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Variation of cone metrics on Riemann surfaces

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Abstract

We discuss the variational properties of the unique conical metric of constant curvature -1 associated to a compact Riemann surface together with a weighted divisor.

Keywords: Hyperbolic cone metric; Riemann surface; Weighted divisor; Weil–Petersson metric; Holomorphic quadratic differentials

1. Introduction

In the present note we study hyperbolic metrics with conical singularities on compact Riemann surfaces. Such metrics have been considered beginning with the work of Picard [12]. Starting from the classical results by Kazdan–Warner [5–7], existence and uniqueness of conical hyperbolic metrics in every conformal class on a compact Riemann surface X equipped with an $\mathbb{R}$-divisor were proved by McOwen [10] and Troyanov [14]. In higher dimensions, results are due to by Cheeger, Tian and others.

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On a Riemann surface, an $\mathbb{R}$-divisor is of the form $\mathbf{a} = \sum_{j=1}^{n} a_j p_j$, where the a_j are real numbers subject to the condition $0 \leqslant a_j \leqslant 1$, and $\chi(X, \mathbf{a}) := \chi(X) - \sum_{j+1}^{n} a_j < 0$. Here, by abuse of language, we say that the metric has a conical singularity of weight a_j at the puncture p_j, if

$a_j = 0$ and g is smooth in a neighborhood of p_j, or
$a_j = 1$ and g satisfies a Poincaré growth condition at p_j or
$0 < a_j < 1$ and the metric is like $\frac{1}{|z|^{2a_j}}$ near p_j (cf. Definition 2.3).

In the view of the recent construction of the moduli space of weighted punctured stable curves [2] and the related hierarchy of compactifications of $\mathcal{M}_{\gamma,n}$, we want to consider the variation of hyperbolic conical metrics in holomorphic families, and introduce a generalized Weil–Petersson metric.

The main difficulty in the application of analytic methods arises from the lack of a Hodge theory for Kähler metrics with conical singularities. It is still possible to introduce "harmonic Beltrami differentials" with respect to a hyperbolic conical metric, together with a Kodaira–Spencer map, needed for the notion of a Weil–Petersson metric. Like in the classical case, harmonic Beltrami differentials are closely related to the variation of hyperbolic conical metrics in a holomorphic family.

It turned out that for $0 < a_j < 1$ the Weil–Petersson metric depends in a smooth monotone way on the weights a_j. For $a_j \to 0$ we recover the Weil–Petersson metric for non-punctured surfaces, assigning norm equal to zero to vectors corresponding to an infinitesimal motion of the puncture.

Later we need to assume that all weights are smaller than $1/2$. This range seems to be relevant. In fact, under this assumption, by a recent result of Wang and Zhu [16], extremal conical metrics are known to be hyperbolic. Properties of the generalized Weil–Petersson metric depend on solving the equation for constant curvature with parameters, and differentiability with respect to the parameter in the family follows, implying C^1 smoothness of the Weil–Petersson metric.

2. The hyperbolic cone metrics

Let X be a connected, compact Riemann surface, let $g = \rho(z)|dz|^2$ be a smooth metric on some open set of X. As we will need the notions of Kähler geometry, we have to use the complex Laplace operator $\Delta_g = \frac{1}{g}\partial^2/(\partial z \partial \bar{z})$, and we denote by ∇ the covariant derivative in z-direction with respect to g. Accordingly, we will prefer to use the scalar curvature instead of the Gaussian one, i.e., the curvature of g is

$$K_g = -\frac{1}{\rho} \cdot \frac{\partial^2 \log(\rho)}{\partial z \partial \bar{z}}.$$

Let us fix some number $p > 1$, and consider the Sobolev spaces $H_k^p(X)$ of complex-valued functions, which possess distributional derivatives up to order k in $L^p(X)$. An operator

$$L : H_2^p(X) \to L^p(X),$$

is defined by $L(\Psi) = \Delta_g(\Psi) - b\Psi$, where $b \in L^p(X)$ is a real function with $b(z) \geqslant b_0 > 0$ almost everywhere.

Note that L is well defined and bounded, since for some positive β, we have $H_2^p(X) \subseteq C^\beta(X)$ with a bounded inclusion. We shall need the following fact.

Lemma 2.1. *The operator L is invertible with bounded inverse.*

The statement can be shown like in [1, pp. 104, 105]: There exist unique weak solutions $\phi \in H_1^2(X)$ of the equation

$$L(\phi) = f \tag{1}$$

for all $f \in L^p(X)$, because of the positivity assumption on b. The rest follows from regularity theorems (cf. [1, p. 55]).

Now let $\mathbf{a} = \sum_{i=1}^n a_i p_i$ be a *real divisor* on X, i.e., a linear combination of distinct points $p_i \in X$ with real coefficients, and assume $0 \leqslant a_i \leqslant 1$. Such a pair $(X, \mathbf{a})$ is called a weighted punctured Riemann surface.

Definition 2.2. The Euler–Poincaré characteristic of a weighted punctured surface $(X, \mathbf{a})$ equals $\chi(X, \mathbf{a}) := \chi(X) - \sum_i a_i$.

In moduli theoretic applications the corresponding log-canonical divisor $K_{(X,\mathbf{a})} := K_X + \mathbf{a}$ is introduced, and the notion of stability corresponds to the negativity of the Euler–Poincaré characteristic.

