KÄHLER STRUCTURE ON MODULI SPACES OF PRINCIPAL BUNDLES

INDRANIL BISWAS AND GEORG SCHUMACHER

Abstract. Let $\mathscr{M}$ be a moduli space of stable principal G-bundles over a compact Kähler manifold (X, ω_X), where G is a reductive linear algebraic group defined over $\mathbb{C}$. Using the existence and uniqueness of a Hermite-Einstein connection on any stable G-bundle P over X, we have a Hermitian form on the harmonic representatives of $H^1(X, \mathrm{ad}(P))$, where $\mathrm{ad}(P)$ is the adjoint vector bundle. Using this Hermitian form a Hermitian structure on $\mathscr{M}$ is constructed; we call this the Petersson-Weil form. The Petersson-Weil form is a Kähler form, a fact which is a consequence of a fiber integral formula that we prove here. The curvature of the Petersson-Weil Kähler form is computed. Some further properties of this Kähler form are investigated.

1. Introduction

The Moduli space of polystable principal bundles of fixed topological type over a compact Riemann surface was constructed by A. Ramanathan in his thesis [RA1, RA2]. Recently T. Gómez and I. Sols constructed the moduli space of polystable principal bundles of fixed topological type over a smooth projective variety by extending the method of Ramanathan [G-S].

Our aim is to investigate differential geometric properties of a moduli space of principal bundles.

We fix a compact connected Kähler manifold X, which is equipped with a Kähler form ω_X. We also fix a reductive linear algebraic group G defined over the field of complex numbers, together with a maximal compact subgroup $K \subset G$.

It is known that a holomorphic principal G-bundle P over X is polystable, if and only if P admits a Hermite-Einstein connection, and furthermore, a polystable principal G-bundle admits a unique Hermite-Einstein connection [A-B]. Also, the connected component, containing the identity element, of the group of all holomorphic automorphisms of a stable G-bundle coincides with the connected component, containing the identity element, of the center of G (Proposition 2.6), and the unique Hermite-Einstein connections depend in a differentiable way upon the parameter in a holomorphic family. From these facts the existence of a moduli space of stable bundles, even in the category of not necessarily reduced complex spaces, follows immediately [S]. Using the above two facts it is possible to construct a moduli space of Hermite-Einstein connections, and a moduli space of polystable G-bundles is naturally identified with a moduli space of Hermite-Einstein connections.

Given a polystable principal G-bundle P over X, using the Hermite-Einstein connection on P, we construct a Hermitian structure on the space of all infinitesimal deformations of P. Recall that the space of infinitesimal deformations of P is parameterized by $H^1(X, \mathrm{ad}(P))$, where $\mathrm{ad}(P)$ is the adjoint vector bundle of P. To construct the above mentioned Hermitian form on $H^1(X, \mathrm{ad}(P))$, we fix a Hermitian structure on the complex Lie algebra $\mathfrak{g}$ of G which is invariant under the adjoint action of K (the maximal compact subgroup) on $\mathfrak{g}$. Using this and the Kähler form ω_X, the space of all $\mathrm{ad}(P)$-valued differential forms on X gets a Hermitian structure. More precisely, for any

$$\alpha, \beta \in H^1(X, \mathrm{ad}(P))$$

their Hermitian pairing is defined to be

$$\int_X \langle \widehat{\alpha}, \widehat{\beta} \rangle \frac{\omega_X^{n-1}}{(n-1)!}\,,$$

where $\widehat{\alpha}, \widehat{\beta} \in C^\infty(X, \Omega_X^{(1,1)} \otimes \mathrm{ad}(P))$ are the harmonic representatives of α and β respectively.

The above defined Hermitian form on infinitesimal deformations of polystable bundles give a Hermitian structure on any moduli space of principal G-bundles. This Hermitian form on the moduli space turns out to be a Kähler form, generalizing the Petersson-Weil form on the moduli spaces of vector bundles.

We calculate the curvature of this Kähler form on a moduli space of polystable principal bundles. If X is of dimension one, then the holomorphic sectional curvature turns out to be non-negative.

Under the assumption that X is a projective manifold, we construct a holomorphic Hermitian line bundle over the moduli space, whose curvature form coincides with the generalized Petersson-Weil form. This construction of a Hermitian line bundle is based on the construction of D. Quillen of a determinant bundle.

Finally, we study the embedding of the moduli space of stable principal bundles into a moduli space of vector bundles, which turns out to be totally geodesic.

2. Preliminaries

We consider principal G-bundles over a compact Kähler manifold (X, ω_X), where G denotes a complex reductive linear algebraic group. If $(z^1, \ldots, z^n)$ is a locally defined holomorphic coordinate system on X, then we will use the expression

$$\omega_X = \sqrt{-1}\, g_{\alpha\overline{\beta}}\, dz^\alpha \wedge dz^{\overline{\beta}}$$

for the Kähler form.

Take a holomorphic principal G-bundle P over X. Let

$$\pi : P \to X$$

denote the bundle projection.

Definition 2.1. A holomorphic principal G-bundle P over X is called *stable* (respectively, *semistable*), if the following holds: Take any triple (Q, U, σ), where $Q \subset P$ is a maximal proper parabolic subgroup, $U \subset X$ is the complement of an analytic (possibly empty) subset of codimension at least two, and $\sigma : U \to P/Q$ is a holomorphic reduction of the structure group on U. Then $\deg \sigma^* T_{\mathrm{rel}}$ is positive (respectively, semi-positive), where T_{rel} denotes the relative tangent bundle for the projection $P/Q \to X$.

Let P be any holomorphic G-bundle over X, and $\widetilde{Q} \subset G$ some parabolic subgroup. A holomorphic reduction of structure group $\widetilde{P} \subset P$ of P to $\widetilde{Q}$ will be called *admissible*, if for any character $\chi : \widetilde{Q} \to \mathbb{C}^*$ which is trivial on the center of G, the line bundle $\widetilde{P} \times_{\widetilde{Q}} \mathbb{C}_\chi$ over X associated to $\widetilde{P}$ for χ is of degree zero.

