Appeared in: Math. Ann. 299, 155-170 (1994)

# Projective Images of Kummer Surfaces

Th. Bauer

April 29, 1993

## 0. Introduction

The aim of this note is to study the linear systems defined by the even resp. odd sections of a symmetric ample line bundle on an abelian surface.

Let $A$ be an abelian surface over the field of complex numbers and let $\mathcal{L}_0$ be an ample symmetric line bundle on $A$. If $\mathcal{L}_0$ is of type $(1,1)$ and if $\mathcal{L}_0$ is not a product polarization, then it is well-known that the linear system $|\mathcal{L}_0^2|$ consists of even divisors only and yields a projective embedding of the Kummer surface of $A$.

Here we will study the following generalized situation: we start with $\mathcal{L}_0$ of type $(1,n)$ for arbitrary $n \geq 1$ and consider for $d \geq 2$ the linear systems $|\mathcal{L}_0^d|^\pm$ defined by the even resp. odd sections of the powers $\mathcal{L}_0^d$. These systems correspond to line bundles $\mathcal{M}_d^+$ and $\mathcal{M}_d^-$ on the smooth Kummer surface $\widetilde{X}$ of $A$ (see section 1 for details). Our aim is to study the maps $\widetilde{X} \longrightarrow \mathbb{P}(H^0(\mathcal{M}_d^\pm))$ defined by these line bundles.

If $\mathcal{L}_0$ is of type $(1,1)$ or $(1,2)$, then $h^0(\mathcal{M}_2^-) = 0$ resp. $h^0(\mathcal{M}_2^-) = 2$. Since $\mathcal{M}_2^-$ does not define a map onto a surface in these cases, *we exclude $\mathcal{M}_2^-$ from our considerations for $n = 1$ and $n = 2$.* We prove:

**Theorem.** *a) $\mathcal{M}_d^\pm$ is free, except for $\mathcal{M}_2^-$ in the product case $\mathcal{L}_0 = \mathcal{O}_A(E_1 + nE_2)$, where $E_1$, $E_2$ are elliptic curves with $E_1E_2 = 1$. In this case the four symmetric translates of $E_2$ yield base curves of $\mathcal{M}_2^-$. Removing these base curves one obtains the line bundle $\mathcal{M}_2^+$ associated to $\mathcal{O}_A(E_1 + (n-2)E_2)$.*

*b) Now let $\mathcal{M}_d^\pm$ be free (i.e. exclude $\mathcal{M}_2^-$ in the product case). Then the morphism $\widetilde{X} \longrightarrow \mathbb{P}(H^0(\mathcal{M}_d^\pm))$ defined by $\mathcal{M}_d^\pm$ is birational onto its image and an isomorphism outside the contracted curves, except for $\mathcal{M}_2^+$ in the product case, where this morphism is of degree 2.*

*c) Again, let $\mathcal{M}_d^\pm$ be free. $\mathcal{M}_d^+$ resp. $\mathcal{M}_d^-$ contracts the exceptional curves associated to even resp. odd halfperiods of $\mathcal{L}_0$. Additional curves are contracted only in the following cases:*

*i) $\mathcal{M}_2^-$ contracts the symmetric elliptic curves $E \subset A$ with $\mathcal{L}_0E = 2$. Such elliptic curves $E$ exist iff*

$$\mathcal{L}_0 \equiv_{\text{alg}} \mathcal{O}_A(kE + E_1 + E_2) \quad \textit{or} \quad \mathcal{L}_0 \equiv_{\text{alg}} \mathcal{O}_A(kE + G),$$

*where $E_1$, $E_2$ are symmetric elliptic curves with $EE_i = 1$, $G$ is a symmetric irreducible curve with $EG = 2$ and $k \geq 1$.*

*ii) If $\mathcal{L}_0$ is of type $(1,1)$, then $\mathcal{M}_3^+$ contracts the unique divisor $\Theta$ in $|\mathcal{L}_0|$.*

*iii) In the product case $\mathcal{L}_0 = \mathcal{O}_A(E_1 + nE_2)$ the additional contractions are given by the following table:*

| line bundle | $n$ | contracted curves |
|---|---|---|
| $\mathcal{M}_4^-$ | | the four symmetric translates of $E_2$ |
| $\mathcal{M}_3^+$ | odd | the curve $E_2$ |
| $\mathcal{M}_3^-$ | even | the four symmetric translates of $E_2$ |
| | odd | the three symmetric translates of $E_2$ different from $E_2$ |

Note: if we have $n = 1$ in iii), then the roles of $E_1$ and $E_2$ can be interchanged. So in this case the corresponding symmetric translates of $E_1$ are contracted as well.

This theorem contains in particular the following special cases:

- $\mathcal{M}_{4k}^+$ embeds the singular Kummer surface (exceptional curves contracted) for $k \geq 1$ (Sasaki [8]).
- $\mathcal{M}_2^+$ embeds the singular Kummer surface in the general case (Khaled [2]).
- $\mathcal{M}_2^-$ is very ample for $n = 3$ in the general case (Naruki [4]).

As to the author's knowledge the other cases have not been considered in the literature so far.

Our method consists in an application of theorems for line bundles on K3-surfaces due to Reider and Saint-Donat.

Throughout this paper the base field is $\mathbb{C}$.

*This work was supported by DFG grant Ba 423/3-4. Furthermore, I would like to thank Prof. Barth for many helpful discussions.*

## 1. Symmetric Line Bundles

In this section we compile some properties of symmetric line bundles on abelian surfaces and of odd and even sections of such bundles.

**Odd and Even Halfperiods.** Let $\mathcal{L}$ be a symmetric line bundle on an abelian surface $A$. By definition the involution $\iota : A \longrightarrow A$, $a \longmapsto -a$, admits a lifting $\iota_{\mathcal{L}}$ to the total space of $\mathcal{L}$. Multiplying by a suitable constant we can achieve that $\iota_{\mathcal{L}}$

is an involution. $\iota_{\mathcal{L}}$ then is uniquely determined up to the sign. From now on we fix one of these two involutions. Let $e_1, \ldots, e_{16}$ be the halfperiods of $A$. $\iota_{\mathcal{L}}$ operates on the fibers $\mathcal{L}(e_i)$ as multiplication by $\pm 1$. We denote this sign of $\iota_{\mathcal{L}}$ on $\mathcal{L}(e_i)$ by $q_i = q_i(\mathcal{L})$. We will call the halfperiods $e_i$ with $q_i = +1$ resp. $q_i = -1$ the *even* resp. *odd* halfperiods of $\mathcal{L}$.

**Odd and Even Sections.** A section $s \in H^0(\mathcal{L})$ is called *even* resp. *odd*, if $\iota_{\mathcal{L}} s \iota = s$ resp. $\iota_{\mathcal{L}} s \iota = -s$. So the even resp. odd sections are just the elements of the eigenspaces $H^0(\mathcal{L})^+$ resp. $H^0(\mathcal{L})^-$ of the linear map $s \longmapsto \iota_{\mathcal{L}} s \iota$ on $H^0(\mathcal{L})$.

