$$S_{n|n} = \frac{\begin{bmatrix}\varphi\\ \psi\end{bmatrix} \in \mathbb{C}_{n|n}}{\varphi\overset{*}{\varphi} = 1 = \psi\overset{*}{\psi}: \quad \varphi\overset{*}{\psi} = 0 = \psi\overset{*}{\varphi}} = \frac{\begin{bmatrix}\varphi\\ \psi\end{bmatrix} \in \mathbb{C}_{n|n}}{\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\begin{bmatrix}\overset{*}{\varphi} & \overset{*}{\psi}\end{bmatrix} = \begin{array}{c|c}1 & 0\\ \hline 0 & 1\end{array} = I} \mathbb{C} \text{ co-sphere bundle}$$

$$U_n^{\mathbb{C}} \text{ -inv}$$

$$S_{n|n} \leftarrow U_2^{\mathbb{C}} \ltimes S_{n|n}$$

$$\underbrace{\begin{array}{c|c}a & b\\ \hline c & d\end{array}\begin{bmatrix}\varphi\\ \psi\end{bmatrix}}_{} \overbrace{\begin{array}{c|c}a & b\\ \hline c & d\end{array}\begin{bmatrix}\varphi\\ \psi\end{bmatrix}}^{*} = \begin{array}{c|c}a & b\\ \hline c & d\end{array} \underbrace{\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\begin{bmatrix}\overset{*}{\varphi} & \overset{*}{\psi}\end{bmatrix}}_{=I} \overset{*}{\begin{array}{c|c}a & b\\ \hline c & d\end{array}} = \begin{array}{c|c}a & b\\ \hline c & d\end{array} \overset{*}{\begin{array}{c|c}a & b\\ \hline c & d\end{array}} = I$$

$$Q\left(\begin{array}{c|c}a & b\\ \hline c & d\end{array}\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\right) = \det \begin{array}{c|c}a & b\\ \hline c & d\end{array} Q\left(\begin{bmatrix}\varphi\\ \psi\end{bmatrix}\right)$$

$$\overbrace{a\varphi + b\psi}^{t} \underbrace{c\varphi + d\psi}_{} - \overbrace{c\varphi + d\psi}^{t} \underbrace{a\varphi + b\psi}_{} = \underbrace{a\overset{t}{\varphi} + b\overset{t}{\psi}}_{} \underbrace{c\varphi + d\psi}_{} - \underbrace{c\overset{t}{\varphi} + d\overset{t}{\psi}}_{} \underbrace{a\varphi + b\psi}_{} = \underbrace{\overset{t}{\varphi}\psi - \overset{t}{\psi}\varphi}_{} \underbrace{ad - bc}_{}$$

$$S_1 = SU_2^{\mathbb{C}} \neg \, S_{n|n}$$

$$S_{2n} = \frac{\begin{bmatrix}\varphi & \psi\end{bmatrix} \in \mathbb{C}_{2n}}{\varphi\overset{*}{\varphi} + \psi\overset{*}{\psi} = 1} = \frac{\begin{bmatrix}\varphi & \psi\end{bmatrix} \in \mathbb{C}_{2n}}{\begin{bmatrix}\varphi & \psi\end{bmatrix}\begin{bmatrix}\overset{*}{\varphi}\\ \overset{*}{\psi}\end{bmatrix} = 1} \text{ sphere}$$

$$S_{2n} \rtimes SU_2^{\mathbb{C}} \to S_{2n}$$

$$\underbrace{\begin{bmatrix}\varphi & \psi\end{bmatrix}\begin{array}{c|c}a & b\\ \hline c & d\end{array}}_{} \overbrace{\begin{bmatrix}\varphi & \psi\end{bmatrix}\begin{array}{c|c}a & b\\ \hline c & d\end{array}}^{*} = \begin{bmatrix}\varphi & \psi\end{bmatrix} \underbrace{\begin{array}{c|c}a & b\\ \hline c & d\end{array} \overset{*}{\begin{array}{c|c}a & b\\ \hline c & d\end{array}}}_{=I} \begin{bmatrix}\overset{*}{\varphi}\\ \overset{*}{\psi}\end{bmatrix} = \begin{bmatrix}\varphi & \psi\end{bmatrix}\begin{bmatrix}\overset{*}{\varphi}\\ \overset{*}{\psi}\end{bmatrix} = 1$$

$$S_{2n} \ulcorner SU_2^{\mathbb{C}} = S_{n|n}$$

$$S_{2n} \ni \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \Longrightarrow \underbrace{\varphi a + \psi c}\,\overbrace{\varphi b + \psi d}^{*} = \underbrace{\varphi a + \psi c}\,\underbrace{\overset{*}{\varphi}\bar{b} + \overset{*}{\psi}\bar{d}} = \varphi\overset{*}{\varphi}\,a\,\bar{b} + \varphi\overset{*}{\psi}\,a\,\bar{d} + \psi\overset{*}{\varphi}\,c\,\bar{b} + \psi\overset{*}{\psi}\,c\,\bar{d}$$

