$$m \in \mathbb{Z}: \quad L_m = \frac{1}{2} \sum_{n}^{\pm 0|m} \alpha^{\mu}_{m-n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n} + \sum_{n}^{m\pm} \overset{*}{\alpha}{}^{\mu}_{n-m} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n}$$

$$m > 0: \quad 2\,L_m = \sum_{n}^{\mathbb{Z}} \alpha^{\mu}_{m-n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n} = \sum_{k<0} \alpha^{\mu}_{m-k} \underset{\mu\nu}{\eta} \alpha^{\nu}_{k} + \sum_{n}^{0|m} \alpha^{\mu}_{m-n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n} + \sum_{n>m} \alpha^{\mu}_{m-n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n}$$

$$= \sum_{n}^{0|m} \alpha^{\mu}_{m-n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n} + \sum_{n>m} \left(\alpha^{\mu}_{n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{\underbrace{m-n}_{=k}} + \alpha^{\mu}_{m-n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n} \right)$$

$$= \sum_{n}^{0|m} \alpha^{\mu}_{m-n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n} + \sum_{n>m} \left(\alpha^{\mu}_{n} \underset{\mu\nu}{\eta} \overset{*}{\alpha}{}^{\nu}_{n-m} + \overset{*}{\alpha}{}^{\mu}_{n-m} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n} \right) = \sum_{n}^{0|m} \alpha^{\mu}_{m-n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n} + 2 \sum_{n>m} \overset{*}{\alpha}{}^{\mu}_{n-m} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n}$$

$$m < 0: \quad 2\,L_m = 2\,\underset{\sim}{L}_{-m} = \sum_{n}^{\mathbb{Z}} \underset{\sim}{\alpha}{}^{\mu}_{n-m} \underset{\mu\nu}{\eta} \underset{\sim}{\alpha}{}^{\nu}_{-n} = \sum_{n<m} \underset{\sim}{\alpha}{}^{\mu}_{n-m} \underset{\mu\nu}{\eta} \underset{\sim}{\alpha}{}^{\nu}_{-n} + \sum_{n}^{m|0} \underset{\sim}{\alpha}{}^{\mu}_{n-m} \underset{\mu\nu}{\eta} \underset{\sim}{\alpha}{}^{\nu}_{-n} + \sum_{k>0} \underset{\sim}{\alpha}{}^{\mu}_{k-m} \underset{\mu\nu}{\eta} \underset{\sim}{\alpha}{}^{\nu}_{-k}$$

$$= \sum_{n}^{m|0} \underset{\sim}{\alpha}{}^{\mu}_{n-m} \underset{\mu\nu}{\eta} \underset{\sim}{\alpha}{}^{\nu}_{-n} + \sum_{n<m} \left(\underset{\sim}{\alpha}{}^{\mu}_{-n} \underset{\mu\nu}{\eta} \underset{\sim}{\alpha}{}^{\nu}_{\underbrace{n-m}_{=-k}} + \underset{\sim}{\alpha}{}^{\mu}_{n-m} \underset{\mu\nu}{\eta} \underset{\sim}{\alpha}{}^{\nu}_{-n} \right)$$

$$= \sum_{n}^{m|0} \underset{\sim}{\alpha}{}^{\mu}_{n-m} \underset{\mu\nu}{\eta} \underset{\sim}{\alpha}{}^{\nu}_{-n} + \sum_{n<m} \left(\underset{\sim}{\alpha}{}^{\mu}_{-n} \underset{\mu\nu}{\eta} \underset{\sim}{\overset{*}{\alpha}}{}^{\nu}_{m-n} + \underset{\sim}{\overset{*}{\alpha}}{}^{\mu}_{m-n} \underset{\mu\nu}{\eta} \underset{\sim}{\alpha}{}^{\nu}_{-n} \right)$$

$$= \sum_{n}^{m|0} \alpha^{\mu}_{m-n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n} + \sum_{n<m} \left(\alpha^{\mu}_{n} \underset{\mu\nu}{\eta} \overset{*}{\alpha}{}^{\nu}_{n-m} + \overset{*}{\alpha}{}^{\mu}_{n-m} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n} \right) = \sum_{n}^{m|0} \alpha^{\mu}_{m-n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n} + 2 \sum_{n<m} \overset{*}{\alpha}{}^{\mu}_{n-m} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n}$$

$$2\,L_0 = \alpha^{\mu}_{0} \underset{\mu\nu}{\eta} \alpha^{\nu}_{0} + 2 \sum_{n>0} \overset{*}{\alpha}{}^{\mu}_{n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n} = \alpha^{\mu}_{0} \underset{\mu\nu}{\eta} \alpha^{\nu}_{0} + 2 \sum_{n<0} \overset{*}{\alpha}{}^{\mu}_{n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n} = \alpha^{\mu}_{0} \underset{\mu\nu}{\eta} \alpha^{\nu}_{0} + \sum_{n \neq 0} \overset{*}{\alpha}{}^{\mu}_{n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n}$$

