The UpMath Notation and Filing System

In elementary arithmetic there are three basic symmetry operations, namely

(1) passing to the negative or reciprocal, within the rational numbers

$$x \to -x \text{ resp. } x \to x^{-1}$$

(2) squaring/taking square root, within the algebraic numbers

$$x \to x^2, x \to \sqrt{x}$$

(3) exponentiating/taking logarithm, within the real or complex numbers

$$x \to \exp x, x \to \log x$$

These basic operations exist also in higher mathematics and mathematical physics.

RATIONAL DUALITY • POSITION-MOMENTUM CORRESPONDENCE

vector space $V \to V^\sharp$ dual vector space

commutative lc group $G \to G^\wedge$ Pontryagin dual group

ALGEBRAIC DUALITY • STATES-OBSERVABLES CORRESPONDENCE

vector space $V \to \mathcal{L}(V) = V \otimes V^\sharp$ endomorphism ring

C*-algebra $A \to A \to \sum_{\text{pure states } \varphi} A_\varphi \otimes A_\varphi^\sharp$ Gelfand-Naimark embedding

complex manifold $M \to M_\mathbb{R}$ underlying real manifold

configuration space $Q \subset \mathbb{R}^n \to T^\sharp(Q)$ phase space

TRANSCENDENTAL DUALITY • CLASSICAL-QUANTUM CORRESPONDENCE

finite set $n \to \mathbb{R}^n$ n-tuples

prime numbers $\mathbb{P} \to \mathbb{N} \ni n = \prod_p p^{n_p}$ natural numbers

manifold $M \to \mathcal{C}^\infty(M)$ smooth functions

Hilbert space $H \to \sum_{0 \le n} \bigvee^n H$ Fock space

The 8-FOLD FILING SYSTEM uses these dualities to classify mathematical objects by a specific type.

Each folder (collection of related documents) has a basic 3-letter type ABC, where

$$\left\{\begin{array}{ll} \text{A denotes the mathematical category} & \text{set, group, vector space, algebra,...} \\ \text{B denotes the size of the object} & \text{finite, 1-dimensional, infinite-dimensional,...} \\ \text{C denotes the relevant ground field} & \text{real numbers, complex numbers, number fields,...} \end{array}\right.$$

$$ABC = \left[\begin{array}{c|c} abc & abC \\ \hline Abc & AbC \\ aBc & aBC \\ \hline ABc & ABC \end{array}\right] \text{ TYPE of document}$$

For example, VER means finite-dimensional real vector spaces.

The dualities have the form

RATIONAL DUALITY

$$\begin{tikzcd} abc \arrow[dr, shift left] & abC \arrow[dl, shift left] \\ Abc \arrow[ur, shift left] & AbC \arrow[ul, shift left] \end{tikzcd} \quad \begin{tikzcd} aBc \arrow[dr, shift left] & aBC \arrow[dl, shift left] \\ ABc \arrow[ur, shift left] & ABC \arrow[ul, shift left] \end{tikzcd}$$

ALGEBRAIC DUALITY

$$\begin{tikzcd} abc \arrow[r, shift left] \arrow[d, shift right] & abC \arrow[l, shift left] \arrow[d, shift right] \\ Abc \arrow[r, shift left] \arrow[u, shift right] & AbC \arrow[l, shift left] \arrow[u, shift right] \end{tikzcd} \quad \begin{tikzcd} aBc \arrow[r, shift left] \arrow[d, shift right] & aBC \arrow[l, shift left] \arrow[d, shift right] \\ ABc \arrow[r, shift left] \arrow[u, shift right] & ABC \arrow[l, shift left] \arrow[u, shift right] \end{tikzcd}$$

TRANSCENDENTAL DUALITY

$$\begin{array}{c|c} abc & abC \\ \hline Abc & AbC \end{array} \iff \begin{array}{c|c} aBc & aBC \\ \hline ABc & ABC \end{array}$$

The RUNIC NOTATION is based on the following principles:

4 basic types of objects $\begin{cases} \bar{\mathsf{h}} & \text{set} \\ \bar{\mathbb{h}} & \text{group} \\ \mathsf{L} & \text{vector space} \\ \mathbb{L} & \text{algebra} \end{cases}$ elements/morphisms $\begin{cases} \mathsf{h} \in \bar{\mathsf{h}} & \bar{\mathsf{h}} \xrightarrow{\sim} \bar{\mathsf{h}}' \\ \mathbb{h} \in \bar{\mathbb{h}} & \bar{\mathbb{h}} \xrightarrow{\sim} \bar{\mathbb{h}}' \\ \mathsf{L} \in \mathsf{L} & \mathsf{L} \xrightarrow{\mathsf{L}} \mathsf{L}' \\ \mathbb{L} \in \mathbb{L} & \mathbb{L} \xrightarrow{\mathbb{L}} \mathbb{L}' \end{cases}$

triangle symbols for homomorphisms/tensor products, with positions reflecting functoriality

$$\begin{cases} {}^{\bar{\mathsf{h}}}\triangle\, \bar{\mathsf{h}}' & \text{set mappings} \\ {}^{\bar{\mathbb{h}}}\triangle\, \bar{\mathbb{h}}' & \text{group homomorphisms} \\ {}^{\mathsf{L}}\triangle\, \mathsf{L}' \bullet \mathsf{l} \triangledown \mathsf{L}' & \text{linear maps} \bullet \text{tensor product} \\ {}^{\mathbb{L}}\triangle\, \mathbb{L}' & \text{algebra homomorphisms} \end{cases}$$

4 orientations for each object $\dfrac{\mathsf{J} \xleftarrow{\mathsf{J}} \mathsf{J} \;\Big|\; \mathsf{L} \xrightarrow{\mathsf{L}} \mathsf{L}}{\mathsf{l} \xleftarrow{\mathsf{l}} \mathsf{l} \;\Big|\; \Gamma \xleftarrow{\Gamma} \Gamma}$ (example for vector spaces)

8 index positions for each object $\vdots\dot{A}\vdots$ plus center index. Examples:

linear map $\mathsf{l} \to {}_i\mathsf{L}^j$ matrix $\bullet$ smooth m-forms over manifold ${}^{\bar{\mathsf{h}}}\triangle_{\infty}\; {}^{\bar{\mathsf{h}}}_{-}\triangle^{m}\mathbb{R}$

composability of objects: vector space bidual $\mathsf{L} = \mathbb{K} \triangledown \overbrace{\mathsf{L} \triangle \mathbb{K}}$

4-fold products $\begin{cases} \times & \text{product} \\ \ast & \text{commutator} \\ \ast & \text{anti-commutator} \\ \ast & \end{cases}$ classical groups $\begin{cases} \mathsf{C}|\mathsf{L} = {}^{\mathsf{C}}_{n}\mathbb{K}^{n} & \text{general linear group} \\ \mathsf{U}|\mathsf{L} = {}^{\mathsf{U}}_{n}\mathbb{C}^{n} & \text{unitary group} \\ \cap|\mathsf{L} = {}^{\cap}_{2n}\mathbb{R}^{2n} & \text{symplectic group} \\ \supset|\mathsf{L} = {}^{\supset}_{n}\mathbb{R}^{n} & \text{orthogonal group} \end{cases}$ $\cdots$

the basic dualities mentioned above take the form

RATIONAL DUALITY: POSITION/MOMENTUM

$$\begin{array}{c|c} & \mathbb{K}\triangle^{0} \ni \mathtt{L} \\ \hline \mathtt{L}\triangle\mathbb{K} \in \triangleleft^{0}\mathbb{K} & \end{array} \quad \text{vector space duality} \quad \begin{array}{c|c} vek & veK \\ \hline Vek & VeK \end{array}$$

$$\begin{array}{c|c} & \triangle^{\omega}_{0} \ni \mathtt{h} \\ \hline \mathtt{h}\triangle_{0}\mathbb{T} \in \triangleleft^{\omega}_{0} & \end{array} \quad \text{Pontryagin duality} \quad \begin{array}{c|c} cek & ceK \\ \hline Cek & CeK \end{array}$$

ALGEBRAIC DUALITY: STATES/OBSERVABLES

$$\begin{array}{c|c} \mathfrak{E}|\mathtt{l} = \mathtt{l}\otimes\mathtt{L} & \mathtt{L} \\ \hline \mathtt{l} = \mathtt{L}\triangle\mathbb{K} & \mathtt{L}|\mathfrak{E} = \mathtt{L}\otimes\mathtt{l} \end{array} \quad \text{endomorphism ring} \quad \begin{array}{c|c} vek & veK \\ \hline Vek & VeK \end{array}$$

$$\begin{array}{c|c} \text{C*-algebra } \mathtt{J} \in \triangle^{\omega}_{0}\mathbb{C} \hookrightarrow \sum_{\Gamma\ \text{pure states}} \mathtt{J}_{\Gamma}\otimes\mathtt{J}^{\sharp}_{\Gamma} & \mathtt{J}_{\Gamma} \text{ GNS Hilbert space} \\ \hline \mathtt{J}^{\sharp}_{\Gamma} \text{ conjugate Hilbert space} & \end{array} \quad \text{GNS-embedding} \quad \begin{array}{c|c} xEc & xEC \\ \hline XEc & XEC \end{array}$$