Let $R_u(\widetilde{Q}) \subset \widetilde{Q}$ be the unipotent radical. We recall that $R_u(\widetilde{Q})$ is the subgroup generated by unipotent elements of the unique largest connected normal solvable subgroup of $\widetilde{Q}$ (see [HU, page 125]). In particular, $R_u(\widetilde{Q})$ is a normal subgroup of $\widetilde{Q}$. The quotient $\widetilde{Q}/R_u(\widetilde{Q})$, which is also the maximal reductive quotient of $\widetilde{Q}$, is called the *Levi quotient*. If we fix a Borel subgroup of G contained in $\widetilde{Q}$ and also fix a maximal torus of the Borel subgroup, then there is a canonically defined subgroup of $\widetilde{Q}$ which projects isomorphically to the Levi quotient $\widetilde{Q}/R_u(\widetilde{Q})$. Such a subgroup of $\widetilde{Q}$ will be called a *Levi subgroup* of $\widetilde{Q}$. Any two Levi subgroups of $\widetilde{Q}$ are conjugate. We will denote by $L(\widetilde{Q})$ a Levi subgroup of $\widetilde{Q}$.

Definition 2.2. A principal G-bundle P is called *polystable*, if there is an admissible reduction of structure group $\widetilde{P} \subset P$ to a parabolic subgroup $\widetilde{Q} \subset G$, and a stable $L(\widetilde{Q})$-bundle $\widehat{P}$ such that $\widetilde{P} \cong \widehat{P} \times_{L(\widetilde{Q})} \widetilde{Q}$, or in other words, the principal $\widetilde{Q}$-bundle $\widetilde{P}$ is obtained by extending the structure group of $\widehat{P}$ using the inclusion of $L(\widetilde{Q})$ in $\widetilde{Q}$.

We will now recall the definition of a connection on a principal bundle.

Definition 2.3. (i) A *connection* on a smooth principal G-bundle $\pi : P \to X$ is a differentiable G-equivariant splitting θ of the differential

$$d\pi : T^{\mathbb{R}} P \to \pi^* T^{\mathbb{R}} X\,,$$

where $T^{\mathbb{R}} P$ and $T^{\mathbb{R}} X$ denote the real tangent bundles.

(ii) A *complex connection* on a holomorphic principal G-bundle $\pi : P \to X$ is a connection

$$\theta : \pi^* T^{\mathbb{R}} X \to T^{\mathbb{R}} P$$

(it is a splitting of $d\pi$), which commutes with the almost complex structures on $T^{\mathbb{R}}P$ and $T^{\mathbb{R}}X$. So giving a complex connection on P is equivalent to giving a differentiable G-equivariant splitting of $d\pi : TP \to \pi^*TX$, where TP and TX denote the holomorphic tangent bundles.

(ii) Let $P_K \subset P$ be a reduction of the structure group of a smooth G-bundle P to a maximal compact subgroup $K \subset G$. A connection on P that is induced by a connection on some such reduction P_K is called a *Hermitian connection*.

Observe that a connection on P_K that induces a given connection on P is uniquely determined by the latter.

Let $P_K \subset P$ be a reduction of the structure group of a holomorphic principal G-bundle P to a maximal compact subgroup $K \subset G$. Then there is a unique complex connection on P, which is induced by a connection on P_K.

In this situation, take any such connection θ on P (so θ is a Hermitian connection). Let $\rho : G \to \mathrm{GL}(V)$ be any representation, and let $K' \subset \mathrm{GL}(V)$ be a maximal compact subgroup containing the image $\rho(K)$. Let $E_{\mathrm{GL}(V)} = P \times_G \mathrm{GL}(V)$ (respectively, $E_{K'} = P_K \times_K K'$) be the principal $\mathrm{GL}(V)$-bundle (respectively, principal K'-bundle) over X obtained by extending the structure group of P (respectively, P_K) using the homomorphism ρ. Note that $E_{K'} \subset E_{\mathrm{GL}(V)}$, or in other words, $E_{K'}$ is a reduction of structure group of $E_{\mathrm{GL}(V)}$ to K'. The Hermitian connection θ on P induces a connection on $E_{\mathrm{GL}(V)}$ that comes from a connection on E_K. This provides the associated vector bundle $E = E_{\mathrm{GL}(V)} \times_{\mathrm{GL}(V)} V$ over X with a Hermitian connection. Using the above reduction of structure group $E_{K'} \subset E_{\mathrm{GL}(V)}$, the vector bundle E has a Hermitian structure. If P is a holomorphic G-bundle and θ the unique complex connection on P induced by P_K, then the above connection on E is the unique Hermitian connection on the holomorphic vector bundle E compatible with its holomorphic structure.

We will denote by $\mathfrak{g}$ the Lie algebra of G. The center of $\mathfrak{g}$ will be denoted by $\mathfrak{z}$. Consider the adjoint action of G on $\mathfrak{g}$. Since $\mathfrak{z} \subset \mathfrak{g}$ is a G-submodule, we get a subbundle

$$P \times_G \mathfrak{z} \subset P \times_G \mathfrak{g} =: \mathrm{ad}(P)$$

of the adjoint bundle $\mathrm{ad}(P)$. Since the adjoint action of G on $\mathfrak{z}$ is trivial, this subbundle is canonically identified with the trivial vector bundle $X \times \mathfrak{z}$ over X with fiber $\mathfrak{z}$.

Let θ be a Hermitian complex connection on P. The curvature of θ, which we will denote by Ω, is a $(1,1)$-form on X with values in $\mathrm{ad}(P)$.

Let Λ be the operator that is adjoint to the exterior multiplication of forms on X with the Kähler form ω_X. Therefore, $\Lambda\Omega$ is a C^∞-section of $\mathrm{ad}(P)$.

Definition 2.4. A Hermitian complex connection θ on a holomorphic principal G-bundle P will be called a *Hermite-Einstein connection* if there exists an element $\lambda \in \mathfrak{z}$ such that the section $\Lambda\Omega$ of $\mathrm{ad}(P)$, where Ω is the curvature of θ, has the constant value λ.

