$$\text{ɿ} \in {}^{\text{ƛ}}\triangle^{\infty}\mathbb{R}$$

$$\text{Untersumme } {}^{\mathcal{I}}\underset{\bullet}{\text{ɿ}} = \sum_{I}^{\mathcal{I}} |I|\, {}^{I}\underset{\bullet}{\text{ɿ}} \leqslant {}^{\mathcal{I}}\overset{\bullet}{\text{ɿ}} = \sum_{I}^{\mathcal{I}} |I|\, {}^{I}\overset{\bullet}{\text{ɿ}} \text{ Obersumme}$$

$$\mathcal{I} \prec \mathcal{J} \Rightarrow {}^{\mathcal{I}}\underset{\bullet}{\text{ɿ}} \leqslant {}^{\mathcal{J}}\underset{\bullet}{\text{ɿ}} \leqslant {}^{\mathcal{J}}\overset{\bullet}{\text{ɿ}} \leqslant {}^{\mathcal{I}}\overset{\bullet}{\text{ɿ}}$$

$${}^{\mathcal{J}}\underset{\bullet}{\text{ɿ}} = \sum_{J}^{\mathcal{J}} |J|\, {}^{J}\underset{\bullet}{\text{ɿ}} = \sum_{I}^{\mathcal{I}} \sum_{J \subset I}^{\mathcal{J}} |J| \underbrace{{}^{J}\underset{\bullet}{\text{ɿ}}}_{\geqslant {}^{I}\underset{\bullet}{\text{ɿ}}} \geqslant \sum_{I}^{\mathcal{I}} {}^{I}\underset{\bullet}{\text{ɿ}} \underbrace{\sum_{J \subset I}^{\mathcal{J}} |J|}_{= |I|} = {}^{\mathcal{I}}\underset{\bullet}{\text{ɿ}}$$

$${}^{\mathcal{J}}\overset{\bullet}{\text{ɿ}} = \sum_{J}^{\mathcal{J}} |J|\, {}^{J}\overset{\bullet}{\text{ɿ}} = \sum_{I}^{\mathcal{I}} \sum_{J \subset I}^{\mathcal{J}} |J| \underbrace{{}^{J}\overset{\bullet}{\text{ɿ}}}_{\leqslant {}^{I}\overset{\bullet}{\text{ɿ}}} \leqslant \sum_{I}^{\mathcal{I}} {}^{I}\overset{\bullet}{\text{ɿ}} \underbrace{\sum_{J \subset I}^{\mathcal{J}} |J|}_{= |I|} = {}^{\mathcal{I}}\overset{\bullet}{\text{ɿ}}$$

$$\bigwedge_{\mathcal{I}:\mathcal{J}}^{\text{part}} {}^{\mathcal{I}}\underset{\bullet}{\text{ɿ}} \leqslant {}^{\mathcal{J}}\overset{\bullet}{\text{ɿ}}$$

$${}^{\mathcal{I}}\underset{\bullet}{\text{ɿ}} \leqslant {}^{\mathcal{I} \curlyvee \mathcal{J}}\underset{\bullet}{\text{ɿ}} \leqslant {}^{\mathcal{I} \curlyvee \mathcal{J}}\overset{\bullet}{\text{ɿ}} \leqslant {}^{\mathcal{J}}\overset{\bullet}{\text{ɿ}}$$

$$\text{lower integral } \underset{\mathcal{I}}{\curlyvee} {}^{\mathcal{I}}\underset{\bullet}{\text{ɿ}} \leqslant \underset{\mathcal{I}}{\curlywedge} {}^{\mathcal{I}}\overset{\bullet}{\text{ɿ}} \text{ upper integral}$$

$$\bigwedge_{\mathcal{I}:\mathcal{J}}^{\text{part}} {}^{\mathcal{I}}\underset{\bullet}{\text{ɿ}} \leqslant {}^{\mathcal{J}}\overset{\bullet}{\text{ɿ}}$$

$$\Rightarrow \text{ ob Schranke } {}^{\mathcal{J}}\overset{\bullet}{\text{ɿ}} \geqslant \underset{\mathcal{I}}{\curlyvee} {}^{\mathcal{I}}\underset{\bullet}{\text{ɿ}} \text{ kleinste ob Schranke}$$