Definition 2.3. By a conical metric on a weighted punctured Riemann surface $(X, \mathbf{a})$ we mean a smooth metric on $X' = X \setminus \bigcup_i p_i$ such that (with respect to a local coordinate z centered at a puncture p_j)

(1) g is smooth in a neighborhood of p_j, if $a_j = 0$,
(2) g satisfies a Poincaré growth condition at p_j, i.e., $g = \frac{\rho(z)}{|z|^2 \log^2(|z|^2)} |dz|^2$, and ρ is a continuous positive function, if $a_j = 1$, and
(3) $g = \frac{\rho(z)}{|z|^{2a_j}} |dz|^2$, where ρ is a continuous positive function, if $0 < a_j < 1$.

Existence of conical metrics holds according to McOwen [10], and Troyanov [14] (cf. also [4,9,11,15]).

Originally, in the definition of a cone metric, the coefficient ρ was supposed to be bounded. However, Heins [3], and McOwen [10] showed Hölder continuity.

Theorem (McOwen, Troyanov). *Let $(X, \mathbf{a})$ be an $\mathbb{R}$-stable weighted punctured Riemann surface. There exists a unique conformal conical metric $g_{\mathbf{a}}$ having constant curvature -1 on X'. Moreover, the following Gauss Bonnet formula holds*:

$$\frac{-1}{\pi} \mathrm{Vol}(g_{\mathbf{a}}) = \chi(X, \mathbf{a}).$$

We will refer to $g_{\mathbf{a}}$ as the *hyperbolic conical metric* of $(X, \mathbf{a})$, and $\bigcup_i p_i$ as the *support* of $\mathbf{a}$.

If $\mathbf{a}$ and $\mathbf{b}$ have the same underlying compact Riemann surface X, such that the support of $\mathbf{a}$ is contained in the support of $\mathbf{b}$, we define $\mathbf{a} \leqslant \mathbf{b}$, if $a_j \leqslant b_j$ for all j.

Proposition 2.4. *Let $(X, \mathbf{a})$, and $(X, \mathbf{b})$ be $\mathbb{R}$ stable weighted punctured Riemann surfaces with $\mathbf{a} \leqslant \mathbf{b}$. Then $g_{\mathbf{a}} \leqslant g_{\mathbf{b}}$.*

Proof. Due to the construction of a hyperbolic conical metric, the function $\Psi = g_{\mathbf{a}}/g_{\mathbf{b}}$ is smooth, and positive on X'. In a neighborhood of a puncture p_j it is of the form $|z|^{2(1-a_j)} \log^2(|z|^2)\rho_j(z)$, if $0 \leqslant a_j < b_j = 1$, or of the form $|z|^{2(b_j - a_j)}\rho_j(z)$ otherwise. Here ρ_j is some continuous positive function on the coordinate neighborhood of p_j. In particular Ψ is continuous, and non-negative on X.

Let $\Delta_{\mathbf{b}} = \frac{1}{g_{\mathbf{b}}}\partial\bar{\partial}$ be the Laplace operator associated to $g_{\mathbf{b}}$ on X'. On the open surface X' the equation for curvature constantly equal to -1 gives us

$$\Delta_{\mathbf{b}}\bigl(\log(\Psi)\bigr) = \Psi - 1.$$

Let $\Psi(x_0)$ be a maximum of Ψ in X. We claim that $\Psi(x_0) \leqslant 1$. If by contradiction we had $\Psi(x_0) > 1$, then by continuity of Ψ the function $\log(\Psi)$ would be subharmonic in a neighborhood of x_0, and it would have an internal maximum, therefore Ψ would be a positive constant. It would follow that $\mathbf{a} = \mathbf{b}$, but then $\Psi \equiv 1$, a contradiction against $\Psi(x_0) > 1$. □

In order to prove continuous dependence on the weights, we need the following fact.

Let $(X, \mathbf{a})$ be a weighted punctured Riemann surface with support $\bigcup p_i$. Let g be a smooth conformal metric on X. We denote by G_i the Green's function of the global Laplace operator Δ_g on X with singularity at p_i (cf. [8, Chapter II, §1]). In a coordinate disc $D = \{|z| < 1\}$ centered at p_i, we know that $G_i + \frac{1}{2\pi}\log|z|$ is smooth in D. From [10] we have that $g_{\mathbf{a}} = e^{u_{\mathbf{a}}}g$ with $u_{\mathbf{a}} = \sum 4\pi a_i G_i + v_{\mathbf{a}}$, and $v_{\mathbf{a}} \in H_2^p(X)$ for all p with $1 < p < \min_i(1/a_i)$. Note that McOwen's result is true also, if $a_i \leqslant 0$ for some i. In this case we have $v_{\mathbf{a}} \in H_2^p(X)$ with $1 \leqslant p < \min_{\{i:\, a_i \geqslant 0\}}(1/a_i)$.

Fix some set of weights $\mathbf{a_0}$ with $0 \leqslant a_{0i} < 1$, and a positive $\varepsilon \in \mathbb{R}$ such that $a_{0i} + \varepsilon < 1$ for all i. Let $I_{\mathbf{a_0},\varepsilon} = \{\mathbf{a} \in \mathbb{R}^n\colon a_i < a_{0i} + \varepsilon\}$. Let $1 < p < \min_i(\frac{1}{a_{0i}+\varepsilon})$.

