$$ {}^{\ell}\!\!\triangle n = \frac{\ell \xrightarrow{\iota} n \text{ streng isoton}}{0 \leqslant {}^{0}\iota < {}^{1}\iota < \cdots < {}^{\ell\text{-}1}\iota < n} = \underbrace{{}^{\ell}\!\!\triangle n}_{\text{isoton}} \cap \underbrace{{}^{\ell}\!\!\blacktriangle n}_{\text{inj}} $$

$$ \sum_{0 \leqslant \ell} t^{\ell} \underbrace{\sharp {}^{\ell}\!\!\triangle n} = \overbrace{1+t}^{n} $$

$$ \underbrace{1+t_0}\underbrace{1+t_1}\cdots\underbrace{1+t_{\ell-1}} = \sum_{\ell}^{0|n} \sigma_{\ell}\left(t_0 \cdots t_{\ell-1}\right) $$

$$ \overbrace{1+t}^{n} = \sum_{\ell}^{0|n} \sigma_{\ell}\,(t \cdots t) = \sum_{\ell}^{0|n} \sum_{S \subset n}^{\sharp S = \ell} \prod_{i \in S} t = \sum_{\ell}^{0|n} \sum_{S \subset n}^{\sharp S = \ell} t^{\sharp S} $$

$$ = \sum_{\ell}^{0|n} \sum_{S \subset n}^{\sharp S = \ell} t^{\sharp S} = \sum_{\ell}^{0|n} t^{\ell} \sum_{S \subset n}^{\sharp S = \ell} 1 = \sum_{\ell}^{0|n} t^{\ell} \underbrace{\sharp {}^{\ell}\!\!\triangle n} $$

$$ \sharp\, {}^{\ell}\!\!\triangle n = \begin{bmatrix} n \\ \ell \end{bmatrix} = \frac{n!}{\ell!\,(n-\ell)!} = \begin{bmatrix} n \\ n-\ell \end{bmatrix} = \begin{bmatrix} n \\ \ell{:}n-\ell \end{bmatrix} \text{ cl binomial} $$

$$ \overbrace{1+t}^{n} \underset{\text{binomi}}{=} \sum_{\ell}^{0|n} \begin{bmatrix} n \\ \ell \end{bmatrix} 1^{n-\ell}\, t^{\ell} = \sum_{\ell} \begin{bmatrix} n \\ \ell \end{bmatrix} t^{\ell} $$

$$ \begin{bmatrix} n+1 \\ \ell \end{bmatrix} = \begin{bmatrix} n \\ \ell \end{bmatrix} + \begin{bmatrix} n \\ \ell-1 \end{bmatrix} $$