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Smooth quartic surfaces with 352 conics

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0. Introduction

The aim of this note is to show the existence of smooth quartic surfaces in $\mathbb{P}_3$ on which there lie

– 16 mutually disjoint smooth conics,

– altogether exactly $352 = 22 \cdot 16$ smooth conics.

Up to now the maximal number of smooth conics, that can lie on a smooth quartic surface, seems not to be known. So our number 352 should be compared with 64, the maximal number of lines that can lie on a smooth quartic [S].

We construct the surfaces as Kummer surfaces of abelian surfaces with a polarization of type $(1, 9)$. Using Saint-Donat's technique [D] we show that they embed in $\mathbb{P}_3$. In this way we only prove their existence and do, unfortunately, not find their explicit equations.

So there are the following obvious questions, which we cannot answer at the moment:

- What is the maximal number of smooth conics (or more general: of smooth rational curves of given degree d) on a smooth quartic surface in $\mathbb{P}_3$?
- What are the equations of the quartics in our (three–dimensional) family of surfaces, which contain 352 smooth conics?
- Using abelian surfaces with other polarizations, it is easy to write down candidates for Kummer surfaces containing 16 skew smooth rational curves of degree $d \geq 2$. Do they embed as smooth quartics in $\mathbb{P}_3$?

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1. Preliminaries

To describe the relation between an abelian surface A and its (desingularized) Kummer surface X we always use the following notation:

$$\begin{array}{ccccc} & \sigma & & \gamma & \\ e_i & \longleftarrow & E_i & \longrightarrow & D_i \\ & & \cap & & \cap \\ A & \longleftarrow & \widetilde{A} & \longrightarrow & X \\ & \text{blow up} & & \text{double} & \\ & \text{of the } e_i & & \text{cover} & \end{array} \tag{$*$}$$

where

A is the abelian surface,

$e_1, ..., e_{16} \in A$ the half–periods,

$E_1, ..., E_{16} \subset \widetilde{A}$ are the blow–ups of $e_1, ..., e_{16}$,

$\widetilde{A} \longrightarrow X$ is the double cover branched over $D_1, ..., D_{16}$, induced by the involution $a \longmapsto -a$ on A,

$E_i \longrightarrow D_i$ is bijective.

If $C \subset X$ is an irreducible curve, not one of the D_i, then its self–intersection is related to the self–intersection of the corresponding curve $F := \sigma\gamma^*(C) \subset A$ as follows: Let $m_i := C.D_i = \gamma^*C.E_i$. Then $\gamma^*C + \sum m_i E_i \subset \widetilde{A}$ descends to A, i.e. $\sigma^*F = \gamma^*C + \sum m_i E_i$ with m_i the multiplicity of F at e_i. This implies

$$F^2 = (\sigma^*F)^2 = (\gamma^*C + \sum m_i E_i)^2 = 2C^2 + \sum m_i^2. \tag{1}$$

We shall consider a line bundle $\mathcal{M}$ on X with $\mathcal{M}.D_i = 2$ for $i = 1, ..., 16$. Then $\gamma^*\mathcal{M} \otimes \mathcal{O}_{\widetilde{A}}(2\sum E_i)$ descends to a line bundle $\mathcal{L}$ on A and

$$\mathcal{L}.F = (\gamma^*\mathcal{M} \otimes \mathcal{O}_{\widetilde{A}}(2\sum E_i)).(\gamma^*C \otimes \mathcal{O}_{\widetilde{A}}(\sum m_i E_i)) = 2(\mathcal{M}.C + \sum m_i). \tag{2}$$

Sometimes we use the sloppy notation $\mathcal{L} - \sum m_i e_i$ to denote the sheaf $\prod \mathcal{I}_{e_i}^{m_i} \cdot \mathcal{L}$ on A, respectively the line bundle $\sigma^*\mathcal{L} \otimes \mathcal{O}_{\widetilde{A}}(\sum m_i E_i)$ on $\widetilde{A}$.

2. Sixteen skew conics

First we analyze the

Situation: $X \subset \mathbb{P}_3$ *is a smooth quartic surface with sixteen mutually disjoint conics* $D_1, ..., D_{16} \subset X$.