$$= a\,\bar{b}\,\underbrace{\varphi\overset{*}{\varphi} - \psi\overset{*}{\psi}} + \varphi\overset{*}{\psi}\,a\,\bar{d} + \psi\overset{*}{\varphi}\,c\,\bar{b}$$

$$\dim_{\mathbb{R}} = 2\,(2n-1) - 2 = 4n - 4$$

$$n = 2r + \varepsilon$$

$$\dim_{\mathbb{R}} S_1 = 1 + 4\,(n-2) = 4n - 7$$

$$\dim_{\mathbb{C}} Z_1 = 2n - 3$$

$$Z_1 = \mathrm{SL}_2^{\mathbb{C}} \neg\, \mathbb{C}_{n|n}$$

$$\begin{array}{ccc} \mathbb{C}_{n|n} & \xrightarrow{\quad Q \quad} & Z_1 \\ \cup & & \cup \\ \mathbb{C}_{n|n}^{=} & \xrightarrow[\quad Q \quad]{} & S_1 \end{array}$$

$$\zeta = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \in \mathbb{C}_{n|n} \xrightarrow[U_n^{\mathbb{C}}\text{-inv}]{Q} Z_1 \ni \overset{t}{\zeta} J\zeta = \begin{bmatrix} \overset{t}{\xi} & \overset{t}{\eta} \end{bmatrix} \left[\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right] \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \overset{t}{\xi}\eta - \overset{t}{\eta}\xi$$

$$\det \zeta\overset{*}{\zeta} = \det \begin{bmatrix} \xi \\ \eta \end{bmatrix} \begin{bmatrix} \overset{*}{\xi} & \overset{*}{\eta} \end{bmatrix} = \underbrace{\xi\overset{*}{\xi}}\,\underbrace{\eta\overset{*}{\eta}} - \underbrace{\xi\overset{*}{\eta}}\,\underbrace{\eta\overset{*}{\xi}}$$

$$\underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi}\ \overbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi}^{*}\ \underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi} = \underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi}\ \det\begin{bmatrix}\xi\\ \eta\end{bmatrix}\begin{bmatrix}\overset{*}{\xi} & \overset{*}{\eta}\end{bmatrix}$$

$$\underbrace{\overset{t}{z}J\zeta}\ \overbrace{\overset{t}{z}J\zeta}^{*}\ \underbrace{\overset{t}{z}J\zeta} = \underbrace{\overset{t}{z}J\zeta}\ \det\zeta\overset{*}{\zeta}$$

$$\text{LHS} = \underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi}\,\underbrace{\overset{*}{\eta}\bar{\xi} - \overset{*}{\xi}\bar{\eta}}\,\underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi} = \overbrace{\underbrace{\eta\overset{*}{\eta}}\overset{t}{\xi}\bar{\xi} - \underbrace{\eta\overset{*}{\xi}}\overset{t}{\xi}\bar{\eta} - \underbrace{\xi\overset{*}{\eta}}\overset{t}{\eta}\bar{\xi} + \underbrace{\xi\overset{*}{\xi}}\overset{t}{\eta}\bar{\eta}}\,\underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi}$$

$$= \underbrace{\eta\overset{*}{\eta}}\,\overset{t}{\xi}\,\bar{\xi}\,\underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi} - \underbrace{\eta\overset{*}{\xi}}\,\overset{t}{\xi}\,\bar{\eta}\,\underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi} - \underbrace{\xi\overset{*}{\eta}}\,\overset{t}{\eta}\,\bar{\xi}\,\underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi} + \underbrace{\xi\overset{*}{\xi}}\,\overset{t}{\eta}\,\bar{\eta}\,\underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi}$$

$$= \underbrace{\eta\overset{*}{\eta}}\,\overbrace{\underbrace{\bar{\xi}\overset{t}{\xi}}\overset{t}{\xi}\eta - \underbrace{\bar{\xi}\overset{t}{\eta}}\overset{t}{\xi}\xi} - \underbrace{\eta\overset{*}{\xi}}\,\overbrace{\underbrace{\bar{\eta}\overset{t}{\xi}}\overset{t}{\xi}\eta - \underbrace{\bar{\eta}\overset{t}{\eta}}\overset{t}{\xi}\xi} - \underbrace{\xi\overset{*}{\eta}}\,\overbrace{\underbrace{\bar{\xi}\overset{t}{\xi}}\overset{t}{\eta}\eta - \underbrace{\bar{\xi}\overset{t}{\eta}}\overset{t}{\eta}\xi} + \underbrace{\xi\overset{*}{\xi}}\,\overbrace{\underbrace{\bar{\eta}\overset{t}{\xi}}\overset{t}{\eta}\eta - \underbrace{\bar{\eta}\overset{t}{\eta}}\overset{t}{\eta}\xi}$$

