$$ {}^{\ell}\blacktriangleleft n = \{\ell \xrightarrow[\text{inj}]{\iota} n\} $$

$$ \begin{array}{c} {}^{\ell}\blacktriangleleft n \\ \Big\downarrow \asymp \\ \mathsf{C}(\ell) \ltimes {}^{\ell}\triangleleft n \end{array} $$

$$ \ell \xrightarrow[\pi]{\text{bij}} \ell \xrightarrow[\jmath]{\text{str iso}} n, \qquad \ell \xrightarrow[\pi \ltimes \jmath]{\text{inj}} n $$

$$ \ell \xrightarrow[\text{inj}]{\iota} n \Longrightarrow \ell \text{ set } {}^{\ell}\iota = \{ {}^{0}\iota \cdots {}^{\ell-1}\iota \} \subset n $$

$$ \Longrightarrow \bigvee_{\text{eind}} \pi \in \mathsf{C}(\ell) \; {}^{\overset{-1}{\pi}0}\iota < {}^{\overset{-1}{\pi}0}\iota < \cdots < {}^{\overset{-1}{\pi}0}\iota $$

$$ {}^{i}\jmath = {}^{\overset{-1}{\pi}i}\iota $$

$$ \sharp\, {}^{\ell}\blacktriangleleft n = {}_{\ell}(n) = n(n-1)\cdots(n+1-\ell) \text{ falling power} $$

$$ \text{LHS} = \underbrace{\mathsf{C}(\ell)}\,\underbrace{{}^{\ell}\triangleleft n} = \ell! \begin{bmatrix} n \\ \ell \end{bmatrix} = \text{RHS} $$