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\iota_{\mathcal{L}}} & \mathcal{L} \\ s \uparrow & & \uparrow \iota_{\mathcal{L}} s \iota \\ A & \xrightarrow{\iota} & A \end{array}$$

Let $\sigma : \widetilde{A} \longrightarrow A$ be the blow-up of $A$ in the sixteen halfperiods. The exceptional divisor $Z$ is a sum of sixteen disjoint rational $(-1)$-curves $Z_1, \ldots, Z_{16}$, corresponding to the points $e_1, \ldots, e_{16}$. We introduce the notations $Z^+ = \sum\limits_{q_i=+1} Z_i$ and $Z^- = \sum\limits_{q_i=-1} Z_i$. Let $\widetilde{\mathcal{L}}$ be the pullback of $\mathcal{L}$ to $\widetilde{A}$. We denote by $\widetilde{\iota}$ the involution on $\widetilde{A}$ induced by $\iota$ and by $\iota_{\widetilde{\mathcal{L}}}$ the corresponding involution of $\widetilde{\mathcal{L}}$. The subspaces $H^0(\widetilde{\mathcal{L}})^+$ and $H^0(\widetilde{\mathcal{L}})^-$ of even and odd sections are defined in the obvious way. We have canonical isomorphisms $H^0(\widetilde{\mathcal{L}})^\pm \cong H^0(\mathcal{L})^\pm$.

**Kummer Surfaces.** The quotient $\widetilde{X} = \widetilde{A}/\widetilde{\iota}$ is a projective K3-surface, the *smooth Kummer surface* of $A$. It is the minimal desingularisation of $X = A/\iota$, the *(singular) Kummer surface* of $A$. Denoting the canonical projections $A \longrightarrow X$ and $\widetilde{A} \longrightarrow \widetilde{X}$ by $\pi$ and $\widetilde{\pi}$ we have the following commutative diagram

$$\begin{array}{ccc} Z_i \subseteq \widetilde{A} & \xrightarrow{\sigma} & A \ni e_i \\ \widetilde{\pi} \downarrow & & \downarrow \pi \\ D_i \subseteq \widetilde{X} & \longrightarrow & X \end{array}$$

The $\widetilde{\pi}$-images $D_1, \ldots, D_{16}$ of $Z_1, \ldots, Z_{16}$ are disjoint rational $(-2)$-curves corresponding to the double points $\pi(e_1), \ldots, \pi(e_{16})$ of $X$. $Z = \sum Z_i$ is the ramification divisor of $\widetilde{\pi}$, $D = \sum D_i$ is the branch locus in $\widetilde{X}$.

Our aim is to study the linear systems $|\mathcal{L}|^\pm$ of even resp. odd divisors in $|\mathcal{L}|$. Here a divisor is called *even* resp. *odd*, if it is defined by an even resp. odd section of $\mathcal{L}$. Along with the subsystems $|\mathcal{L}|^\pm$ we will consider certain line bundles $\mathcal{M}^+$ and $\mathcal{M}^-$ on $\widetilde{X}$ associated to $\mathcal{L}$. These bundles are given by the following proposition:

**Proposition 1.1** *The direct image sheaf $\mathcal{M} = \widetilde{\pi}_* \widetilde{\mathcal{L}}$ is locally free of rank 2. It admits a decomposition $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^-$ into line bundles $\mathcal{M}^+$ and $\mathcal{M}^-$ such that $H^0(\mathcal{M}^\pm) \cong H^0(\mathcal{L})^\pm$.*

*Proof.* For an open set $U \subset \widetilde{X}$ we define $\mathcal{M}^+(U)$ and $\mathcal{M}^-(U)$ to be the eigenspaces of the linear map

$$\begin{array}{ccc} \widetilde{\mathcal{L}}\left(\widetilde{\pi}^{-1}(U)\right) & \longrightarrow & \widetilde{\mathcal{L}}\left(\widetilde{\pi}^{-1}(U)\right) \\ s & \longmapsto & \iota_{\widetilde{\mathcal{L}}} s \widetilde{\iota}. \end{array}$$

Since multiplication of a section in $\mathcal{M}^\pm(U)$ by a function in $\mathcal{O}_{\widetilde{X}}(U)$ preserves the parity of the section, the subvector spaces $\mathcal{M}^\pm(U)$ are in fact $\mathcal{O}_{\widetilde{X}}(U)$-submodules. Since these submodules are easily seen to be of rank 1 and since by definition $H^0(\mathcal{M}^\pm) = H^0(\widetilde{\mathcal{L}})^\pm$, our assertion is proved. □

**Proposition 1.2** *Let $F$ be a symmetric divisor in $|\mathcal{L}|$. Then the multiplicity $m_i$ of $F$ in the halfperiod $e_i$ is even resp. odd according to the following table:*

| | $e_i$ *even* | $e_i$ *odd* |
|---|---|---|
| $F$ *even* | $m_i$ *even* | $m_i$ *odd* |
| $F$ *odd* | $m_i$ *odd* | $m_i$ *even* |

*Proof.* It will be enough to consider the case that $e_i$ is an even halfperiod. First we see that odd sections $s \in H^0(\widetilde{\mathcal{L}})^-$ vanish on $Z_i$, because $s = \iota_{\widetilde{\mathcal{L}}} s \widetilde{\iota} = -s$ on $Z_i$. Now let $s \in H^0(\widetilde{\mathcal{L}})^\pm$ be a section defining the pullback $\sigma^* F$. Then $m_i$ is the order of vanishing of $s$ along $Z_i$. We can find a local equation $f$ of $Z_i$ such that $f\widetilde{\iota} = -f$. Then the section

$$\frac{s}{f^{m_i}}$$

is even (because an odd section would vanish on $Z_i$). We conclude that $m_i$ must be even resp. odd iff $s$ is even resp. odd. □

If $\mathcal{L}$ is effective, then the line bundles $\mathcal{M}^\pm$ can also be described in the following way: Let $\Theta$ be a divisor in $|\mathcal{L}|^+$. We write

$$\sigma^*\Theta = \widehat{\Theta} + \sum_{i=1}^{16} m_i Z_i,$$

where $\widehat{\Theta}$ denotes the proper transform of $\Theta$ and $m_i$ the multiplicity of $\Theta$ in $e_i$. Now we define a divisor $C$ on $\widetilde{X}$ by

$$C = \widetilde{\pi}(\widehat{\Theta}) + \sum \left[\frac{m_i}{2}\right] D_i,$$

where by $\widetilde{\pi}(\widehat{\Theta})$ we mean the image divisor, whose multiplicities at irreducible components are the same as those of $\widehat{\Theta}$. This procedure gives a bijection $|\mathcal{L}|^+ \longrightarrow |\mathcal{M}^+|$, where $\mathcal{M}^+ = \mathcal{O}_{\widetilde{X}}(C)$. Using the fact that $\Theta$ has odd multiplicities in the odd halfperiods we obtain

$$\widetilde{\pi}^* C = \widehat{\Theta} + \sum 2\left[\frac{m_i}{2}\right] Z_i = \sigma^*\Theta - Z^-,$$

hence $\widetilde{\pi}^*\mathcal{M}^+ = \widetilde{\mathcal{L}} - Z^-$. (Here and at similar occasions we use the notation $\widetilde{\mathcal{L}} - Z^-$ as a short form for $\widetilde{\mathcal{L}} \otimes \mathcal{O}_{\widetilde{X}}(-Z^-)$). Since we can proceed in the same way with $\mathcal{M}^-$, we obtain

**Proposition 1.3**

$$\widetilde{\pi}^*\mathcal{M}^\pm = \widetilde{\mathcal{L}} - Z^\mp$$

Now we can determine the intersection numbers of $\mathcal{M}^\pm$ with the curves $D_i$ by calculating on $\widetilde{A}$. Using $\widetilde{\pi}^* D_i = 2Z_i$ we get

**Proposition 1.4**

$$\begin{aligned} \mathcal{M}^+ \cdot D_i &= \begin{cases} 0, & \text{if } q_i = +1 \\ 1, & \text{if } q_i = -1 \end{cases} \\ \mathcal{M}^- \cdot D_i &= \begin{cases} 1, & \text{if } q_i = +1 \\ 0, & \text{if } q_i = -1 \end{cases} \end{aligned}$$

Consider the following sets:

i) symmetric effective divisors $F$ on $A$

ii) effective divisors $C$ on $\widetilde{X}$ such that none of the exceptional curves $D_i$, $i = 1, \ldots, 16$, is a component of $C$

Clearly, the map $F \longmapsto C = \widetilde{\pi}(\widehat{F})$, which maps $F$ to the image in $\widetilde{X}$ of the proper transform $\widehat{F}$ of $F$, is a bijection between i) and ii). We will need some formulas relating the intersection numbers of $F$ and $C$. Here and in the sequel we denote by $m_i = m_i(F)$ the multiplicities of the divisor $F$ in the halfperiods $e_i$.

