KÄHLER GEOMETRY OF DOUADY SPACES

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1. Abstract

We consider the generalized Petersson-Weil metric on the moduli space of compact submanifolds of a Kähler manifold or a projective variety. It is extended as a positive current to the space of points corresponding to reduced fibers, and estimates are shown. For moduli of projective varieties the Petersson-Weil form is the curvature of a certain determinant line bundle equipped with a Quillen metric. We investigate its extension to the compactified moduli space.

2. Introduction

The complex structure of a moduli space reflects the variation of complex structures on the fibers of a holomorphic family. On the level of deformations, the Kodaira-Spencer map usually identifies the tangent space of the base of a universal family with a certain cohomology group defined intrinsically on the fibers. Once these groups can be equipped with an inner product in a functorial way, the induced Hermitian structure is expected to induce a Kähler metric on the moduli space, which will be called a *generalized Petersson-Weil metric*. Well-known cases are moduli of Kähler-Einstein manifolds (cf. [F-S1]), and stable holomorphic vector bundles (cf. [F-S2, S-T]).

In the classical situation of moduli of compact Riemann surfaces Wolpert showed in [WO] that the Petersson-Weil metric extends to the Deligne-Mumford compactification as a strictly positive closed current. This current is in fact the Chern form of a certain determinant bundle, which was studied before by Mumford in [M], giving rise to an embedding into a projective space. Here the techniques of Richberg [RI] could be used for a purely analytic proof of the ampleness. These results were extended by Zograf and Takhtadzhyan in [Z-T].

In this paper we will study the generalized Petersson-Weil metric on the Douady space of submanifolds of a fixed Kähler manifold Z.

The Kähler property for both the Douady, and Barlet spaces of compact cycles had been previously established by Fujiki [F1, F2] and Varouchas [VA1, VA2], where a certain fiber integral was used. In [B-S2] this fiber integral is tied to the geometric situation, and recovered as a generalized Petersson-Weil metric ω^{PW}.

If Z is projective, the generalized Grothendieck-Hirzebruch-Riemann-Roch theorem by Bismut, Gillet and Soulé [BGS] can be used to define a hermitian line bundle λ on the Douady space of smooth submanifolds such that the curvature form of λ equals ω^{PW}.

We use the Grothendieck-Hirzebruch-Riemann-Roch theorem for not necessarily smooth mappings of complex manifolds by O'Brian-Toledo-Tong [O-T-T1, O-T-T2] to show that the determinant line bundle extends to the whole Douady space, i.e. to the Hilbert scheme, as determinant bundle up to a divisor arising from the non-smooth locus. From these two ingredients, we construct a hermitian line bundle, whose curvature is the generalized Petersson-Weil form (cf. Theorem 7.2). This line bundle is somewhat different from the invertible sheaf used in Algebraic Geometry (cf. [V]).

The refined Riemann-Roch theorem by Bismut [BI2] extends the Quillen metric to determinant bundles for singular fibrations, where nodal singularities are allowed.

If generic singularities for the given component of the Douady space are nodal, which is the case for plane curves or a large class of space curves, we can say more: The above determinant bundle extends to the whole component of the Douady space as a singular hermitian bundle, which possesses a continuous $\partial\overline{\partial}$-potential, and on the set of smooth points the approximation techniques of Demailly for singular hermitian metrics with vanishing Lelong numbers [D] are applicable.

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3. Kähler Geometry of Douady Spaces

Let Z be a complex manifold, and

$$\begin{tikzcd} \mathscr{X} \arrow[r, hook, "i"] \arrow[dr, "f"'] & Z \times S \arrow[d, "\mathrm{pr}_2"] \\ & S \end{tikzcd} \tag{1}$$

a flat holomorphic family of complex submanifolds of Z, parameterized by a complex space S of dimension n. Let s_0 be a distinguished point of S with fiber $X = \mathscr{X}_{s_0}$. The above holomorphic maps define morphisms of tangent and normal bundles. Since $\mathscr{N}_{X|\mathscr{X}}$ is trivial, we get

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{T}_X \arrow[r] \arrow[d, equal] & \mathscr{T}_{\mathscr{X}}|X \arrow[r, "\nu_{\mathscr{X}}"] \arrow[d, "q_*"'] & T_{s_0}S \otimes_{\mathbb{C}} \mathscr{O}_X \arrow[r] \arrow[d, "\rho"'] & 0 \\ 0 \arrow[r] & \mathscr{T}_X \arrow[r] & \mathscr{T}_Z|X \arrow[r, "\nu_Z"] & \mathscr{N}_{X|Z} \arrow[r] & 0 \end{tikzcd} \tag{2}$$

where $q = \mathrm{pr}_1 \circ i$.

The morphism ρ applied to global sections defines the Kodaira-Spencer map:

$$\rho_{s_0} : T_{s_0}S \to H^0(X, \mathscr{N}_{X|Z}).$$

For closed subspaces rather than submanifolds the Kodaira-Spencer map can according to [DO] also be defined, but then $\mathscr{N}_{X|Z}$ has to be generalized to the sheaf of sections of the normal linear space of X in Z.

From now on, (Z, ω_Z) denotes a Kähler manifold. Denote by ω_X the induced Kähler form on X. The Kähler form ω_Z induces a hermitian inner product $\langle \cdot, \cdot \rangle_{\omega_Z}$ on the normal bundle $\mathscr{N}_{X|Z}$, and by integration a natural inner product on the space of its global holomorphic sections.

Definition 3.1. *The Petersson-Weil inner product for $v, w \in T_{s_0}S$ equals*

$$\langle v, w \rangle_{PW} = \int_X \langle \rho(v), \rho(w) \rangle_{\omega_Z} \omega_X^n.$$

It follows from the definition that the Petersson-Weil norm is strictly positive in effective directions of the family, in particular it provides the base space with a (positive definite) hermitian structure.

