$$\bigvee_{J}^{I} = {}^{I>J}-1$$

$$\hbar_{\sim} \triangle_{\infty} \underbrace{\bar{\hbar}_{-} \triangle^{\pm} \mathbb{K}^{\mathbb{N}}} \xrightarrow[\Vdash \vDash]{\Vdash \ltimes} \hbar_{\sim} \triangle_{\infty} \underbrace{\bar{\hbar}_{-} \triangle^{\pm} \mathbb{K}^{\mathbb{N}}}$$

$${}^{z}(\Vdash \vDash \dashv) = \Vdash_{z} \vDash {}^{z}\dashv$$

$$\Vdash \ltimes = (\Vdash \vDash)\, d + d\, (\Vdash \vDash)$$

$${}_{M}\dashv\Vdash {}^{z}(\Vdash \ltimes \dashv) = {}_{M}\dashv\Vdash \Vdash_{z} {}^{z}\underline{\dashv} + \sum_{i}^{M} \bigvee_{M \llcorner i}^{i} \frac{{}_{i}\dashv\Vdash \Vdash_{-z}}{{}_{M \llcorner i}\dashv\Vdash} {}^{z}\dashv \quad {}_{M}\dashv\Vdash_{\sim} (\Vdash \ltimes \dashv) = \Vdash \ltimes ({}_{M}\dashv\Vdash \dashv) + \sum_{i}^{M} \bigvee_{M \llcorner i}^{i} \frac{{}_{i}\dashv\Vdash \times \Vdash}{{}_{M \llcorner i}\dashv\Vdash} \dashv$$

$${}_{M}\dashv\Vdash \left(\Vdash \vDash (d\dashv) + d\, (\Vdash \vDash \dashv)\right) = \frac{\Vdash_{z}}{{}_{M}\dashv\Vdash} {}^{z}(d\dashv) + \sum_{i}^{M} \bigvee_{M \llcorner i}^{i} {}_{M \llcorner i}\dashv\Vdash \left({}_{i}\dashv\Vdash {}^{z}\underline{\Vdash \vDash \dashv}\right)$$

$$= {}_{M}\dashv\Vdash \Vdash_{z} {}^{z}\underline{\dashv} - \sum_{i}^{M} \bigvee_{M \llcorner i}^{i} \frac{\Vdash_{z}}{{}_{M \llcorner i}\dashv\Vdash} {}_{i}\dashv\Vdash {}^{z}\underline{\dashv} + \sum_{i}^{M} \bigvee_{M \llcorner i}^{i} \left(\frac{{}_{i}\dashv\Vdash \Vdash_{-z}}{{}_{M \llcorner i}\dashv\Vdash} {}^{z}\dashv + \frac{\Vdash_{z}}{{}_{M \llcorner i}\dashv\Vdash} {}_{i}\dashv\Vdash {}^{z}\underline{\dashv} \right) = {}_{M}\dashv\Vdash {}^{z}(\Vdash \ltimes \dashv)$$

$${}_{M}\dashv\Vdash_{z} {}^{z}(\Vdash \ltimes \dashv) - \Vdash_{z} {}^{z}\underline{{}_{M}\dashv\Vdash_{\sim} {}^{\sim}\dashv} = {}_{M}\dashv\Vdash \Vdash_{z} {}^{z}\underline{\dashv} + \sum_{i}^{M} \bigvee_{M \llcorner i}^{i} \frac{{}_{i}\dashv\Vdash_{z} \Vdash_{-z}}{{}_{M \llcorner i}\dashv\Vdash_{z}} {}^{z}\dashv - {}_{M}\dashv\Vdash \Vdash_{z} {}^{z}\underline{\dashv} - \sum_{i}^{M} \bigvee_{M \llcorner i}^{i} \frac{\Vdash_{z\, i}\dashv\Vdash_{-z}}{{}_{M \llcorner i}\dashv\Vdash_{z}} {}^{z}\dashv$$