By Nikulin's theorem [N] there is a diagram $(*)$ representing X as the Kummer surface of an abelian surface A. We denote by $\widetilde{\mathcal{L}}$ on $\widetilde{A}$ the pull–back of the line bundle $\mathcal{O}_X(1)$. Then the self–intersection numbers are

$$(\mathcal{O}_X(1).\mathcal{O}_X(1)) = 4, \quad (\widetilde{\mathcal{L}}.\widetilde{\mathcal{L}}) = 8.$$

Since

$$(E_i.E_i) = -1 \quad \text{and} \quad (\widetilde{\mathcal{L}}.E_i) = (\mathcal{O}_X(1).D_i) = 2,$$

the line bundle $\widetilde{\mathcal{L}} \otimes \mathcal{O}_{\widetilde{A}}(2E_1 + ... + 2E_{16})$ descends to a symmetric line bundle $\mathcal{L}$ on A with self–intersection

$$(\mathcal{L}.\mathcal{L}) = (\widetilde{\mathcal{L}} \otimes \mathcal{O}_{\widetilde{A}}(2\sum E_i).\widetilde{\mathcal{L}} \otimes \mathcal{O}_{\widetilde{A}}(2\sum E_i)) = 8 + 8 \cdot 16 - 4 \cdot 16 = 72.$$

The general linear polynomial in $H^0(\mathcal{O}_X(1))$ induces a section in $\mathcal{L}$ vanishing at each e_i to the second order. Therefore the line bundle $\mathcal{L}$ is totally symmetric. So $\mathcal{L} = \mathcal{O}_A(2\Theta)$ where $\mathcal{O}_A(\Theta)$ is a symmetric line bundle on A of type

$$(3, 3) \quad \text{or} \quad (1, 9).$$

The map

$$A \leftarrow \widetilde{A} \rightarrow X \subset \mathbb{P}_3$$

is given by a linear system consisting of (symmetric or anti–symmetric) sections in $\mathcal{L}$ vanishing at the half–periods to the order two precisely. This implies that these sections are *symmetric*. The map therefore is given by some linear subsystem of

$$H^0(\mathcal{L}^{\otimes 2} - 2(e_1 + ... + e_{16}))^+.$$

First we exclude the case $(3, 3)$:

Claim 1: *Assume that $\Theta = 3T$ with a symmetric divisor $T \subset A$ defining a principal polarization on A. Then the linear system $|\mathcal{L}^{\otimes 2} - 2\sum e_i|$ induces a linear system on the (nonsingular) Kummer surface X, which is not very ample.*

Proof. We show, that the linear system is not ample on the translates of T by half–periods. In fact, if T is irreducible, then it contains six half–periods, hence

$$(\mathcal{L}^{\otimes 2} - 2\sum e_i).T = 12 - 12 = 0.$$

And if $T = T_1 + T_2$ with two elliptic curves T_j, then

$$(\mathcal{L}^{\otimes 2} - 2\sum e_i).T_j = 6 - 8 < 0.$$

□

3. Abelian surfaces of type (1,9)

Here we show, that the general surface of type $(1,9)$ indeed leads to a smooth quartic surface with 16 skew conics. To be precise, we assume: *A is an abelian surface with Néron–Severi group of rank 1, generated by the class of the (symmetric) line bundle $\mathcal{L}$ of type* $(1,9)$. We use the notation of diagram $(*)$.

Claim 2: *The linear system $|\mathcal{L}^{\otimes 2} - 2\sum e_i|^+$ is free of (projective) dimension three.*

Proof. Since $h^0(\mathcal{L}^{\otimes 2})^+ = 20$ we have

$$h^0(\mathcal{L}^{\otimes 2} - 2\sum e_i)^+ = h^0(\mathcal{L}^{\otimes 2} - \sum e_i)^+ \geq 20 - 16 = 4.$$

On the (nonsingular) Kummer surface X of A there is a line bundle $\mathcal{M}$ with

$$\sigma^*(\mathcal{L}^{\otimes 2} - 2\sum e_i) = \gamma^*(\mathcal{M}), \quad \sigma^* H^0(\mathcal{L}^{\otimes 2} - 2\sum e_i)^+ = \gamma^* H^0(\mathcal{M}).$$

If $|\mathcal{M}|$ has base points, then by [D, Corollary 3.2] it also has a base curve. This corresponds to a base curve $B \subset A$ of the linear system $|\mathcal{L}^{\otimes 2} - 2\sum e_i|^+$. Since the linear system is symmetric and invariant under all half–period translations, so is B. This implies $B \simeq 2k\Theta$. If $k > 0$, then the class $\mathcal{L}^{\otimes 2} - 2\sum e_i - B = -2(k-1)B - 2\sum e_i$ cannot be effective. So $B = 0$ and the base locus on X can consist of curves D_i only. Since it is invariant under half–period translations, it is of the form $k \cdot \sum D_i$, i.e.

$$h^0(\mathcal{L}^{\otimes 2} - 2\sum e_i)^+ = h^0(\mathcal{L}^{\otimes 2} - (2+k)\sum e_i)^+ \geq 4.$$

But this is impossible for $k \geq 1$, because then the bundle $\mathcal{L}^{\otimes 2} - (2+k)\sum e_i$ has negative self–intersection.