We now give a geometric description of the Petersson-Weil inner product.

Let $\mathrm{pr}_1 : Z \times S \to Z$ denote the projection onto the first factor. The d-closed, real $(1,1)$-form $\widetilde{\omega} = \mathrm{pr}_1^*(\omega_Z)$ is positive semi-definite, in particular when restricted to $\mathscr{X}$, and positive definite when restricted to fibers of f, in particular to X. Observe that this property is sufficient to define what it means that a tangent vector of $\mathscr{X}$ at a point of X is orthogonal to X.

The positive semidefinite form $\widetilde{\omega}|\mathscr{X} = \omega_{\mathscr{X}}$ induces a hermitian structure $\langle \cdot, \cdot \rangle_{\omega_{\mathscr{X}}}$ on $\mathscr{T}_{\mathscr{X}}|X$, and we realize $\mathscr{T}_{\mathscr{X}}|X$ as a C^∞ orthogonal sum of $\mathscr{T}_X$ and $T_{s_0}S \otimes_{\mathbb{C}} \mathscr{O}_X$ induced by a section $\sigma_{\mathscr{X}}$ of $\nu_{\mathscr{X}}$ of class C^∞. In a similar way $\mathscr{T}_Z|X$ decomposes as an orthogonal sum of $\mathscr{T}_X$ and $\mathscr{N}_{X|Z}$ with C^∞ section σ_Z of ν_Z.

Lemma 3.2. *Let $v, w \in T_{s_0}S$. Then*

$$\langle v, w \rangle_{PW} = \int_X \langle \sigma_{\mathscr{X}}(v), \sigma_{\mathscr{X}}(w) \rangle_{\omega_{\mathscr{X}}} \omega_X^n.$$

Proof. Observe that $q_* \circ \sigma_{\mathscr{X}} = \sigma_Z \circ \rho$ and that $q^* \omega_Z = \omega_{\mathscr{X}}$. □

Now we describe the Petersson-Weil metric in terms of local coordinates. Let k be the embedding dimension of S at s_0. Denote by $S \subset U = \{(s_1, \ldots, s_k)\} \subset \mathbb{C}^k$ a local embedding of S into s smooth ambient space near s_0. Then $(\partial/\partial s^1)|_{s_0}, \ldots, (\partial/\partial s^k)|_{s_0}$ define a basis of the tangent space.

We need to describe the infinitesimal deformation in *adapted* local coordinates, namely $Z \times S = \{(z, w, s)\}$ where the components of z

are z^α; $\alpha = 1, \ldots, n$, the components of w are w^i; $i = 1, \ldots, m$, and the components of s are $s^1, \ldots, s^k$; that the coordinates are adapted means that $\mathscr{X} = \{(z, 0, s)\}$, and $f(z, w, s) = s$.

Here it is sufficient to replace S by the first infinitesimal neighborhood of $s_0 = 0$ in S, in order to treat singularities of S.

With respect to the above local coordinates we write the coefficients of $\omega_{\mathscr{X}}$ as $g_{\alpha\overline{\beta}}; g_{i\overline{\beta}}, g_{\alpha\overline{\jmath}}$ etc. We pay special attention to $\omega_X = \widetilde{\omega}|X = \sqrt{-1}\sum g_{\alpha\overline{\beta}} dz^\alpha \wedge dz^{\overline{\beta}}$, and use raising and lowering of indices with respect to this tensor.

We consider a tangent vector $v_i = \frac{\partial}{\partial s^i}\big|_{s_0} \in T_0 S$. We restrict the given family to the corresponding subspace S_i of S, having the embedding dimension one. Let $\mu_i = \sigma_{\mathscr{X}}(v_i)$ be the induced differentiable section of $\mathscr{T}_{\mathscr{X}}|X$.

Explicitly, the orthogonal, differentiable vector field on X reads

$$\mu_i = \frac{\partial}{\partial s^i} - g_{i\overline{\beta}} g^{\overline{\beta}\alpha} \frac{\partial}{\partial z^\alpha}. \tag{3}$$

Its pointwise norm equals

$$\|\mu_i(z)\|^2_{PW} = g_{i\bar{i}} - g_{i\overline{\beta}} g_{\alpha\bar{i}} g^{\overline{\beta}\alpha} \geq 0,$$

and the Petersson-Weil inner product $\langle \cdot, \cdot \rangle_{PW}$ on tangent vectors of the base is defined by

$$\langle \frac{\partial}{\partial s^i}|_{s_0}, \frac{\partial}{\partial s^j}|_{s_0} \rangle = \langle \mu_i, \mu_j \rangle = \int_X (g_{i\bar{\jmath}} - g_{i\overline{\beta}} g_{\alpha\bar{\jmath}} g^{\overline{\beta}\alpha})\, g\, dV \geq 0. \tag{4}$$

Thus, a positive semi-definite $(1,1)$-form ω_S^{PW} on S is constructed, in the sense of a collection of positive definite, hermitian forms on all tangent spaces $T_s S$:

$$\omega_S^{PW} := \sqrt{-1} \sum_{i,j} \langle \mu_i, \mu_j \rangle_{PW}\, ds^i \wedge \overline{ds^j}.$$

In the sequel, we use the convention that m-th powers of differential forms always carry a factor $1/m!$.

A direct calculation similar to the case of deformations of canonically polarized varieties (cf. [S]) yields (in any dimension of S) a fiber integral formula.