$$\Rightarrow \text{ unt Schranke } \int_{-}^{\text{ƛ}} \text{ɿ} \leqslant \underset{\mathcal{J}}{\curlywedge} {}^{\mathcal{J}}\overset{\bullet}{\text{ɿ}} = \int_{+}^{\text{ƛ}} \text{ɿ} \text{ groesste unt Schranke}$$

$${}^{\mathbb{I}}\triangle^{\infty}\mathbb{R} \ni \text{ɿ} \text{ integrable } \Leftrightarrow \underset{\mathcal{I}}{\curlyvee} {}^{\mathcal{I}}\underset{\bullet}{\text{ɿ}} = \int^{\text{ƛ}} \text{ɿ} = \underset{\mathcal{I}}{\curlywedge} {}^{\mathcal{I}}\overset{\bullet}{\text{ɿ}} \text{ integral}$$

$$\mathbb{I}^{\infty}_{\triangle_i}\mathbb{R} = \frac{\text{ገ} \in {}^{\mathbb{I}}\triangle^{\infty}\mathbb{R}}{\text{ገ} \text{ integrable}}$$

$$\text{ገ int} \Leftrightarrow \bigwedge_{\varepsilon}^{>0} \bigvee_{\mathcal{I}}^{\text{part}} {}^{\mathcal{I}}\dot{\text{ገ}} - {}^{\mathcal{I}}\underset{\bullet}{\text{ገ}} \leqslant \varepsilon$$

$$\text{ገ int} \wedge \int^{\text{ፕ}} \text{ገ} = S \Leftrightarrow \bigwedge_{\varepsilon}^{>0} \bigvee_{\mathcal{I}}^{\text{part}} {}^{\mathcal{I}}\dot{\text{ገ}} - \varepsilon \leqslant S \leqslant {}^{\mathcal{I}}\underset{\bullet}{\text{ገ}} + \varepsilon$$

$$\Rightarrow: \quad \bigwedge_{\varepsilon}^{>0} \bigvee_{\dot{\mathcal{I}}}^{\text{part}} S - \varepsilon \leqslant {}^{\mathcal{I}}\underset{\bullet}{\text{ገ}} \leqslant {}^{\dot{\mathcal{I}}}\dot{\text{ገ}} \leqslant S + \varepsilon \Rightarrow S - \varepsilon \leqslant {}^{\mathcal{I}}\underset{\bullet}{\text{ገ}} \leqslant {}^{\mathcal{I}\curlyvee\dot{\mathcal{I}}}\underset{\bullet}{\text{ገ}} \leqslant {}^{\mathcal{I}\curlyvee\dot{\mathcal{I}}}\dot{\text{ገ}} \leqslant {}^{\dot{\mathcal{I}}}\dot{\text{ገ}} \leqslant S + \varepsilon$$

$$\Leftarrow: \quad \int_{+}^{\text{ፕ}} \text{ገ} - \varepsilon \leqslant {}^{\mathcal{I}}\dot{\text{ገ}} - \varepsilon \leqslant S \leqslant {}^{\mathcal{I}}\underset{\bullet}{\text{ገ}} + \varepsilon \leqslant \int_{-}^{\text{ፕ}} \text{ገ} + \varepsilon \Rightarrow 0 \leqslant \int_{+}^{\text{ፕ}} \text{ገ} - \int_{-}^{\text{ፕ}} \text{ገ} \leqslant 2\varepsilon \underset{\varepsilon \curvearrowright 0}{\Rightarrow} \int_{+}^{\text{ፕ}} \text{ገ} = \int_{-}^{\text{ፕ}} \text{ገ}$$

$$\mathbb{I}^{\infty}_{\triangle}\mathbb{R} \ni \dot{\text{ገ}} \text{ integrable} \quad \overset{\text{Sum}}{\underset{\text{Rule}}{\Rightarrow}} \begin{cases} \mathbb{I}^{\infty}_{\triangle}\mathbb{R} \ni \text{ገ} + \text{ተ} \text{ integrable} \\ \int^{\text{ፕ}} \text{ገ} + \text{ተ} = \int^{\text{ፕ}} \text{ገ} + \int^{\text{ፕ}} \text{ተ} \end{cases}$$