**Proposition 1.5** *Let $F$ be a symmetric effective divisor on $A$ and let $C$ be its image on $\widetilde{X}$. Then we have*

*a)* $F^2 = 2C^2 + \sum_{i=1}^{16} m_i^2$

*b)* $\mathcal{M}^\pm C = \frac{1}{2}(\mathcal{L}F - \sum_{q_i=\mp 1} m_i)$

*c)* $m_i = CD_i$

*d)* $\sum_{i=1}^{16} m_i$ *is even.*

*Proof.* a), b) and c) are shown by obvious calculations, whereas d) follows from the fact that $\chi(\widehat{F}) = \chi(\mathcal{O}_{\widetilde{A}}) + C^2 - \frac{1}{2}C\sum D_i$ is an integer. $\square$

## 2. Line Bundles on the Smooth Kummer Surface

Studying the line bundles $\mathcal{M}^\pm$ on the K3-surface $\widetilde{X}$ we will apply the following theorem of Saint-Donat ([7], Cor. 3.2, Thm. 5.2 and Thm. 6.1(iii)):

**Theorem 2.1** (Saint-Donat) *Let $S$ be a K3-surface and let $\mathcal{B}$ be a line bundle on $S$ such that $\mathcal{B}^2 \geq 4$, $|\mathcal{B}| \neq \emptyset$ and such that $|\mathcal{B}|$ has no fixed components. Then $|\mathcal{B}|$ has no base points. Furthermore the morphism $S \longrightarrow \mathbb{P}_N$ defined by $\mathcal{B}$ is birational, except in the following cases:*

*i) There exists an irreducible curve $E$ such that $p_a(E) = 1$ and $\mathcal{B}E = 2$.*

*ii) There exists an irreducible curve $H$ such that $p_a(H) = 2$ and $\mathcal{B} = \mathcal{O}_S(2H)$.*

*If the morphism is birational, then it is an isomorphism outside the contracted curves.*

The following intersection property of the elliptic curve $E$ in Saint-Donat's theorem will turn out to be essential:

**Proposition 2.2** *Assume case i) of Saint-Donat's Theorem and let $D$ be an irreducible curve such that $\mathcal{B}D = 1$. Then $ED \leq 1$.*

*Proof.* First note that we may assume $E$ to be smooth. Now let $\Phi$ be the morphism defined by $\mathcal{B}$. It follows from $\mathcal{B}E = 2$ that the restricted morphism $\Phi|E$ is of degree 2. We conclude that the image $\Phi(E)$ is a line. Since $\Phi(D)$ is also a line, we see that $E$ and $D$ have at most one point in common. This proves our assertion, because by Bertini's Theorem we may assume $E$ and $D$ to intersect transversally. $\square$

If the selfintersection numbers of $\mathcal{M}^\pm$ are sufficiently high, we will use Reider's method to show that the linear systems $|\mathcal{M}^\pm|$ are base point free. For K3-surfaces his theorem takes the following form ([6], Thm. 1 and Prop. 5):

**Theorem 2.3** (Reider) *Let $S$ be a K3-surface and let $L$ be a nef divisor on $S$ with $L^2 \geq 6$. Then the linear system $|L|$ has base points iff there is a divisor $E$ on $S$ with*

$$LE = 1 \quad \text{and} \quad E^2 = 0.$$

The intersection matrix of $E$ then is necessarily negative semidefinite. We need:

**Proposition 2.4** *a) In the situation of Reider's Theorem the divisor $E$ can always be chosen irreducible.*

*b) If $D$ is a $(-2)$-curve such that $LD = 1$, then $ED \leq 1$.*

*Proof.* a) If the linear system $|E|$ has no base component, then it follows from Bertini's theorem that $\mathcal{O}_{\widetilde{X}}(E) = \mathcal{O}_{\widetilde{X}}(kC)$, where $C$ is an irreducible elliptic curve in $\widetilde{X}$. From $LE = 1$ we conclude $k = 1$.

Now if $|E|$ has fixed part $B$, we write $E = B + E'$ where $E'$ is free of base components, thus $E'^2 \geq 0$. On the other hand we have $E'^2 \leq 0$, because the intersection matrix of $E$ is negative semidefinite. Hence $E'^2 = 0$. Further, if $LE' = 0$, then $E'$ would be numerically trivial by the Hodge Index Theorem. So from $1 = LE' = LB + LE'$ we conclude $LE' = 1$, because $L$ is nef. This shows that we can replace $E$ by $E'$ and argue as above.

b) From the Hodge Index Theorem we obtain the inequality $L^2(E+D)^2 \le (L(E+D))^2$, which immediately yields our assertion. □

## 3. The Line Bundles $\mathcal{M}^{\pm}$

There are formulas for the dimensions $h^0(\mathcal{L})^{\pm} = h^0(\mathcal{M}^{\pm})$, where $\mathcal{L}$ is an ample symmetric line bundle on an abelian variety of arbitrary dimension ([1], Thm. 5.4). Here we give a simple formula in the case of abelian surfaces:

**Theorem 3.1** *Let $\mathcal{L}$ be an ample symmetric line bundle of type $(d_1, d_2)$ on an abelian surface $A$ and let $n^{\pm}$ be the number of even resp. odd halfperiods of $\mathcal{L}$. Then we have the following formula for the associated line bundles $\mathcal{M}^{\pm}$ on the smooth Kummer surface $\widetilde{X}$ of $A$:*

$$h^0(\mathcal{M}^{\pm}) = 2 + \frac{d_1 d_2}{2} - \frac{n^{\mp}}{4}\ .$$

*Proof.* First note that it suffices to prove the formula for one of the line bundles $\mathcal{M}^+$ or $\mathcal{M}^-$, because for the other line bundle the formula then follows from $h^0(\mathcal{M}^+) + h^0(\mathcal{M}^-) = h^0(\mathcal{L}) = d_1 d_2$. Further, by Riemann-Roch the Euler-Poincaré-Characteristic $\chi(\mathcal{M}^{\pm})$ equals the right hand side of the asserted formula. So it is enough to prove that the higher cohomology groups of $\mathcal{M}^{\pm}$ vanish. We proceed in three steps.

*Step 1: We have $h^1(\mathcal{M}^{\pm}) = h^2(\mathcal{M}^{\pm}) = 0$ in each of the following cases:*

*i) $\mathcal{L}$ is totally symmetric, i.e. $n^+ = 16$ or $n^- = 16$.*

*ii) $|\mathcal{L}|^+ \neq \emptyset$ or $|\mathcal{L}|^- \neq \emptyset$.*

Let i) be fulfilled. We may assume $n^+ = 16$, hence $(\mathcal{M}^+)^2 = d_1 d_2 > 0$ and $\mathcal{M}^+ C > 0$ for any irreducible curve $C \subset \widetilde{X}$ different from the exceptional curves $D_1, \dots, D_{16}$. Thus $\mathcal{M}^+$ is nef and our assertion follows from Ramanujam's Vanishing Theorem [5].

Now suppose ii). Since $\mathcal{M}^{\pm}$ is not trivial, we have $h^2(\mathcal{M}^{\pm}) = 0$. Thus Riemann-Roch gives

$$h^0(\mathcal{M}^{\pm}) - h^1(\mathcal{M}^{\pm}) = 2 + \frac{d_1 d_2}{2} - \frac{n^{\mp}}{4}.$$

and the assertion follows by simply adding up these two equations.