Lemma 3.3. *For any smooth family $\mathscr{X} \hookrightarrow Z \times S \to S$ over S, S not necessarily reduced, the Petersson-Weil form on S equals the fiber integral*

$$\omega_S^{PW} = \int_{\mathscr{X}/S} \widetilde{\omega}^{n+1}|\mathscr{X}, \tag{5}$$

where $n = \dim X$.

The construction is compatible with base changes of families of manifolds embedded in a fixed ambient manifold Z.

From now on, we restrict ourselves to families over *reduced* complex spaces. By definition, a Kähler form is given locally in terms of a strictly plurisubharmonic $\partial\overline{\partial}$-potential of class C^∞ on a smooth ambient space. According to [VA2] the existence of a $(1,1)$-current possessing locally strongly plurisubharmonic $\partial\overline{\partial}$-potentials already implies the Kähler property. (For the more refined notion of a Kähler metric on a singular space, see [ibid].)

At this point, we denote by S an irreducible component of the Douady space such that the generic fiber is smooth, and by $\mathscr{X} \hookrightarrow Z \times S \to S$ the restriction of the universal family (of not necessarily smooth subvarieties). By general theory the fiber integral (5) defines a closed, positive, real $(1,1)$-current ω_S^{PW} on all of S, and the integral stands for the push-forward of a current. Exterior derivatives are taken in the sense of currents. As above positivity of currents stands for semi-positivity. We use (5) to prove the following:

Theorem 3.4. (1) *The Petersson-Weil form can be extended as a real, positive* $(1,1)$*-current possessing locally a continuous* $\partial\overline{\partial}$*-potential to those irreducible components of the (reduced) Douady space that contain points corresponding to non-singular fibers. In particular the Lelong numbers of the extended Petersson-Weil form vanish.*

(2) *The extended Petersson-Weil form is a strictly positive, real* $(1,1)$*-current on the space of points with reduced fibers. In particular it possesses a Kähler form.*

We denote the extended Petersson-Weil form again by ω^{PW}.

Proof. (1) The proof uses methods of Varouchas. In order to apply these, we need a Kähler form on $\mathscr{X}$. Since the statement of the Lemma is local with respect to the base, we can replace the base space S by an open subset U having a Kähler form ω_U (with $\partial\overline{\partial}$-potentials) such that $\widetilde{\omega}|_{f^{-1}U} + f^*\omega_U$ is a Kähler form on $f^{-1}U$. We consider the push-forward

$$\int_{f^{-1}(U)/U} (\widetilde{\omega}|_{f^{-1}U} + f^*\omega_U)^{n+1}.$$

For holomorphic mappings of complex manifolds (with equidimensional fibers) it was shown in [VA1, Théorème 2] that such $(1,1)$-current possesses a continuous $\partial\overline{\partial}$-potential χ_U. For reduced complex spaces this fact is contained in [VA2, Theorem 2, 3, see also Section II.3.6].

Since $\int_{\mathscr{X}_s} \widetilde{\omega}^n|\mathscr{X}_s$ does not depend on $s \in S$, including those parameters with singular fibers, we have

$$\int_{\mathscr{X}/S} \widetilde{\omega}^n|_{f^{-1}U} \wedge f^*\omega_U = \mathrm{vol}(\mathscr{X}_s)\, \omega_U.$$

Now

$$\left(\int_{\mathscr{X}/S} \widetilde{\omega}^{n+1}|\mathscr{X} \right)\Big|_U = \sqrt{-1}\partial\overline{\partial}\chi_U - \mathrm{vol}(\mathscr{X}_s)\, \omega_U.$$

Since ω_U has a differentiable $\sqrt{-1}\partial\overline{\partial}$-potential, the form given by the fiber integral possesses a continuous $\sqrt{-1}\partial\overline{\partial}$-potential.

In order to show plurisubharmonicity of the potential we can, by definition, assume that the base S is smooth and of dimension one. Let $\rho \in C_0^\infty(S, \mathbb{R})$ be a nowhere negative function. Then

$$\int_S \rho \cdot \omega^{PW} = \int_{\mathscr{X}} (\rho \circ f) \cdot \widetilde{\omega}^{n+1} | \mathscr{X} \geq 0.$$

(2) Let S be a component of the Douady space satisfying the condition of the Lemma, and $\mathscr{X} \hookrightarrow Z \times S \to S$ be the restriction of the universal family of (not necessarily smooth) subvarieties.

We have to show that locally $\omega^{PW}|U - \epsilon \cdot \omega_U$ is positive in the sense of currents for some $\epsilon > 0$ and some Kähler form ω_U on U. We need to show that

$$\int_{\mathscr{X}/S} (\widetilde{\omega}|_{f^{-1}U} - \widetilde{\epsilon} f^* \omega_U)^{n+1} \geq 0 \tag{6}$$

holds in the sense of currents for some $\widetilde{\epsilon} > 0$, or equivalently that the difference of the corresponding $\partial\overline{\partial}$-potentials is plurisubharmonic. At this point, we do not restrict the family to an analytic curve but restrict it to an arbitrary double point $D \subset (S, s_0)$ corresponding to a direction $v \in T_{s_0} S$ in order to realize all tangent directions. Let $\mathscr{X}_D = \mathscr{X} \times_S D$.

The fiber integral

$$\int_{\mathscr{X}/S} (p_1^* \omega_Z) | \mathscr{X})^{n+1} = \int_{\mathscr{X}/S} \widetilde{\omega}^{n+1} | \mathscr{X}$$

restricted to D is given by the pointwise non-negative (semi-)norm $\int_{\mathscr{X}_{s_0}} \|\mu(z,s)\|^2_{PW}\, g\, dV$ of the horizontal lift μ of the tangent vector v on the regular part of the fibers.

Since we already have plurisubharmonicity, it is sufficient to have a positive contribution from the regular locus of a given fiber, which depends continuously upon the direction v.