A principal G-bundle admits a Hermite-Einstein connection, if and only if it is polystable. Furthermore, a polystable G-bundle admits a unique Hermite-Einstein connection (cf. [R-S, A-B]).

Let $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, and $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ be the adjoint representations. Take a holomorphic principal G-bundle $P \to X$ together with the adjoint bundle $\mathrm{ad}(P) = P \times_G \mathfrak{g}$, and the corresponding endomorphism bundle $\mathrm{End}(\mathrm{ad}(P))$.

Lemma 2.5. [A-B, Corollary 3.8] *A holomorphic principal G-bundle P is polystable, if and only if* $\mathrm{ad}(P)$ *is polystable. The Hermite-Einstein connection on P induces the unique Hermite-Einstein connection on the vector bundle* $\mathrm{ad}(P)$.

Fix a Hermitian form on the Lie algebra $\mathfrak{g}$ which is left invariant by the adjoint action of the maximal compact subgroup $K \subset G$. Let P be a polystable G-bundle over X. Let $P_K \subset P$ be a reduction of structure group of P to K such that the Hermite-Einstein connection on P is induced by a connection on P_K. Since the adjoint vector bundle $\mathrm{ad}(P)$ is canonically identified with $P_K \times_K \mathfrak{g}$, the vector bundle over X associated to P_K for the adjoint action of K on $\mathfrak{g}$, the K-invariant Hermitian form on $\mathfrak{g}$ gives a Hermitian structure on the vector bundle $\mathrm{ad}(P)$. It follows immediately from Lemma 2.5 that this is a Hermite-Einstein metric on $\mathrm{ad}(P)$.

Let $Z^0(G) \subset G$ be the connected component of the center of G containing the identity element.

For a principal G-bundle P over X, let $\mathrm{Ad}(P) := P \times_G G$ be the fiber bundle over X associated to P for the adjoint action of G on itself. Each fiber of $\mathrm{Ad}(P)$ is a Lie group isomorphic to G. The space of all smooth (respectively, holomorphic) automorphisms of the G-bundle P is identified with the space of all smooth (respectively, holomorphic) sections of $\mathrm{Ad}(P)$. Since the adjoint action of G on itself preserves the subgroup $Z^0(G)$, we have a subbundle

$$P \times_G Z^0(G) \subset P \times_G G =: \mathrm{Ad}(P)$$

of the fiber bundle $\mathrm{Ad}(P)$. Since G acts trivially on $Z^0(G)$, this subbundle is canonically identified with the trivial fiber bundle $X \times Z^0(G)$ over X.

Let P be a holomorphic principal G-bundle over X. Let $\mathrm{Aut}^0(P)$ denote the connected component, containing the identity element, of the holomorphic automorphism group of P. from the above remarks it follows that $Z^0(G)$ is a subgroup of $\mathrm{Aut}^0(P)$.

Proposition 2.6. *If P is a stable principal G-bundle over X, then $Z^0(G)$ coincides with $\mathrm{Aut}^0(P)$.*

Proof. Let P be a stable principal G-bundle over X. The proposition is clearly equivalent to the assertion that $H^0(X, \mathrm{ad}(P)) = \mathfrak{z}$, where $\mathfrak{z}$, as before, is the center of $\mathfrak{g}$. To prove this assertion, take any holomorphic section

$$\phi \in H^0(X, \mathrm{ad}(P))\,.$$

Let θ denote the unique Hermite-Einstein connection on P. The connection on $\mathrm{ad}(P)$ induced by θ will be denoted by $\mathrm{ad}(\theta)$. We know that the vector bundle $\mathrm{ad}(P)$ is polystable, and furthermore, $\mathrm{ad}(\theta)$ is the Hermite-Einstein connection on $\mathrm{ad}(P)$ (see Lemma 2.5).

Since $\deg \mathrm{ad}(P) = 0$ (note that $\mathrm{ad}(P)^* \cong \mathrm{ad}(P)$ and $\mathfrak{g} \cong \mathfrak{g}^*$ as a G-module), and $\mathrm{ad}(\theta)$ is a Hermite-Einstein connection on $\mathrm{ad}(P)$, we conclude that the above section ϕ is flat with respect to $\mathrm{ad}(\theta)$. (Any holomorphic section of a vector bundle of degree zero equipped with a Hermite-Einstein connection is a flat section.)

Let

$$\phi = \phi_s + \phi_n$$

be the Jordan decomposition, where ϕ_s (respectively, ϕ_n) is a holomorphic section of $\mathrm{ad}(P)$ such that for each point $x \in X$ the evaluation $\phi_s(x)$ (respectively, $\phi_n(x)$) is a semisimple (respectively, nilpotent) element of the Lie algebra $\mathrm{ad}(P)_x$ ($\cong \mathfrak{g}$).

Since ϕ_s is a flat section with respect to the connection $\mathrm{ad}(\theta)$ on $\mathrm{ad}(P)$, if for some point $x_0 \in X$ the evaluation $\phi_s(x_0)$ lies in the subspace $\mathfrak{z} \subset \mathrm{ad}(P)_{x_0}$, then the section ϕ_s coincides everywhere with $\phi_s(x_0) \in \mathfrak{z}$ (the manifold X is connected). Hence, if the section ϕ_s is not given by a constant element of $\mathfrak{z}$, then $\phi_s(x) \in \mathrm{ad}(P)_x \setminus \mathfrak{z}$ for each point $x \in X$.

This implies that, if the section ϕ_s is not given by a constant element of $\mathfrak{z}$, then ϕ gives a holomorphic reduction of structure group of P to the Levi subgroup of some proper parabolic subgroup of G. The details of the construction of this reduction of structure group are given in [B-B-N]. It is easy to see from the definition of a stable principal bundle that a stable G-bundle does not admit reduction of structure group to a Levi subgroup of any proper parabolic subgroup of G. Therefore, we conclude that the section ϕ_s is given by an element of $\mathfrak{z}$.