*Step 2: If the type of $\mathcal{L}$ is different from $(1,1)$ and $(1,2)$, then i) or ii) is fulfilled.*

Suppose the contrary and w.l.o.g. assume $|\mathcal{L}| = |\mathcal{L}|^+$. Thus the odd halfperiods of $\mathcal{L}$ are base points of the full linear system $|\mathcal{L}|$. By [3], Prop. 4.1.6 and Lemma 10.1.2.a), then necessarily $d_1 = 1$ and $(A, \mathcal{L})$ is a polarized product of elliptic curves. But in this case it is easy to see that there are both odd and even divisors in $|\mathcal{L}|$, contradicting our assumption.

*Step 3: If $\mathcal{L}$ is of type $(1,1)$ or $(1,2)$, then $h^0(\mathcal{M}^\pm) = \chi(\mathcal{M}^\pm)$ holds as well.*

First let $\mathcal{L}$ be of type $(1,1)$. If the unique divisor $\Theta$ in $|\mathcal{L}|$ is irreducible, then $\Theta$ is a smooth hyperelliptic curve, the odd halfperiods of $\mathcal{L}$ being the Weierstraß points of $\Theta$. Thus $n^- = 6$ and $h^0(\mathcal{M}^+) = 1$, hence $h^0(\mathcal{M}^+) = \chi(\mathcal{M}^+)$. The product case can be treated analogously.

Now let $\mathcal{L}$ be of type $(1,2)$. By Step 1 we may assume $h^0(\mathcal{L})^+ = 2$, $h^0(\mathcal{L})^- = 0$ and $n^- > 0$. We have to show that $n^- = 4$. Since this is obvious in the product case, let $(A, \mathcal{L})$ not be a product of elliptic curves. The odd halfperiods are base points of $|\mathcal{L}|$. Thus we have $n^- \leq 4$, because $\mathcal{L}^2 = 4$. Since $n^-$ must be a multiple of 4, equality $n^- = 4$ holds. □

The formula in (3.1) shows in particular that

- $n^\pm$ is even, and
- $n^\pm$ is a multiple of 4 iff the product $d_1 d_2$ is even.

From now on let $\mathcal{L}_0$ be an ample symmetric line bundle on $A$ of type $(1,n)$, where $n \geq 1$. For $d \geq 2$ we denote by $\mathcal{M}_d^\pm$ the line bundles on $\widetilde{X}$ associated to the powers $\mathcal{L}_0^d$.

We begin our study of the line bundles $\mathcal{M}_d^\pm$ by getting rid of an exceptional case:

**Proposition 3.2** *Let $\mathcal{L}_0 = \mathcal{O}_A(E_1 + nE_2)$, where $E_1$ and $E_2$ are elliptic curves with $E_1E_2 = 1$ and $n \geq 3$. Then the four symmetric translates of $E_2$ yield base curves of $\mathcal{M}_2^-$. Removing these base curves one obtains the bundle $\mathcal{M}_2^+$ associated to $\mathcal{O}_A(E_1 + (n-2)E_2)$.*

*Proof.* If $C \subset \widetilde{X}$ is the curve corresponding to $E_2$ or to one of its symmetric translates $E_2'$, $E_2''$, $E_2'''$, then $\mathcal{M}_2^- C = -1$. So $C$ is a base curve of $\mathcal{M}_2^-$. The divisor

$$2(E_1 + nE_2) - (E_2 + E_2' + E_2'' + E_2''')$$

is linearly equivalent to $2(E_1 + (n-2)E_2)$. So the proposition follows from the fact that adding the sum $E_2 + E_2' + E_2'' + E_2'''$ obviously changes the parity of a divisor. □

Apart from this exception we will proceed as follows:

- We show that $\mathcal{M}_d^\pm$ is nef and we determine the curves $C$ on $\widetilde{X}$ such that $\mathcal{M}_d^\pm C = 0$.
- We show that the linear system $|\mathcal{M}_d^\pm|$ has no base points.
- The morphism defined by $\mathcal{M}_d^\pm$ is either birational or of degree 2. We show that the latter case occurs only with $\mathcal{M}_2^+$ in case $(A, \mathcal{L}_0)$ is a polarized product of elliptic curves.

**The Property Nef.** First we consider the case $n \geq 2$. The principally polarized case will be postponed to section 4, because it needs a somewhat different discussion. By (1.5) $\mathcal{M}_2^+ C \geq 0$, where equality holds iff $C$ is one of the exceptional curves $D_1, \ldots, D_{16}$. Let us consider $\mathcal{M}_2^-$ now:

**Lemma 3.3** *Let $\mathcal{L}_0$ be of type $(1, n)$ with $n \geq 3$ and assume that $\mathcal{L}_0$ is not a product polarization. Then $\mathcal{M}_2^-$ is nef. For an irreducible curve $C \subset \widetilde{X}$ we have $\mathcal{M}_2^- C = 0$ iff the corresponding symmetric curve $F \subset A$ is elliptic and $\mathcal{L}_0 F = 2$. Then necessarily*

$$\mathcal{L}_0 \equiv_{\text{alg}} \mathcal{O}_A(kF + E_1 + E_2) \quad \textit{or} \quad \mathcal{L}_0 \equiv_{\text{alg}} \mathcal{O}_A(kF + G),$$

*where $E_1$, $E_2$ are symmetric elliptic curves with $FE_i = 1$, $G$ is a symmetric irreducible curve with $FG = 2$ and $k \geq 1$.*

*Proof.* The last assertion follows easily by considering intersection numbers. For the proof of the other assertions let $C$ be an irreducible curve with $\mathcal{M}_2^- C \leq 0$ and let $F$ be the corresponding symmetric curve on $A$.

1) First we show that $F$ is contained in a symmetric divisor $\Theta \in |\mathcal{L}_0|$: By (1.5) the assumption $\mathcal{M}_2^- C \leq 0$ means $\mathcal{L}_0^2 F \leq \sum_{i=1}^{16} m_i$, where $m_i = m_i(F)$. Because of $n \geq 3$ the linear systems $|\mathcal{L}_0|^+$ and $|\mathcal{L}_0|^-$ are of dimension $\geq 0$ and at least one of them is of positive dimension. Now our claim follows from the fact that—apart from the halfperiods—we may prescribe additional points of intersection for $\Theta \in |\mathcal{L}_0|^\pm$ with $F$.

2) By the Hodge Index Theorem we have $C^2 = -2$, hence $F^2 = \sum m_i^2 - 4$. Now consider the inequalities

$$(F^2)^2 \leq \mathcal{L}_0^2 F^2 \leq (\mathcal{L}_0 F)^2 \leq (\frac{1}{2} \sum m_i)^2,$$

where the first one follows from our claim above. We obtain $4 \leq \sum m_i \leq 8$ from $(F^2)^2 \leq (\frac{1}{2} \sum m_i)^2$. Then we see from $(\mathcal{L}_0^2 F^2) \leq (\frac{1}{2} \sum m_i)^2$ and $(\mathcal{L}_0)^2 \geq 6$ that necessarily $F^2 = 0$ and $\sum m_i = 4$, thus $\mathcal{L}_0 F \leq 2$. Since the product case was excluded, we have $\mathcal{L}_0 F = 2$, i.e. $\mathcal{M}_2^- C = 0$, and $F$ is a symmetric elliptic curve. This proves the lemma. $\square$

We know that for $i = 1, \ldots, 16$ we have $\mathcal{M}_d^\pm D_i = 0$, if $q_i = +1$. Now we are interested in curves $C$ different from the $D_i$ with $\mathcal{M}_d^\pm C = 0$. For brevity we will simply call them *additional contractions* in the sequel.