Suppose that $\|\mu(z, s_0)\|_{PW}$ vanishes identically on the regular part X' of $X = \mathscr{X}_{s_0}$. Let $v = \partial/\partial s_i|_{s_0}$ for some coordinates s_k on a smooth ambient space of S.

Then the horizontal lift of $\partial/\partial s_i|_{s_0}$ equals $\partial/\partial s_i|_s$ with respect to the coordinates $\{(z, w, s)\}$ used in (3) at the regular locus, in particular, the horizontal lift is *holomorphic*.

We restrict the given family to the double point $D \subset S$, where $\mathscr{O}_D = \mathbb{C} \oplus \varepsilon\mathbb{C}$, with $\varepsilon^2 = 0$. We have

$$\begin{tikzcd} \mathscr{X}_D \arrow[r, hook, "i"] \arrow[dr, "f"'] & Z \times D \arrow[d, "\mathrm{pr}_2"] \\ & D \end{tikzcd} .$$

Now for a Zariski open subset $\mathscr{X}'_D$ of $\mathscr{X}_D$, where $\mathscr{X}'_D \cap X = X'$, we know that

$$\mathscr{X}'_X = X' \times D \subset Z \times D \tag{7}$$

as space over D. Here we have equality of subspaces, rather than an isomorphism. As a last step we show that the given family has to be trivial; this, however, was excluded.

In fact, from (7) we obtain the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{I} \arrow[r] \arrow[d, "i"] & \mathscr{J} \arrow[r] \arrow[d, "j"] & \mathscr{I} \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & \mathscr{O}_Z \arrow[r] \arrow[d, "\nu"] & \mathscr{O}_Z \oplus \varepsilon \mathscr{O}_Z \arrow[r] \arrow[d, "\mu"] & \mathscr{O}_Z \arrow[r] \arrow[l, bend right, "\tau"'] \arrow[d, "\nu"] & 0 \\
0 \arrow[r] & \mathscr{O}_X \arrow[r, "\alpha"] \arrow[d] & \mathscr{O}_{\mathscr{X}} \arrow[r, "\rho"'] \arrow[d] & \mathscr{O}_X \arrow[r] \arrow[l, bend right, "\sigma"'] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The morphism τ defines the canonical splitting of the second line. Over X' it descends to a morphism σ. The morphism $\mu \circ \tau \circ i$ has values in $\alpha(\mathscr{O}_X)$, i.e. $\mu \circ \tau \circ i = \alpha \circ \lambda$, where $\lambda : \mathscr{I} \to \mathscr{O}_X$ is $\mathscr{O}_Z$-linear. Now λ amounts to an $\mathscr{O}_X$-linear morphism $\overline{\lambda} : \mathscr{I}/\mathscr{I}^2 \to \mathscr{O}_X$, which describes the given infinitesimal deformation of the embedded subspace $X \subset Z$.

Over X' we have $\mu \circ \tau \circ i = 0$, i.e. $\lambda|_{X'} = 0$. Since the support of $\mathscr{O}_X/\lambda(\mathscr{I})$ is closed in X and contains X', the map λ must be zero. □

Remark 3.5. *The existence of local, continuous $\partial\overline{\partial}$-potentials, in particular the absence of residues, means that the null extension of the restriction of the Petersson-Weil current to the complement of an analytically thin subset equals the Petersson-Weil current itself. Also all Lelong numbers vanish for the current ω^{PW}. Its restriction to any subspace exists as a current with vanishing Lelong numbers, provided some fiber is smooth. On one dimensional subspaces of this kind, ω^{PW} is absolutely continuous.*

4. Asymptotics for One Dimensional Families

In this section, we include an estimate for the generalized Petersson-Weil metric for families of embedded manifolds, which is similar to estimates for degenerating Hodge metrics. The estimate will not be used in the following sections.

Let $\Delta = \{s \in \mathbb{C}; |s| < 1\}$, and $\mathscr{Y} \to \Delta$ be a proper, holomorphic map of complex manifolds such that the general fiber $\mathscr{Y}_s$ is smooth and of dimension n and such that the central fiber $\mathscr{Y}_0$ is a normal crossings divisor. Let $k+1$ be the maximum number of local components of $\mathscr{Y}_0$ which intersect in one point. Let $\eta \geq 0$ be a smooth, real $(1,1)$-form on $\mathscr{Y}$, and

$$\omega_\Delta = \int_{\mathscr{Y}/\Delta} \eta^{n+1}.$$

Lemma 4.1. *There exists a constant $C > 0$ such that the estimate*

$$\omega_\Delta \leq C \cdot \log^k(1/|s|) \frac{\sqrt{-1}}{2} ds \wedge \overline{ds} \tag{8}$$

holds near 0 in Δ.

Proof. Since the statement is local, we can assume that $\mathscr{Y}$ is the unit polycylinder $\Delta^{n+1} = \{(z_1, \ldots, z_{n+1})\} \subset \mathbb{C}^{n+1}$ with respect to suitable local coordinates, and $f(z_1, \ldots, z_{n+1}) = s = z_1 \cdot \ldots \cdot z_{k+1}$. Furthermore, there exists a constant $C_1 > 0$ such that $\eta^{n+1} \leq C_1 \cdot (\sqrt{-1}/2)^{n+1} dz_1 \wedge \overline{dz_1} \wedge \ldots \wedge dz^{z+1} \wedge \overline{dz^{n+1}}$. The fiber integral of the right hand side equals

$$C_1 (2\pi)^n \left(\int_{|s|}^1 \int_{|s|/r_2}^1 \cdots \int_{|s|/r_2 \cdot \ldots \cdot r_k}^1 \frac{dr_1 dr_2 \ldots dr_k}{r_1 \cdot r_2 \cdot \ldots \cdot r_k} \right) \frac{\sqrt{-1}}{2} ds \wedge \overline{ds}$$
$$\leq C \log^k(1/|s|) \frac{\sqrt{-1}}{2} ds \wedge \overline{ds}.$$

□

Proposition 4.2. *For any one dimensional, generically smooth, family of embedded varieties, after a finite base change the Petersson-Weil form satisfies an estimate of the type* (8).