$$\sum_{n>0} \overset{*}{\alpha}{}^{\mu}_{n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n} = \sum_{n<0} \overset{*}{\alpha}{}^{\mu}_{n} \underset{\mu\nu}{\eta} \alpha^{\nu}_{n}$$

$$M \subset \mathbb{Z}: \quad \underset{M}{\imath} = \bigotimes_{M}^{m} \underset{m}{\imath}$$

$$L_0 = \frac{1}{2} \overset{0}{\imath} \,⧗\, \underbrace{P^{\mu} \eta_{\mu\nu} P^{\nu}} \,⧗\, \underset{0}{\imath} + \sum_{n>0}^{n} \overset{n}{\imath} \,⧗\, \underbrace{\overset{*}{\partial}{}^{\mu} \eta_{\mu\nu} \partial^{\nu}} \,⧗\, \underset{n}{\imath}$$

$$\bar{L}_0 = \frac{1}{2} \overset{0}{\imath} \bowtie \underbrace{P^\mu \eta_{\mu\nu} P^\nu} \bowtie \underset{0}{\imath} + \sum_{n>0}^{n} \overset{-n}{\imath} \bowtie \underbrace{\overset{*}{\partial}^\mu \eta_{\mu\nu} \partial^\nu} \bowtie \underset{-n}{\imath}$$

$$L_0 - \bar{L}_0 = \sum_{n>0}^{n} \left(\overset{n}{\imath} \bowtie \underbrace{\overset{*}{\partial}^\mu \eta_{\mu\nu} \partial^\nu} \bowtie \underset{n}{\imath} - \overset{-n}{\imath} \bowtie \underbrace{\overset{*}{\partial}^\mu \eta_{\mu\nu} \partial^\nu} \bowtie \underset{-n}{\imath} \right)$$

$$= \sum_{n>0}^{n} \left(\overset{-n}{\imath} \bowtie \underset{-n}{\overset{-n}{\imath}} \bowtie \underset{-n}{\overset{n}{\imath}} \bowtie \underbrace{\overset{*}{\partial}^\mu \eta_{\mu\nu} \partial^\nu} \bowtie \underset{n}{\imath} - \overset{-n}{\imath} \bowtie \underbrace{\overset{*}{\partial}^\mu \eta_{\mu\nu} \partial^\nu} \bowtie \underset{-n}{\overset{n}{\imath}} \bowtie \underset{n}{\overset{n}{\imath}} \bowtie \underset{n}{\imath} \right)$$

$$z^{L_0} = \overset{<}{\imath}{}^0 \bowtie \sqrt{z}^{P^\mu \eta_{\mu\nu} P^\nu} \bowtie \underset{n}{\overset{>0}{\bigotimes}} z^{n \overset{*}{\partial}^\mu \eta_{\mu\nu} \partial^\nu}$$

$$\bar{z}^{\bar{L}_0} = \underset{n}{\overset{<0}{\bigotimes}} \bar{z}^{-n \overset{*}{\partial}^\mu \eta_{\mu\nu} \partial^\nu} \bowtie \sqrt{\bar{z}}^{P^\mu \eta_{\mu\nu} P^\nu} \bowtie \overset{>}{\imath}{}^0$$

$$z^{L_0} \, \bar{z}^{\bar{L}_0} = \underset{n}{\overset{<0}{\bigotimes}} \bar{z}^{-n \overset{*}{\partial}^\mu \eta_{\mu\nu} \partial^\nu} \bowtie \overline{z}^{P^\mu \eta_{\mu\nu} P^\nu} \bowtie \underset{n}{\overset{>0}{\bigotimes}} z^{n \overset{*}{\partial}^\mu \eta_{\mu\nu} \partial^\nu}$$

$$\underline{L}_n = \frac{1}{2} \sum_{m}^{\mathbb{Z}} \underline{X}^\mu_{n-m} \, \eta_{\mu\nu} \, \underline{X}^\nu_m$$

$$\bar{L}_n = \frac{1}{2} \sum_{m}^{\mathbb{Z}} \bar{X}^\mu_{n-m} \, \eta_{\mu\nu} \, \bar{X}^\nu_m$$

$$\underline{L}_m \times \underline{L}_n = (m-n) \, \underline{L}_{m+n} + \frac{d}{12} \left(m^3 - m \right) \delta_{m+n}$$

$$\bar{L}_m \times \bar{L}_n = (m-n) \, \bar{L}_{m+n} + \frac{d}{12} \left(m^3 - m \right) \delta_{m+n}$$

$${}^2\mathbb{C}_2^{\mathsf{C}} \ni \begin{array}{c|c} a & b \\ \hline c & d \end{array} \ \text{unbroken symm}$$