Proposition 2.5. *The map $\mathbf{a} \to v_{\mathbf{a}}$ defined on $I_{\mathbf{a}_0,\varepsilon}$ with values in $H_2^p(X)$ is smooth. In particular, on X' we have continuity of $g_{\mathbf{a}}$ with respect to $\mathbf{a}$ in the compact-open topology.*

Proof. For simplicity we assume $n = 1$, i.e., with puncture p_0, weight a, and Green's function G singular at p_0. The proof in the general case follows along the same lines.

Let K be the curvature of g, the equation of constant curvature -1 for the metric $g_{\mathbf{a}}$ gives

$$\Delta_g(u_{\mathbf{a}}) - e^{u_{\mathbf{a}}} = K,$$

i.e.,

$$\Delta_g(v_{\mathbf{a}}) - e^{4\pi a G}e^{v_{\mathbf{a}}} = K$$

on X'.

We consider the map $\Psi : I_{a_0,\varepsilon} \times H_2^p(X) \to L^p(X)$ given by

$$\Psi(a, v) = \Delta_g(v) - e^{4\pi a G} e^v - K.$$

In order to show that all partial derivatives of Ψ with respect to a and v exist and depend continuously on these variables, we make the following observation. Fix a coordinate neighborhood U centered at p_0. For $a \in I_{a_0,\varepsilon}$, and any $k \in \mathbb{R}$, we have $\frac{\log(|z|)^k}{|z|^{2a}} \in L^p(U)$, moreover any function in $H_2^p(X)$ is in some $C^\beta(X)$. So Ψ has actually values in $L^p(X)$, and moreover it is of class C^∞. On the other hand, by Lemma 2.1 the partial derivative

$$D_2\Psi|_{(a_1,v_1)}(w) = \Delta_g(w) - e^{4\pi a_1 G} e^{v_1} w$$

is invertible, so we can conclude the proof by means of the implicit function theorem. □

3. The generalized Weil–Petersson metric

We first choose **a** such that $0 < a_i \leqslant 1$ for all $1 \leqslant i \leqslant n$. Let S be an open set of some $\mathbb{C}^k$, by definition a holomorphic family $(\mathcal{X}, \mathbf{a}) \to S$ of weighted punctured Riemann surfaces of genus $\gamma \geqslant 0$, is a holomorphic family $\mathcal{X} \to S$ together with n holomorphic sections $\sigma_1 \ldots \sigma_n$ such that for all $s \in S$ the points $\sigma_j(s)$ are pairwise distinct. From a deformation theoretic standpoint the Teichmüller space of such objects is just the usual Teichmüller space $\mathcal{T}_{\gamma,n}$, where γ is the genus of the compact surface. Given a point in $\mathcal{T}_{\gamma,n}$ with induced n-punctured Riemann surface X, we assign the weight a_j to the puncture p_j. Denote by $H^0(X, \Omega^2{}_{(X,\mathbf{a})})$ the space of holomorphic quadratic differentials with at most simple poles at the punctures, identified with the cotangent space at $\mathcal{T}_{\gamma,n}$ at the given point.

Definition 3.1. The Weil–Petersson inner product on $H^0(X, \Omega^2{}_{(X,\mathbf{a})})$ is given by

$$\langle \phi, \psi \rangle_{WP,\mathbf{a}} = \int_X \frac{\phi\bar{\psi}}{g_{\mathbf{a}}^2}\, dA_{\mathbf{a}},$$

where $g_{\mathbf{a}}$ is the hyperbolic conical metric, with surface element $dA_{\mathbf{a}}$.

Observe that the above integrals are finite, because $0 < a_i \leqslant 1$ for all i.

It follows from Theorem 2.5 that for $0 < a_i < 1$ the Weil–Petersson inner product depends continuously on the weights.

We use an ad hoc definition of the space of *harmonic Beltrami differentials* with respect to the hyperbolic conical metric $g_{\mathbf{a}}$.

Definition 3.2. Let $H^1(X, \mathbf{a})$ be the space of Beltrami differentials of the form

$$\mu = \mu(z)\frac{\partial}{\partial z}\,\overline{dz} = \frac{\overline{\phi(z)}}{g_{\mathbf{a}}(z)}\frac{\partial}{\partial z}\,\overline{dz},$$

where $\phi = \phi(z)\,dz^2 \in H^0(X, \Omega^2_{(X,\mathbf{a})})$, and $g_{\mathbf{a}} = g_{\mathbf{a}}(z)\,dz\,\overline{dz}$ is the hyperbolic conical metric.

Lemma 3.3. *There is a natural non-degenerate pairing*

$$\Phi : H^0\big(X, \Omega^2_{(X,\mathbf{a})}\big) \times H^1(X, \mathbf{a}) \to \mathbb{C},$$

where

$$\Phi\left(\phi(z)\,dz^2, \mu(z)\frac{\partial}{\partial z}\,\overline{dz}\right) = \int\limits_X \phi(z)\mu(z)\,dz\,d\bar{z}.$$

Proof. We have

$$\Phi\left(\phi(z)\,dz^2, \frac{\overline{\phi(z)}}{g_{\mathbf{a}}(z)}\frac{\partial}{\partial z}\,\overline{dz}\right) = \|\phi\|^2_{WP,\mathbf{a}}. \qquad \square$$

The Weil–Petersson metric on the cotangent space to $\mathcal{T}_{\gamma,n}$ together with the above duality defines a Weil–Petersson metric $G_{WP,\mathbf{a}}$ on the tangent space identified with $H^1(X, \mathbf{a})$.