In a sense the above Proposition is more precise that Theorem 3.4. However, by the Theorem, Lelong numbers vanish, without assuming any base change.

Observe that we already have the stronger statement of the existence of a continuous (plurisubharmonic) potential, which implies the vanishing of Lelong numbers for the Petersson-Weil form, before applying a finite base change.

Proof of the proposition. After s sequence of blow-ups of the total space, we are in a position to apply Lemma 4.1, where $\eta \geq 0$ stands for the pull-back of the Kähler form on the ambient space. □

5. Chern Classes of Determinant Bundles and Extended Petersson-Weil Class

In this section, we state consequences of the generalized Grothendieck-Hirzebruch-Riemann-Roch theorem. For any holomorphic proper holomorphic map $f : \mathscr{X} \to S$ of complex spaces $f_! = \underline{\underline{R}} f_* : K_0^{\mathrm{hol}}(\mathscr{X}) \to K_0^{\mathrm{hol}}(S)$ denotes the direct image functor in the derived category (extended to the Grothendieck group).

Our basic situation is the following.

Assumption 5.1. *Let*

(9)
$$\begin{tikzcd} \mathscr{X} \arrow[r, hook, "i"] \arrow[dr, "f"'] & Z \times S \arrow[d, "\mathrm{pr}_2"] \\ & S \end{tikzcd}$$

be a proper, flat family, where S is a locally irreducible, reduced complex space, and the restriction to the complement S^0 of an analytically thin subset in S is smooth.

Let $\mathscr{L}$ be an invertible sheaf on $\mathscr{X}$. We denote by $\mathscr{F}$ the virtual vector bundle $(\mathscr{L} - \mathscr{L}^{-1})^{n+1}$ (of rank zero), where n is the dimension of the fibers of f, and by $\lambda = \det f_! \mathscr{F}$ the determinant line bundle on S.

Let $\nu : \widetilde{S} \to S$ be a desingularization of the base space, and $\mu : \mathscr{Y} \to \widetilde{\mathscr{X}} = \mathscr{X} \times_S \widetilde{S}$ a desingularization of the pull-back of the given family. (Both desingularizations are achieved by a finite sequence of blow-ups of smooth local ambient spaces with smooth centers). Then we have a commutative diagram

(10)
$$\begin{tikzcd} \mathscr{Y} \arrow[r, "\mu"] \arrow[rr, bend left, "\widetilde{\mu}"] \arrow[dr, "\varphi"'] & \mathscr{X} \times_S \widetilde{S} \arrow[r, "\widetilde{\nu}"] \arrow[d, "\widetilde{f}"] & \mathscr{X} \arrow[d, "f"] \\ & \widetilde{S} \arrow[r, "\nu"] & S \end{tikzcd}$$

where $\widetilde{\nu}$ and $\widetilde{f}$ are the canonical projections, $\varphi = \widetilde{f} \circ \mu$ and $\widetilde{\mu} = \widetilde{\nu} \circ \mu$.

Set $\widetilde{\mathscr{L}} = \widetilde{\nu}^* \mathscr{L}$ and $\widetilde{\mathscr{F}} = \widetilde{\nu}^* \mathscr{F}$. Let $\mathscr{L}'$ stand for any power $\mathscr{L}^k$, $k \in \mathbb{Z}$. Then $\underline{\underline{R}}\varphi_*(\widetilde{\mu}^* \mathscr{L}')$ does not depend upon the choice of the desingularization $\mathscr{Y}$: This follows from the fact that any two desingularizations are dominated by a third one, and since for any blow-up $\sigma : M \to N$ of manifolds with smooth center and any invertible sheaf $\mathscr{L}''$ on N the equality $\underline{\underline{R}}\sigma_*(\sigma^* \mathscr{L}'') = \mathscr{L}''$ holds (cf. [EL]). (Observe that $\underline{\underline{R}}\varphi_*(\widetilde{\mu}^* \mathscr{F})$ can also be described in terms of $\mathscr{L}$ and the dualizing complex on $\mathscr{X}$).

We consider the line bundles

$$\begin{aligned} \widetilde{\lambda}_d &= \det \underline{\underline{R}}\varphi_*(\widetilde{\mu}^*(\mathscr{F})), \\ \widetilde{\lambda} &= \det \underline{\underline{R}}\widetilde{f}_* \widetilde{\mathscr{F}} = \nu^* \lambda \end{aligned}$$

on $\widetilde{S}$. Let A be the locus of all points in S where f is not smooth.

Proposition 5.2. *There exists an effective divisor $\widetilde{D}$ on $\widetilde{S}$, whose support is contained in $\nu^{-1}A$, such that*

(11)
$$\nu^* \lambda = \widetilde{\lambda}_d(-\widetilde{D}).$$

Proof. The canonical morphism $\widetilde{\mathscr{F}} \to \underline{\underline{R}}\mu_*(\mu^*\widetilde{\mathscr{F}})$ defines a morphism $\widetilde{\lambda} \to \widetilde{\lambda}_d$, which is an isomorphism of invertible sheaves at all points of $\widetilde{S} \setminus \nu^{-1}A$. □

Assumption 5.3. *Assume that $Z \subset \mathbb{P}_N$ is a projective variety, equipped with the Fubini-Study form ω_Z, and that $\mathscr{L} = \mathscr{O}_{\mathscr{X}}(1)$ is the pull-back of the hyperplane section bundle on the projective space $\mathbb{P}_N$, equipped with the natural Fubini-Study hermitian metric h_{FS}. In particular, $\omega_{\mathscr{X}} = c_1(\mathscr{L}, h_{FS})$.*

Now we are in a position to apply the Hirzebruch-Grothendieck-Riemann-Roch-Theorem for proper holomorphic maps of complex manifolds by O'Brian, Toledo, and Tong [O-T-T1, O-T-T2] to (9) (cf. [L]).