$$= \sum_{i}^{M} \bigvee_{M \llcorner i}^{i} \frac{{}_{i}\dashv\Vdash_{z} \Vdash_{-z} - \Vdash_{z\, i}\dashv\Vdash_{-z}}{{}_{M \llcorner i}\dashv\Vdash_{z}} {}^{z}\dashv$$

$$(\Vdash \ltimes) \times \left(\Vdash' \ltimes\right) = \left(\Vdash \times \Vdash'\right) \ltimes$$

$${}_{M}\dashv\vdash \llcorner^{z} \underline{\Vdash \ltimes \dashV} = {}_{M}\dashv\vdash \left(\llcorner \Vdash_{\underline{\ }z}{}^{z}\underline{\dashV} + (\llcorner \Vdash_{z})^{z}\underset{\varnothing}{\dashV}\right) + \sum_{i}^{M} \overset{i}{\underset{M\llcorner i}{\bigvee}} \left(\frac{{}_{i}\dashv\vdash\llcorner\dashV_{\varnothing}z}{{}_{M\llcorner i}\dashv\vdash}{}^{z}\dashV + \frac{{}_{i}\dashv\vdash\Vdash_{\underline{\ }z}}{{}_{M\llcorner i}\dashv\vdash}\llcorner^{z}\underline{\dashV}\right) {}_{M}\dashv\vdash^{z}\left(\Vdash \ltimes \left(\dot{\Vdash} \ltimes \dashV\right)\right)$$

$$= {}_{M}\dashv\vdash \Vdash_{z}{}^{z}\underline{\dot{\Vdash} \ltimes \dashV} + \sum_{i}^{M} \overset{i}{\underset{M\llcorner i}{\bigvee}} \frac{{}_{i}\dashv\vdash\Vdash_{\underline{\ }z}}{{}_{M\llcorner i}\dashv\vdash}{}^{z}\left(\dot{\Vdash} \ltimes \dashV\right) = {}_{M}\dashv\vdash \left(\left(\Vdash_{z}\dot{\Vdash}_{\underline{\ }z}\right)^{z}\underline{\dashV} + \left(\Vdash_{z}\dot{\Vdash}_{z}\right)^{z}\underset{\varnothing}{\dashV}\right)$$

$$+ \sum_{i}^{M} \overset{i}{\underset{M\llcorner i}{\bigvee}} \left(\frac{\left({}_{i}\dashv\vdash\Vdash_{z}\right)\dot{\Vdash}_{\varnothing}z}{{}_{M\llcorner i}\dashv\vdash}{}^{z}\dashV + \frac{{}_{i}\dashv\vdash\dot{\Vdash}_{\underline{\ }z}}{{}_{M\llcorner i}\dashv\vdash}\Vdash_{z}{}^{z}\underline{\dashV} + \frac{{}_{i}\dashv\vdash\Vdash_{\underline{\ }z}}{{}_{M\llcorner i}\dashv\vdash}\dot{\Vdash}_{z}{}^{z}\underline{\dashV}\right) + \sum_{i}^{M} \overset{i}{\underset{M\llcorner i}{\bigvee}} \left(\frac{{}_{i}\dashv\vdash\Vdash_{\underline{\ }z}\dot{\Vdash}_{\underline{\ }z}}{{}_{M\llcorner i}\dashv\vdash}{}^{z}\dashV - \sum_{j}^{M\llcorner i} \overset{j}{\underset{M\llcorner i:j}{\bigvee}} \frac{{}_{j}\dashv\vdash\dot{\Vdash}_{\underline{\ }z}}{{}_{i}\dashv\vdash\Vdash_{\underline{\ }z}}{}^{z}\dashV\right)$$