Lemma 3.4. *Let μ_1, and μ_2 in $H^1(X, \mathbf{a})$, then*

$$\langle \mu_1, \mu_2\rangle_{WP,\mathbf{a}} = \int\limits_X \mu_1\,\overline{\mu_2}\,dA_{\mathbf{a}}.$$

Proof. By polarization it is sufficient to prove the lemma for $\mu_1 = \mu_2$.

We have by definition

$$\|\mu_1\|^2_{WP,\mathbf{a}} = \sup_{\|\psi\|\neq 0} \frac{|\Phi(\psi, \mu_1)|^2}{\|\psi\|^2_{WP,\mathbf{a}}}.$$

It follows from the Cauchy–Schwarz inequality that the right-hand side is smaller or equal to $\int |\mu_1|^2\,dA_{\mathbf{a}}$. With $\mu_1 = \frac{\overline{\phi_1}}{g_{\mathbf{a}}}\frac{\partial}{\partial x}\,\overline{dz}$, setting $\psi = \phi_1$, we get equality. $\square$

Denoting by $(\mathbf{b}, 0)$ the weighted punctured surface with fake punctures $p_{m+1}, \ldots, p_n$, whose weights are zero, the notion of convergence $\mathbf{a} \to (\mathbf{b}, 0)$ becomes meaningful. Because of the regularity theorem for an elliptic non-linear equation, the hyperbolic conical metric $g_{\mathbf{b}}$ can be identified with $g_{(\mathbf{b},\mathbf{0})}$.

Denote by $\chi : \mathcal{T}_{\gamma,n} \to \mathcal{T}_{\gamma,m}$ the holomorphic map, forgetting punctures. We equip $\mathcal{T}_{\gamma,n}$, and $\mathcal{T}_{\gamma,m}$ with $G_{WP,\mathbf{a}}$, and $G_{WP,\mathbf{b}}$ respectively, then we have

Theorem 3.5. *For $\mathbf{a} \to \mathbf{b}$ the metric $G_{WP,\mathbf{a}}$ converges to the degenerate metric $\chi^*(G_{WP,\mathbf{b}})$.*

Proof. Because of Theorem 2.5, the metric $g_{\mathbf{a}}$ depends in a continuous way upon the a_j. For $a_j > 0$ this implies that $G_{WP,\mathbf{a}}$ is continuous in a_j. For simplicity we let $a_n \to 0$, and fix the rest. We choose local coordinates $(t_1, \ldots, t_N)$ around $(X, \mathbf{a})$ in $\mathcal{T}_{\gamma,n}$ such that $N = 3\gamma - 3 + n$, and the coordinates $t_{3\gamma-3+j}$ corresponds to moving the puncture p_j. Therefore $dt_{3\gamma-3+j}$ on the cotangent space to $\mathcal{T}_{\gamma,n}$ corresponds to a holomorphic quadratic differential $\phi_{3\gamma-3+j}$ with a simple pole at p_j whereas ϕ_k is holomorphic for $k \leqslant 3\gamma - 3$.

Now for $\mathbf{a} \to (\mathbf{b}, \mathbf{0})$, i.e., $a_n \to 0$, the hyperbolic cone metric $g_{\mathbf{a}}$ converges to a metric which is smooth around p_n. So $\|\phi_N\|_{WP,\mathbf{a}} \to +\infty$. Therefore $\langle \phi_j, \phi_N\rangle_{WP,\mathbf{a}}$ has a finite limit as $a_n \to 0$. On the tangent space we get that $\langle \frac{\partial}{\partial t_N}, \frac{\partial}{\partial t_j} \rangle = 0$ for all $1 \leqslant j \leqslant N$. Furthermore, we see that for $i, j < N$ the functions t_i, and t_j can be considered as coordinates in $\mathcal{T}_{\gamma,n-1}$, so that $\langle dt_i, dt_j \rangle_{WP,(\mathbf{b},0)} = \langle dt_i, dt_j \rangle_{WP,\mathbf{b}}$. This implies the claim. □

4. Variation with respect to parameters in a family

Let $\Pi : (\mathcal{X}, \mathbf{a}) \to S$ be a holomorphic family of weighted punctured Riemann surfaces. Let us take $S = \{s \in \mathbb{C}:\ |s| < 1\}$. Let $X = \Pi^{-1}(0)$. We will consider local coordinates (z, s) in $\mathcal{X}$ such that Π is the projection onto the second factor. Punctures are defined by holomorphic sections σ_j with weights a_j, $0 \leqslant a_j < 1$, and local equation $t_j(s)$. Denote by $\mathcal{X}'$ the open subset $\mathcal{X} \setminus \bigcup_j \sigma_j(S)$. For each j choose a function ψ_j which is smooth on $\mathcal{X} \setminus \sigma_j(S)$, and of the form $\log(|z - t_j(s)|^2)$ near $\sigma_j(S)$. For simplicity let us assume that $n = 1$, $a_1 = a$, $t_1(s) = t(s)$, and $\psi_1 = \psi$.