To complete the proof of the proposition we need to show that $\phi_n = 0$.

Assume that $\phi_n \neq 0$. Take any nonzero element nilpotent element v' of the Lie algebra $\mathfrak{g}'$ of a reductive linear algebraic group G' defined over the field of complex numbers. This element v' canonically defines a proper parabolic subgroup of G' [HU, page 186, Corollary A].

We know that the section ϕ_n is flat with respect to the Hermite-Einstein connection $\text{ad}(\theta)$ on $\text{ad}(P)$, and the connection $\text{ad}(\theta)$ is induced by a connection on P (it is induced by θ). Therefore, the conjugacy class of $\phi_n(x)$, $x \in X$, is independent of x. To explain this, since the vector bundle $\text{ad}(P)$ is associated to the principal G-bundle P for the G-module $\mathfrak{g}$, for each point $x \in X$ the fiber $\text{ad}(P)_x$ is identified with $\mathfrak{g}$ up to a conjugation action of G; hence the element $\phi_n(x)$ defines an element in the space of all conjugacy classes of $\mathfrak{g}$. Flatness of ϕ_n implies that this element is independent of x.

Consequently, using the above mentioned construction of a parabolic subgroup from a nilpotent element of the Lie algebra we conclude that the section ϕ_n gives a reduction of structure group $P_Q \subset P$ of the G-bundle P to some proper parabolic subgroup $Q \subset G$, and furthermore, the connection θ on P is induced by a connection on the principal Q-bundle P_Q.

On the other hand, the connection θ on P is a Hermitian connection. If Q' is any parabolic subgroup of G and K' any maximal compact subgroup of G, then the intersection $Q' \cap K'$ is a maximal compact subgroup of a Levi subgroup of Q'. Therefore, the connection on P_Q which induces θ gives a holomorphic reduction of structure group of P_Q to a Levi subgroup of Q. Using again the fact that a stable G-bundle does not admit reduction of structure group to a Levi subgroup of any proper parabolic subgroup of G we get a contradiction.

Hence $\phi_n = 0$, and the proof of the proposition is complete. $\square$

3. Infinitesimal deformations and the generalized Petersson-Weil inner product

The existence of a semi-universal deformation of a holomorphic principal G-bundle $P \to X$ was shown in [PO, DO1, DO2]. To any holomorphic family $\mathscr{P} \to X \times S$ of principal G-bundles we associate the adjoint vector bundle $\text{ad}(\mathscr{P}) \to X \times S$. This amounts to a morphism of deformation functors. For any $s \in S$, and $P = \mathscr{P}|X \times \{s\}$, it induces a morphism

$$H^1(X, \text{ad}(P)) \to H^1(X, \text{End}(\text{ad}(P)))$$

of spaces of infinitesimal deformations. This homomorphism coincides with the one obtained from the homomorphism of vector bundles $\text{ad}(P) \to \text{End}(\text{ad}(P))$ given by the Lie algebra structure of the fibers of $\text{ad}(P)$.

Infinitesimal deformations of a principal bundle can be characterized in terms of complex Hermitian connections. We will describe below the Kodaira-Spencer map in terms of the Atiyah exact sequence.

Let (S, s_0) be a pointed complex analytic space, and

$$\widehat{\pi} : \mathscr{P} \to X \times S \tag{1}$$

a holomorphic principal G-bundle over $X \times S$. Denote by $AT(\mathscr{P})$ the sheaf on $X \times S$ defined by the space of G-invariant holomorphic vector fields on $\mathscr{P}$. So for each open subset $U \subset X \times S$ this sheaf associates the space of all G-invariant holomorphic vector fields on the inverse image $\widehat{\pi}^{-1}(U)$, where $\widehat{\pi}$ is the projection in (1). Since G acts transitively on the fibers of $\widehat{\pi}$, this sheaf is a locally free coherent analytic one. Therefore, it defines a holomorphic vector bundle over $X \times S$. Consequently, $AT(\mathscr{P})$ is also a holomorphic vector bundle over $X \times S$.

It is easy to see that the adjoint vector bundle $\mathrm{ad}(\mathscr{P})$ corresponds to the sheaf on $X \times S$ defined by the space of G-invariant vertical holomorphic vector fields on $\mathscr{P}$ (vertical with respect to the projection $\widehat{\pi}$). Therefore, $\mathrm{ad}(\mathscr{P})$ is canonically a holomorphic subbundle of $AT(\mathscr{P})$. In fact, we have an exact sequence of vector bundles

$$0 \longrightarrow \mathrm{ad}(\mathscr{P}) \longrightarrow AT(\mathscr{P}) \longrightarrow T(X \times S) \longrightarrow 0 \ ,$$

where the projection $AT(\mathscr{P}) \to T(X \times S)$ is defined by the differential $d\widehat{\pi}$.

Let $p : X \times S \to S$ be the canonical projection. We consider the diagram

(2)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{ad}(\mathscr{P}) & \longrightarrow & AT(\mathscr{P}) \times_{T(X\times S)} p^*TS & \longrightarrow & p^*TS & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{ad}(\mathscr{P}) & \longrightarrow & AT(\mathscr{P}) & \longrightarrow & T(X \times S) & \longrightarrow & 0
\end{array}$$

obtained from the above exact sequence. Let $P = \mathscr{P}|X \times \{s_0\}$. The Kodaira-Spencer map

$$\rho : T_{s_0} \to H^1(X, \mathrm{ad}(P)) \tag{3}$$

is induced by the connecting homomorphism in the long exact sequence of cohomologies for the top short exact sequence of sheaves in (2). More precisely, the short exact sequence of sheaves is restricted to $X \times \{s_0\}$, and then the connecting homomorphism is considered, to get the Kodaira-Spencer map in (3).