**Lemma 3.4** *Let $\mathcal{L}_0$ be of type $(1, n)$ with $n \geq 2$ and let $d \geq 3$.*

*a) If $\mathcal{L}_0$ is not a product polarization, then $\mathcal{M}_d^\pm$ is nef and has no additional contractions.*

*b) If $\mathcal{L}_0 = \mathcal{O}_A(E_1 + nE_2)$ is a product polarization, then $\mathcal{M}_d^\pm$ is nef, too. The additional contractions are given by the table in section 0.*

*Proof.* For an irreducible curve $C \subset \widetilde{X}$ different from the exceptional curves $D_1, \ldots, D_{16}$ we have

$$\mathcal{M}_d^{\pm} C \geq \frac{1}{2}(3\mathcal{L}_0 F - \sum_{i=1}^{16} m_i) = \mathcal{M}_2^{-} C + \frac{1}{2}\mathcal{L}_0 F.$$

We see that for a curve with $\mathcal{M}_d^{\pm} C \leq 0$ necessarily $\mathcal{M}_2^{-} C < 0$ holds.

As for a): For $n \geq 3$ $\mathcal{M}_2^{-}$ is nef by (3.3) and we are done. If $n = 2$, then $\mathcal{M}_2^{-}$ is free, hence nef, by (3.6) below.

As for b): Only the symmetric translates of $E_2$ can yield curves with $\mathcal{M}_2^{-} C < 0$. Thus our claim follows by simply calculating $\mathcal{M}_d^{\pm} C$ for each of these curves. □

**Base Points.** First we consider those cases, where we can apply Reider's theorem. We will then have to deal separately with the line bundles defining maps into $\mathbb{P}_3$.

**Lemma 3.5** *Let $\mathcal{L}_0$ be of type $(1, n)$ with $n \geq 1$ and let $d \geq 2$. If $\mathcal{M}_d^{\pm}$ is nef and $(\mathcal{M}_d^{\pm})^2 \geq 6$, then $\mathcal{M}_d^{\pm}$ is free.*

*Proof.* Assume that $\mathcal{M}_d^{\pm}$ has base points. The assumptions on $\mathcal{M}_d^{\pm}$ allow us to apply Reider's Theorem. Thus there is a curve $C$ on $\widetilde{X}$ such that $C^2 = 0$ and $\mathcal{M}_d^{\pm} C = 1$. By (2.4) we can assume $C$ to be irreducible. So we get a symmetric curve $F$ on $A$ such that

$$F^2 = \sum_{i=1}^{16} m_i^2 d\mathcal{L}_0 F = 2 + \sum_{q_i = \mp 1} m_i,$$

where the $m_i$ denote the multiplicities of $F$ in the halfperiods. By (2.4) we have $m_i \leq 1$, if $q_i = \mp 1$. Now consider the following inequality for the sum $s = \sum_{q_i = \mp 1} m_i$:

$$\mathcal{L}_0^2 s \leq \mathcal{L}_0^2 F^2 \leq (\mathcal{L}_0 F)^2 = \frac{1}{d^2}(2 + s)^2$$

Using $s \leq 16$ and $(\mathcal{M}_d^{\pm})^2 \geq 6$ we find that the only solution is $s = 0$. Hence $d = 2$ and $\mathcal{L}_0 F = 1$. But then $F^2 = 0$ by the Index Theorem, i.e. $F$ must be a sum of two algebraically equivalent elliptic curves, contradicting $\mathcal{L}_0 F = 1$. □

It remains to consider the line bundles $\mathcal{M}_d^{\pm}$ of selfintersection smaller than 6. These are

i) $\mathcal{M}_2^{+}$, if $\mathcal{L}_0$ is of type $(1, 1)$,

ii) $\mathcal{M}_2^{-}$, if $\mathcal{L}_0$ is of type $(1, 2)$ or $(1, 3)$ and

iii) $\mathcal{M}_3^{-}$, if $\mathcal{L}_0$ is of type $(1, 1)$.

Case iii) will be considered in section 4. Here we turn to the bundles $\mathcal{M}_2^{+}$ and $\mathcal{M}_2^{-}$. We give a proof that these bundles are free, which works for all polarizations $(1, n)$, $n \geq 1$ resp. $n \geq 2$. This shows that for $\mathcal{M}_2^{\pm}$ we could have done without Reider's Theorem.

**Lemma 3.6** *Let $\mathcal{L}_0$ be of type $(1,n)$. Then*

*a) $\mathcal{M}_2^+$ is free for all $n \geq 1$.*

*b) $\mathcal{M}_2^-$ is free for $n \geq 2$, if $\mathcal{L}_0$ is not a product polarization.*

*Proof.* According to Saint-Donat's Theorem it is enough to prove that the systems $|\mathcal{M}_2^\pm|$ have no base *curves*.

1) First suppose that there are base curves different from the exceptional curves $D_1, \ldots, D_{16}$. Then the system $|\mathcal{L}_0^2|^\pm$ on $A$ has a fixed part $B$. Since this system is invariant under translation by halfperiods, so is $B$. Hence $\mathcal{O}_A(B)$ is totally symmetric. If $B^2 > 0$, then we have $h^0(B)^\pm = \frac{1}{B^2} \pm 2$, which never equals 1, because $B^2$ is a multiple of 8. Thus necessarily $B^2 = 0$, i.e. $B$ is a sum of algebraically equivalent elliptic curves. $B$ has odd multiplicities in the halfperiods. Now let $\Theta$ be a divisor in the system $|\mathcal{L}_0^2|^\pm - B$. Then $\mathcal{O}_A(\Theta)$ is totally symmetric and

$$|\mathcal{L}_0^2|^\pm - B = |\Theta|^\mp.$$

As for b): Since we are not in the product case, we have $\Theta^2 > 0$. Using (3.1) we see that $h^0(\Theta)^+ = h^0(\mathcal{L}_0^2)^-$ implies $\Theta B = 8$. But this occurs in the product case only, because $\Theta$ is a square and $B$ is at least a 4-th power in the Néron-Severi group of $A$.

As for a): If $\Theta^2 > 0$, then $h^0(\Theta)^- = h^0(\mathcal{L}_0^2)^+$ gives the contradiction $\Theta^2 > (\mathcal{L}_0^2)^2$. If $\Theta^2 = 0$, i.e. $\Theta$ is a sum of $k$ elliptic curves, $k > 0$, then we compute

$$h^0(\Theta)^- = \frac{\Theta B}{2} + 2 \geq 2k + 2,$$

which again is impossible.

2) Now suppose that one of the exceptional curves $D_i$ is a base curve of $\mathcal{M}_2^\pm$. Using the translation argument again, we see that then all the curves $D_1, \ldots, D_{16}$ must be base curves. Thus the line bundles $\mathcal{N}^\pm = \mathcal{M}_2^\pm - \sum_{i=1}^{k} D_i$ are nef for $k = 1, \ldots, 16$. Now we compute the selfintersection numbers

$$\begin{aligned} (\mathcal{N}^+)^2 &= (\mathcal{M}_2^+)^2 - 2k \\ (\mathcal{N}^-)^2 &= (\mathcal{M}_2^-)^2 - 4k \end{aligned}$$

If $(\mathcal{N}^\pm)^2$ is positive, then by Ramanujam's Vanishing Theorem [5] and Riemann-Roch this number must be equal to $(\mathcal{M}_2^\pm)^2$, which is impossible. Hence $(\mathcal{N}^\pm)^2 = 0$ for all $k$, a contradiction again. $\square$

**Degree of the Morphisms.** Now we want to determine the degree of the maps defined by the line bundles $\mathcal{M}_d^\pm$. We always exclude $\mathcal{M}_2^-$ in the product case. According to the results above this is the same as requiring $\mathcal{M}_d^\pm$ to be nef.

