

$$2 \frac{\partial \mathcal{V}^j}{\partial z^i} = \frac{\partial \mathcal{V}^j}{\partial x^i} + \frac{1}{i} \frac{\partial \mathcal{V}^j}{\partial y^i} = \underbrace{\frac{\partial u^j}{\partial x^i} + i \frac{\partial v^j}{\partial x^i}}_{\partial \mathcal{V}^j / \partial z^i} + \frac{1}{i} \underbrace{\frac{\partial u^j}{\partial y^i} + i \frac{\partial v^j}{\partial y^i}}_{\partial \mathcal{V}^j / \partial z^i} = \overline{\frac{\partial u^j}{\partial x^i} + \frac{\partial v^j}{\partial y^i}} + i \overline{\frac{\partial v^j}{\partial x^i} - \frac{\partial u^j}{\partial y^i}}$$

$$2 \frac{\partial \mathcal{V}^j}{\partial \bar{z}^i} = \frac{\partial \mathcal{V}^j}{\partial x^i} - \frac{1}{i} \frac{\partial \mathcal{V}^j}{\partial y^i} = \underbrace{\frac{\partial u^j}{\partial x^i} + i \frac{\partial v^j}{\partial x^i}}_{\partial \mathcal{V}^j / \partial \bar{z}^i} - \frac{1}{i} \underbrace{\frac{\partial u^j}{\partial y^i} + i \frac{\partial v^j}{\partial y^i}}_{\partial \mathcal{V}^j / \partial \bar{z}^i} = \overline{\frac{\partial u^j}{\partial x^i} - \frac{\partial v^j}{\partial y^i}} + i \overline{\frac{\partial v^j}{\partial x^i} + \frac{\partial u^j}{\partial y^i}}$$

$$\frac{\partial (\mathcal{V} : \bar{\mathcal{V}})}{\partial (z : \bar{z})} := \frac{\partial \mathcal{V}^j / \partial z^i}{\partial \mathcal{V}^j / \partial \bar{z}^i} \Big| \frac{\partial \bar{\mathcal{V}}^j / \partial z^i}{\partial \bar{\mathcal{V}}^j / \partial \bar{z}^i}$$

$$\frac{\partial (u : v)}{\partial (x : y)} := \frac{\partial u^j / \partial x^i}{\partial u^j / \partial y^i} \Big| \frac{\partial v^j / \partial x^i}{\partial v^j / \partial y^i}$$

$$\frac{\partial (\mathcal{V} : \bar{\mathcal{V}})}{\partial (z : \bar{z})} = \frac{I_n}{i I_n} \Big|^{-1}_{-i I_n} \frac{\partial (u : v)}{\partial (x : y)} \frac{I_m}{i I_m} \Big|^{-1}_{-i I_m}$$

$$\begin{aligned} \frac{\partial \mathcal{V}^j / \partial z^i}{\partial \mathcal{V}^j / \partial \bar{z}^i} \Big| \frac{\partial \bar{\mathcal{V}}^j / \partial z^i}{\partial \bar{\mathcal{V}}^j / \partial \bar{z}^i} &= \frac{I_n}{i I_n} \Big|^{-1}_{-i I_n} \frac{\partial u^j / \partial x^i}{\partial u^j / \partial y^i} \Big| \frac{\partial v^j / \partial x^i}{\partial v^j / \partial y^i} \frac{I_m}{i I_m} \Big|^{-1}_{-i I_m} \\ &= \frac{I_n}{I_n} \Big|^{-1}_{i I_n} / \sqrt{2} \frac{\partial u^j / \partial x^i}{\partial u^j / \partial y^i} \Big| \frac{\partial v^j / \partial x^i}{\partial v^j / \partial y^i} \frac{I_m}{i I_m} \Big|^{-1}_{-i I_m} / \sqrt{2} \\ &\quad \text{unitary} \end{aligned}$$

