathbb{R}^n}\mathcal{RV}_{c,\lambda}\subset E_\lambda\subset\{(x,\xi)\in\mathbf{X}\times\mathbb{R}^n : \chi_\lambda=\eta_{\lambda,-2}=1\}$, due to the fact that $W_\lambda\cap B_{c,\lambda}=\emptyset$. Furthermore, $(\eta^2_{\lambda,-2}\chi_\lambda)^2(3-2\eta^2_{\lambda,-2}\chi_\lambda)$ vanishes on $B_{c,\lambda}$, since for large c, $(x,\xi)\in B_{c,\lambda}$ implies $(x,\xi)\notin\mathcal{M}_\lambda(1,h^{-2\delta}g)$, by the proof of the previous lemma. Taking into account that W_λ and $\mathcal{RV}_{c,\lambda}$ are subsets of $A_{c,\lambda}$, we therefore obtain for sufficiently large c

$$\begin{aligned}&\int\int((\eta^2_{\lambda,-2}\chi_\lambda)^2(3-2\eta^2_{\lambda,-2}\chi_\lambda))(x,\xi)dx\,d\!\!{}^-\xi\\ &=\frac{\operatorname{vol}\left(W_\lambda\cap\complement_{A_{c,\lambda}}\mathcal{RV}_{c,\lambda}\right)}{(2\pi)^n}+\int\int_{A_{c,\lambda}-(W_\lambda\cap\complement_{A_{c,\lambda}}\mathcal{RV}_{c,\lambda})}((\eta^2_{\lambda,-2}\chi_\lambda)^2(3-2\eta^2_{\lambda,-2}\chi_\lambda))(x,\xi)dx\,d\!\!{}^-\xi.\end{aligned}$$

Now, since $\complement_{A_{c,\lambda}}\mathcal{RV}_{c,\lambda}\subset W_\lambda$, one has $A_{c,\lambda}-W_\lambda\cap\complement_{A_{c,\lambda}}\mathcal{RV}_{c,\lambda}=\mathcal{RV}_{c,\lambda}$. The estimate (57) now follows, and together with (50) we obtain (48). Finally, if in the previous computations $\mathcal{E}_\lambda$ is replaced by $\mathcal{E}^2_\lambda$, we obtain a similar estimate for the trace of $P_\chi\mathcal{E}_\lambda\cdot P_\chi\mathcal{E}_\lambda=P_\chi\mathcal{E}^2_\lambda$. This concludes the proof of the proposition. □

As a consequence, we get the following

Theorem 7. *Let $N^{\mathcal{E}_\lambda}_\chi$ be the number of eigenvalues of $\mathcal{E}_\lambda$ which are $\geq 1/2$ and whose eigenfunctions are contained in the χ-isotypic component $\mathcal{H}_\chi$ of $\mathrm{L}^2(\mathbb{R}^n)$. Then*

$$|N^{\mathcal{E}_\lambda}_\chi - V_\chi(\mathbf{X}\times\mathbb{R}^n,a_\lambda)|\leq c\,vol\mathcal{RV}_{c,\lambda} \tag{58}$$

for some sufficiently large $c>0$.

Proof. From the preceding proposition, and the estimate (18), one deduces that for some sufficiently large $c>0$

$$\begin{aligned}N^{\mathcal{E}_\lambda}_\chi&\leq 3\operatorname{tr}P_\chi\mathcal{E}_\lambda-2\operatorname{tr}P_\chi\mathcal{E}_\lambda\cdot P_\chi\mathcal{E}_\lambda+c_2\leq V_\chi(\mathbf{X}\times\mathbb{R}^n,a_\lambda)+c\operatorname{vol}\mathcal{RV}_{c,\lambda},\\ N^{\mathcal{E}_\lambda}_\chi&\geq 2\operatorname{tr}P_\chi\mathcal{E}_\lambda\cdot P_\chi\mathcal{E}_\lambda-\operatorname{tr}P_\chi\mathcal{E}_\lambda-c_1\geq V_\chi(\mathbf{X}\times\mathbb{R}^n,a_\lambda)-c\operatorname{vol}\mathcal{RV}_{c,\lambda},\end{aligned}$$

which completes the proof of (58). □

In analogy to the previous considerations, one proves the following

Theorem 8. *For sufficiently large $c>0$ one has the estimate*

$$|M^{\mathcal{F}_\lambda}_\chi - V_\chi(\mathbf{X}\times\mathbb{R}^n,a_\lambda)|\leq c\,vol\mathcal{RV}_{c,\lambda},$$

where $M^{\mathcal{F}_\lambda}_\chi$ is the number of eigenvalues of $\mathcal{F}_\lambda$, counting multiplicities, greater or equal $1/2$, and whose eigenfunctions are contained in the χ-isotypic component $\mathcal{H}_\chi$ of $\mathrm{L}^2(\mathbb{R}^n)$.