The holomorphic family $\mathcal{X} \to S$ of the underlying compact Riemann surfaces can be chosen to be a trivial differentiable family. Therefore the various Sobolev spaces $H_k^q(\mathcal{X}_s)$ with $\mathcal{X}_s = \Pi^{-1}(s)$ can be identified with $H_k^q(X)$. Let the differential operators $\partial/\partial z$ depend on the parameter s in such a way that the induced conformal structure is the given one. So we can choose a differentiable lift V of $\partial/\partial s$ so that $[V, \bar{V}] \equiv 0$. Explicitly, we let $V(s) = \frac{\partial}{\partial s} + \eta(z,s)\frac{\partial}{\partial z} + \theta(z,s)\frac{\partial}{\partial \bar{z}}$, and we assume that $\eta \equiv 0$, and $\theta \equiv 0$, i.e., $V = (\partial/\partial s)|_{(z,s)}$ in a neighborhood of the section σ, which defines the punctures depending on s.

Choose a smooth family $g(s,z)|dz|^2$ of smooth conformal metrics on $\mathcal{X}_s$ of curvature $K(s,z)$. Let $g_a(z,s)$ be the hyperbolic conical metric on $\mathcal{X}_s$. Then $g_a(z,s)|dz|^2 = e^{u(z,s)}g(z,s)$. Here $u(z,s) = a\psi + w(z,s)$, where for fixed s, and fixed p with $1 < p < 1/a$ the function $w(z,s)$ belongs to $H_2^p(X)$.

We use the implicit function theorem to produce (the unique) hyperbolic cone metrics on neighboring fibers in a holomorphic family from the one on the central fiber. This approach yields an understanding, and estimates of derivatives of $g_a(z,s)$ with respect to s.

The equation for hyperbolicity on X gives

$$\Delta u - e^u = K, \tag{2}$$

where Δ is the Laplace operator with respect to the metric g, and K the curvature. It is equivalent to

$$\Delta w - e^{a\psi} e^w = K - a\Delta(\psi). \tag{3}$$

Since ψ is harmonic in a neighborhood of the set $\sigma_j(S)$, in such a neighborhood the above equation is locally of the form

$$\Delta w - \frac{e^w}{|z-t|^{2a}} = K. \tag{4}$$

We want to study the solutions of (3) depending on the parameter s.

Let us first assume for simplicity that the holomorphic family $\mathcal{X} \to S$ is trivial. Then (3) is of the form

$$\Delta w(z,s) - b(z,s)e^{w(z,s)} = f(z,s).$$

(This is a global equation expressed in a local coordinate z.)

Define

$$\Phi : S \times H_2^p(X) \to L^p(X)$$

by

$$\Phi(s, w) = \Delta(w) - e^{a\psi} e^w - K + a\Delta(\psi) = \Delta(w) - b \cdot e^w - f, \tag{5}$$

where $\Delta(\psi)$ is considered as a function that is identically zero near the punctures. Our solution $w(z, s)$ satisfies the equation $\Phi(s, w(z, s)) \equiv 0$.

At this point we assume $0 \leqslant a < 1/2$, and chose p such that $1 < p < \frac{2}{2a+1}$. We claim that Φ is a map of class C^1: The only singularity of $b = e^{a\psi}$ is at the punctures, where it equals $\frac{e^w}{|z-t(s)|^{2a}}$, and where $\frac{\partial b}{\partial s} = -\frac{t'(s)}{z-t(s)} \cdot \frac{e^w}{|z-t(s)|^{2a}}$. So $\frac{\partial b}{\partial s} \in L^p(X)$, and it depends continuously on s and w. The function f is smooth in s (and independent of w). Furthermore, the coefficients of the Laplacian are smooth functions in s, and

$$\frac{\partial}{\partial s}(\Delta w) = -\frac{\partial}{\partial s}\big(\log g(z, s)\big) \cdot \Delta(w) \in L^p(X)$$

for any fixed $s \in S$, $w \in H_2^p(X)$, and it depends continuously on s and w. So $(D_1\Phi)_{(s,w)} \in L^p(X)$ for all $(s, w) \in S \times H_2^p(X)$. Furthermore, the map

$$(D_2\Phi)_{(s,w)} : H_2^p(X) \to L^p(X)$$

is given by

$$(D_2\Phi)_{(s,w)}(W) = \Delta(W) - e^w \cdot W.$$

This map depends continuously on s and w, and by Lemma 2.1 it is a continuous linear map with a bounded inverse. By the implicit function theorem the solution $w(z, s) : S \to H_2^p(X)$ is of class C^1 in s. In particular it is differentiable in s as a function of z and s, and as $H_2^p(X) \subset C^\beta(X)$ for some $\beta > 0$ it is continuous in z and s. So we can also apply local regularity theorems on the complement of the punctures, and see that it is of class C^∞ there.