Theorem 5.4. *The first (real) Chern class of $\widetilde{\lambda}_d$ equals*

$$c_1(\widetilde{\lambda}_d) = 2^{n+1}\widetilde{f}_* c_1(\widetilde{\mathscr{L}})^{n+1} = 2^{n+1}[\nu^*\omega_S^{PW}] \in H^1(\widetilde{S}, \Omega^1_{\widetilde{S}}), \tag{12}$$

where ω_S^{PW} denotes the extended Petersson-Weil form.

Proof. According to the main theorem of [O-T-T1, O-T-T2], we have

$$\mathrm{ch}(\underline{\underline{R}}\varphi_*(\mu^*\widetilde{\mathscr{F}})) = \varphi_*\Big(\mathrm{ch}(\mu^*\widetilde{\mathscr{F}})\mathrm{td}(\mathscr{Y}/S)\Big),$$

where td and ch denote the Todd and Chern characters respectively.

Let $k = \dim S$, and thus $k + n = \dim \mathscr{Y}$. We regard the push forward morphism of Hodge cohomology groups $\varphi_* : H^{n+\ell}(\mathscr{Y}, \Omega^{n+\ell}_{\mathscr{Y}}) \to H^\ell(\widetilde{S}, \Omega^\ell_{\widetilde{S}})$, for $\ell \in \mathbb{Z}$, which will be interpreted as follows: Let $j_\varphi : \mathscr{Y} \to \mathscr{Y} \times \widetilde{S}$ be the embedding onto the graph of φ, and $\pi : \mathscr{Y} \times \widetilde{S} \to \widetilde{S}$ be the projection. Then $\varphi_* = \pi_* \circ (j_\varphi)_*$. Here, for any ℓ,

$$(j_\varphi)_* : H^{n+\ell}(\mathscr{Y}, \Omega^{n+\ell}_{\mathscr{Y}}) \to H^{n+\ell+k}(\mathscr{Y} \times \widetilde{S}, \Omega^{n+\ell+k}_{\mathscr{Y}\times\widetilde{S}})$$

is the Gysin morphism, which we interpret analytically as given by the push-forward of currents, and

$$\pi_* : H^{n+\ell+k}(\mathscr{Y} \times \widetilde{S}, \Omega^{n+\ell+k}_{\mathscr{Y}\times\widetilde{S}}) \to H^\ell(\widetilde{S}, \Omega^\ell_{\widetilde{S}})$$

is given by fiber integration, which in terms of currents is again the push-forward.

As the degree of the virtual bundle $\mu^*(\widetilde{\mathscr{L}} - \widetilde{\mathscr{L}}^{-1})^{n+1}$ equals zero, the lowest order term of its Chern character is $2^{n+1}c_1(\mu^*\widetilde{\mathscr{L}})^{n+1}$. Since no higher terms contribute to the term in degree two,

$$c_{1,\mathbb{R}}(\widetilde{\lambda}_d) = c_{1,\mathbb{R}}(\underline{\underline{R}}\varphi_*(\mu^*(\widetilde{\mathscr{L}} - \widetilde{\mathscr{L}}^{-1})^{n+1})) = 2^{n+1}\varphi_* c_1(\mu^*\widetilde{\mathscr{L}})^{n+1}. \tag{13}$$

We observe that $\widetilde{f}_*(c_1(\widetilde{\mathscr{L}})^{n+1})$ (due to functoriality) is represented by the extended Petersson-Weil current $\nu^*\omega_S^{PW} = \omega_{\widetilde{S}}^{PW}$ according to Theorem 3.4.

Let $\eta = c_1(\widetilde{\mathscr{L}}, h_{FS})$. Since $\widetilde{\mathscr{X}}$ is allowed to be singular, we need to interpret (13) in the sense of currents: Let $\chi \in \mathscr{A}_0^{k-1,k-1}(\widetilde{S}, \mathbb{R})$ be a differential form with compact support. Then, by definition, $\int_{\widetilde{S}} \varphi_*(\mu^*\eta^{n+1}) \wedge \chi$ equals $\int_{\mathscr{Y}} \mu^*\eta^{n+1} \wedge \varphi^*\chi$, which again by definition equals $\int_{\widetilde{\mathscr{X}}} \eta^{n+1} \wedge \widetilde{f}^*\chi = \int_{\widetilde{S}} \widetilde{f}_*\eta^{n+1} \wedge \chi$, i.e. $\varphi_* c_1(\widetilde{\mu}^*\mathscr{L}^{n+1}) = [\nu^*\omega_S^{PW}]$. $\square$

The existence of a continuous, plurisubharmonic local $\partial\overline{\partial}$-potential according to Theorem 3.4, i.e. the absence of residues, means that the null extension of the restriction of the Petersson-Weil current to the complement of an analytically thin subset equals the Petersson-Weil current.

