$$q^m \blacktriangle q^n = \frac{q^m \xrightarrow{\Lsh} q^n \quad \text{q-lin}}{\Lsh \text{ inj}} = \left\{ q^m \xrightarrow[\text{inj lin}]{\Lsh^{\cdot}} q^n \right\} \ni \Lsh^{\cdot} = \begin{bmatrix} \Lsh_9 0 \cdot \\ {}_{m-}\Lsh^{\cdot} \end{bmatrix} \text{ free}$$

$$\sharp\, {}^{q^m} \blacktriangle q^n = \underbrace{q^n - 1}\,\underbrace{q^n - q}\,\underbrace{q^n - q^2} \cdots \underbrace{q^n - q^{m-1}} = \prod_i^m \underbrace{q^n - q^i} = \prod_i^{0|m-} \underbrace{q^n - q^i}$$

$$q^m \xrightarrow[\text{inj}]{\Lsh} q^n$$

$$\Lsh_9 0 \cdot \in q^n \llcorner 0 \Longrightarrow \sharp\, {}_0\Lsh^{\cdot} = q^n - 1$$

$${}_1\Lsh^{\cdot} \in q^n \llcorner q\, {}_0\Lsh^{\cdot} \Longrightarrow \sharp\, {}_1\Lsh^{\cdot} = q^n - q$$

$${}_2\Lsh^{\cdot} \in q^n \llcorner \underbrace{q\, {}_0\Lsh^{\cdot} + q\, {}_2\Lsh^{\cdot}} \Longrightarrow \sharp\, {}_2\Lsh^{\cdot} = q^n - q^2$$

$${}_{m-}\Lsh^{\cdot} \in q^n \llcorner \underbrace{q\, {}_0\Lsh^{\cdot} + q\, {}_2\Lsh^{\cdot} + \cdots + q\, {}_{m-2}\Lsh^{\cdot}} \Longrightarrow \sharp\, {}_{m-}\Lsh^{\cdot} = q^n - q^{m-}$$

$$\Longrightarrow \sharp\, {}^{q^m} \blacktriangle q^n = \sharp\, \Lsh^{\cdot} = \underbrace{q^n - 1}\,\underbrace{q^n - q}\,\underbrace{q^n - q^2} \cdots \underbrace{q^n - q^{m-}} = \text{ RHS}$$

$$\text{basis} \quad \begin{matrix} {}_0\Rsh^{\cdot} \\ + \\ {}_{m\text{-}1}\Rsh^{\cdot} \end{matrix} \in q^m \Longrightarrow \text{ frei } \begin{matrix} {}_0\Rsh^{\cdot}\Lsh \\ + \\ {}_{m\text{-}1}\Rsh^{\cdot}\Lsh \end{matrix} \in q^m$$

$${}_0\Rsh\ \Lsh \in q^n \llcorner \underbrace{q < 0 >}_{\sharp = 1} \Longrightarrow \sharp = q^n - 1 = q^n - q^0$$

$${}_1\Rsh\ \Lsh \in q^n \llcorner \underbrace{q < {}_0\Rsh\Lsh >}_{\sharp = q} \Longrightarrow \sharp = q^n - q = q^n - q^1$$

$${}_2\Rsh\ \Lsh \in q^n \llcorner \underbrace{q < {}_0\Rsh\Lsh : {}_1\Rsh\Lsh >}_{\sharp = q^2} \Longrightarrow \sharp = q^n - q^2$$

$${}_{m\text{-}1}\Rsh\ \Lsh \in q^n \llcorner \underbrace{q < {}_0\Rsh\Lsh : {}_1\Rsh\Lsh : \cdots : {}_{m-2}\Rsh\Lsh >}_{\sharp = q^{m\text{-}1}} \Longrightarrow \sharp = q^n - q^{m\text{-}1}$$

$$\frac{\sharp\, {}^{q^m} \blacktriangle q^n}{q^{nm}} = \frac{q^n - 1}{q^n} \frac{q^n - q}{q^n} \frac{q^n - q^2}{q^n} \cdots \frac{q^n - q^{m-1}}{q^n}$$

$$= \underbrace{1 - q^{-n}}\,\underbrace{1 - q^{1-n}}\,\underbrace{1 - q^{2-n}} \cdots \underbrace{1 - q^{m-1-n}}$$

$$\frac{\sharp\, q^{m} \blacktriangleright q^{n}}{q^{nm}} = \sum_{k}^{0|m} \left(-1\right)^{k} q^{k\,(1\text{-}n)} \begin{bmatrix} m \\ k \end{bmatrix} = \sum_{k}^{0|m} \left(-q^{1\,-\,n}\right)^{k} \begin{bmatrix} m \\ k \end{bmatrix}$$

$$\underbrace{q^{n} - 1}\,\underbrace{q^{n} - q}\cdots\underbrace{q^{n} - q^{m\text{-}1}} = q^{nm} \sum_{k}^{0|m} \left(-1\right)^{k} q^{k\,(1\text{-}n)} \begin{bmatrix} m \\ k \end{bmatrix}$$