$$= \overbrace{\underbrace{\eta\overset{*}{\eta}}\underbrace{\bar{\xi}\overset{t}{\xi}} - \underbrace{\eta\overset{*}{\xi}}\underbrace{\bar{\eta}\overset{t}{\xi}}}\ \overset{t}{\xi}\,\eta + \overbrace{\underbrace{\eta\overset{*}{\xi}}\underbrace{\bar{\eta}\overset{t}{\eta}} - \underbrace{\eta\overset{*}{\eta}}\underbrace{\bar{\xi}\overset{t}{\eta}}}^{=0}\ \overset{t}{\xi}\,\xi + \overbrace{\underbrace{\xi\overset{*}{\xi}}\underbrace{\bar{\eta}\overset{t}{\xi}} - \underbrace{\xi\overset{*}{\eta}}\underbrace{\bar{\xi}\overset{t}{\xi}}}^{=0}\ \overset{t}{\eta}\,\eta + \overbrace{\underbrace{\xi\overset{*}{\eta}}\underbrace{\bar{\xi}\overset{t}{\eta}} - \underbrace{\xi\overset{*}{\xi}}\underbrace{\bar{\eta}\overset{t}{\eta}}}\ \overset{t}{\eta}\,\xi$$

$$= \overbrace{\underbrace{\eta\overset{*}{\eta}}\underbrace{\bar{\xi}\overset{t}{\xi}} - \underbrace{\eta\overset{*}{\xi}}\underbrace{\bar{\eta}\overset{t}{\xi}}}\ \underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi} = \text{RHS}$$

$$k \in U_n^{\mathbb{C}}:\quad \overset{t}{k}\,\underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi}\,k = \underbrace{\overset{t}{k}\overset{t}{\xi}}\,\underbrace{\eta k} - \underbrace{\overset{t}{k}\overset{t}{\eta}}\,\underbrace{\xi k} = \overbrace{\xi k}^{t}\,\eta k - \overbrace{\eta k}^{t}\,\xi k = \overbrace{\underbrace{\xi{:}\eta}\,k}^{\sharp}$$

$$\overset{t}{\xi}\,\eta - \overset{t}{\eta}\,\xi \in S_1 \Leftrightarrow \det\begin{bmatrix}\xi\\ \eta\end{bmatrix}\begin{bmatrix}\overset{*}{\xi} & \overset{*}{\eta}\end{bmatrix} = 1$$

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array}\ \begin{bmatrix}\xi\\ \eta\end{bmatrix} = \begin{bmatrix}a\xi + b\eta\\ c\xi + d\eta\end{bmatrix}$$

$$\underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi} \mathbin{\blacktriangledown} \underbrace{\overset{t}{\sigma}\tau - \overset{t}{\tau}\sigma} = 2\det\begin{bmatrix}\xi\\ \eta\end{bmatrix}\begin{bmatrix}\overset{*}{\sigma} & \overset{*}{\tau}\end{bmatrix}$$

$$\text{LHS} = \operatorname{tr}\ \underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi}\ \overbrace{\overset{t}{\sigma}\tau - \overset{t}{\tau}\sigma}^{*} = \operatorname{tr}\ \underbrace{\overset{t}{\xi}\eta - \overset{t}{\eta}\xi}\,\underbrace{\overset{*}{\tau}\bar{\sigma} - \overset{*}{\sigma}\bar{\tau}} = \operatorname{tr}\ \underbrace{\overset{t}{\xi}\eta}\,\underbrace{\overset{*}{\tau}\bar{\sigma}} - \operatorname{tr}\ \underbrace{\overset{t}{\xi}\eta}\,\underbrace{\overset{*}{\sigma}\bar{\tau}} - \operatorname{tr}\ \underbrace{\overset{t}{\eta}\xi}\,\underbrace{\overset{*}{\tau}\bar{\sigma}} + \operatorname{tr}\ \underbrace{\overset{t}{\eta}\xi}\,\underbrace{\overset{*}{\sigma}\bar{\tau}}$$

$$= \underbrace{\eta\overset{*}{\tau}}\,\underbrace{\overset{t}{\xi}\bar{\sigma}} - \underbrace{\eta\overset{*}{\sigma}}\,\underbrace{\overset{t}{\xi}\bar{\tau}} - \underbrace{\xi\overset{*}{\tau}}\,\underbrace{\overset{t}{\eta}\bar{\sigma}} + \underbrace{\xi\overset{*}{\sigma}}\,\underbrace{\overset{t}{\eta}\bar{\tau}} = 2\,\overbrace{\underbrace{\xi\overset{*}{\sigma}}\,\underbrace{\eta\overset{*}{\tau}} - \underbrace{\eta\overset{*}{\sigma}}\,\underbrace{\xi\overset{*}{\tau}}} = \text{RHS}$$