So far we showed that our linear system is free. I.e., as a linear system on X it is big and nef. Then by Ramanujam's vanishing theorem [R] it has no higher cohomology and from Riemann–Roch we find:

$$h^0(\mathcal{L}^{\otimes 2} - 2 \cdot \sum e_i)^+ = 4.$$

□

Claim 3: *The line bundle $\mathcal{M}$ on X is ample.*

Proof. We have to show that there is no irreducible curve $C \subset X$ with intersection number $\mathcal{M}.C = 0$. Any such curve would be a (-2)–curve different from $D_1, ..., D_{16}$. For each $i = 1, ..., 16$ we use the Hodge index inequality

$$\begin{aligned}
\mathcal{M}^2(C + D_i)^2 &\leq (\mathcal{M}C + \mathcal{M}D_i)^2 \\
&= (\mathcal{M}D_i)^2 \\
&= 4, \\
-4 + 2C.D_i &\leq 1
\end{aligned}$$

to find

$$m_i := C.D_i \leq 2.$$

Let $F \subset A$ be the curve $\sigma\gamma^*(C)$. It is symmetric and has at $e_i \in A$ the multiplicity m_i. This implies

$$\begin{aligned} F^2 &= 2C^2 + \sum m_i^2 \\ &= -4 + \sum m_i^2 \\ F.\Theta &= \sum m_i. \end{aligned}$$

by (1) and (2). Since Θ generates the Néron–Severi group of A, the curve F is homologous to $d\Theta$ for some $1 \leq d \in \mathbb{Z}$. From

$$18 \cdot d = F.\Theta = \sum m_i \leq 32$$

we conclude $d = 1$ and

$$\sum m_i = 18, \quad \sum m_i^2 = 22.$$

This implies that two of the multiplicities are 2, while the other fourteen are 1. The symmetric line bundle $\mathcal{O}_A(F)$ would have 14 odd half–periods, a contradiction with [LB, Proposition 4.7.5] □

Now we finally can prove

Claim 4: *The bundle $\mathcal{M}$ on X is very ample.*

Proof. By [D, Theorem 6.1.iii] it remains to show that $\mathcal{M}$ defines a morphism of degree 1. By [D, Theorem 5.2] we have to exclude the possibilities that there is

either an elliptic curve $C \subset X$ with $\mathcal{M}.C = 2$,

or an irreducible curve $H \subset X$ with $H^2 = 2$ and $\mathcal{M} = \mathcal{O}_X(2H)$.

The latter, however, cannot happen because $\mathcal{M}^2 = 4$. So let $C \subset X$ be elliptic with $\mathcal{M}.C = 2$ and $F \subset A$ the symmetric curve $\sigma\gamma^*(C)$. Let again $m_i = C.D_i$ be the multiplicity of F at e_i. For each i we use the Hodge index inequality

$$4(2C + D_i)^2 = \mathcal{M}^2(2C + D_i)^2 \leq (2\mathcal{M}.C + \mathcal{M}.D_i)^2 = 36$$

to conclude again $m_i \leq 2$.

As above we find

$$F.\Theta = 2 + \sum m_i \quad \text{and} \quad F^2 = \sum m_i^2.$$

Again we assume F is homologous with $d\Theta, 1 \leq d \in \mathbb{Z}$. Hence

$$18d = 2 + \sum m_i \leq 34 \quad \text{and} \quad d = 1.$$

So we find

$$\sum m_i = 16 \quad \text{and} \quad \sum m_i^2 = 18.$$

This implies that one of the multiplicities is 2, while one is 0 and the other fourteen ones are 1. This leads to the same kind of contradiction as above. □

4. Conics on the surface

Here we assume that $X = Km(A)$ is a surface as considered in the preceding section, by the linear system $|\mathcal{M}|$ embedded in $\mathbb{P}_3$ as a smooth quartic surface.