$$\mathcal{I} \succ (\varepsilon/2)^{\text{ፕ}}_{\text{ገ}} \curlyvee (\varepsilon/2)^{\text{ፕ}}_{\text{ተ}}$$

$$\Rightarrow {}^{\mathcal{I}}\overset{\bullet}{\overbrace{\text{ገ} + \text{ተ}}} - \varepsilon = \sum_{I}^{\mathcal{I}} |I| \, {}^{I}\overset{\bullet}{\overbrace{\text{ገ} + \text{ተ}}} - \varepsilon \leqslant \sum_{I}^{\mathcal{I}} |I| \underbrace{{}^{I}\dot{\text{ገ}} + {}^{I}\dot{\text{ተ}}}_{\text{ob Schr}} - \varepsilon = {}^{\mathcal{I}}\dot{\text{ገ}} + {}^{\mathcal{I}}\dot{\text{ተ}} - \varepsilon$$

$$= \underbrace{{}^{\mathcal{I}}\dot{\text{ገ}} - \frac{\varepsilon}{2}} + \underbrace{{}^{\mathcal{I}}\dot{\text{ተ}} - \frac{\varepsilon}{2}} \leqslant \int^{\text{ፕ}} \text{ገ} + \int^{\text{ፕ}} \text{ተ} \leqslant \underbrace{{}^{\mathcal{I}}\underset{\bullet}{\text{ገ}} + \frac{\varepsilon}{2}} + \underbrace{{}^{\mathcal{I}}\underset{\bullet}{\text{ተ}} + \frac{\varepsilon}{2}}$$

$$= {}^{\mathcal{I}}\underset{\bullet}{\text{ገ}} + {}^{\mathcal{I}}\underset{\bullet}{\text{ተ}} + \varepsilon = \sum_{I}^{\mathcal{I}} |I| \underbrace{{}^{I}\underset{\bullet}{\text{ገ}} + {}^{I}\underset{\bullet}{\text{ተ}}}_{\text{unt Schr}} + \varepsilon \leqslant \sum_{I}^{\mathcal{I}} |I| \, {}^{I}\underset{\bullet}{\underbrace{\text{ገ} + \text{ተ}}} + \varepsilon = {}^{\mathcal{I}}\underset{\bullet}{\overbrace{\text{ገ} + \text{ተ}}} + \varepsilon$$

$$\mathbb{I}\overset{\infty}{\text{◺}}\mathbb{R} \ni \text{ɿ} \text{ integrable } \overset{\text{Scalar}}{\underset{\text{Rule}}{\Rightarrow}} \begin{cases} \mathbb{I}\overset{\infty}{\text{◺}}\mathbb{R} \ni \text{ɿ}\alpha \text{ integrable} \\ \int^{\text{ƙ}} \text{ɿ}\alpha = \alpha \int^{\text{ƙ}} \text{ɿ} \end{cases}$$

$$\mathcal{I} \succ (\varepsilon/\overline{\alpha})^{\text{ƙ}}_{\text{ɿ}}$$

$$\Rightarrow \begin{cases} {}^{\mathcal{I}}\overbrace{\dot{\text{ɿ}}\alpha} - \varepsilon = \underbrace{{}^{\mathcal{I}}\dot{\text{ɿ}}}\alpha - \varepsilon = \underbrace{{}^{\mathcal{I}}\dot{\text{ɿ}} - \frac{\varepsilon}{\alpha}}\alpha & \leqslant \underbrace{\int^{\text{ƙ}}}\text{ɿ}\alpha \leqslant \underbrace{{}^{\mathcal{I}}\underset{\bullet}{\text{ɿ}} + \frac{\varepsilon}{\alpha}}\alpha = \underbrace{{}^{\mathcal{I}}\underset{\bullet}{\text{ɿ}}}\alpha + \varepsilon = {}^{\mathcal{I}}\overbrace{\underset{\bullet}{\text{ɿ}}\alpha} + \varepsilon \\ {}^{\mathcal{I}}\overbrace{\text{ɿ}\,\dot{\underbrace{-\alpha}}} - \varepsilon = \underbrace{-{}^{\mathcal{I}}\underset{\bullet}{\text{ɿ}}}\alpha - \varepsilon = \underbrace{{}^{\mathcal{I}}\underset{\bullet}{\text{ɿ}} + \frac{\varepsilon}{\alpha}}(-\alpha) & \leqslant \underbrace{\int^{\text{ƙ}}}\text{ɿ}(-\alpha) \leqslant \underbrace{{}^{\mathcal{I}}\dot{\text{ɿ}} - \frac{\varepsilon}{\alpha}}(-\alpha) = \underbrace{-{}^{\mathcal{I}}\dot{\text{ɿ}}}\alpha + \varepsilon = \sum_{-}^{\mathcal{I}} \underbrace{\text{ɿ}(-\alpha)} + \varepsilon \end{cases}$$