6. Determinant Bundles and Positivity

We first recall some results on the Quillen metric on determinant bundles: According to Bismut, Gillet, and Soulé [BGS], the Grothendieck-Riemann-Roch theorem holds, in the case of a proper, smooth family $f : \mathscr{X} \to S$ over a smooth base space S, for distinguished differential forms in degree 2, rather than cohomology classes. Namely for those that are induced by the given Kähler metric on $\mathscr{X}$, a Hermitian metric h on a locally free sheaf $\mathscr{F}$ on one hand, and by the Quillen metric h^Q on the other hand:

$$c_1(\lambda, h^Q) = \left[\int_{\mathscr{X}/S} \mathrm{td}(\mathscr{X}/S, \omega_{\mathscr{X}}) \cdot \mathrm{ch}(\mathscr{F}, h)\right]^{(2)}. \tag{14}$$

For the generalization to reduced complex spaces (and smooth proper mappings) cf. [F-S1, Appendix].

In [BI2] Bismut extended the above theorem (14) to the class of nodal singular mappings $f : \mathscr{X} \to S$ of complex manifolds: With respect to suitable local coordinates,

$$f(z_1, \ldots, z_{k+1+n}) = (z_1 \cdot z_2, z_3, \ldots, z_{k+1}) = (s_1, \ldots, s_k). \tag{15}$$

The set of singularities of f is denoted by $\Sigma = \{s_1 = s_2 = 0\} \subset \mathscr{X}$ with $\Delta = \{s_1 = 0\} \subset S$. Then by [BI2], the Quillen metric extends from $S \setminus \Delta$ into Δ as a singular hermitian metric with an extra term of the form

$$-\frac{1}{2}\Big[\int_{\Sigma} \mathrm{td}(\Sigma) E(\nu) \mathrm{ch}(\mathscr{F})\Big]^{(0)} \delta_\Delta \tag{16}$$

on the right-hand side of (14). Here, δ_Δ denotes a delta distribution (i.e. the $(1,1)$-current associated to Δ), the characteristic class E is associated to the function $E(x) = (x - \sinh x)[2x(1 - \cosh x)]^{-1}$, and ν is the normal bundle of Σ.

Now let

$$
\begin{tikzcd}
\mathscr{X} \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{P}_M \times S \arrow[d, "\mathrm{pr}_2"] \\
& S
\end{tikzcd}
\;; \qquad M \geq 2 \tag{17}
$$

denote a family of embedded subvarieties with the above general assumption (9).

In our case, again $\mathscr{F}$ will stand for a virtual hermitian bundle of the form $((\mathscr{L}-\mathscr{L}^{-1})^{n+1}, h_{FS})$, where the hermitian metric h_{FS} is induced by the Fubini-Study hermitian metric on a very ample line bundle $\mathscr{L}$. We apply our previous argument for virtual bundles of rank 0. The contribution of the Chern character in (14) equals $2^{n+1}c_1(\mathscr{L})^{n+1}$, while the term (16) vanishes.

For families with at most nodal singularities of the type (15) we obtain

$$
c_1(\lambda, h^Q) = 2^{n+1} \cdot \omega_S^{PW}. \tag{18}
$$

The case of normal crossings divisors with more than two components through one point seems to be still open. We call the subset B of S of points whose fibers have this property the *non-nodal locus*. We consider desingularizations like in (10). The non-nodal locus of the family $\widetilde{f} : \widetilde{\mathscr{X}} \to \widetilde{S}$ is $\widetilde{B} = \nu^{-1}B$.

The construction of both the determinant line bundle and the Quillen metric is functorial, i.e. compatible with base change. In particular h^Q is defined for any point of S where f is non-nodal, and continuity of h^Q on $S \setminus B$ follows from the base change property and its continuity on $\widetilde{S}$.

The generalized Petersson-Weil form is defined for the whole base of any embedded family through the fiber integral (5), and as such it is compatible with base change.

Theorem 6.1. *Let S be a connected component of the Douady space, containing at least one point corresponding to a smooth fiber, with universal family*

$$
\begin{tikzcd}
\mathscr{X} \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{P}_M \times S \arrow[d, "\mathrm{pr}_2"] \\
& S
\end{tikzcd}
\;.
$$

Set $\mathscr{L} = \mathscr{O}_{\mathscr{X}}(1)$.

Assume that the codimension of the non-nodal locus $\widetilde{B} \subset \widetilde{S}$ is at least two. Then the Quillen metric h^Q on the determinant line bundle $\lambda = \det f_!((\mathscr{L}-\mathscr{L}^{-1})^{n+1})$ extends as a continuous metric from $S \setminus B$ to S, and its Chern form $c_1(\lambda, h^Q)$ is equal to a constant multiple of the extended Petersson-Weil form ω^{PW} on S.

The extended Petersson-Weil form is strongly positive on the complement of the non-reduced locus by Theorem 3.4.

Remark 6.2. *The condition on the codimension of the non-nodal locus is always satisfied for universal families of plane curves.*

Proof of Theorem 6.1. Let $\nu : \widetilde{S} \to S$ be a desingularization and $\widetilde{f} : \widetilde{X} \to \widetilde{S}$ be the pull-back. Let U be an open subset of S such that $\omega_S^{PW}|U = \sqrt{-1}\partial\overline{\partial}u$ for some continuous function u. Set $\widetilde{U} = \nu^{-1}U$. Now $\widetilde{\lambda}|(\widetilde{U}\setminus\widetilde{B})$ carries the singular hermitian metric $e^{-u\circ\nu}\cdot(h^Q|(\widetilde{U}\setminus\widetilde{B}))$, whose curvature form vanishes. As $\widetilde{\lambda}|\widetilde{U}$ is *trivial*, we can say that $\log h^Q - u\circ\nu$ is harmonic on $\widetilde{U}\setminus\widetilde{B}$, and hence extendable to a harmonic function on $\widetilde{U}$. So the Quillen metric h^Q possesses a continuous extension to $\widetilde{S}$ which is the pull-back of a continuous metric on λ over S, which we denote again by h^Q. By definition, we have a singular, positive metric on the reduced complex space, and by continuity of h^Q it also follows that its curvature form is a constant multiple of the extended Petersson-Weil form, which was defined as a fiber integral. □