In our situation, the holomorphic family is usually non-trivial, and derivatives with respect to a parameter are meaningful with respect to a differentiable trivialization. This process is equivalent to taking Lie derivatives with respect to the vector field V. Let us denote by L the operator of derivation in the V-direction. We have

$$[L, V] = L(\log g) \cdot \Delta + R,$$

where R is a differential operator of order two in fiber direction, whose coefficients are first and second order derivatives of η and θ. In particular, R vanishes on a neighborhood of the punctures. Up to inserting the differentiable trivialization of $\mathcal{X} \to S$ coming from an integration of V, a map

$$\Phi : S \times H_2^p(X) \to L^p(X)$$

is given by the same formula (5). The derivative $D_1(\Phi)_{(s,w)}$ is defined either by applying $\partial/\partial s$ on $X \times S$ or L on $\mathcal{X}$. So

$$D_1(\Phi)_{(s,w)} = e^{a\psi} e^w \big(L(\log g) - aL(\psi)\big) + L(K) \\ - L(\log g)(K - a\Delta\psi) - R(w) - aL\big(\Delta(\psi)\big).$$

This element represents a function on $\mathcal{X}_s \simeq X$, whose restriction to a neighborhood of the punctures equals

$$-e^{a\psi} e^w L(\log g) - e^{a\psi} e^w aL(\psi) - L(\log g) \cdot K + L(w).$$

The previous argument applies again: The derivative $D_1(\Phi)_{(s,w)}$ is in $L^p(X)$ for any $s \in S$, $w \in H_2^p(X)$ and depends continuously on s and w. Furthermore $(D_2\Phi)_{(s,w)}(W) = \Delta(W) - e^{a\psi} e^w W$.

Theorem 4.1. *Let $0 \leqslant a_i < 1/2$ for all i, and $1 < p < \min_i \frac{2}{2a_i+1}$. Consider a holomorphic family of punctured Riemann surfaces with conical hyperbolic metrics $g_{\mathbf{a}}(s)$, and write $g_{\mathbf{a}}(s) = e^u(s)g(s)$, where $g(s)$ is a smooth family of smooth metrics on the $\mathcal{X}_s$. Let $u = \sum_i a_i\psi_i + w_{\mathbf{a}}(s)$.*

Then $w_{\mathbf{a}}(s)$ and $L_V(w_{\mathbf{a}}(s))$ are contained in $H_2^p(X)$ for all $s \in S$, and they depend in a C^1 smooth way on s. Moreover $w_{\mathbf{a}}(s)$, and $L_V(w_{\mathbf{a}}(s))$ are C^∞ on $\mathcal{X}'$.

Corollary 4.2. *The generalized Weil–Petersson metric is also C^1 smooth with respect to s.*

5. The generalized Kodaira–Spencer map

We now want to compute harmonic Beltrami differentials in the sense of Section 3 from the variation of hyperbolic cone metrics.

In the case of compact Riemann surfaces with no punctures, the harmonic Beltrami differential $\mu = \mu(z)\frac{\partial}{\partial z}\,d\bar{z}$ associated to $\frac{\partial}{\partial s}|_{s=0}$ equals

$$\mu = -\frac{\partial}{\partial \bar{z}}\left(\frac{1}{g}\frac{\partial^2 \log(g(z,s))}{\partial \bar{z}\partial s}\right)\bigg|_{s=0}\frac{\partial}{\partial z}\,d\bar{z}$$

(see [13]).

From now on all values are taken at $s = 0$.

A formal calculation shows that $\phi = -g\bar{\mu}$ is a *holomorphic* quadratic differential. On the other hand, the above Beltrami differential is the $\bar{\partial}$ exterior derivative of a smooth lift of the tangent vector $\frac{\partial}{\partial s}$ to the total space $\mathcal{X}$ of the family (restricted to the central fiber $X = \Pi^{-1}(0)$).

In the case of conical hyperbolic metrics we define the smooth Beltrami differential on X' given by

$$\mu_{\mathbf{a}}\left(\frac{\partial}{\partial s}\right) = -\frac{\partial}{\partial \bar{z}}\left(\frac{1}{g_{\mathbf{a}}}\frac{\partial^2 \log g_{\mathbf{a}}}{\partial \bar{z}\partial s}\right)\frac{\partial}{\partial z}\,d\bar{z}. \tag{6}$$

and the quadratic differential $\phi_{\mathbf{a}}(\frac{\partial}{\partial s}) = g_{\mathbf{a}}\overline{\mu_{\mathbf{a}}}(\frac{\partial}{\partial s})$.

Lemma 5.1. *The quadratic differential $\phi_{\mathbf{a}}(\frac{\partial}{\partial s})$ is in $L^1(X)$.*

Proof. Assume again $n = 1$. From formula (6) we derive that

$$\phi_{\mathbf{a}}\left(\frac{\partial}{\partial s}\right) = \frac{\partial \log g_{\mathbf{a}}}{\partial z} \cdot \frac{\partial^2 \log g_{\mathbf{a}}}{\partial z \partial \bar{s}} - \frac{\partial^3 \log g_{\mathbf{a}}}{\partial z^2 \partial \bar{s}}$$

By the Sobolev embedding theorem $H_1^p(X) \subseteq L^h(X)$ for all $h < p' := \frac{2p}{2-p}$. However, as p can been chosen between 1 and $\frac{2}{1+2a}$, the exponent p' can be any number between 2 and $\frac{1}{a}$. Recall that $\frac{\partial^2 \log(|z-t(s)|^2)}{\partial z \partial \bar{s}} \equiv 0$, and that $\frac{\partial \log(|z-t(s)|^2)}{\partial z} = \frac{1}{z-t(s)} \in L^q(D)$ for all $1 \leqslant q < 2$, where $D \subset \mathbb{C}$ denotes the unit disk.