$$\mathbb{I}\overset{\circ}{\text{◺}}\mathbb{R} \sqsupset \mathbb{I}\underset{i}{\overset{\infty}{\text{◺}}}\mathbb{R} \xrightarrow[\text{lin}]{\int^{\text{ƙ}}} \mathbb{R}$$

Monotony rule / $\mathbb{I}\overset{\infty}{\text{◺}}\mathbb{R} \ni \dot{\text{ɿ}}$ integrable

$$\text{ɿ} \leqslant \text{ɟ} \Rightarrow \int^{\text{ƙ}} \text{ɿ} \leqslant \int^{\text{ƙ}} \text{ɟ}$$

$$\int^{\text{ƙ}} \text{ɟ} - \int^{\text{ƙ}} \text{ɿ} = \int^{\text{ƙ}} \underbrace{\text{ɟ} - \text{ɿ}} \geqslant {}^{\mathcal{I}}\overbrace{\text{ɿ} \underset{\bullet}{-} \text{ɟ}} = \sum_{I}^{\mathcal{I}} |I| \overset{I}{\text{人}} \underbrace{\text{ɿ} - \text{ɟ}}_{\geqslant 0} \geqslant 0$$

$$\mathbb{I}\overset{\infty}{\text{◺}}\mathbb{R} \ni \text{ɿ} \text{ integrable } \Rightarrow |\text{ƙ}|\,{}^{\text{ƙ}}\underset{\bullet}{\text{ɿ}} \leqslant \int^{\text{ƙ}} \text{ɿ} \leqslant |\text{ƙ}|\,{}^{\text{ƙ}}\dot{\text{ɿ}}$$

$${}^{\text{ƙ}}\underset{\bullet}{\text{ɿ}} \leqslant \text{ɿ} \leqslant {}^{\text{ƙ}}\dot{\text{ɿ}} \Rightarrow |\text{ƙ}|\,{}^{\text{ƙ}}\underset{\bullet}{\text{ɿ}} = \int^{\text{ƙ}} {}^{\text{ƙ}}\underset{\bullet}{\text{ɿ}} \leqslant \int^{\text{ƙ}} \text{ɿ} \leqslant \int^{\text{ƙ}} {}^{\text{ƙ}}\dot{\text{ɿ}} = |\text{ƙ}|\,{}^{\text{ƙ}}\dot{\text{ɿ}}$$

$$\mathbb{I}^{\infty}_{\triangle}\mathbb{R} \ni \text{ʔ} \text{ integrable } \Longrightarrow \overline{\int^{\text{ƛ}} \text{ʔ}} \leqslant \int^{\text{ƛ}} \overline{\text{ʔ}} \leqslant |\text{ƛ}|^{\text{ƛ}}\dot{\overline{\text{ʔ}}}$$

$$-^{\text{ƛ}}\dot{\overline{\text{ʔ}}} \leqslant -\overline{\text{ʔ}} \leqslant \text{ʔ} \leqslant \overline{\text{ʔ}} \leqslant {}^{\text{ƛ}}\dot{\overline{\text{ʔ}}} \Longrightarrow -|\text{ƛ}|^{\text{ƛ}}\dot{\overline{\text{ʔ}}} = -\int^{\text{ƛ}} {}^{\text{ƛ}}\dot{\overline{\text{ʔ}}} \leqslant -\int^{\text{ƛ}} \overline{\text{ʔ}} \leqslant \int^{\text{ƛ}} \text{ʔ} \leqslant \int^{\text{ƛ}} \overline{\text{ʔ}} \leqslant \int^{\text{ƛ}} {}^{\text{ƛ}}\dot{\overline{\text{ʔ}}} = |\text{ƛ}|^{\text{ƛ}}\dot{\overline{\text{ʔ}}}$$