First we show that case ii) of Saint-Donat's Theorem 2.1 never occurs for the line bundles $\mathcal{M}_d^\pm$:

**Proposition 3.7** *There is no irreducible curve $H$ on $\widetilde{X}$ such that $p_a(H) = 2$ and $\mathcal{M}_d^\pm = \mathcal{O}_{\widetilde{X}}(2H)$.*

*Proof.* We have $H^2 = 2$ by the adjunction formula, hence $(\mathcal{M}_d^{\pm})^2 = 8$. Furthermore, from $2HD_i = \mathcal{M}_d^{\pm} D_i \leq 1$ we conclude $\mathcal{M}_d^{\pm} D_i = 0$ for $i = 1, \ldots, 16$. This is possible only for the bundle $\mathcal{M}_2^{+}$ associated to a $(1,2)$-polarization. In this case we consider the symmetric curve $F$ on $A$ corresponding to $H$. We have $F^2 = 4 + \sum_{i=1}^{16} m_i(F)^2$ and $4 = \mathcal{L}_0 F$. It follows from $\mathcal{L}_0^2 F^2 \leq (\mathcal{L}_0 F)^2$ that $F^2 = 4$, thus $\sum m_i(F)^2 = 0$. This is impossible, because $\mathcal{O}_A(F)$ cannot be totally symmetric. $\square$

**Lemma 3.8** *Let $\mathcal{L}_0$ be an ample symmetric line bundle of type $(1,n)$ with $n \geq 1$ and let $\mathcal{M}_d^{\pm}$ be nef, $d \geq 2$. Then the morphism defined by $\mathcal{M}_d^{\pm}$ is birational onto its image and an isomorphism outside the contracted curves, except for $\mathcal{M}_2^{+}$ in the product case, where this morphism is of degree 2.*

*Proof.* According to Saint-Donat's Theorem and (3.7) we have to check whether there is an elliptic curve $C$ on $\widetilde{X}$ with $\mathcal{M}_d^{\pm} C = 2$. Equivalently we ask whether there is a symmetric curve $F \subset A$ having at most two components such that

$$F^2 = \sum_{i=1}^{16} m_i^2 d\mathcal{L}_0 F = 4 + \sum_{q_i = \mp 1} m_i. \qquad (*)$$

By (2.2) we have $m_i \leq 1$, if $q_i = \mp 1$. Denote the sum $\sum_{q_i=\mp 1} m_i$ by $s$.

1) First we consider the case $s = 0$. Then obviously $d = 2$ or $d = 4$. If $d = 4$, then we have $\mathcal{L}_0 F = 1$, hence $F^2 = 0$ by the Index Theorem. Thus $F$ is a sum of two algebraically equivalent elliptic curves, contradicting $\mathcal{L}_0 F = 1$. If $d = 2$, then $\mathcal{L}_0 F = 2$, and again this implies that $F$ must be the sum of two algebraically equivalent elliptic curves. This occurs if and only if $\mathcal{L}_0$ is a product polarization, as claimed.

2) Now suppose $s > 0$. From (*) we get the inequalities

$$\mathcal{L}_0^2 s \leq \mathcal{L}_0^2 F^2 \leq (\mathcal{L}_0 F)^2 = \frac{1}{d^2}(4+s)^2.$$

These inequalities can be satisfied only if $d$ and $(\mathcal{L}_0)^2$ are small enough. Using $s \leq 16$ we arrive at:

i) the case $\mathcal{M}_2^{-}$, where $\mathcal{L}_0$ is of type $(1,3)$ and $s = 16$, and

ii) the case $\mathcal{M}_3^{\pm}$, where $\mathcal{L}_0$ is of type $(1,1)$ and $s = 8$ or $s = 2$.

It remains to show that in these situations there is no curve $F$ satisfying (*). But this follows from (4.4) and (5.2) below. $\square$

## 4. Special Case: Principal Polarizations

In this section we study the line bundles $\mathcal{M}_d^{\pm}$ in case $\mathcal{L}_0$ is a principal polarization. We have to consider two cases:

| | | | |
|---|---|---|---|
| $e_7$ $\Theta_1$ | $e_4$ $\Theta_8$ | $e_5$ $\Theta_9$ | $e_6$ $\Theta_{10}$ |
| $e_3$ $\Theta_2$ | $e_{14}$ $\Theta_{11}$ | $e_{15}$ $\Theta_{12}$ | $e_{16}$ $\Theta_{13}$ |
| $e_2$ $\Theta_3$ | $e_{11}$ $\Theta_{14}$ | $e_{12}$ $\Theta_{15}$ | $e_{13}$ $\Theta_{16}$ |
| $e_1$ $\Theta_7$ | $e_8$ $\Theta_4$ | $e_9$ $\Theta_5$ | $e_{10}$ $\Theta_6$ |

Table 1: The $16_6$-configuration on a principally polarized abelian surface. The sixteen symmetric translates of $\Theta$ are denoted by $\Theta_i = t_{e_i}^*\Theta$, $i = 1, \dots, 16$. The table is organised such that $e_i$ lies on $\Theta_j$ iff $e_i$ appears in the same line or column as $\Theta_j$ unless $e_i$ and $\Theta_j$ appear at the same position of the table.

I) **The irreducible case.** $\mathcal{L}_0 = \mathcal{O}_A(\Theta)$, where $\Theta$ is a smooth hyperelliptic curve. Then $\Theta$ contains six halfperiods $e_1, \dots, e_6$.

II) **The product case.** $\mathcal{L}_0 = \mathcal{O}_A(E_1 + E_2)$, where $E_1$ and $E_2$ are elliptic curves with $E_1 E_2 = 1$. Then $E_1 + E_2$ contains six halfperiods as smooth points and one halfperiod as the intersection of $E_1$ and $E_2$.

First we turn to the irreducible case. We will make use of the well-known $(16_6)$-configuration. On $A$ this means the following:

- every halfperiod lies on exactly six symmetric translates of $\Theta$.
- every symmetric translate of $\Theta$ contains exactly six halfperiods.

**The Property Nef.** Since obviously $\mathcal{M}_2^+$ has no additional contractions, we only have to consider $\mathcal{M}_d^\pm$ for $d \geq 3$ here.

**Lemma 4.1** *Let $\mathcal{L}_0 = \mathcal{O}_A(\Theta)$ be an irreducible principal polarization and let $d \geq 3$. Then $\mathcal{M}_d^\pm$ is nef. $\mathcal{M}_d^\pm$ has no additional contractions, except for $\mathcal{M}_3^+$, which contracts the image of $\Theta$.*

*Proof.* Let $C \subset \widetilde{X}$ be an irreducible curve different from the exceptional curves $D_1, \dots, D_{16}$ and let $F$ be the corresponding symmetric curve on $A$. We have

$$\mathcal{M}_d^\pm C = \frac{1}{2}(d\Theta F - \sum_{q_i = \mp 1} m_i),$$

where $m_i = m_i(F)$ and $q_i = q_i(d\Theta)$. It is enough to consider the bundles $\mathcal{M}_3^+$, $\mathcal{M}_3^-$ and $\mathcal{M}_4^-$.

1) Let us consider $\mathcal{M}_3^+$ first. Here $\sum\limits_{q_i=-1} m_i = \sum\limits_{i=1}^{6} m_i$. If $F = \Theta$, then we get $\mathcal{M}_3^+ C = 0$. Now let $F \neq \Theta$. In case $\sum\limits_{i=1}^{6} m_i = 0$ we clearly have $\mathcal{M}_3^+ C > 0$. In case $\sum\limits_{i=1}^{6} m_i > 0$ we have

$$3\Theta F \geq 3 \sum_{i=1}^{6} m_i > \sum_{i=1}^{6} m_i,$$

hence $\mathcal{M}_3^+ C > 0$ as well.