Proof of Remark 6.2. The statement about nodal singularities for families of plane curves is a simple count of dimensions: In the affine situation, we are given a curve $C = V(F(w_0, w_1))$ with $F(w_0, w_1) = \sum a_{j,k} w_0^j w_1^k$. If the line $L = V(w_0)$ is a double tangent to C in $(0, 0)$, we must have $a_{0,0} = a_{1,0} = a_{0,1} = a_{0,2} = a_{1,1} = 0$ so that the codimension of the space of curves passing through a given point with (at least) a double tangent in a given direction is five. Now the isotropy subgroup of the group of affine transformations that fixes a point together with a line through the point is of codimension three, so that the space of curves with some double tangent at some point is of codimension two. The cases of a simple triple point etc. are handled in a similar way. □

In the situation of Theorem 6.1 the base space S is quasi-projective. By Hironaka's flattening theorem there exists an extension

$$
\begin{tikzcd}
\overline{\mathscr{X}} \arrow[r, "\overline{i}"] \arrow[dr, "\overline{f}"'] & \mathbb{P}_M \times \overline{S} \arrow[d, "\mathrm{pr}_2"] \\
& \overline{S}
\end{tikzcd}
$$

of the universal family for some compactification $\overline{S}$ of S, where $\overline{f}$ is flat, but $\overline{i}$ need not be an embedding. Denote by $\overline{\lambda}$ the extended determinant line bundle. The fiber integral formula (5) yields an extension $\omega_{\overline{S}}^{PW}$ of the Petersson-Weil form, which is in general only positive semi-definite. After a suitable choice of $\overline{S}$ and $\overline{\lambda}$ the following holds under the assumptions of Theorem 6.1.

Theorem 6.3. *There exists a power of $\overline{\lambda}$, whose global sections define a rational map from $\overline{S}$ to some projective space, which yields an embedding, when restricted to S.*

We indicate an analytic proof. Basically, the Quillen metric is extended as a continuous, singular hermitian metric on an extension of λ; its curvature form is in general only semi-positive at the boundary. The construction follows from the extension of the Chern class, and may involve further blow ups of the boundary. Finally we are in a position to apply [S-TS, Theorem 6]. Note that this theorem is not affected by the error in the first part of [S-TS] (vanishing of Lelong numbers). The sections of λ can be considered as sections of λ over S that are square integrable with respect to the Quillen metric. □

7. General Case

With no assumption on the generic singularities of the embedded families, we need to deal with the contribution of the divisor that was introduced in Proposition 5.2. Our aim is to combine the above statement with the statement of (12).

We will need the following well-known fact.

Let $\widetilde{S}$ be a complex manifold, and $\cup_j D_j \subset \widetilde{S}$ a normal crossings divisor. Set $S' = \widetilde{S} \setminus \cup_j D_j$. Let h' be a hermitian metric of class C^∞ on the trivial line bundle over S'.

Lemma 7.1. *There exist numbers $a_j \in \mathbb{R}$, and a flat line bundle $\gamma \in H^1(\widetilde{S}, U(1))$, whose restriction to S' is trivial, such that after applying an automorphism of the trivial bundle over S', the metric h' extends to $\widetilde{S}$ as a singular hermitian metric on the generalized line bundle associated to the $\mathbb{R}$-divisor $\sum a_j D_j$, tensorized with γ. The curvature current of h' equals the current induced by D.*

Proof. Since the local monodromy is abelian, we may assume that D consists of one smooth component. Let $\widetilde{S} = \cup_\alpha U_\alpha$ be a suitable open covering such that $-\log h'|U'_\alpha = 2\,\mathrm{Re}(a \cdot \log(z_1^{(\alpha)}) + f_\alpha(z))$, where $U'_\alpha = U_\alpha \cap S'$, $a \in \mathbb{R}$, $z_1^{(\alpha)}$ is a coordinate function with $V(z_1^{(\alpha)}) = D \cap U_\alpha$ (or equal to 1), and $f_\alpha \in \mathscr{O}_{\widetilde{S}}(U'_\alpha)$. If $a = 0$, the functions $\exp(f_\alpha - f_\beta)$ extend to S as an $U(1)$-cocycle. Let $a \neq 0$. Then, again we find a $U(1)$-valued 1-cocycle γ such that $z_1^{(\alpha)}/z_1^{(\beta)} = \exp((f_\alpha - f_\beta)/a) \cdot \gamma_{\alpha\beta}$. □

We put ourselves in our general situation of (9) and use the desingularization procedure (10). In particular $\nu : \widetilde{S} \to S$ denotes a desingularization of the base space.

Theorem 7.2. *Let $\widetilde{\lambda} = 2^{n+1}\nu^*(\det f_!((\mathscr{L} - \mathscr{L}^{-1})^{n+1}))$. Then there exists a divisor D on $\widetilde{S}$, whose support is contained in the union of the singular locus $\widetilde{A}$ of the family $\widetilde{f}$, and the exceptional divisor of the* *desingularization, and a flat line bundle* $\gamma \in H^1(\widetilde{S}, U(1))$ *together with a singular hermitian metric* $\widetilde{h}$ *on* $\widetilde{\lambda}(D) \otimes \gamma$*, whose Chern form equals the generalized Petersson-Weil form. The generalized Petersson-Weil form on* S *is strictly positive on the reduced locus of the family.*

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