Proof. The proof is similar to the one of Theorem 7, and uses Lemma 12. In particular, as in equation (51), one has

$$\sigma^\tau(\mathcal{F}_\lambda)=(\eta^2_2\chi^+_\lambda)^2(3-2\eta^2_2\chi^+_\lambda)+f_\lambda+r_\lambda, \tag{59}$$

where $r_\lambda\in S^{-\infty}(h^{-2\delta}g,1)$, and $f_\lambda\in S(h^{-2\delta}g,h^{1-2\delta})$, everything uniformly in λ. The aymptotic analysis for $\operatorname{tr}P_\chi\mathcal{F}_\lambda$ and $\operatorname{tr}(P_\chi\mathcal{F}_\lambda)^2$ now follows the lines of the proof of Proposition 6. □

7. Proof of Theorem 1

We collect all the results obtained so far in the following

Proposition 7. *There exist constants $C_1, C_2 > 0$ which do not depend on λ, such that for all λ*

$$|\mathcal{N}(A_0 - \lambda \mathbf{1}, \mathcal{H}_\chi \cap \mathrm{C_c^\infty}(\mathbf{X})) - V_\chi(\mathbf{X} \times \mathbb{R}^n, a_\lambda)| \leq C_1 vol \mathcal{RV}_{C_1,\lambda} + C_2.$$

Proof. By Theorems 5 and 6, there exist constants $C_i > 0$ independent of λ such that $N_\chi^{\mathcal{E}_\lambda} - C_1 \leq \mathcal{N}(A_0 - \lambda \mathbf{1}, \mathcal{H}_\chi \cap \mathrm{C_c^\infty}(\mathbf{X})) \leq M_\chi^{\mathcal{F}_\lambda} + C_2$. Theorems 7 and 8 then yield the estimate

$$-c \,\mathrm{vol}\, \mathcal{RV}_{c,\lambda} - C_1 \leq \mathcal{N}(A_0 - \lambda \mathbf{1}, \mathcal{H}_\chi \cap \mathrm{C_c^\infty}(\mathbf{X})) - V_\chi(\mathbf{X} \times \mathbb{R}^n, a_\lambda) \leq c \,\mathrm{vol}\, \mathcal{RV}_{c,\lambda} + C_2$$

for some sufficiently large $c > 0$. □

In order to formulate the main result, we need two last lemmata.

Lemma 17. *Let $\gamma = \frac{1}{n} \int_\mathbf{X} \int_{S^{n-1}} \big(a_{2m}(x,\xi)\big)^{-n/2m} dx\, d\xi$, where $2m$ is the order of A_0. Then*

$$V_\chi(\mathbf{X} \times \mathbb{R}^n, a_\lambda) = \frac{d_\chi^2}{|G|} \gamma \cdot \lambda^{n/2m} + C$$

for some constant $C > 0$ independent of λ.

Proof. The reduced Weyl volume was defined in (49) as

$$V_\chi(\mathbf{X} \times \mathbb{R}^n, a_\lambda) = \frac{d_\chi^2}{|G|} \int\int_{\mathbf{X} \times \mathbb{R}^n} \iota_{(-\infty,0]}(a_\lambda(x,\xi)) dx\, đ\xi = \frac{d_\chi^2}{(2\pi)^n |G|} \mathrm{vol}\, W_\lambda,$$

where $W_\lambda = \{(x,\xi) \in \mathbf{X} \times \mathbb{R}^n : a_\lambda(x,\xi) < 0\}$. Now, for some sufficiently small $\varrho > 0$, on $\mathbf{X}_\varrho \times \{\xi : |\xi| > 1\}$, a_λ is given by

$$a_\lambda(x,\xi) = \frac{1}{1 + \lambda |\xi|^{-2m}} \Big(1 - \frac{\lambda}{a_{2m}(x,\xi)}\Big).$$