$$\begin{aligned} \frac{I_n}{i I_n} \Big|^{-1}_{-i I_n} \frac{\partial \mathcal{V}^j}{\partial z^i} \Big| \frac{\partial \bar{\mathcal{V}}^j}{\partial \bar{z}^i} &= \frac{\partial \mathcal{V}^j}{\partial z^i} + \frac{\partial \mathcal{V}^j}{\partial \bar{z}^i} \Big| \frac{\partial \bar{\mathcal{V}}^j}{\partial z^i} + \frac{\partial \bar{\mathcal{V}}^j}{\partial \bar{z}^i} \Big| = \frac{\partial \mathcal{V}^j}{\partial x^i} \Big| \frac{\partial \bar{\mathcal{V}}^j}{\partial x^i} \\ &= \frac{\frac{\partial u^j}{\partial x^i} + i \frac{\partial v^j}{\partial x^i}}{\frac{\partial u^j}{\partial y^i} + i \frac{\partial v^j}{\partial y^i}} \Big| \frac{\frac{\partial u^j}{\partial x^i} - i \frac{\partial v^j}{\partial x^i}}{\frac{\partial u^j}{\partial y^i} - i \frac{\partial v^j}{\partial y^i}} = \frac{\frac{\partial u^j}{\partial x^i}}{\frac{\partial u^j}{\partial y^i}} \Big| \frac{\frac{\partial v^j}{\partial x^i}}{\frac{\partial v^j}{\partial y^i}} \Big|^{-1}_{-i I_m} \end{aligned}$$

$$\partial_z \mathcal{V} + \partial_{z^*} \mathcal{V} = \partial_x \mathcal{V} \partial_z \mathcal{V} - \partial_{z^*} \mathcal{V} = -i \partial_y \mathcal{V}$$

$${}^{x^* + iy^*} \mathcal{V}^j = {}^{x:y} u^j + i {}^{x:y} v^j : {}^{u^* + iv^*} \mathbf{h}^k = {}^{u:v} p^k + i {}^{u:v} q^k$$

$$\Rightarrow {}^{x^* + iy^*} \widehat{\mathcal{V} \mathbf{h}}^k = {}^{x:y} p^k + i {}^{x:y} q^k$$

$$\begin{aligned}\frac{\partial \widehat{\mathcal{V} \mathfrak{H}}^k}{\partial z^i} &= \frac{\partial \mathcal{V}^j}{\partial z^i} \frac{\partial \mathfrak{H}^k}{\partial w^j} + \frac{\partial \bar{\mathcal{V}}^j}{\partial z^i} \frac{\partial \mathfrak{H}^k}{\partial \bar{w}^j} \\ \frac{\partial \widehat{\mathcal{V} \mathfrak{H}}^k}{\partial \bar{z}^i} &= \frac{\partial \mathcal{V}^j}{\partial \bar{z}^i} \frac{\partial \mathfrak{H}^k}{\partial w^j} + \frac{\partial \bar{\mathcal{V}}^j}{\partial \bar{z}^i} \frac{\partial \mathfrak{H}^k}{\partial \bar{w}^j} \\ \frac{\partial \left( \widehat{\mathcal{V} \mathfrak{H}} : \widehat{\mathcal{V} \mathfrak{H}} \right)}{\partial (z : \bar{z})} &= \frac{\partial \left( \mathcal{V} : \bar{\mathcal{V}} \right)}{\partial (z : \bar{z})} \frac{\partial \left( \mathfrak{H} : \bar{\mathfrak{H}} \right)}{\partial (w : \bar{w})}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{I_n}{iI_n} \left| \begin{array}{c|c} I_n & -1 \\ \hline -iI_n & \end{array} \right. \frac{\partial (u:v)}{\partial (x:y)} \frac{I_m}{iI_m} \left| \begin{array}{c|c} I_m & -1 \\ \hline -iI_m & \end{array} \right. \frac{I_m}{iI_m} \left| \begin{array}{c|c} I_m & -1 \\ \hline -iI_m & \end{array} \right. \frac{\partial (p:q)}{\partial (u:v)} \frac{I_\ell}{iI_\ell} \left| \begin{array}{c|c} I_\ell & -1 \\ \hline -iI_\ell & \end{array} \right. \\ &= \frac{I_n}{iI_n} \left| \begin{array}{c|c} I_n & -1 \\ \hline -iI_n & \end{array} \right. \frac{\partial (u:v)}{\partial (x:y)} \frac{\partial (p:q)}{\partial (u:v)} \frac{I_\ell}{iI_\ell} \left| \begin{array}{c|c} I_\ell & -1 \\ \hline -iI_\ell & \end{array} \right. = \frac{I_n}{iI_n} \left| \begin{array}{c|c} I_n & -1 \\ \hline -iI_n & \end{array} \right. \frac{\partial (p:q)}{\partial (x:y)} \frac{I_\ell}{iI_\ell} \left| \begin{array}{c|c} I_\ell & -1 \\ \hline -iI_\ell & \end{array} \right. = \text{LHS}\end{aligned}$$