Any complex connection θ on $\mathscr{P}$ gives a C^∞-splitting

$$\sigma : p^*TS \to AT(\mathscr{P}) \times_{T(X\times S)} p^*TS$$

of the top exact sequence in (2). Given a tangent vector v of S at s_0, extended locally to a holomorphic vector field V, the form $\overline{\partial}(\sigma(V))|X \times \{s_0\}$ represents the element $\rho(v) \in H^1(X, \mathrm{ad}(P))$ in terms of Dolbeault cohomology. Since the projection $AT(\mathscr{P}) \times_{T(X\times S)} p^*TS \to p^*TS$ in (2) is holomorphic, the form $\overline{\partial}(\sigma(V))|X \times \{s_0\}$ is actually a $(0,1)$-form with values in $\mathrm{ad}(P)$. It is easy to see that this form coincides with the form given by the contraction $\Omega \llcorner v|X \times \{s_0\}$, where Ω is the curvature of the connection θ.

We put this down as the following proposition:

Proposition 3.1. *The following equality holds:*

$$\rho(v) = [\Omega \llcorner v] \in H^1(X, \mathrm{ad}(P)) ,$$

where ρ is the Kodaira-Spencer homomorphism in (3).

Now assume S to be smooth. Let $(s_1, \dots, s_r)$ denote local holomorphic coordinates on S around s_0. Then

$$\rho\left(\left.\frac{\partial}{\partial s^i}\right|_{s=s_0}\right) = \left[R_{i\overline{\beta}}\, dz^{\overline{\beta}}\right] ,$$

where R denotes the curvature tensor of Ω (the curvature of the connection θ).

We now assume the principal G-bundle $P \to X$ to be stable. According to Proposition 2.6, the group $\mathrm{Aut}^0(P)$ coincides with $Z^0(G)$. So in a holomorphic family of stable G-bundles, any automorphism of the central fiber P which lies in $\mathrm{Aut}^0(P)$ can be extended to the neighboring bundles. General deformation theory now implies that semi-universal deformations are universal. Since Hermite-Einstein connections are unique and depend in a C^∞ way on the parameter, the induced analytic equivalence relation on the base spaces of universal deformations is proper. This fact implies the existence of a moduli space for stable principal G-bundles over a Kähler manifold [S]. We denote by $\mathscr{M}_s$ a connected component of a moduli space of stable G-bundles over X. This construction of $\mathscr{M}_s$ from local universal families $\mathscr{P} \to X \times S$ yields a global adjoint bundle on $X \times \mathscr{M}_s$ in the orbifold sense, which comes from the adjoint bundles of local families. For short, we denote the adjoint bundle on $X \times \mathscr{M}_s$ by $\mathrm{ad}(\mathscr{P})$. Since bundles and metrics will descend to $X \times \mathscr{M}_s$ in the orbifold sense (from local universal families), it will be sufficient to consider local families of stable G-bundles $\mathscr{P} \to X \times S$.

We recall that the Hermite-Einstein condition reads

$$\Lambda\Omega_s = c \in \mathfrak{z} \hookrightarrow C^\infty(X \times \{s\}, \mathrm{ad}(\mathscr{P}|X \times \{s\})). \tag{4}$$

Here the element $c \in \mathfrak{z}$ is a topological constant, or in other words, it remains fixed for all $s \in S$, provided S is connected. The vector bundles $\mathrm{ad}(\mathscr{P})|X \times \{s\}$ carry Hermite-Einstein metrics, whose curvature forms $\mathrm{ad}(\Omega_s)$ satisfy

$$\Lambda\, \mathrm{ad}(\Omega_s) = 0 .$$

Since

$$\Lambda\Omega_s = g^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}(z, s) ,$$

the above condition (4) implies that

$$- g^{\alpha\overline{\beta}} R_{i\overline{\beta};\alpha} = -g^{\alpha\overline{\beta}} R_{\alpha\overline{\beta};i} = 0 . \tag{5}$$

We use here the semi-colon notation for covariant derivatives with respect to the Hermite-Einstein metric on the vector bundle $\mathrm{ad}(\mathscr{P})$ and the Kähler structure on $X \times S$, flat in S-direction. The connection form θ and the curvature tensor R have values in the vector bundle $\mathrm{ad}(\mathscr{P})$. However, when we consider the induced Hermite-Einstein structure on the vector bundle $\mathrm{ad}(\mathscr{P})$, these act as $\mathrm{ad}(\theta)$ and $\mathrm{ad}(R)$ on differential forms with values in $\mathrm{End}(\mathrm{ad}(\mathscr{P}))$.

The decomposition $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ of the Lie algebra gives a decomposition of the adjoint bundle

$$\mathrm{ad}(\mathscr{P}) = (\mathscr{P} \times_G \mathfrak{z}) \oplus [\mathrm{ad}(\mathscr{P}), \mathrm{ad}(\mathscr{P})] \, .$$

In terms of Hermitian metrics on the fibers of $\mathrm{ad}(\mathscr{P})$ the above is an orthogonal decomposition. Since $\mathscr{P} \times_G \mathfrak{z}$ is trivial, and since the action of G on $\mathfrak{z}$ is trivial, the induced connection on $\mathscr{P} \times_G \mathfrak{z}$ is trivial.

From (4) and Proposition 3.1 we infer

Proposition 3.2. *The form*

$$\eta_i = R_{i\overline{\beta}}\, dz^{\overline{\beta}} \in \mathscr{A}^{0,1}(X \times \{s\}, \mathrm{ad}(\mathscr{P})|_{X \times \{s\}})$$

is the harmonic *representative of the Dolbeault cohomology class*

$$\rho\left(\frac{\partial}{\partial s^i}\bigg|_{s=s_0} \right) \in H^1(X, \mathrm{ad}(P))$$

where ρ is the homomorphism in (3).

Remark 3.3. Like in the case of Hermite-Einstein vector bundles, the above proposition includes the case where S is singular and possibly non-reduced.