By [12], Lemma 13.1, condition (11) implies that $a_{2m}(x,\xi) \geq \iota > 0$ for all $(x,\xi) \in \mathbf{X} \times S^{n-1}$, so that a_λ is strictly negative on $\mathbf{X}_\varrho \times \{\xi : |\xi| > 1\}$ if, and only if, $a_{2m}(x,\xi) - \lambda < 0$, which in turn is equivalent to

$$|\xi| < [\lambda\, a_{2m}^{-1}(x, \xi/|\xi|)]^{1/2m},$$

due to the homogeneity of the principal symbol. From this one concludes

$$\begin{aligned} V_\chi(\mathbf{X} \times \mathbb{R}^n, a_\lambda) = {} & \frac{d_\chi^2}{(2\pi)^n |G|} \big[\mathrm{vol}\, \{(x,\xi) \in \mathbf{X} \times \mathbb{R}^n : |\xi| \leq 1\} \\ & + \mathrm{vol}\, \big\{(x,\xi) \in \mathbf{X} \times \mathbb{R}^n : |\xi|^{2m} < \lambda\, a_{2m}^{-1}(x, \xi/|\xi|)\big\}\big] \\ & = O(1) + \frac{d_\chi^2}{(2\pi)^n |G|} \int_\mathbf{X} \int_{S^{n-1}} \int_0^{(\lambda/a_{2m}(x,\eta))^{1/2m}} r^{n-1} dr\, dS^{n-1}(\eta) dx \\ & = O(1) + \frac{d_\chi^2}{(2\pi)^n |G|} \int_\mathbf{X} \int_{S^{n-1}} \frac{1}{n} (\lambda/a_{2m}(x,\eta))^{n/2m} dS^{n-1}(\eta) dx. \end{aligned}$$

□

Lemma 18. *Assume that for some sufficiently small $\varrho > 0$ there exists a constant $C > 0$ such that $vol(\partial\, \mathbf{X})_\varrho \leq C\varrho$. Then $vol \mathcal{RV}_{c,\lambda} = \mathrm{O}(\lambda^{(n-\varepsilon)/2m})$, where $\varepsilon \in (0, \frac{1}{2})$.*

Proof. According to the definition of $\mathcal{RV}_{c,\lambda}$ at the beginning of Section 6, we have

$$\begin{aligned}\operatorname{vol}\mathcal{RV}_{c,\lambda} \leq \operatorname{vol}\left\{(x,\xi)\in \mathbf{X}\times\mathbb{R}^n : |a_\lambda| - c(h^{\delta-\omega}+d)<0\right\}\\ +\operatorname{vol}\left\{(x,\xi)\in D_c : x\in\mathbf{X}, a_\lambda < c(h^{\delta-\omega}+d)\right\},\end{aligned}$$

where $D_c=\{(x,\xi):\operatorname{dist}(x,\partial\,\mathbf{X})<\sqrt{c}(1+|x|^2+|\xi|^2)^{-\delta/2}\}$, and $0<\delta-\omega<1/2$. In what follows, let us assume that $\lambda\geq 1$. It is not difficult to see that, for $|\xi|>1$, there exists a constant $c_1>0$ independent of λ such that

$$a_\lambda(x,\xi)-c(h^{\delta-\omega}+d)(x,\xi)<0 \quad\Longrightarrow\quad |\xi|<c_1\lambda^{1/2m}. \tag{60}$$

Indeed, let c_1 be such that

$$c_1^{2m}\geq \max\big(2,2/\iota\big), \qquad \sup_{x\in\mathbf{X},|\xi|>c_1} c(h^{\delta-\omega}+d)(x,\xi)\leq\frac{1}{3},$$

where $\iota>0$ is a lower bound for $a_{2m}(x,\xi)$ on $\mathbf{X}\times S^{n-1}$. Since

$$1-\frac{\lambda}{a_{2m}(x,\xi)}\geq\frac{1}{2}\quad\Longleftrightarrow\quad |\xi|^{2m}\geq\frac{2\lambda}{a_{2m}(x,\xi/|\xi|)},$$