Around the puncture,

$$\frac{\partial^2 \log g_{\mathbf{a}}}{\partial z \partial \bar{s}} = \frac{\partial^2 w}{\partial z \partial \bar{s}}.$$

By Theorem 4.1, we have

$$\frac{\partial w}{\partial \bar{s}} \in H_s^p(X)$$

for all $1 \leqslant p < \frac{2}{2a+1}$. So $\frac{\partial^2 \log g_{\mathbf{a}}}{\partial z \partial \bar{s}} \in H_1^p(X) \hookrightarrow L^h(X)$ for all $1 \leqslant h < 1/a$.

Furthermore, $\frac{\partial \log g_{\mathbf{a}}}{\partial z} \in L^q(X)$ for all $1 \leqslant q < 2$.

As $0 \leqslant a < 1/2$, we can find q, h in the above range with $1/q + 1/h = 1$. By Hölder inequality,

$$\frac{\partial \log g_{\mathbf{a}}}{\partial z} \cdot \frac{\partial^2 \log g_{\mathbf{a}}}{\partial z \partial \bar{s}} \in L^1(X).$$

On the other hand, Theorem 4.1 gives us that

$$\frac{\partial^3 \log g_a}{\partial z^2 \partial \bar{s}}$$

is also in $L^1(X)$ so that $\phi_a(\frac{\partial}{\partial s}) \in L^1(X)$. □

Proposition 5.2. *Let $(\mathcal{X}, \mathbf{a}) \to S$ be a holomorphic family of punctured Riemann surfaces, with $0 \leqslant a_i < 1/2$ for all $1 \leqslant i \leqslant n$. Then the quadratic differential $\phi_{\mathbf{a}}(\frac{\partial}{\partial s})$ is holomorphic on X' with at most simple poles at the punctures. In particular $\mu_{\mathbf{a}} \in H^1(X, \mathbf{a})$.*

Proof. The proof that $\phi_{\mathbf{a}}(\frac{\partial}{\partial s})$ is holomorphic in X' follows as in [13] from hyperbolicity of the metric. To show that $\phi_{\mathbf{a}}$ has at most simple poles at the punctures we use the following well-known fact. □

Lemma 5.3. *Let $D \subset \mathbb{C}$ be the unit disk, and let f be holomorphic on $D \backslash \{0\}$, let D_r be a disk of radius $0 < r < 1$. The function f is in $L^1(D_r)$, if and only if it has at most a simple pole at 0. In this case $f \in L^p(D_r)$ for all $1 \leqslant p < 2$. Moreover, f is $L^2(D_r)$, if and only if it extends holomorphically to D.*

Proof. As $1/z \in L^p(D_r)$ for all $1 \leqslant p < 2$, any f with at most a simple pole is in $L^p(D_r)$. In order to prove the converse, let $f(z) = \sum_{j\geqslant 2} a_j z^{-j}$, and assume that $0 \not\equiv f \in L^1(D_r)$. Write $f(z) = \frac{1}{z^k} h(\frac{1}{z})$ with h holomorphic on $\mathbb{C}$, $h(0) \neq 0$, and $k \geqslant 2$. So

$$\infty > \int\limits_{|w| > \frac{1}{r}} \left|h(w)\right| \cdot |w|^{k-4} \frac{i}{2}\, dw \wedge d\bar{w},$$

in particular

$$I = \int\limits_{|w| > \frac{1}{r}} h(w) \cdot |w|^{k-4} \frac{i}{2}\, dw \wedge d\bar{w}$$

is finite. As for any ρ the integral $\int_0^{2\pi} h(\rho e^{i\theta})\, d\theta$ equals $2\pi h(0)$, we get $I = 2\pi h(0) \times \int_{\frac{1}{r}}^{\infty} \rho^{k-3}\, d\rho$, which is not finite for $k \geqslant 2$.

Now let $f(z) = \sum_{j\geqslant 1} a_j z^{-j}$. Then

$$\left(\|f\|_2\right)^2 = 2\pi \sum_{j\geqslant 1} |a_j|^2 \int\limits_0^1 \frac{d\rho}{\rho^{2j-1}},$$

which equals ∞, unless all coefficients vanish. □

Let $\mathcal{X} \to S$ be a holomorphic family of weighted punctured Riemann surfaces not necessarily over a disc. If $\mathbf{v} \neq 0$ is a tangent vector of S at the distinguished base point, we can find a local embedding of S into some ambient space U, such that $\mathbf{v}$ equals $\partial/\partial s$, where s is one of the holomorphic coordinates. Since in our case tangent directions are not obstructed (in the sense of deformation theory), we can always assume that S is a disc.

Proposition 5.2 allows us to define the *generalized Kodaira Spencer map* $\rho : T_0 S \to H^1(X, \mathbf{a})$, which associates to the tangent vector $\partial/\partial s|_{s=0}$ of a holomorphic one parameter family in S, the Beltrami differential $\mu_{\mathbf{a}}(\frac{\partial}{\partial s})$ defined in (6). By Proposition 5.2 the Beltrami differential $\mu_{\mathbf{a}}(\frac{\partial}{\partial s})$ is harmonic in the sense of Definition 3.2.