$$= {}_{M}\dashv\vdash \left(\Vdash_{z}\dot{\Vdash}_{\underline{\ }z}\right)^{z}\underline{\dashV} + \sum_{i}^{M} \overset{i}{\underset{M\llcorner i}{\bigvee}} \frac{{}_{i}\dashv\vdash\Vdash_{\underline{\ }z}\dot{\Vdash}_{\underline{\ }z} + \left({}_{i}\dashv\vdash\Vdash_{z}\right)\dot{\Vdash}_{\varnothing}z}{{}_{M\llcorner i}\dashv\vdash}{}^{z}\dashV = {}_{M}\dashv\vdash \left(\Vdash \times \Vdash'\right)_{z}{}^{z}\underline{\dashV} + \sum_{i}^{M} \overset{i}{\underset{M\llcorner i}{\bigvee}} \frac{{}_{i}\dashv\vdash\underline{\Vdash \times \Vdash'}_{z}}{{}_{M\llcorner i}\dashv\vdash}{}^{z}\dashV = {}_{M}\dashv\vdash^{z}\left(\left(\Vdash \times \Vdash'\right) \ltimes \dashV\right)$$

$$\Vdash \ltimes \left(\dashV \boxtimes \dashV'\right) = (\Vdash \ltimes \dashV) \boxtimes \dashV' + \dashV \boxtimes \left(\Vdash \ltimes \dashV'\right)$$

$${}_{M}\dashv\vdash \Vdash_{z}{}^{z}\underline{\dashV \boxtimes \dashV'} + \sum_{i}^{M} \overset{i}{\underset{M\llcorner i}{\bigvee}} \frac{{}_{i}\dashv\vdash\Vdash_{\underline{\ }z}}{{}_{M\llcorner i}\dashv\vdash}\left({}^{z}\dashV \boxtimes {}^{z}\dashV'\right) = \sum_{P \subset M} \overset{P}{\underset{M\llcorner P}{\bigvee}} \left(\left({}_{P}\dashv\vdash\Vdash_{z}{}^{z}\underline{\dashV}\right)\left({}_{M\llcorner P}\dashv\vdash^{z}\dashV'\right) + \left({}_{P}\dashv\vdash^{z}\dashV\right)\left({}_{M\llcorner P}\dashv\vdash\Vdash_{z}\right)^{z}\underline{\dashV'}\right)$$

$$+ \sum_{i}^{M} \overset{i}{\underset{M\llcorner i}{\bigvee}} \left(\sum_{Q \subset M\llcorner i} \overset{\bullet:Q}{\underset{O:M\llcorner Q \cup i}{\bigvee}} \frac{{}_{i}\dashv\vdash\Vdash_{\underline{\ }z}}{{}_{Q}\dashv\vdash}{}^{z}\dashV \left({}_{M\llcorner Q \cup i}\dashv\vdash^{z}\dashV'\right) + \overset{O:Q}{\underset{\bullet:M\llcorner Q \cup i}{\bigvee}} \left({}_{Q}\dashv\vdash^{z}\dashV\right) \frac{{}_{i}\dashv\vdash\Vdash_{\underline{\ }z}}{{}_{M\llcorner Q \cup i}\dashv\vdash}{}^{z}\dashV'\right)$$

$$= (*) + \sum_{P \subset M} \overset{P}{\underset{M\llcorner P}{\bigvee}} \left(\sum_{i}^{P} \overset{i}{\underset{P\llcorner i}{\bigvee}} \frac{{}_{i}\dashv\vdash\Vdash_{\underline{\ }z}}{{}_{P\llcorner i}\dashv\vdash}{}^{z}\dashV \left({}_{M\llcorner P}\dashv\vdash^{z}\dashV'\right) + \sum_{i}^{M\llcorner P} \overset{i}{\underset{M\llcorner P \cup i}{\bigvee}} \left({}_{P}\dashv\vdash^{z}\dashV\right) \frac{{}_{i}\dashv\vdash\Vdash_{\underline{\ }z}}{{}_{M\llcorner P \cup i}\dashv\vdash}{}^{z}\dashV'\right)$$

$$= \sum_{P \subset M} \overset{P}{\underset{M\llcorner P}{\bigvee}} \left(\left({}_{P}\dashv\vdash^{z}(\Vdash \ltimes \dashV)\right)\left({}_{M\llcorner P}\dashv\vdash^{z}\dashV'\right) + \left({}_{P}\dashv\vdash^{z}\dashV\right) {}_{M\llcorner P}\dashv\vdash^{z}\left(\Vdash \ltimes \dashV'\right)\right) = {}_{M}\dashv\vdash^{z}\mathrm{RHS}$$