First we prove

Claim 5: *There are no lines on a quartic surface X as above.*

Proof. Assume that $C \subset X$ is a line, i.e. $\mathcal{M}C = 1$. This implies for the symmetric pre–image $F = \sigma\gamma^* C \subset A$

$$\Theta F = 1 + \sum m_i.$$

As F is homologous to some $d\Theta$, $d \geq 1$, the intersection number $\Theta F = 18d$ is even and $\sum m_i$ is odd. But on the other hand, by Riemann–Roch on $\widetilde{A}$ the Euler–Poincare–characteristic of $\gamma^* C$ is

$$\chi(\gamma^* C) = \frac{1}{2}\gamma^* C(\gamma^* C - \sum E_i) + \chi(\mathcal{O}_{\widetilde{A}}) = C^2 - \frac{1}{2}\sum CD_i + \chi(\mathcal{O}_{\widetilde{A}}),$$

which implies that $\sum m_i = \sum CD_i$ is even, a contradiction. □

Now we specify several divisors on X:

i) For each $i = 1, ..., 16$ the exceptional curve E_i over e_i maps bijectively into $\mathbb{P}_3$ Because of

$$(\mathcal{L}^{\otimes 2} - 2 \cdot \sum_1^{16} E_i).E_i = 2$$

the image curve D_i is a conic.

ii) That a divisor $L \in |\mathcal{L}^{\otimes 2} - 2\sum e_j|^+$ may have not only a double point, but a triple point in e_i, this imposes three additional conditions on L. So for each $i = 1, ..., 16$ there is a divisor

$$L_i \in |(\mathcal{L}^{\otimes 2} - 2\sum e_j) - 2 \cdot e_i| = |\mathcal{L}^{\otimes 2} - 2 \cdot \sum_{j \neq i} E_j - 4 \cdot E_i|.$$

Because of

$$(\mathcal{L}^{\otimes 2} - 2 \cdot \sum E_j).L_i = 72 - 4 \cdot 15 - 8 = 4$$

the proper transform of L_i in $\widetilde{A}$ maps two–to–one to a conic in $\mathbb{P}_3$, which we denote by C_i.

iii) Let $e_1, ..., e_6 \in A$ be the odd half–periods and $e_7, ..., e_{16}$ be the even ones. All odd sections from $H^0(\mathcal{L})^-$ vanish in the ten even half–periods. As $h^0(\mathcal{L})^- = 4$, we may impose three conditions on such a section. So for each triplet $i, j, k \subset \{1, ..., 6\}$ of numbers there is a divisor $L_{i,j,k} \in |\mathcal{L}|^-$ passing through e_i, e_j and e_k, and having then double points in these three half–periods. Because of

$$[\mathcal{L}^{\otimes 2} - 2 \cdot \sum E_i].[\mathcal{L} - (E_7 + ... + E_{16}) - 2 \cdot (E_i + E_j + E_k)] = 36 - 2 \cdot 10 - 4 \cdot 3 = 4$$

the proper transform of $L_{i,j,k}$ in $\widetilde{A}$ maps two–to–one to a conic $C_{i,j,k} \subset \mathbb{P}_3$.

Claim 6: *The curves $C_{ijk} \subset X$ are uniquely determined by the triplet $\{i,j,k\}$. For $\{i,j,k\} \neq \{l,m,n\}$ the curves C_{ijk} and C_{lmn} are different.*

Proof. If there would be two different curves $L_{ijk} \in |\mathcal{L}|^-$ through the same odd half–periods e_i, e_j, e_k, or if $L_{ijk} = L_{lmn}$ for $\{i,j,k\} \neq \{l,m,n\}$, then there would be some divisor $L \in |\mathcal{L}|^-$ passing through four odd half–periods e_i, e_j, e_k, e_l. Choose some half–period e such that $e_j = e_i + e$. The divisor $L + e$ then passes

- twice through e_i and e_j,
- once through the four odd half–periods $e_m, i, j \neq m = 1, ..., 6$,
- twice through the even half–periods $e_k + e, e_l + e$,
- once through six more even half–periods.

This shows

$$L.(L+e) \geq \underbrace{2 \cdot 4}_{e_i,e_j} + \underbrace{2}_{e_k,e_l} + \underbrace{2 \cdot 2}_{e_k+e,e_l+e} + 6 = 20.$$

Since L is irreducible, we conclude $L = L + e$ is invariant under translation by e. So L would descend to some curve L' on A/e of self–intersection $18/2 = 9$, a contradiction. □

By construction

$$L_i + 2E_i \equiv L_{ijk} + L_{lmn} \in |\mathcal{L}^{\otimes 2} - 2\sum E_\nu|^+$$

for $\{i,j,k,l,m,n\} = \{1,...,6\}$. So the pairs of conics $C_i + D_i$ and $C_{ijk} + C_{lmn}$ lie in the same plane.