Theorem 5.4. *Let* $\mathbf{a}$ *be an* $\mathbb{R}$*-divisor such that* $0 < a_i < \frac{1}{2}$ *for all* $1 \leqslant i \leqslant n$*. A tangent vector* $\mathbf{v}$ *to* S *at* 0*, is in the kernel of the generalized Kodaira-Spencer map, if and only if the family of punctured Riemann surfaces is infinitesimally trivial in the direction of* $\mathbf{v}$*.*

Proof. As explained above, we can assume that S is a disc. If the given family is infinitesimally trivial in the direction of $\mathbf{v} = \frac{\partial}{\partial s}|_{s=0}$, as far as the computation of $\mu_{\mathbf{a}}(\frac{\partial}{\partial s})$ concerns, we can replace our family with the trivial family over a disc tangent to $\mathbf{v}$ at 0, because there are no obstructions. Hence $\mu_{\mathbf{a}}(\frac{\partial}{\partial s}) \equiv 0$.

Assume conversely that for some tangent vector $0 \neq \frac{\partial}{\partial s}$ the Beltrami differential $\mu_{\mathbf{a}}(\frac{\partial}{\partial s})$ vanishes.

In a first step, we show that the given family with punctures disregarded has to be trivial.

Let us set

$$\gamma(z)\frac{\partial}{\partial z} = -\left(\frac{1}{g_{\mathbf{a}}}\frac{\partial^2 \log(g_{\mathbf{a}}(z,s))}{\partial \bar{z}\partial s}\right)\bigg|_{s=0}\frac{\partial}{\partial z}.$$

The vector field $W = \frac{\partial}{\partial s} + \gamma(z)\frac{\partial}{\partial z}$ is a smooth lifting to $\mathcal{X}'$ of the vector field $\frac{\partial}{\partial s}$ on S (cf. [13]). By the above assumption W is a holomorphic vector field on X', where $X = \mathcal{X}_0$. To prove that it extends holomorphically to X, we show that it is $L^2(X)$, and apply Lemma 5.3. According to Theorem 4.1, and by the same reasoning as in Lemma 5.1 we get at $s = 0$:

$$\frac{\partial^2 \log g_a}{\partial z \partial \bar{s}} \in L^2(X).$$

Since $\frac{1}{g_{\mathbf{a}}}$ is continuous, hence bounded, the claim follows.

Now since $W = \frac{\partial}{\partial s} + \gamma\frac{\partial}{\partial z}$ is holomorphic on the first infinitesimal neighborhood of X in $\mathcal{X}$, the family of compact Riemann surfaces is infinitesimally trivial in the direction of $\frac{\partial}{\partial s}$.

With no loss of generality we can assume $\mathcal{X} \simeq X \times S \to S$ is actually trivial, and also that the family g of smooth reference metrics to be independent on s. In this geometric situation, $\gamma(z)\frac{\partial}{\partial z}$ is a holomorphic vector field on X.

It is sufficient to show that the direction $t_j'(0)$ of the movement of a puncture p_j with respect to the parameter s is equal to the value of γ at the puncture. Therefore, the vector field W provides an infinitesimal holomorphic trivialization of the punctured family.

Around any puncture p_j, according to Section 4 we have

$$g_{\mathbf{a}} = e^u g = \frac{e^w g}{|z - t_j(s)|^{2a_j}}$$

with $w \in H_2^p(X)$. On the other hand, $\partial\bar{\partial}\log g_{\mathbf{a}} = g_{\mathbf{a}}$, and since γ is holomorphic, we have at $s = 0$,

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial \log g_{\mathbf{a}}}{\partial s} + \gamma\frac{\partial \log g_{\mathbf{a}}}{\partial z}\right) \equiv 0.$$

Hence, the function

$$\chi = \frac{\partial w}{\partial s} + \gamma\frac{\partial w}{\partial z} + \gamma\frac{\partial \log(g)}{\partial z}$$

is holomorphic on a punctured disc around p_j. Since $\frac{\partial w}{\partial s} \in H_2^p(X)$ and $\frac{\partial w}{\partial z} \in H_1^p(X)$, the function χ is in $L^2(X)$. So by Lemma 5.3 it is holomorphic in a neighborhood of the puncture, and finally the function

$$\chi_0 = \frac{\partial w}{\partial s} + \gamma\frac{\partial w}{\partial z}$$

is smooth in a neighborhood of the puncture. By differentiating the hyperbolicity equation (4), we find that up to additive terms of class $C^\infty(X)$ the function $\Delta(\chi_0)$ equals

$$e^u\left(\frac{\partial u}{\partial s} + \frac{\partial \gamma}{\partial z} + \gamma\frac{\partial \log g}{\partial z} + \gamma\frac{\partial u}{\partial z}\right).$$

Since e^{-u} is continuous around p_j, we conclude that

$$\frac{\partial u}{\partial s} + \gamma \frac{\partial u}{\partial z}$$

is also continuous around p_j. It follows immediately that $\gamma(p_j) = t_j'(0)$. $\square$

Corollary 5.5. *The generalized Kodaira–Spencer map $\rho : T_0S \to H^1(X, \mathbf{a})$ is an isomorphism for universal families of punctured Riemann surfaces.*

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