2) Now we turn to $\mathcal{M}_3^-$. In this case $\sum\limits_{q_i=+1} m_i = \sum\limits_{i=7}^{16} m_i$. We can assume $\sum\limits_{i=7}^{16} m_i > 0$, $m_{i_0} > 0$ say. Further, we may assume $F \neq \Theta$. Using table 1 we see that there are three symmetric translates $\Theta^{(1)}$, $\Theta^{(2)}$, $\Theta^{(3)}$ of $\Theta$ such that

i) all of the ten even halfperiods $e_7, \ldots, e_{16}$ lie on the divisor $\Theta^{(1)} + \Theta^{(2)} + \Theta^{(3)}$, and

ii) $e_{i_0}$ lies on at least two of the three translates.

So we have

$$3\Theta F = \sum_{k=1}^{3} \Theta^{(k)} F \geq \sum_{k=1}^{3} \sum_{e_i \in \Theta^{(k)}} m_i > \sum_{i=7}^{16} m_i,$$

hence $\mathcal{M}_3^- C > 0$.

3) Finally we consider $\mathcal{M}_4^-$. Here we can proceed as in 2) using four symmetric translates covering all the sixteen halfperiods to conclude $\mathcal{M}_4^- C > 0$. $\square$

In the product case we can proceed similarly, now using suitable symmetric translates of $E_1$ and $E_2$, to obtain

**Lemma 4.2** *Let $\mathcal{L}_0 = \mathcal{O}_A(E_1 + E_2)$ be a product principal polarization and let $d \geq 3$. Then $\mathcal{M}_d^{\pm}$ is nef. The additional contractions are given by the following table:*

| line bundle | contracted curves |
|---|---|
| $\mathcal{M}_4^-$ | the symmetric translates of $E_1$ and $E_2$ |
| $\mathcal{M}_3^+$ | the curves $E_1$ and $E_2$ |
| $\mathcal{M}_3^-$ | the symmetric translates of $E_1$ and $E_2$ different from $E_1$ and $E_2$ |

**Base Points.** In the discussion on base points in section 3 we left out $\mathcal{M}_3^-$ in the principally polarized case. We will fill this gap now:

**Lemma 4.3** *If $\mathcal{L}_0$ is a principal polarization, then $\mathcal{M}_3^-$ is free.*

*Proof.* It is sufficient to show that there are no base curves. First assume that we are in the irreducible case and let $\mathcal{L}_0 = \mathcal{O}_A(\Theta)$.

1) For $i = 1, \dots, 16$ we denote the image in $\widetilde{X}$ of the translate $t_{e_i}^*\Theta$ by $C_i$. By (1.5) the $C_i$ are $(-2)$-curves. Now we claim

$$\begin{aligned} \mathcal{M}_3^- &= \mathcal{O}_{\widetilde{X}}(C_2 + C_3 + C_7 + D_1 + D_2 + D_3 + D_7) \\ &= \mathcal{O}_{\widetilde{X}}(C_8 + C_9 + C_{10} + D_4 + D_5 + D_6 + D_7). \end{aligned} \tag{$*$}$$

Indeed, an immediate calculation shows that the pullbacks to $\widetilde{A}$ of all the three line bundles above are numerically equivalent. Then the line bundles themselves are numerically equivalent on $\widetilde{X}$. But on a K3-surface numerically equivalent line bundles are isomorphic.

2) From (*) we conclude that the only possible base curve for $\mathcal{M}_3^-$ is the exceptional curve $D_7$. In this case $\mathcal{M}_3^- - D_7$ is free. Further we have $(\mathcal{M}_3^- - D_7)^2 = 0$. As a consequence of Bertini's theorem then $\mathcal{M}_3^- - D_7 = \mathcal{O}_{\widetilde{X}}(kE)$, where $E$ is an elliptic curve and $k = h^1(\mathcal{M}_3^- - D_7) + 1$. By Riemann-Roch we compute $k = 3$, contradicting $(\mathcal{M}_3^- - D_7)D_8 = 1$.

The product case can be treated similarly, using translates of the elliptic curves $E_1$ and $E_2$ instead of $\Theta$-translates. We omit the details. $\square$

By the method used in (4.3) one can show that all the line bundles $\mathcal{M}_d^\pm$ are free, if $\mathcal{L}_0$ is of type $(1,1)$. So we do not really need to apply Reider's Theorem in the principally polarized case.

**Degree of the Morphisms.** In order to prove that the morphisms defined by $\mathcal{M}_3^\pm$ are birational, we had to know that the abelian surface $A$ cannot contain curves with certain properties. The following proposition completes the proof of (3.8) in the principally polarized case:

**Proposition 4.4** *Let $\mathcal{L}_0$ be of type $(1,1)$, let $F \subset A$ be a symmetric curve and let $s = \sum_{q_i = \mp 1} m_i$, where $q_i = q_i(\mathcal{L}_0^3)$ and $m_i = m_i(F)$. Then the following conditions cannot be fulfilled at the same time:*

*i)* $F^2 = \sum_{i=1}^{16} m_i^2$

*ii)* $3\mathcal{L}_0 F = 4 + s$

*iii)* $s = 2$ *or* $s = 8$.

*Proof.* Assume that there is a curve $F$ satisfying i), ii) and iii). First suppose $s = 2$. Then $F^2 \geq 2$ and $\mathcal{L}_0 F = 2$. It follows from the Index Theorem that then $F^2 = 2$, i.e. $\mathcal{O}_A(F)$ is of type $(1,1)$, contradicting i).

Now suppose $s = 8$. Here $F^2 \geq 8$ and $\mathcal{L}_0 F = 4$. Applying the Index Theorem again, we conclude that $\mathcal{O}_A(F)$ is algebraically equivalent to $\mathcal{L}_0^2$. Further, $F$ contains exactly eight halfperiods (as smooth points), all of which are even halfperiods of $\mathcal{L}_0$. According to (4.5) below this is impossible. $\square$

**Proposition 4.5** *Let $\mathcal{L}_0 = \mathcal{O}_A(\Theta)$ be of type $(1,1)$ and let $\mathcal{B}$ be a symmetric line bundle algebraically equivalent to $\mathcal{O}_A(2\Theta)$, $\mathcal{B} \not\cong \mathcal{O}_A(2\Theta)$. Then $\mathcal{B}$ has eight even and eight odd halfperiods. Neither of these two sets is contained in the set of ten even halfperiods of $\mathcal{L}_0$.*

*Proof.* We have $\mathcal{B} \cong t_a^*\mathcal{O}_A(2\Theta)$ for some $a \in A$. Necessarily $2a$ is a halfperiod, $e_i$ say. Thus by the Theorem of the Square ([3], 2.3.3)

$$t_a^* 2\Theta \equiv_{\text{lin}} \Theta + t_{e_i}^*\Theta,$$

i.e. $\Theta + t_{e_i}^*\Theta$ is a symmetric divisor in $|\mathcal{B}|$. It contains eight halfperiods to an even order and eight halfperiods to an odd order. Six resp. four of these are even halfperiods of $\mathcal{O}_A(\Theta)$. This proves our assertion. □

## 5. Special Case: Polarizations of Type $(1,3)$

In this section we complete our study of the line bundles $\mathcal{M}_d^{\pm}$ in case $\mathcal{L}_0$ is of type $(1,3)$. First we have a closer look at the line bundle $\mathcal{L}_0$. A straightforward calculation with intersection numbers shows that there are five cases:

I) **The irreducible case.** All divisors in $|\mathcal{L}_0|^+$ and $|\mathcal{L}_0|^-$ are irreducible.