In order to prove the assertion made in the above remark, we make the following observation: In a holomorphic family of stable, holomorphic vector bundles over a smooth base space S, a Hermite-Einstein connection on the central fiber can be extended uniquely to the neighboring fibers using the implicit function theorem on appropriate function spaces. If $S \subset U = \{(s_1, \ldots, s_N)\} \subset \mathbb{C}^N$ is a singular, closed subspace of some open subset of a complex number space, one extends the differential operators, which are defined on the underlying differentiable complex vector bundle, in some C^∞ way to the neighboring fibers over points of U, applies the implicit function theorem, and restricts the result to S afterwards by applying the integrability conditions for holomorphic structures. In this way Proposition 3.2 becomes valid for general S.

On the vector bundle $\mathrm{ad}(\mathscr{P})$ we denote the inner product by parentheses.

Definition 3.4. Let $\mathscr{P} \to X \times S$ be a local, universal family of stable vector bundles over a general complex space $S \subset U = \{(s_i, \ldots, s_N)\}$, equipped with a family of Hermite-Einstein metrics. Then the *Petersson-Weil inner product* is defined by

$$G^{PW}_{i\bar{\jmath}} = G^{PW}\left(\frac{\partial}{\partial s_i}\bigg|_s, \frac{\partial}{\partial s_{\bar{\jmath}}}\bigg|_s\right) = \int_{X\times\{s\}} g^{\alpha\overline{\beta}}(R_{i\overline{\beta}}, R_{\alpha\bar{\jmath}})g\, dV,$$

and

$$\omega^{PW} = \sqrt{-1}G^{PW}_{i\bar{\jmath}}(s)ds^i \wedge ds^{\bar{\jmath}}$$

is the corresponding Hermitian form.

Once the generalized Petersson-Weil metric is set up in the above way, the proof of a fiber integral formula, the curvature formula, and the construction of a determinant line bundle with a Quillen metric, whose curvature form equals ω^{PW} up to a numerical constant, follow like in the case of holomorphic vector bundles.

Let $\Omega \in \mathscr{A}^{1,1}(X \times S, \mathrm{ad}(\mathscr{P}))$ be the curvature form given by the family of Hermite-Einstein connections. We use the concept of a fiber integral $\int_{\mathscr{Y}/S} \tau$ of a differential form τ over the fibers of a family $f : \mathscr{Y} \to S$ of compact $2n$-dimensional fibers, resulting in a differential form of degree $m - 2n$, where m is the degree of τ. In our case the family is the trivial family $f : X \times S \to S$. We will write $\int_X \tau$ instead of $\int_{(X\times S)/S} \tau$.

Let $c \in \mathfrak{z}$ be the topological constant from (4).

Proposition 3.5. *The generalized Petersson-Weil form on a complex space S equals*

$$\omega^{PW} = \frac{1}{2}\int_X (\Omega \wedge \Omega) \wedge \frac{\omega_X^{n-1}}{(n-1)!} + c\int_X \sqrt{-1}\Omega \wedge \frac{\omega_X^n}{n!}, \tag{6}$$

in particular, it is Kähler, and possesses locally a $\partial\overline{\partial}$-potential.

Proof. We have

$$\begin{aligned}\frac{1}{2}\int_X (\Omega \wedge \Omega) \wedge \frac{\omega_X^{n-1}}{(n-1)!} &= -\frac{1}{2}\int_X (\sqrt{-1}\Omega \wedge \sqrt{-1}\Omega) \wedge \frac{\omega_X^{n-1}}{(n-1)!}\\ &= \int_X \big((R_{\alpha\bar{\jmath}}R_{i\overline{\beta}}) - (R_{\alpha\overline{\beta}}R_{i\bar{\jmath}})\big)\, g^{\alpha\overline{\beta}} g\, dV \sqrt{-1}ds^i \wedge ds^{\bar{\jmath}}.\end{aligned}$$

Now $g^{\alpha\overline{\beta}}R_{\alpha\overline{\beta}}$ is the image of the fixed element c of the center $\mathfrak{z} \subset \mathfrak{g}$ so that the corresponding term is cancelled by the second integral. □

Next, we interpret $\mathrm{ad}(\Omega)$ as the curvature form for the Hermite-Einstein bundle $\mathrm{ad}(\mathscr{P})$. The orthogonal decomposition $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}$ as G-modules gives rise to a decomposition

$$\mathrm{ad}(\mathscr{P}) = (\mathscr{P} \times_G [\mathfrak{g}, \mathfrak{g}]) \oplus (X \times \mathfrak{z}).$$

From this decomposition we obtain a decomposition of the tangent spaces of the moduli space

$$H^1(X, \mathrm{ad}(\mathscr{P})) = H^1(X, \mathscr{P} \times_G [\mathfrak{g}, \mathfrak{g}]) \oplus H^1(X, \mathscr{O}_X) \otimes \mathfrak{z}$$

which is orthogonal with respect to the Petersson-Weil metric. Note that $\mathscr{P} \times_G [\mathfrak{g}, \mathfrak{g}] = [\mathrm{ad}(\mathscr{P}), \mathrm{ad}(\mathscr{P})]$. Accordingly the Petersson-Weil form decomposes into

$$\omega^{PW} = \omega'_{PW} + \omega^{\mathfrak{z}}_{PW} \,.$$

It follows immediately that $\omega^{\mathfrak{z}}_{PW}$ is flat.

Concerning the moduli spaces, up to taking finite quotients, we have an orthogonal decomposition into a product of Jacobians and a "trace-free part". The latter means that we are looking at the semisimple quotient

$$G' := G/Z^0(G)$$

of G and the principal G'-bundles obtained by extending the structure group of principal G-bundles using the quotient homomorphism $G \rightarrow G'$. Note that the group $Z^0(G)$ is a product of copies of $\mathbb{G}_m = \mathbb{C}^*$. So the description of $\omega^{\mathfrak{z}}_{PW}$ as a curvature form of a determinant line bundle is contained in the study of holomorphic line bundles. Hence from here on, we can restrict ourselves to the case where G itself is semisimple.