The sixteen conics C_i as well as the sixteen conics D_i form an orbit under the half–period translation group of A. Each conic C_{klm} however creates a whole orbit of sixteen conics C^i_{klm}. All curves in the orbit are different, because the line bundle $\mathcal{L}$ does not admit half–period translations. Altogether we found

$$(2 + \binom{6}{3}) \cdot 16 = 22 \cdot 16 = 352$$

smooth conics on the quartic surface X, falling into 22 orbits of 16 ones.

It is a natural question to ask, whether the 16 conics $C^i_{klm}, i = 1, ..., 16$ in the same orbit are skew or not. In fact we have:

Claim 7: *In the orbit of sixteen conics $C^i_{klm}, i = 1, ..., 16$ each conic is disjoint from three other ones and meets 12 other ones in two points.*

Proof. After reordering of subscripts we may assume $\{k,l,m\} = \{1,2,3\}$. It suffices to consider $C_{123} \cap C^i_{123}$ for all half–periods $e_i \neq 0$. Now translation by e_i maps the sixtuplet $e_1,...,e_6$ of odd half–periods to a sixtuplet $e_1+e_i,...,e_6+e_i$ containing two odd and four even half–periods. Then there are the following two possibilities:

1) The triplet $e_1+e_i, e_2+e_i, e_3+e_i$ meets the triplet e_1, e_2, e_3 in two points, say

$$e_2 = e_1+e_i, \quad \{e_7,...,e_{10}\} = \{e_3+e_i,...,e_6+e_i\}, \quad \{e_{11},...,e_{16}\} = \{e_{11}+e_i,...,e_{16}+e_i\}$$

up to reordering. (This happens for three different e_i). Then the curves L_{123} and L^i_{123} have the following multiplicities at the half–periods

	L_{123}	L^i_{123}	intersection
e_1, e_2	2	2	$2\cdot 4$
e_3	2	1	2
e_7	1	2	2
$e_{11},...,e_{16}$	1	1	$6\cdot 1$

.

The intersection multiplicities add up to $18 = L_{123}.L^i_{123}$. The proper transforms of these curves on $\widetilde{A}$ therefore are disjoint.

2) The triplets $e_1+e_i, e_2+e_i, e_3+e_i$ and e_1, e_2, e_3 are disjoint, say

$$e_1+e_i = e_4, \quad e_2+e_i = e_7, e_3+e_i = e_8, e_5+e_i = e_9, e_6+e_i = e_{10},$$
$$\{e_{11}+e_i,...,e_{16}+e_i\} = \{e_{11},...,e_{16}\}$$

up to renumbering. Now the multiplicities

	L_{123}	L^i_{123}	intersection
e_2, e_3	2	1	$2\cdot 2$
e_7, e_8	1	2	$2\cdot 2$
$e_{11},...,e_{16}$	1	1	$6\cdot 1$

add up to 14. This implies that the conics C_{123} and C^i_{123} meet in two points. □

The 352 conics we found so far are all the conics which there are on the surface:

Claim 8: *A quartic surface X as considered above contains exactly 352 smooth conics.*

Proof. Let $C \subset X$ be some smooth conic. We show that C is one of the curves $D_i, C_i, C^i_{k,l,m}$. The conic C satisfies

$$\mathcal{M}C = 2 \quad \text{and} \quad C^2 = -2.$$

If C is different from $D_1,...,D_{16}$, then by (1) and (2), for its symmetric pre–image $F = \sigma\gamma^* C \subset A$ we find

$$\Theta F = 2 + \sum m_i \quad \text{and} \quad F^2 = -4 + \sum m_i^2.$$

Using that F is homologous to $d\Theta$ for some $d \geq 1$ we get

$$18d = 2 + \sum m_i \quad \text{and} \quad 18d^2 = -4 + \sum m_i^2.$$

Both C and D_i are conics, so $m_i = CD_i \leq 4$. If $m_i \geq 3$, then C and D_i lie in the same plane, hence $C = C_i$. Therefore we may assume $m_i \leq 2$. This implies $d = 1$ and we find

$$\sum m_i = 16 \quad \text{and} \quad \sum m_i^2 = 22 \ .$$

Then necessarily three of the multiplicities m_i are 2, while ten of them are 1 and the other three are 0. Since $\mathcal{O}_A(F)$ is one of the 16 symmetric translates of $\mathcal{O}_A(\Theta)$ this implies that F is one of the curves $C^i_{k,l,m}$. □

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