

$$\mathcal{E}_{\downarrow}^{n\mathbb{Z}} = \sum_{\uparrow}^{n\mathbb{Z}} \mathcal{E}^{-\uparrow \uparrow^* \downarrow + \uparrow \downarrow} 2\pi i$$

$$\mathcal{I} \downarrow > 0: \quad \mathbb{R}_n \ni \Gamma$$

$$\mathcal{E}_{\uparrow}^{n\mathbb{Z}} = \sum_{\downarrow}^{n\mathbb{Z}} \mathcal{E}^{\pi i \downarrow \uparrow \downarrow + 2\pi i \Gamma}$$

$$\uparrow: \downarrow \in \mathbb{Z}_n \Rightarrow \mathcal{E}_{\uparrow + \downarrow + \uparrow}^{n\mathbb{Z}} = \mathcal{E}^{-\pi i \downarrow \uparrow \downarrow - 2\pi i \Gamma^*} \mathcal{E}_{\uparrow}^{n\mathbb{Z}}$$

$$\begin{aligned} \mathcal{E}_{\uparrow + \downarrow + \uparrow}^{n\mathbb{Z}} &= \sum_{\downarrow}^{n\mathbb{Z}} \mathcal{E}^{2\pi i \downarrow \uparrow \downarrow + 2\pi i \Gamma + \uparrow \downarrow} = \underbrace{\mathcal{E}^{2\pi i \downarrow}}_{=1} \sum_{\downarrow}^{n\mathbb{Z}} \mathcal{E}^{\pi i \downarrow \uparrow \downarrow + 2\pi i \Gamma + 2\pi i \downarrow} \\ &= \sum_{\downarrow}^{n\mathbb{Z}} \mathcal{E}^{\pi i \downarrow + \downarrow \uparrow \downarrow + \downarrow - \pi i \downarrow \uparrow \downarrow + 2\pi i \Gamma + \downarrow - 2\pi i \Gamma^*} = \mathcal{E}^{-\pi i \downarrow \uparrow \downarrow - \Gamma^*} \underbrace{\sum_{\downarrow}^{n\mathbb{Z}} \mathcal{E}^{\pi i \downarrow + \downarrow \uparrow \downarrow + 2\pi i \Gamma + \downarrow}}_{= \mathcal{E}_{\uparrow}^{n\mathbb{Z}}} \end{aligned}$$

$$\mathbb{I}_{\uparrow \downarrow}^T \Theta = \frac{-1/2}{\mathbb{I}^{\downarrow}} \mathcal{E}_{\uparrow} \Theta$$

$$\mathcal{E}_{\uparrow + \downarrow} \Theta = \mathcal{E}_{\uparrow} \Theta$$