one computes for $|\xi|\geq c_1\lambda^{1/2m}$ that

$$a_\lambda(x,\xi)\geq\frac{1}{2}\frac{1}{1+\lambda|\xi|^{-2m}}\geq\frac{1}{2}\frac{1}{1+c_1^{-2m}}\geq\frac{1}{3},$$

while, on the other hand, $c(h^{\delta-\omega}+d)(x,\xi)\leq\frac{1}{3}$, so that $a_\lambda(x,\xi)-c(h^{\delta-\omega}+d)(x,\xi)\geq 0$. This proves (60). As a consequence, we obtain the estimate

$$\begin{aligned}&\operatorname{vol}\{(x,\xi)\in D_c : x\in\mathbf{X}, a_\lambda<c(h^{\delta-\omega}+d)\}\\ &\qquad\leq \operatorname{vol}\left\{(x,\xi)\in\mathbf{X}\times\mathbb{R}^n : |\xi|\geq K, \operatorname{dist}(x,\partial\,\mathbf{X})<c_2|\xi|^{-\delta}, a_\lambda<c(h^{\delta-\omega}+d)\right\}\\ &\qquad+\operatorname{vol}\left\{(x,\xi)\in\mathbf{X}\times\mathbb{R}^n : |\xi|\leq K\right\}\\ &\qquad\leq\int_{K\leq|\xi|\leq c_1\lambda^{1/2m}}\operatorname{vol}\big((\partial\,\mathbf{X})_{c_2|\xi|^{-\delta}}\cap\mathbf{X}\big)\,d\xi+c_3,\end{aligned}$$

where $\delta\in(\frac{1}{4},\frac{1}{2})$, and $K\geq 1$ is some sufficiently large constant; here and in all what follows, $c_i>0$ denote suitable, positive constants independent of λ. Now, since $\operatorname{vol}(\partial\,\mathbf{X})_\varrho\leq C\varrho$, for some $\varrho>0$,

$$\begin{aligned}\operatorname{vol}\{(x,\xi)\in D_c : x\in\mathbf{X}, a_\lambda<c(h^{\delta-\omega}+d)\}&\leq c_2\,c_4\int_{S^{n-1}}\int_{K\leq r\leq c_1\lambda^{1/2m}} r^{n-1-\delta}dr dS^{n-1}(\eta)+c_3\\ &=c_5(\lambda^{(n-\delta)/2m}-K^{n-\delta})+c_3.\end{aligned}$$

Next, let $|\xi|\geq K$, and assume that the inequality $|a_\lambda(x,\xi)|\leq c(h^{\delta-\omega}+d)(x,\xi)$ is fulfilled. As before, we have $|\xi|^{2m}<c_1^{2m}\lambda$, as well as

$$\left|1-\frac{\lambda}{a_{2m}(x,\xi)}\right|\leq c(1+\lambda|\xi|^{-2m})(d+h^{\delta-\omega})(x,\xi)\leq c_6(1+\lambda|\xi|^{-2m})|\xi|^{-(\delta-\omega)}. \tag{61}$$

Combining (60) and (61), one deduces for sufficiently large K that

$$|\xi|^{2m}\geq -c_6(|\xi|^{2m}+\lambda)|\xi|^{-(\delta-\omega)}+\frac{\lambda}{a_{2m}(x,\xi/|\xi|)}\geq c_7\lambda.$$

Let us now introduce the variable $R(x,\xi) = \lambda/a_{2m}(x,\xi) = \lambda|\xi|^{-2m}/a_{2m}(x,\xi/|\xi|)$. Performing the corresponding change of variables one computes

