$$
{}^{\ell}\triangle n = \frac{\ell \xrightarrow{\iota} n \text{ weak isoton}}{0 \leqslant {}^{0}\iota \leqslant {}^{1}\iota \leqslant \cdots \leqslant {}^{\ell\text{-}1}\iota < n}
$$

$$
\sum_{0 \leqslant \ell} t^{\ell} \sharp\, {}^{\ell}\triangle n = \overbrace{1-t}{}^{-n}
$$

$$
\overbrace{1-t_{0}}{}^{-1}\, \overbrace{1-t_{1}}{}^{-1} \cdots \overbrace{1-t_{n-1}}{}^{-1} = \underbrace{1+t_{0}+t_{0}^{2}+\cdots}\,\underbrace{1+t_{1}+t_{1}^{2}+\cdots}\cdots\underbrace{1+t_{n-1}+t_{n-1}^{2}+\cdots}
$$

$$
= \sum_{0 \leqslant {}^{0}\alpha} \cdots \sum_{0 \leqslant {}^{n-1}\alpha} t_{0}^{{}^{0}\alpha} \cdots t_{n-1}^{{}^{n-1}\alpha}
$$

$$
|\alpha| = {}^{0}\alpha + \cdots + {}^{n-1}\alpha
$$

$$
\overbrace{1-t}{}^{-n} = \sum_{0 \leqslant {}^{0}\alpha} \cdots \sum_{0 \leqslant {}^{n-1}\alpha} t^{|\alpha|} = \sum_{0 \leqslant \ell} t^{\ell} \sum_{0 \leqslant {}^{0}\alpha : \cdots : 0 \leqslant {}^{n-1}\alpha}^{|\alpha| = \ell} 1 = \sum_{0 \leqslant \ell} t^{\ell} \sharp\, {}^{\ell}\triangle n
$$

$$
\sharp\, {}^{\ell}\triangle n = \begin{bmatrix} n+\ell-1 \\ \ell \end{bmatrix} = \begin{bmatrix} n+\ell-1 \\ n-1 \end{bmatrix}
$$

$$
(1-t)^{-n} = \sum_{0 \leqslant \ell} \begin{bmatrix} -n \\ \ell \end{bmatrix} (-t)^{\ell} = \sum_{0 \leqslant \ell} (-1)^{\ell} \begin{bmatrix} -n \\ \ell \end{bmatrix} t^{\ell}
$$

$$
(-1)^{\ell} \begin{bmatrix} -n \\ \ell \end{bmatrix} = (-1)^{\ell} \frac{(-n)(-n-1)\cdots(-n+1-\ell)}{\ell!} = \frac{n(n+1)\cdots(n-1+\ell)}{\ell!} = \frac{(n+1-\ell)!}{\ell!\,(n-1)!}
$$