$$\begin{array}{c|c} \text{underlying real manifold } \mathtt{h}\otimes\bar{\mathtt{h}} \in \mathbb{R}\triangle^{\omega}_{\infty} & \text{complex manifold } \mathtt{h} \in \mathbb{C}\triangle^{\omega}_{\omega} \\ \hline & \end{array} \quad \begin{array}{c|c} sec & seC \\ \hline Sec & SeC \end{array}$$

$$\begin{array}{c|c} \text{phase space } \mathtt{h}\times^{\mathtt{h}}_{-}\triangle\mathbb{R} & \text{configuration space } \mathtt{h} \subset \mathbb{R}^{n} \\ \hline & \end{array} \quad \begin{array}{c|c} ser & seR \\ \hline Ser & SeR \end{array}$$

TRANSCENDENTAL DUALITY: CLASSICAL/QUANTUM

$$\left[\begin{array}{c} \begin{array}{c|c} & \text{finite set } n \\ \hline & \end{array} \\ \begin{array}{c|c} & \\ \hline \text{n-columns } {}_{n}\mathbb{R} \in \triangleleft\mathbb{R} & \end{array} \end{array}\right] \left[\begin{array}{c|c} sor & soR \\ \hline Sor & SoR \\ sOr & sOR \\ \hline SOr & SOR \end{array}\right]$$

$$\left[\begin{array}{c} \begin{array}{c|c} & \text{manifold } \mathtt{h} \in \mathbb{R}\triangle^{\omega}_{\infty} \\ \hline & \end{array} \\ \begin{array}{c|c} \text{diff operators } \mathfrak{E}|^{\mathtt{h}}\triangle_{\infty}\mathbb{R} & \text{distributions } \mathbb{R}\triangleleft_{-\infty}\mathtt{h} \\ \hline \text{smooth functions } {}^{\mathtt{h}}\triangle_{\infty}\mathbb{R} & \end{array} \end{array}\right] \left[\begin{array}{c|c} ser & seR \\ \hline Ser & SeR \\ sEr & sER \\ \hline SEr & SER \end{array}\right]$$

$$\left[\begin{array}{c|c} & \text{dual Hilbert space } \bar{\llcorner} \in \mathbb{C}\,\overset{2}{\underset{0}{\triangle}} \\ \hline \text{Hilbert space } \underline{\urcorner} \in \overset{2}{\underset{0}{\triangledown}}\mathbb{C} & \\ \hline \text{symmetric Fock space } \underset{\mathbb{N}}{\mathbb{C}}\overset{+}{\triangledown}\underline{\urcorner} = \bar{\llcorner}\overset{+}{\triangle}\overset{\mathbb{N}}{\mathbb{C}} & \end{array}\right]
\left[\begin{array}{c|c} vic & viC \\ \hline Vic & ViC \\ vIc & vIC \\ \hline VIc & VIC \end{array}\right]$$

RELATIONS TO SEARCHING

The UpMath database, being a highly organized and relatively small system can of course serve only as a toy model for searching. However it has specifically search-oriented features:

SEARCH OPERATION: ABSTRACT CONCEPTS $\Longleftrightarrow$ CONCRETE EXAMPLES

Every abstract concept (e.g. non-commutative algebra) is linked to all examples realizing this concept, and vice-versa. This manual marking process should be supported by automatic searching. A similar search operation is planned linking every major theorem to all of its relevant consequences and, conversely, every mathematical argument to its relevant definitions or justifications.

RESTRICTED SEACHING VIA TYPE ASSIGNMENT

The assignment of a type to each document, as described above, makes it easy to search for formulas under specified restrictions. For example, searching for an algebraic formula like

$$a^2 + b^2 = c^2$$

can be restricted to discrete settings (number theory) or continuous settings (analysis) by specifying the type ABC. More restrictive searching (e.g. finite dimensional/infinite-dimensional) is easily implemented. A more sophisticated example concerns the Cauchy-Riemann differential equations in complex analysis

$$\bar{\partial} f = 0\cdot$$

Its solution space has quite distinct features viewed as a classical object (field equation) or as a quantum object (quantization Hilbert space). These features are separated by their different type. Alternatively, the runic symbols themselves contain enough graphical information for restricted searching.