

$$p + q = 12$$

$$\text{bi-quaternion } D_6^6: U_2^{\mathbb{H}} \times U_2^{\mathbb{H}}: \frac{L_5^{\mathbb{R}}}{L_5^{\mathbb{Z}}} \supset \frac{O_{4:4}^{\mathbb{R}}}{O_{4:4}^{\mathbb{Z}}} \sqsubset \frac{O_{4:20}^{\mathbb{R}}}{O_{4:20}^{\mathbb{Z}}} \sqsubset \frac{O_{4:20}^{\mathbb{R}}}{O_{4:20}^{\mathbb{Z}}} \times \frac{O_{1:1}^{\mathbb{R}}}{O_{1:1}^{\mathbb{Z}}}: U_1^{\mathbb{H}} \times U_1^{\mathbb{H}}$$

$$\text{quaternion } D_5^7 \left\{ \begin{array}{l} U_2^{\mathbb{H}}: \frac{O_{5:5}^{\mathbb{R}}}{O_{5:5}^{\mathbb{Z}}} \supset \frac{O_{3:3}^{\mathbb{R}}}{O_{3:3}^{\mathbb{Z}}} \sqsubset \frac{O_{3:19}^{\mathbb{R}}}{O_{3:19}^{\mathbb{Z}}} \sqsubset \frac{O_{3:19}^{\mathbb{R}}}{O_{3:19}^{\mathbb{Z}}} \times \frac{O_{1:1}^{\mathbb{R}}}{O_{1:1}^{\mathbb{Z}}}: U_1^{\mathbb{H}} \\ U_4^{\mathbb{H}}: \frac{E_6^{\mathbb{R}}}{E_6^{\mathbb{Z}}} \supset \frac{O_{5:5}^{\mathbb{R}}}{O_{5:5}^{\mathbb{Z}}} \sqsubset \frac{O_{5:21}^{\mathbb{R}}}{O_{5:21}^{\mathbb{Z}}} \sqsubset \frac{O_{5:21}^{\mathbb{R}}}{O_{5:21}^{\mathbb{Z}}} \times \frac{O_{1:1}^{\mathbb{R}}}{O_{1:1}^{\mathbb{Z}}}: U_2^{\mathbb{H}} \end{array} \right.$$

$$\text{complex } D_4^8 \left\{ \begin{array}{l} U_2^{\mathbb{C}}: \frac{L_3^{\mathbb{R}}}{L_3^{\mathbb{Z}}} \times \frac{L_2^{\mathbb{R}}}{L_2^{\mathbb{Z}}} \supset \frac{O_{2:2}^{\mathbb{R}}}{O_{2:2}^{\mathbb{Z}}} \sqsubset \frac{O_{2:18}^{\mathbb{R}}}{O_{2:18}^{\mathbb{Z}}} \sqsubset \frac{O_{2:18}^{\mathbb{R}}}{O_{2:18}^{\mathbb{Z}}} \times \frac{O_{1:1}^{\mathbb{R}}}{O_{1:1}^{\mathbb{Z}}}: U_1^{\mathbb{C}} \\ U_8^{\mathbb{C}}: \frac{E_7^{\mathbb{R}}}{E_7^{\mathbb{Z}}} \supset \frac{O_{6:6}^{\mathbb{R}}}{O_{6:6}^{\mathbb{Z}}} \sqsubset \frac{O_{6:22}^{\mathbb{R}}}{O_{6:22}^{\mathbb{Z}}} \sqsubset \frac{O_{6:22}^{\mathbb{R}}}{O_{6:22}^{\mathbb{Z}}} \times \frac{L_2^{\mathbb{R}}}{L_2^{\mathbb{Z}}}: U_4^{\mathbb{C}} \end{array} \right.$$

$$\text{real } D_3^9 \left\{ \begin{array}{l} U_2^{\mathbb{R}}: \frac{O_{1:1}^{\mathbb{R}}}{O_{1:1}^{\mathbb{Z}}} \times \frac{L_2^{\mathbb{R}}}{L_2^{\mathbb{Z}}} \supset \frac{O_{1:1}^{\mathbb{R}}}{O_{1:1}^{\mathbb{Z}}} \sqsubset \frac{O_{1:17}^{\mathbb{R}}}{O_{1:17}^{\mathbb{Z}}} \sqsubset \frac{O_{1:17}^{\mathbb{R}}}{O_{1:17}^{\mathbb{Z}}} \times \frac{O_{1:1}^{\mathbb{R}}}{O_{1:1}^{\mathbb{Z}}}: U_1^{\mathbb{R}} \\ U_{16}^{\mathbb{R}}: \frac{E_8^{\mathbb{R}}}{E_8^{\mathbb{Z}}} \supset \frac{O_{7:7}^{\mathbb{R}}}{O_{7:7}^{\mathbb{Z}}} \sqsubset \frac{O_{7:23}^{\mathbb{R}}}{O_{7:23}^{\mathbb{Z}}} \sqsubset \frac{O_{8:24}^{\mathbb{R}}}{O_{8:24}^{\mathbb{Z}}}: U_8^{\mathbb{R}} \end{array} \right.$$

$$D_2^{10} \left\{ \begin{array}{l} \frac{O_{1:1}^{\mathbb{R}}}{O_{1:1}^{\mathbb{Z}}} / \frac{L_2^{\mathbb{R}}}{L_2^{\mathbb{Z}}} \supset \frac{O_{0:0}^{\mathbb{R}}}{O_{0:0}^{\mathbb{Z}}} \sqsubset \frac{O_{0:16}^{\mathbb{R}}}{O_{0:16}^{\mathbb{Z}}} \sqsubset \frac{O_{0:16}^{\mathbb{R}}}{O_{0:16}^{\mathbb{Z}}} \times \frac{O_{1:1}^{\mathbb{R}}}{O_{1:1}^{\mathbb{Z}}} \\ E_9^{\mathbb{R}} \supset \frac{O_{8:8}^{\mathbb{R}}}{O_{8:8}^{\mathbb{Z}}} \sqsubset \frac{O_{8:24}^{\mathbb{R}}}{O_{8:24}^{\mathbb{Z}}} \sqsubset \frac{O_{8:24}^{\mathbb{R}}}{O_{8:24}^{\mathbb{Z}}} \end{array} \right.$$

$$D_1^{11} \left\{ \begin{array}{l} E_{10}^{\mathbb{R}} \supset \frac{O_{9:9}^{\mathbb{R}}}{O_{9:9}^{\mathbb{Z}}} \sqsubset \frac{O_{8:24}^{\mathbb{R}}}{O_{8:24}^{\mathbb{Z}}} \sqsubset \frac{O_{8:24}^{\mathbb{R}}}{O_{8:24}^{\mathbb{Z}}} \\ E_{10}^{\mathbb{Z}} \supset \frac{O_{9:9}^{\mathbb{R}}}{O_{9:9}^{\mathbb{Z}}} \sqsubset \frac{O_{8:24}^{\mathbb{R}}}{O_{8:24}^{\mathbb{Z}}} \sqsubset \frac{O_{8:24}^{\mathbb{R}}}{O_{8:24}^{\mathbb{Z}}} \end{array} \right.$$

$$D_0^{12}$$