$$(I)\, Q = P\llcorner i \subset M\llcorner i \Rightarrow \overset{P}{\underset{M\llcorner P \cup i}{\bigvee}} \overset{i}{\underset{P\llcorner i}{\bigvee}} = \overset{Q \cup i}{\underset{M\llcorner Q \cup i}{\bigvee}} \overset{i}{\underset{Q}{\bigvee}} = \overset{Q}{\underset{M\llcorner Q \cup i}{\bigvee}} \overset{i}{\underset{M\llcorner Q \cup i}{\bigvee}} \overset{i}{\underset{Q}{\bigvee}} = \overset{\bullet:Q}{\underset{\varnothing:M\llcorner Q \cup i}{\bigvee}} \overset{i}{\underset{M\llcorner i}{\bigvee}}$$

$$(\Vdash \ltimes) \times \left(\Vdash' \vDash\right) = \left(\Vdash \times \Vdash'\right) \vDash$$

$$ {}_{M}\dashv\!\!\Vdash{}^{z}(\Vdash \vDash \dashv) = \frac{\Vdash_z}{{}_{M}\dashv\!\!\Vdash}{}^{z}\dashv \implies {}_{M}\dashv\!\!\Vdash \mathsf{L}\,{}^{z}\underline{\Vdash \vDash \dashv} = \frac{\mathsf{L}\Vdash_{\underline{\ }z}}{{}_{M}\dashv\!\!\Vdash}{}^{z}\dashv + \frac{\Vdash_z}{{}_{M}\dashv\!\!\Vdash}\mathsf{L}\,{}^{z}\underset{-}{\dashv} \implies $$

$$ {}_{M}\dashv\!\!\Vdash{}^{z}\left(_{\Vdash}\left(d\left(\Vdash' \vDash \dashv\right)\right) - \Vdash' \vDash (\Vdash \ltimes \dashv)\right) = {}_{M}\dashv\!\!\Vdash \Vdash_z{}^{z}\underline{\Vdash' \vDash \dashv} + \sum_{i}^{M} \bigvee_{M\llcorner i}^{i} \frac{{}_{i}\dashv\!\!\Vdash \Vdash_{\underline{\ }z}}{{}_{M\llcorner i}\dashv\!\!\Vdash}{}^{z}\left(\Vdash' \vDash \dashv\right) - \frac{\Vdash'_z}{{}_{M}\dashv\!\!\Vdash}{}^{z}(\Vdash \ltimes \dashv) $$

$$ = \frac{\Vdash_z \Vdash'_{\underline{\ }z}}{{}_{M}\dashv\!\!\Vdash}{}^{z}\dashv + \frac{\Vdash'_z}{{}_{M}\dashv\!\!\Vdash}\Vdash_z{}^{z}\underset{-}{\dashv} + \sum_{i}^{M} \bigvee_{M\llcorner i}^{i} \begin{bmatrix} \Vdash'_z \\ {}_{i}\dashv\!\!\Vdash \Vdash_{\underline{\ }z} \\ {}_{M\llcorner i}\dashv\!\!\Vdash \end{bmatrix}{}^{z}\dashv - \frac{\Vdash'_z}{{}_{M}\dashv\!\!\Vdash}\Vdash_z{}^{z}\underset{-}{\dashv} - \frac{\Vdash'_z \Vdash_{\underline{\ }z}}{{}_{M}\dashv\!\!\Vdash}{}^{z}\dashv - \sum_{i}^{M} \bigvee_{M\llcorner i}^{i} \begin{bmatrix} {}_{i}\dashv\!\!\Vdash \Vdash_{\underline{\ }z} \\ \Vdash'_z \\ {}_{M\llcorner i}\dashv\!\!\Vdash \end{bmatrix}{}^{z}\dashv = \frac{\Vdash_z \Vdash'_{\underline{\ }z} - \Vdash'_z \Vdash_{\underline{\ }z}}{{}_{M}\dashv\!\!\Vdash}{}^{z}\dashv $$