Let G be a semisimple linear algebraic group, defined over the field of complex numbers, and let $\mathscr{P} \rightarrow X \times S$ be a universal family of stable principal G-bundles, equipped with the family of Hermite-Einstein connections. The curvature of the connection on $\mathscr{P}$ will be denoted by Ω. Then $\mathrm{ad}(\mathscr{P})$ is (relative) Hermite-Einstein with curvature form $\mathrm{ad}(\Omega)$.

Let $\mathrm{Ad} : \mathfrak{g} \rightarrow \mathrm{End}(\mathfrak{g})$ be the linear representation defined by the Lie algebra structure of $\mathfrak{g}$. Since G is now semisimple, this homomorphism Ad is injective. We may normalize the K-invariant Hermitian form on $\mathfrak{g}$ so that the homomorphism Ad takes the K-invariant Hermitian form on $\mathfrak{g}$ to the Hermitian form on $\mathrm{End}(\mathfrak{g})$ defined by $(A, B) \mapsto \mathrm{tr}(AB^*)$.

Lemma 3.6. *The Petersson-Weil form for* $\mathrm{ad}(\mathscr{P}) \rightarrow X \times S$*, i.e.,*

$$\omega^{\mathrm{ad}(P)}_{PW} = \frac{1}{2}\int_X \mathrm{tr}(\mathrm{ad}(\Omega) \wedge \mathrm{ad}(\Omega)) \wedge \frac{\omega_X^{n-1}}{(n-1)!}$$

equals ω^{PW} *defined in Definition 3.4.*

We draw conclusions: Let $r = \dim_{\mathbb{C}} \mathfrak{g}$. Then the term of lowest degree of the Chern character form of the virtual vector bundle

$$\mathrm{ad}(\mathscr{P}) - (X \times S) \times \mathfrak{g}$$

equals

$$ch_2(\mathrm{ad}(\mathscr{P})) = \mathrm{tr}\frac{1}{2}\left(\frac{\sqrt{-1}}{2\pi}\Omega \wedge \frac{\sqrt{-1}}{2\pi}\Omega\right).$$

Next we assume that X is a projective manifold such that the Kähler form ω_X is the Chern form $c_1(\mathscr{L}, h)$ of a positive Hermitian line bundle.

We consider the product of the Chern character forms and Todd forms with respect to the given metrics

$$ch\left((\mathrm{ad}(\mathscr{P}) - (X \times S) \times \mathfrak{g}) \otimes (\mathscr{L} - \mathscr{L}^{-1})^{\otimes(n-1)}\right) td(X \times S/S).$$

The term of degree $(n+1, n+1)$ in the above expression equals

$$4\pi^2 \cdot 2^n \frac{1}{2} \mathrm{tr}(\mathrm{ad}(\Omega) \wedge \mathrm{ad}(\Omega)) \wedge \omega_X^{n-1}.$$

The Riemann-Roch theorem by Bismut, Gillet and Soulé [B-G-S] implies the following theorem (for singular S cf. [F-S]).

Theorem 3.7. *Let (δ, h^Q) be the determinant line bundle*

$$\delta = \det \underline{\underline{R}} f_* \left((\mathrm{ad}(\mathscr{P}) - \mathfrak{g} \times (X \times S)) \otimes (\mathscr{L} - \mathscr{L}^{-1})^{\otimes(n-1)}\right)$$

equipped with the Quillen metric h^Q. Then

$$\omega_{PW} = \frac{1}{4\pi^2 \cdot 2^n \cdot (n-1)!}\, c_1(\delta, h^Q).$$

Next, we give the formula for the curvature tensor of the generalized Petersson-Weil metric. We note that Lemma 3.6 provides the reduction to the case of moduli space of vector bundles with Hermite-Einstein connection [B-S, S-T]. We use the Hermite-Einstein structure of the adjoint vector bundles of the form $\mathrm{ad}(P)$ and use the curvature form $\mathrm{ad}(\Omega)$ on a G-bundle $\mathscr{P} \to X \times S$. In this sense we identify the forms $\eta_i = R_{i\overline{\beta}} dz^{\overline{\beta}}$ with their images under ad that have values in $\mathrm{End}(\mathrm{ad}(\mathscr{P}))$. We denote by G_0 the Green's operator on differentiable sections of $\mathrm{End}(\mathrm{ad}(P))$, and by G_2 the Green's operator on $(2,0)$-forms (or skew-symmetric tensors) with values in the vector bundle $\mathrm{End}(\mathrm{ad}(\mathscr{P}))$.

Theorem 3.8. *The curvature tensor of the generalized Petersson-Weil metric equals*

$$\begin{aligned} R^{PW}_{i\bar{j}k\bar{l}} \quad = \quad & \int_X \mathrm{tr}\ (\mathrm{G}_0([R_{i\bar{\beta}}, R_{\alpha\bar{j}}]g^{\bar{\beta}\alpha})[R_{k\bar{\delta}}, R_{\gamma\bar{l}}]g^{\bar{\delta}\gamma})\ g\ dV \\ & \int_X \mathrm{tr}\ (\mathrm{G}_0([R_{i\bar{\beta}}, R_{\alpha\bar{l}}]g^{\bar{\beta}\alpha})[R_{k\bar{\delta}}, R_{\gamma\bar{j}}]g^{\bar{\delta}\gamma})\ g\ dV \\ & -\frac{1}{2}\int_X \mathrm{tr}\ ([R_{i\bar{\beta}}, R_{k\bar{\delta}}]\mathrm{G}_2([R_{\alpha\bar{j}}, R_{\gamma\bar{l}}]))(g^{\bar{\beta}\alpha}g^{\bar{\delta}\gamma} - g^{\bar{\beta}\gamma}g^{\bar{\delta}\alpha})\ g\ dV \end{aligned}$$

Remark 3.9. If the dimension of X is one, the third term in the formula in Theorem 3.8 is not present, and we get non-negative holomorphic sectional curvature.

4. Embedding of the moduli space of stable principal bundles

Let G be a simple linear algebraic group defined over the field of complex numbers together with a faithful representation $G \hookrightarrow \mathrm{GL}(V_0)$. If P is a polystable principal G-bundle over X, then the vector bundle $P_{V_0} := P \times_G V_0$ associated to P for the G-module V_0 is also polystable [A-B, Theorem 3.9].