II) **The quasi-product case.** $\mathcal{L}_0 = \mathcal{O}_A(E_1+E_2)$, where $E_1$ and $E_2$ are elliptic curves with $E_1E_2 = 3$.

III) **The hyperelliptic case.** $\mathcal{L}_0 = \mathcal{O}_A(H+E)$, where $E$ is an elliptic curve, $H$ a hyperelliptic curve and $EH = 2$.

IV) **The diagonal case.** $\mathcal{L}_0 = \mathcal{O}_A(E_1+E_2+E_3)$, where $E_1$, $E_2$ and $E_3$ are elliptic curves with $E_iE_j = 1$ for $i \neq j$.

V) **The product case.** $\mathcal{L}_0 = \mathcal{O}_A(E_1+3E_2)$, where $E_1$ and $E_2$ are elliptic curves with $E_1E_2 = 1$.

We have six odd halfperiods, $e_1, \ldots, e_6$ say, hence $h^0(\mathcal{L}_0)^- = 1$ and $h^0(\mathcal{L}_0)^+ = 2$. by (3.1). Let $\Theta^-$ denote the unique divisor in $|\mathcal{L}_0|^-$ and for $j = 7, \ldots, 16$ let $\Theta_j^+$ denote a divisor in $|\mathcal{L}_0|^+$ through $e_j$. Then we have

**Proposition 5.1** *a) If $\Theta^-$ is irreducible, then it is smooth.*

*b) If $\Theta_j^+$ is irreducible, then it has a double point in $e_j$ and is smooth away from $e_j$.*

*Proof.* a) Let $C$ be the image of $\Theta^-$ in $\widetilde{X}$. By (1.5) then

$$2C^2 = (\Theta^-)^2 - \sum_{i=1}^{16} m_i^2,$$

where $m_i = m_i(\Theta^-)$. $C$ is irreducible, hence $C^2 \geq -2$ by the adjunction formula. We conclude $\sum_{i=1}^{16} m_i^2 \leq 10$. Since $\Theta^-$ vanishes in the halfperiods $e_7, \ldots, e_{16}$, we must have $m_1 = \ldots = m_6 = 0$ and $m_7 = \ldots = m_{16} = 1$, thus $C^2 = -2$. So $C \cong \mathbb{P}_1$ is smooth. We conclude that $\Theta^-$ is smooth too, because the halfperiods are smooth points of $\Theta^-$.

b) Again we obtain $\sum_{i=1}^{16} m_i^2 \leq 10$. Since $\Theta_j^+$ vanishes in $e_1, \ldots, e_6$ and in $e_j$, we must have $m_1 = \ldots = m_6 = 1$ and $m_j = 2$, hence $C^2 = -2$ and the assertion follows as above. $\square$

Let us sum up what the results of section 3 mean for a non-product $(1,3)$-polarization: For $d \geq 2$ the line bundle $\mathcal{M}_d^{\pm}$ defines a birational morphism, which is an isomorphism outside the contracted curves. $\mathcal{M}_2^+$ and $\mathcal{M}_d^{\pm}$, $d \geq 3$, have no additional contractions. For $\mathcal{M}_2^-$ we have:

- In cases I) and II) $\mathcal{M}_2^-$ defines an embedding of $\widetilde{X}$.
- In case III) $\mathcal{M}_2^-$ contracts the four symmetric translates of $E$.
- In case IV) $\mathcal{M}_2^-$ contracts the 12 symmetric translates of $E_1$, $E_2$ and $E_3$.

In order to complete the proof of (3.8) it remains to show that $A$ does not contain curves with certain properties:

**Proposition 5.2** *Let $\mathcal{L}_0$ be of type $(1,3)$ and assume that we are not in the product case. Then there is no symmetric curve $F$ on $A$ such that*
*i)* $F^2 = 16$
*ii)* $m_1 = \ldots = m_{16} = 1$
*iii)* $\mathcal{L}_0 F = 10$

*Proof.* We consider the four cases arising if $\mathcal{L}_0$ is not a product polarization.

1) *The irreducible case.* We have $h^0(F)^- = 2$, so there is a divisor $F' \in |F|^-$ through an arbitrary point of $\Theta^-$. We conclude from $\Theta^- F = 10$ that $\Theta^- \subset F'$. For $j = 7, \ldots, 16$ we have $(F' - \Theta^-)\Theta_j^+ = 4$, hence $\Theta_j^+$ is contained in $F'$, because $F' - \Theta^-$ and $\Theta_j^+$ have six halfperiods in common. But obviously $F'$ cannot contain all the curves $\Theta_j^+$.

2) *The quasi-product case.* For this and the remaining cases note that the symmetric line bundle $\mathcal{O}_A(F) \otimes \mathcal{L}_0^{-1}$ is ample. Indeed, it has positive selfintersection and positive intersection with $\mathcal{L}_0$.

Using the fact that $\mathcal{O}_A(F)$ is totally symmetric, hence a square, we conclude from $(F - E_1 - E_2)E_i > 0$ that $FE_i \geq 4$ for $i = 1, 2$. We can assume $FE_1 = 4$, $FE_2 = 6$. Because of $h^0(F)^+ = 6$ there is a divisor $F' \in |F|^+$ through the four halfperiods of $E_2$. Since $F'$ then has even multiplicities in these halfperiods, $E_2$ must be contained in $F'$. We calculate $(F' - E_2)E_1 = 1$, hence $F' - E_2$ is algebraically equivalent to a divisor $4E_1 + E'$, where $E'$ is an elliptic curve with $E_1 E' = 1$. Intersecting $F' - E_2$ with $E_2$ we arrive at a contradiction.

3) *The hyperelliptic case.* We obtain $FH = 6$ and $FE = 4$ from $(F-H-E)H > 0$ and $(F - H - E)E > 0$. By the Index Theorem $\mathcal{O}_A(F - H - E)$ and $\mathcal{O}_A(H)$ are

algebraically equivalent. But this contradicts the fact that $\mathcal{O}_A(F-H-E)$ and $\mathcal{L}_0$ must have the same set of odd halfperiods.

4) *The diagonal case.* Here we get $FE_i \geq 4$ from $(F-E_1-E_2-E_3)E_i > 0$, which again is impossible. □

Combining the results of section 3 with those of of the principally polarized case (section 4) and with proposition 5.2 above, we obtain the theorem stated in the introduction.

## References


[1] Birkenhake, Ch., Lange, H.: Symmetric Theta-Structures. Manuscripta Math. **70**, 67-91 (1990)

[2] Khaled, A.: Projective normality and equations of Kummer varieties. Prépubl. Univ. Paris (1992)

[3] Lange, H., Birkenhake, Ch.: Complex Abelian Varieties. Berlin Heidelberg New York: Springer 1992

[4] Naruki, Isao: On Smooth Quartic Embedding of Kummer Surfaces. Proc. Japan Acad. **67**, Ser. A, 223-225 (1991)

[5] Ramanujam, C.P.: Supplement to the article 'Remarks on the Kodaira vanishing theorem'. Journal of the Indian Math. Soc. **38**, 121-124 (1974)

[6] Reider, Igor: Vector bundles of rank 2 and linear systems on algebraic surfaces. Annals of Mathematics **127**, 309-316 (1988)

[7] Saint-Donat, B.: Projective models of K3 surfaces. Amer. J. of Math. **96**, 602-639 (1974)

[8] Sasaki, R.: Bounds on the degree of the equations defining Kummer varieties. J. Math. Soc. Japan, Vol. 33, No. 2, 323-333 (1981)



Thomas Bauer
Mathematisches Institut der Universität
Bismarckstr. $1\frac{1}{2}$
W-8520 Erlangen