$$
\begin{aligned}
&\operatorname{vol}\{(x,\xi) \in \mathbf{X} \times \mathbb{R}^n : |a_\lambda| - c(h^{\delta-\omega} + d) < 0\} \\
&\quad\leq \operatorname{vol}\left\{(x,\xi) \in \mathbf{X} \times \mathbb{R}^n : c_1\lambda^{1/2m} > |\xi| \geq K,\, |1 - R(x,\xi)| \leq c_6(1 + \lambda|\xi|^{-2m})|\xi|^{-(\delta-\omega)}\right\} + c_8 \\
&\quad\leq \int_{\mathbf{X}} \int_{S^{n-1}} \int_{\{r \geq K : |1-R| \leq c_9\lambda^{-(\delta-\omega)/2m}\}} r^{n-1}\, dr\, dS^{n-1}(\eta)\, dx + c_8 \\
&\quad\leq c_{10} \int_{\mathbf{X}} \int_{S^{n-1}} \int_{\{R : |1-R| \leq c_9\lambda^{-(\delta-\omega)/2m}\}} R^{-1} \left(\frac{\lambda}{Ra_{2m}(x,\eta)}\right)^{\frac{n}{2m}} dR\, dS^{n-1}(\eta)\, dx + c_8 \\
&\quad\leq c_{11}\lambda^{\frac{n}{2m}} \int_{\{R : |1-R| \leq c_9\lambda^{-(\delta-\omega)/2m}\}} R^{-\frac{n}{2m}-1}\, dR + c_8 = O(\lambda^{(n-(\delta-\omega))/2m}) + c_8.
\end{aligned}
$$

Hereby we made use of the fact that $(1+z)^\beta - (1-z)^\beta = O(|z|)$ for arbitrary $z \in \mathbb{C}$, $|z| < 1$, and $\beta \in \mathbb{R}$. □

We are now in position to prove the main result of Part I, which generalizes Theorem 13.1 of [12] to bounded domains with symmetries in the finite group case.

Theorem 1. *Let G be a finite group of isometries in Euclidean space $\mathbb{R}^n$, and $\mathbf{X} \subset \mathbb{R}^n$ a bounded domain which is invariant under G such that, for some sufficiently small $\varrho > 0$, $vol(\partial\mathbf{X})_\varrho \leq C\varrho$. Let further A_0 be a symmetric, classical pseudodifferential operator in $\mathrm{L}^2(\mathbb{R}^n)$ of order $2m$ with G-invariant Weyl symbol $\sigma^w(A_0) \in S(g, h^{-2m})$ and principal symbol a_{2m}, and assume that $(A_0\, u,\, u) \geq c\,\|u\|_m^2$ for some $c > 0$ and all $u \in \mathrm{C_c^\infty}(\mathbf{X})$. Consider further the Friedrichs extension of the operator*

$$\operatorname{res} \circ A_0 \circ \operatorname{ext} : \mathrm{C_c^\infty}(\mathbf{X}) \longrightarrow \mathrm{L}^2(\mathbf{X}),$$

and denote it by A. Finally, let $N_\chi(\lambda)$ be the number of eigenvalues of A less or equal λ and with eigenfunctions in the χ-isotypic component $\operatorname{res}\mathcal{H}_\chi$ of $\mathrm{L}^2(\mathbf{X})$, if $(-\infty, \lambda)$ contains no points of the essential spectrum, and equal to ∞, otherwise. Then, for all $\varepsilon \in (0, \frac{1}{2})$,

$$N_\chi(\lambda) = \frac{d_\chi^2}{|G|}\gamma\lambda^{n/2m} + O(\lambda^{(n-\varepsilon)/2m}),$$

where d_χ denotes the dimension of the irreducible representation of G corresponding to the character χ, and

$$\gamma = \frac{1}{n}\int_{\mathbf{X}}\int_{S^{n-1}} (a_{2m}(x,\xi))^{-n/2m} dx\, d\xi.$$

In particular, A has discrete spectrum.

Proof. By Lemma 8 and Proposition 7 we have

$$|N_\chi(\lambda) - V_\chi(\mathbf{X} \times \mathbb{R}^n, a_\lambda)| \leq C_1 \operatorname{vol} \mathcal{RV}_{C_1,\lambda} + C_2$$

for some suitable constants $C_1, C_2 > 0$ independent of λ. Lemma 17 and 18 then imply

$$-\mathrm{O}(\lambda^{(n-\varepsilon)/2m}) \leq N_\chi(\lambda) - \frac{d_\chi^2}{|G|}\gamma\lambda^{n/2m} \leq \mathrm{O}(\lambda^{(n-\varepsilon)/2m})$$

with arbitrary $\varepsilon \in (0, 1/2)$. In particular, $N_\chi(\lambda)$ remains finite for $\lambda < \infty$, so that the essential spectrum of A must be empty. The assertion of the theorem now follows. □

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