As above, we associate to any stable principal G-bundle P the polystable vector bundle $P_{V_0} := P \times_G V_0$. This gives rise to an immersion of moduli spaces $\mathscr{M} \to \mathscr{M}_V$, which is an embedding in the orbifold sense, i.e. an embedding on the level of universal deformations. We equip both spaces with the Petersson-Weil metrics ω_{PW} and ω^V_{PW} respectively.

We fix a decomposition

$$\mathrm{End}(V_0) = \mathfrak{g} \oplus \mathfrak{w} \tag{7}$$

of G-modules (since G is simple, such a decomposition exists). Let $P \to X$ be a stable principal G-bundle. The decomposition of G-modules in (7) induces a decomposition of vector bundles

$$\mathrm{End}(P_{V_0}) = \mathrm{ad}(P) \oplus (P \times_G \mathfrak{w})\,, \tag{8}$$

which is clearly an orthogonal decomposition with respect to the Hermite-Einstein connection on $\mathrm{End}(P_{V_0})$ (the Hermite-Einstein connection on the polystable vector bundle P_{V_0} induces a Hermite-Einstein connection on $\mathrm{End}(P_{V_0})$). Now (8) yields a decomposition

$$H^1(X, \mathrm{ad}(P)) \oplus H^1(X, P \times_G \mathfrak{w}) = H^1(X, \mathrm{End}(P_{V_0}))\,, \tag{9}$$

which is an orthogonal decomposition with respect to the Petersson-Weil inner product for moduli of polystable vector bundles, and the restriction to $H^1(X, \mathrm{ad}(P))$ gives the Petersson-Weil inner product for stable principal G-bundles. Using the decomposition (9) we obtain embeddings $S \hookrightarrow V$ for the base spaces of universal deformations of stable principal G-bundles and polystable vector bundles, which descend to the moduli spaces.

Therefore, we have the following theorem:

Theorem 4.1. *The second fundamental form of the immersion of orbifold spaces $\mathscr{M} \hookrightarrow \mathscr{M}_V$ vanishes, in particular, $\mathscr{M}$ is totally geodesic in $\mathscr{M}_V$.*

References

[B-B-N] Balaji, V.; Biswas, I.; Nagaraj, D.S.: Krull-Schmidt reduction for principal bundles. Jour. Reine. Angew. Math. 578 (2005), 1–9.

[A-B] Anchouche, B.; Biswas, I.: Einstein-Hermitian connections on polystable principal bundles over a compact Kähler manifold. Amer. Jour. Math. 123 (2001), 207–228.

[B-G-S] Bismut, J.-M.; Gillet, H.; Soulé, C.: Analytic torsion and holomorphic determinant bundles. I: Bott-Chern forms and analytic torsion. II: Direct images and Bott-Chern forms. III: Quillen metrics on holomorphic determinants. Commun. Math. Phys. 115 (1988), 49–78, 79–126, 301–351.

[B-S] Biswas, I.; Schumacher, G.: Determinant bundle, Quillen metric, and Petersson-Weil form on moduli spaces. Geom. Funct. Anal. 9 (1999), 226–256.

[DO1] Donin, I.F.: Versal families of deformations of holomorphic bundles. Uspehi Mat. Nauk 28 (1973), 239–240.

[DO2] Donin, I.F.: Construction of a versal family of deformations for holomorphic principal bundles over a compact complex space. Mat. Sb. (N.S.) 94 (1974), 430–443.

[F-K] Forster, O.; Knorr, K.: Über die Deformationen von Vektorraumbündeln auf kompakten komplexen Räumen. Math. Ann. 209 (1974), 291–346.

[F-S] Fujiki, A.; Schumacher, G.: The moduli space of extremal compact Kähler manifolds and generalized Weil-Petersson metrics. Publ. Res. Inst. Math. Sci. 26 (1990), 101–183.

[G-S] Gómez, T.L.; Sols, I.: Moduli space of principal sheaves over projective varieties. Ann. of Math. 161 (2005), 1033–1088.

[HU] Humphreys, J.E.: *Linear algebraic groups.* Graduate Texts in Mathematics, Vol. 21, Springer-Verlag, New York, Heidelberg, Berlin (1987).

[L-T] Lübke, M., Teleman, A.: The universal Kobayashi-Hitchin correspondence on Hermitian manifolds, math.DG/0402341.

[PO] Ponomarev, D.A.: Deformations of principal G-bundles on compact analytic spaces. (Russian) Uspehi Mat. Nauk 28 (1973), no. 1 (169), 245–246.

[RA1] Ramanathan, A.: Moduli for principal bundles over algebraic curves: I. Proc. Ind. Acad. Sci. (Math. Sci.) 106 (1996), 301–328.

[RA2] Ramanathan, A.: Moduli for principal bundles over algebraic curves: II. Proc. Indian Acad. Sci. (Math. Sci.) 106 (1996), 421–449.

[R-S] Ramanathan, A.; Subramanian, S.: Einstein-Hermitian connections on principal bundles and stability. J. Reine Angew. Math. 390 (1988), 21–31.

[S] Schumacher, G.: Moduli as algebraic spaces. Complex analysis in several variables—Memorial Conference of Kiyoshi Oka's Centennial Birthday, 283–288, Adv. Stud. Pure Math., 42, Math. Soc. Japan, Tokyo, 2004.

[S-T] Schumacher, G.; Toma, M.: On the Petersson-Weil metric for the moduli space of Hermite-Einstein bundles and its curvature. Math. Ann. 293 (1992), 101–107.

[VA] Varouchas, J.: Stabilité de la classe des variétés Kählériennes par certains morphismes propres. Invent. Math. 77 (1984), 117–127.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India
E-mail address: indranil@math.tifr.res.in

Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Lahnberge, Hans-Meerwein-Strasse, D-35032 Marburg, Germany
E-mail address: schumac@mathematik.uni-marburg.de